Some remarks on set theory. V.

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Let E be an arbitrary set of power $\mathfrak{m}$ and suppose that with every element x of E there is associated a *non empty* subset of E. Two distinct elements of E, x and y, are called *independent*, if $x \notin f(y)$ and $y \notin f(x)$. A subset F of E is called *free* if F has only one element or if F has at least two elements and any two of their distinct elements are independent. We say that the subset Γ of E has the property $T(\mathfrak{q}, \mathfrak{p})$, where $\mathfrak{q}$ and $\mathfrak{p}$ are two cardinal numbers such that $\mathfrak{q} \leqq \mathfrak{m}$, $\mathfrak{p} \leqq \mathfrak{m}$, if

$$\overline{\overline{\Gamma}} = \overline{\overline{\bigcup_{x \in \Gamma} f(x)}} = \mathfrak{q} \quad \text{and} \quad \overline{\overline{\bigcup_{\substack{x, y \in \Gamma \\ x \neq y}} (f(x) \cap f(y))}} < \mathfrak{p}.$$

A subset C of E is called closed, if for every element x of C, $f(x) \subseteqq C$.

We assume that $\overline{\overline{\bigcup_{x \in E} f(x)}} = \mathfrak{m}$ and one of the following conditions hold for the sets $f(x)$:

(A) There is a cardinal number $\mathfrak{n} < \mathfrak{m}$ such that, for every $x \in E$, $\overline{\overline{f(x)}} < \mathfrak{n}$.

(B) There is a cardinal number $\mathfrak{n} < \mathfrak{m}$ such that, for every pair of distinct elements x and y of E, $\overline{\overline{f(x) \cap f(y)}} < \mathfrak{n}$.

(C) If $x, y \in E$ and $x \neq y$, then $f(x) \not\subseteq f(y)$ and $f(y) \not\subseteq f(x)$.

(D) For every $x \in E$, the power of the set of elements y, for which $f(x) \cap f(y) \neq 0$, is smaller than $\mathfrak{m}$.

We deal in this paper first with the following two questions.

1. Whether or not these conditions imply the existence of subsets with the property $T(\mathfrak{q}, \mathfrak{p})$ of E.

2. Whether or not these conditions imply the existence of free sets of certain cardinalities.

If the condition (A) is satisfied, then both questions are investigated (in some cases by supposing the generalised continuum hypothesis) (see [1],

[2], [4]). For instance E has a free subset of power $\mathfrak{m}$ and a subset with the property $T(\mathfrak{m}, \mathfrak{m})$ (if $\mathfrak{m}$ is the sum of $\mathfrak{n}$ cardinal numbers smaller than $\mathfrak{m}$, the generalised continuum hypothesis is assumed).

In the sections I, II, III a number of results is given with respect to the questions 1 und 2, if one of the conditions (B), (C), (D) is satisfied.

Our most interesting unsolved problem is the following one: Let $\mathfrak{m}$ be any cardinal, $\overline{\overline{f(x)}} < \mathfrak{m}, \overline{\overline{f(x) \cap f(y)}} < \mathfrak{n} < \mathfrak{m}$. Does there then exist a free subset of power $\mathfrak{m}$? We can only prove (without the generalised continuum hypothesis) that there always exists an infinite free subset (theorem 8). Perhaps the most striking formulation of our unsolved problem is the case $\mathfrak{m} = \aleph_1$, $\mathfrak{n} = \aleph_0$. If $\mathfrak{m} = \aleph_1, \mathfrak{n} = k < \aleph_0$ we can prove (without the continuum hypothesis) the existence of a free subset of power $\aleph_1$ (theorem 6).

Finally we deal with the following two questions:

a) If the condition (A) is satisfied, does there exist a closed proper subset of E, of power $\mathfrak{m}$?

b) If the condition (A) is satisfied, do there exist two almost disjoint closed subsets of E, of power $\mathfrak{m}$?

