A remark on the theorem of Simmons.

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The theorem of SIMMONS in question [1] can be formulated as follows: *If n and h are positive integers, and if we put for* $0 \leqq p \leqq 1$, $q = 1-p$

$$f_{n,h}(p) = \sum_{r=0}^{h-1} \binom{n}{r} p^r q^{n-r} - \sum_{r=h+1}^{n} \binom{n}{r} p^r q^{n-r}, \tag{1}$$

then we have

$$f_{n,h}\left(\frac{h}{n}\right) > 0 \quad \textit{if} \quad p = \frac{h}{n} < \frac{1}{2}. \tag{2}$$

An ingenious and simple proof of this theorem has been given by E. FELDHEIM ([2] and [3]; the proof is reproduced also in the text book [4], p. 171—172).

The generalization of the inequality of SIMMONS, for the case when np is not an integer, has been considered in this journal by CH. JORDAN[1]) [5] and recently by I. B. HAÁZ [6].

HAÁZ tried to generalize the inequality of Simmons in that he has shown that for fixed values of n and h

$$f_{n,h}(p) > 0 \quad \text{if} \quad 1 \leqq h \leqq \frac{n+1}{2} \quad \text{and} \quad \frac{h-1}{n} \leqq p < \min\left(\frac{1}{2}, \frac{h}{n}\right). \tag{3}$$

The aim of this note is to show that the apparent generalization given by HAÁZ is really a consequence of the original inequality of SIMMONS if $\frac{h}{n} < \frac{1}{2}$, and for the remaining cases $n = 2h$ resp. $n = 2h-1$ it follows

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[1]) One of JORDAN's results expressed by the notations of the present paper runs as follows:

$$f_{n,h}(p) > \binom{n}{h} p^h q^{n-h} \quad \text{if} \quad p < \frac{1}{2} \quad \text{and} \quad \frac{h-1}{n} \leqq p \leqq \frac{h-\frac{1}{2}}{n};$$

further for $\frac{h}{n+1} \leqq p \leqq \frac{h}{n}$ and $p < \frac{1}{2}$ the reversed inequality is valid.

from the evident relations

(4) $$f_{2h,h}\left(\frac{1}{2}\right)=0 \quad\text{and}\quad f_{2h-1,h}\left(\frac{1}{2}\right)>0.$$

To prove our assertions we need nothing else than the well known formula

(5) $$\sum_{r=0}^{s}\binom{n}{r}p^r q^{n-r}=(n-s)\binom{n}{s}\int_p^1 t^s(1-t)^{n-s-1}\,dt$$

(see e. g. [2] p. 110 or [4] p. 133). It follows from (1) and (5) that

(6) $$f_{n,h}(p)=\binom{n}{h}\int_p^1 (h(1-t)+(n-h)t)t^{h-1}(1-t)^{n-h-1}\,dt-1.$$

It can be seen from (6) without any calculations that $f_{n,h}(p)$ is a *decreasing* function of p $(0\leqq p\leqq 1)$. Thus it follows from (2) that

(7) $$f_{n,h}(p)>0 \quad\text{for}\quad p\leqq\frac{h}{n} \quad\text{if}\quad \frac{h}{n}<\frac{1}{2},$$

further it follows from (4) resp. (5) that

(8) $$f_{2h,h}(p)>0 \quad\text{and}\quad f_{2h-1,h}(p)>0 \quad\text{for}\quad p<\frac{1}{2}.$$

Evidently (7) and (8) contain (3) which is thus shown to be a consequence of (2) resp. (4).

We have at the same time shown that for $\frac{h}{n}<\frac{1}{2}$ (3) can be replaced by the stronger inequality

(3') $$f_{n,h}(p)>f_{n,h}\left(\frac{h}{n}\right) \quad\text{for}\quad p<\frac{h}{n}<\frac{1}{2}.$$

References.

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