These questions are completely solved in section IV.

Notation and definitions. Throughout this paper, the symbols $\overline{\overline{F}}$ and $\overline{\beta}$ denote the cardinal number of the set F and the ordinal number β, respectively. For any subset Γ of E let

$$\sum\nolimits_\Gamma^f = \bigcup_{x \in \Gamma} f(x) \quad \text{and} \quad \Pi_\Gamma^f = \bigcup_{\substack{x, y \in \Gamma \\ x \neq y}} (f(x) \cap f(y)).$$

For any $x \in E$, let $f^{-1}(x) = \{y : x \in f(y)\}$. For any cardinal number $\mathfrak{r}$ we denote by $\varphi_\mathfrak{r}$ the initial number of $\mathfrak{r}$, by $\mathfrak{r}^*$ the smallest cardinal number for which $\mathfrak{r}$ is the sum of $\mathfrak{r}^*$ cardinal numbers each of which is smaller than $\mathfrak{r}$, by $\mathfrak{r}^-$ the immediate predecessor of $\mathfrak{r}$ provided that such a predecessor exists. We say that $\mathfrak{r}$ is singular if $\mathfrak{r}$ can be represented in the form $\mathfrak{r} = \sum_{\nu \in F} \mathfrak{r}_\nu$, where $\overline{\overline{F}} < \mathfrak{r}, \mathfrak{r}_\nu < \mathfrak{r}$, and regular if no such representation exists.

We say that the sets F_1 and F_2 are almost disjoint if $\overline{\overline{F_1 \cap F_2}} < \min(\overline{\overline{F_1}}, \overline{\overline{F_2}})$.

I.

We assume in this section that the condition (B) holds on the sets $f(x)$ and we give some results concerning to the questions 1 and 2.

We begin by proving two lemmas.

Lemma 1. *Let A be a set of power $\mathfrak{m}, \mathfrak{m} \geqq \aleph_0$. There is a sequence $\{A_\xi\}_{\xi < \varphi_\mathfrak{m}}$ of the type $\varphi_\mathfrak{m}$, of subsets of A such that*

1. $A = \bigcup_{\xi<\varphi_{\mathfrak{m}}} A_\xi$,

2. $\overline{\overline{A_\xi}} = \mathfrak{m}$ *for every* $\xi < \varphi_{\mathfrak{m}}$,

3. $\overline{\overline{A_\nu \cap A_\mu}} = 1$ *for every* $\nu, \mu, \nu < \varphi_{\mathfrak{m}}, \mu < \varphi_{\mathfrak{m}}$ *and* $\nu \neq \mu$,

4. $\overline{\overline{A_\alpha - \bigcup_{\xi<\alpha} A_\xi}} = \mathfrak{m}$ *for every* $\alpha < \varphi_{\mathfrak{m}}$,

5. *if* $x \in A$, *then there are at most two ordinal numbers* ν *and* μ, *such that* $x \in A_\nu$ *and* $x \in A_\mu$,

6. *if* $\overline{\overline{\bigcup_{\substack{\mu,\nu\in\Gamma \\ \mu\neq\nu}} (A_\nu \cap A_\mu)}} < \mathfrak{m}$, *then* $\overline{\overline{\Gamma}} < \mathfrak{m}$.[1]

Proof. Let $\{B_\xi\}_{\xi<\varphi_{\mathfrak{m}}}$ be a sequence of subsets of A such that $\overline{\overline{B_\xi}} = \mathfrak{m}$, $A = \bigcup_{\xi<\varphi_{\mathfrak{m}}} B_\xi$ and $B_\nu \cap B_\mu = 0$ for every μ, ν with $\nu < \varphi_{\mathfrak{m}}, \mu < \varphi_{\mathfrak{m}}$ and $\nu \neq \mu$. We define the sequence $\{A_\xi\}_{\xi<\varphi_{\mathfrak{m}}}$ by transfinite induction as follows: Let $A_0 = B_0$. Let now β be an ordinal number, $0 < \beta < \varphi_{\mathfrak{m}}$, and suppose that all sets A_ξ, where $0 \leqq \xi < \beta$, have been already defined such that the conditions 2, 3, 4 hold for $\xi < \beta$; $\mu, \nu < \beta$; and $\alpha < \beta$. Let $A_\beta = B_\beta \cup \{x_\xi\}_{\xi<\beta}$, where $x_\xi \in A_\xi - \bigcup_{\zeta<\xi} A_\zeta - \{x_\zeta\}_{\zeta<\xi}$. It is easy to see that the conditions 1—6 are satisfied.

Lemma 2. *If* A *is a set of power* $\mathfrak{m}$, $\mathfrak{m} > \aleph_0$, $\mathfrak{m}$ *has immediate predecessor and* $\mathfrak{m}^-$ *is regular, then there is a sequence* $\{A_\xi\}_{\xi<\varphi_{\mathfrak{m}}}$ *of the type* $\varphi_{\mathfrak{m}}$ *of subsets of* A *such that*

1. $A = \bigcup_{\xi<\varphi_{\mathfrak{m}}} A_\xi$,

2. $\overline{\overline{A_\xi}} = \mathfrak{m}^-$ *for every* $\xi < \varphi_{\mathfrak{m}}$,

3. $\overline{\overline{A_\nu \cap A_\mu}} < \mathfrak{m}^-$ *for every distinct* $\nu, \mu, \nu < \varphi_{\mathfrak{m}}$ *and* $\mu < \varphi_{\mathfrak{m}}$,

4. $\overline{\overline{A_\alpha - \bigcup_{\xi<\alpha} A_\xi}} = \mathfrak{m}^-$ *for every* $\alpha < \varphi_{\mathfrak{m}}$,

5. *if* $\overline{\overline{\bigcup_{\substack{\nu,\mu\in\Gamma \\ \nu\neq\mu}} (A_\nu \cap A_\mu)}} < \mathfrak{m}$, *then* $\overline{\overline{\Gamma}} < \mathfrak{m}$.

Proof. Let $\{B_\xi\}_{\xi<\varphi_{\mathfrak{m}}}$ be a sequence of subsets of A, such that $\overline{\overline{B_\xi}} = \mathfrak{m}^-$, $A = \bigcup_{\xi<\varphi_{\mathfrak{m}}} B_\xi$ and $B_\nu \cap B_\mu = 0$ for every distinct $\nu, \mu < \varphi_{\mathfrak{m}}$. We define the sequence $\{A_\xi\}_{\xi<\varphi_{\mathfrak{m}}}$ by transfinite induction in the following manner: Let $A_0 = B_0$. Let now β be an ordinal number, $0 < \beta < \varphi_{\mathfrak{m}}$, and suppose that all sets A_ξ, where $0 \leqq \xi < \beta$, have been already defined such that 2, 3, and 4 are satisfied for $\xi < \beta$; $\mu, \nu < \beta$; and $\alpha < \beta$.

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[1]) It is clear that 6 follows from 3 and 5.

If $\beta \leqq \varphi_{\mathfrak{m}^-}$, then we define A_β in the same way as in the proof of lemma 1. If $\beta > \varphi_{\mathfrak{m}^-}$, then let $\{C_\xi^{(\beta)}\}_{\xi < \varphi_{\mathfrak{m}^-}}$ be a wellordering of the set $\{A_\xi\}_{\xi < \beta}$. For every $\nu < \beta$ there is a $\xi_\nu < \varphi_{\mathfrak{m}^-}$ such that $A_\nu = C_{\xi_\nu}^{(\beta)}$. Let $A_\beta = B_\beta \cup \{x_\nu\}_{\nu < \beta}$, where $x_\nu \in (A_\nu - \bigcup_{\xi < \nu} A_\xi) - \bigcup_{\xi < \xi_\nu} C_\xi^{(\beta)}$. It is easy to see that the conditions 1—5 are satisfied.

We shall now prove some negative results concerning the question 1.

Theorem 1. *If* $\mathfrak{m}$ *is an arbitrary infinite cardinal number,* $\mathfrak{n} = 2$, *and, for every* $x \in E, \overline{\overline{f(x)}} = \mathfrak{m}$, *then* (*B*) *does not imply the existence of a subset of E with the property* $T(\mathfrak{m}, \mathfrak{m})$.

Proof. By the lemma 1 there is a sequence $\{E_\xi\}_{\xi < \varphi_{\mathfrak{m}}}$ of subsets of E with the properties 1—6 in the lemma 1. Let $\{x_\xi\}_{\xi < \varphi_{\mathfrak{m}}}$ be any wellordering of E. Let now $f(x_\xi) = E_\xi$ for every $\xi < \varphi_{\mathfrak{m}}$.

Theorem 2. *If* $\mathfrak{m}$ *is a singular cardinal number and for every* $x \in E, f(x) < \mathfrak{m}$, *then* (B) *does not imply the existence of a subset of E with the property* $T(\mathfrak{m}, \mathfrak{m})$.

Proof. There exist cardinal numbers $\mathfrak{m}_0, \mathfrak{m}_1, \ldots, \mathfrak{m}_\xi, \ldots (\xi < \varphi_{\mathfrak{m}^*})$ such that $\mathfrak{m}_\beta > \mathfrak{m}_\alpha$ for $\beta > \alpha$ and $\mathfrak{m} = \sum_{\xi < \varphi_{\mathfrak{m}^*}} \mathfrak{m}_\xi$. Let $\{E_\xi\}_{\xi < \varphi_{\mathfrak{m}^*}}$ be a sequence of mutually disjoint subsets of E such that $E = \bigcup_{\xi < \varphi_{\mathfrak{m}^*}} E_\xi$ and $\overline{\overline{E_\xi}} = \mathfrak{m}_\xi$. By the lemma 1 there is, for every ξ, a sequence $\{E_\nu^\xi\}_{\nu < \varphi_{\mathfrak{m}_\xi}}$ with the properties 1—6 in the lemma 1. Let $\{x_\nu^\xi\}_{\nu < \varphi_{\mathfrak{m}_\xi}}$ be any wellordering of E_ξ and $f(x_\nu^\xi) = E_\nu^\xi$ for every $\xi < \varphi_{\mathfrak{m}^*}$ and $\nu < \varphi_{\mathfrak{m}_\xi}$. Obviously there is no subset of E with the property $T(\mathfrak{m}, \mathfrak{m})$.

Theorem 3. *If* $\mathfrak{m} > \aleph_0$ *and* $\mathfrak{m}$ *has regular immediate predecessor, and for every* $x \in E, f(x) = \mathfrak{m}^-$, *then* (B) *does not imply the existence of a subset of E with the property* $T(\mathfrak{m}, \mathfrak{m})$.

Proof. Using the lemma 2, the proof is similar to the proof of theorem 1.

We shall now prove a positive result concerning to question 1.

Theorem 4. *If* $\overline{\overline{f(x)}} < \mathfrak{m}$, $\mathfrak{m} = \aleph_1$ *and* $\mathfrak{n} < \aleph_0$ *or* $2^{\aleph_\beta} = \aleph_{\beta+1}$ *for every ordinal number* β, $\mathfrak{m} = \aleph_{\alpha+1}$, $\mathfrak{r} = \aleph_\alpha$ $(\alpha > 1)$ *and* $\mathfrak{n} < \mathfrak{r}^*$, *then there exists a subset of E with the property* $T(\mathfrak{m}, \mathfrak{m})$.

Proof. Suppose that the theorem is false, i. e. if M is a subset of E for which $\overline{\overline{M}} < \mathfrak{m}$, then, for every subset Γ of E for which $\prod_\Gamma^f \subseteq M$, the power of the set $\sum_\Gamma^f$ is smaller than $\mathfrak{m}$. Define the sets M_β and K_β by transfinite induction as follows. Let M_0 be a subset of E, of power less than $\mathfrak{m}$, and et $K_0=0$. Let now β be an ordinal number, $1\leqq\beta<\varphi_{\mathfrak{m}}$, and suppose that all sets M_ξ and K_ξ, where $0\leqq\xi<\beta$, have been already defined such that $\overline{\overline{M}}_\xi<\mathfrak{m}$. Let $N_\beta=\bigcup_{\xi<\beta}M_\xi$. Obviously $\overline{\overline{N}}_\beta<\mathfrak{m}$. Let K_β be a subset of E such that:

1. $f(x)\cap(E-N_\beta)\neq 0$, *if* $x\in K_\beta$,
2. $\prod^f_{K_\beta}\subseteq N_\beta$ and
3. for every $x\in E-K_\beta$ there is an element y of K_β such that $f(x)\cap f(y)$ is not a subset of N_β.

Let

$$M_\beta=\sum{}^f_{K_\beta}-N_\beta.$$

Obviously $M_\beta\neq 0$ and $\overline{\overline{M}}_\beta<\mathfrak{m}$. Let $M=\bigcup_{\xi<\varphi_\mathfrak{n}}M_\xi$. Clearly $\overline{\overline{M}}<\mathfrak{m}$ and $\overline{\overline{\bigcup_{\xi<\varphi_\mathfrak{n}}K}}<\mathfrak{m}$. Let F be the set of all sets which have one and only one common element with every M_ξ $(\xi<\varphi_\mathfrak{n})$. If $x\in E-\bigcup_{\xi<\varphi_\mathfrak{n}}K_\xi$, then for every ξ there exists an element $y\in K_\xi$ such that $f(x)\cap f(y_\xi)\neq 0$, i. e. $M_\xi\cap f(x)\neq 0$. Thus for every $x\in E-\bigcup_{\xi<\varphi_\mathfrak{n}}K_\xi$ there exists a set $g(x)\in F$ such that $g(x)\subseteq f(x)$. Since $\overline{\overline{F}}<\mathfrak{r}^{\mathfrak{r}^*}<\mathfrak{m}$, there exists a $g\in F$ and two distinct elements x and y of $E-\bigcup_{\xi<\varphi_\mathfrak{n}}K_\xi$ such that $g\subseteq f(x)$ and $g\subseteq f(y)$, which is impossible, since $\overline{\overline{f(x)\cap f(y)}}<\mathfrak{n}$.

We prove now some results concerning to the question 2.

Theorem 5. *If there is an element* $x_0\in E$ *for which* $\overline{\overline{f(x_0)}}=\mathfrak{m}$, *then there exists a free subset of E, of power* $\mathfrak{m}$.

Proof. By the condition (B), for every element $y\in f(x_0)$, $\overline{\overline{f(y)\cap f(x_0)}}<\mathfrak{n}$. Let $g(x)=f(x)\cap f(x_0)$ for $x\in f(x_0)$. By the theorem V of [2] (with $f(x_0)=S$ and $f(x)=g(x)$ $(x\in S)$) there exists a free subset of power $\mathfrak{m}$ of E with respect to $g(x)$. This subset is a free subset of E with respect to $f(x)$.

Lemma 3. *If the condition* (B) *on the sets* $f(x)$ *implies the existence of a subset of E with the property* $T(\mathfrak{m},\mathfrak{m})$, *then the same condition implies the existence of a free subset of E, of power* $\mathfrak{m}$.

Proof. Let $g(x)=\{x\}\cup f(x)$ for every $x\in E$. Clearly the sets $g(x)$ satisfy the condition (B) for every $x\in E$. By the hypothesis there exists a subset Γ of E, of power $\mathfrak{m}$, for which

$$\overline{\overline{\sum{}^g_\Gamma}}=\mathfrak{m}\quad\text{and}\quad\overline{\overline{\prod{}^g_\Gamma}}<\mathfrak{m}.$$

Put $G=\Gamma-\Pi^g_\Gamma$. Obviously $\overline{\overline{G}}=\mathfrak{m}$. G is a free set. Indeed let x and y be two distinct elements of G. Then $x\notin f(y)$, since in the opposite case $x\in g(x)\cap g(y)\subset\Pi^g_\Gamma$, which is impossible. Similarly $y\notin f(x)$.

From lemma 3, and theorem 4 we deduce

Theorem 6. *If* $\overline{\overline{f(x)}} < \mathfrak{m}$, $\mathfrak{m} = \aleph_1$ *and* $\mathfrak{n} < \aleph_0$ *or* $2^{\aleph_\beta} = \aleph_{\beta+1}$ *for every ordinal number* β, $\mathfrak{m} = \aleph_{\alpha+1}$, $\mathfrak{r} = \aleph_\alpha$ *and* $\mathfrak{n} < \mathfrak{r}^*$, *then there exists a free subset of* G, *of power* $\mathfrak{m}$.

Theorem 7. *If* $\mathfrak{m}$ *is singular,* $\overline{\overline{f(x)}} < \mathfrak{m}$ *for every* $x \in E$ *and* $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ *for every ordinal number* α, *then there exists a free subset of* E, *of power* $\mathfrak{m}$.

The proof of this theorem is analogous to the proof of the second part of the theorem V of [2], if we use theorem 6 of this paper instead of the first part of theorem V of [2].

Now we prove the following

Lemma 4. *Let* F *be an arbitrary subset of* E, *of power* $\mathfrak{m}$. *The condition* (B) *on the sets* $f(x)$ *implies the existence of an element* x *of* F *such that* $\overline{\overline{F - f^{-1}(x)}} = \mathfrak{m}$, *where* $f^{-1}(x) = \{y\colon x \in f(y)\}$.

Proof. Suppose that the lemma is false. Then there is a subset L of E, of power $\mathfrak{m}$, such that for every $x \in L$

$$\overline{\overline{F - f^{-1}(x)}} < \mathfrak{m}.$$

There is no loss of generality in assuming that $L = E$. We consider two cases. First suppose that $\mathfrak{m}$ is regular. Let N be an arbitrary subset of E, of power greater than $\mathfrak{n}$. Since $\mathfrak{m}$ is regular by the hypothesis, we have

$$\overline{\overline{\bigcup_{x \in N} (E - f^{-1}(x))}} < \mathfrak{m}.$$

Suppose now that $\mathfrak{m}$ is singular. There exist regular cardinal numbers $\mathfrak{m}_0, \mathfrak{m}_1 \ldots, \mathfrak{m}_\xi, \ldots (\xi < \varphi_{\mathfrak{m}^*})$ such that $\mathfrak{m}_\beta > \mathfrak{m}_\alpha > \max(\mathfrak{m}^*, \mathfrak{n})$ for $\beta > \alpha$ and

$$\mathfrak{m} = \sum_{\xi < \varphi_{\mathfrak{m}^*}} \mathfrak{m}_\xi .$$

Consider an arbitrary subset M of E, of power $\mathfrak{m}_0$. Let M_ξ be the set of all elements of M for which $\overline{\overline{E - f^{-1}(x)}} < \mathfrak{m}_\xi$. Obviously

$$M = \bigcup_{\xi < \varphi_{\mathfrak{m}^*}} M_\xi .$$

Since $\mathfrak{m}_0$ is regular and $\mathfrak{m}_0 > \mathfrak{m}^*$, there exists an ordinal number ξ_0 such that $\overline{\overline{M}}_{\xi_0} = \mathfrak{m}_0$. Obviously the power of the set

$$\bigcup_{x \in M_{\xi_0}} (E - f^{-1}(x))$$

is not greater than $\mathfrak{m}_0 \mathfrak{m}_{\xi_0}$ $(< \mathfrak{m})$. Let now $H = N$ if $\mathfrak{m}$ is regular and $H = M_{\xi_0}$ if $\mathfrak{m}$ is singular. Put $K = \bigcup_{x \in H} (E - f^{-1}(x))$. Clearly $\overline{\overline{E - (K \cup H)}} = \mathfrak{m}$ and by the definition

$$E - (H \cup K) \subseteq f^{-1}(x)$$

for every $x \in H$. It follows that

$$H \subseteq f(y)$$

for every $y \in E-(H \cup K)$ which is impossible, because $\overline{\overline{f(x) \cap f(y)}} < \mathfrak{n}$ for every distinct $x, y \in E$ and $\overline{\overline{H}} \geqq \mathfrak{n}$. This contradiction proves the lemma.

Without using the generalised continuum hypothesis we now prove

Theorem 8. *If* $\mathfrak{m}$ *is an arbitrary infinite cardinal number and* $f(x) < \mathfrak{m}$ *for every* $x \in E$, *then there exists a free subset of* E, *of power* $\aleph_0$.

Proof. Let x_0 be an element of E for which $\overline{\overline{E-f^{-1}(x_0)}} = \mathfrak{m}$ and k a natural number, $k > 0$, and suppose that all elements x_j, where $0 \leqq j < k$, have been already defined such that the power of the set

$$E_k = E - \bigcup_{j<k} f(x_j) - \bigcup_{j<k} f^{-1}(x_j)$$

is equal to $\mathfrak{m}$. By the lemma 4 there is an element y of E_k, such that $\overline{\overline{E_k - f^{-1}(y)}} = \mathfrak{m}$. Let $x_k = y$. The set $\{x_j\}_{j<\omega}$ is obviously free.

II.

We assume in this section that the sets $f(x)$ satisfy condition (C).

Theorem 9. (C) *does not imply the existence of a subset of* E *with the property* $T(2, \mathfrak{m})$ *and it does not imply the existence of an independent pair.*

Proof. It is sufficient to consider the case where $f(x) = E - \{x\}$.

The theorems 2 and 3 show that the additional assumption that $\overline{\overline{f(x)}} < \mathfrak{m}$ for every $x \in E$ does not imply the existence of a subset of E with the property $T(\mathfrak{m}, \mathfrak{m})$.

We prove now the following

Lemma 5. *If* $\mathfrak{m}$ *is regular,* $\mathfrak{m} \geqq \aleph_0$, *and* $\overline{\overline{f(x)}} < \mathfrak{m}$ *for every* $x \in E$, *then* (C) *implies the existence of an element* $x \in E$ *such that* $\overline{\overline{E-f^{-1}(x)}} = \mathfrak{m}$.

Proof. Suppose that the lemma is false. Then for every $x \in E$, $\overline{\overline{E-f^{-1}(x)}} < \mathfrak{m}$. Let $A = \bigcup_{y \in f(x)} (E-f^{-1}(y))$. Obviously $\overline{\overline{A}} < \mathfrak{m}$, because $\overline{\overline{f(x)}} < \mathfrak{m}$ and $\mathfrak{m}$ is regular. If $z \in E-A$, then $f(z) \supset f(x)$, which contradicts the condition (C).

Theorem 10. *If* $\mathfrak{m}$ *is regular,* $\mathfrak{m} \geqq \aleph_0$, *and* $\overline{\overline{f(x)}} < \mathfrak{m}$ *for every* $x \in E$, *then* (C) *implies the existence of an independent pair.*

P r o o f. By the lemma 5, there is an element x of E such that $\overline{\overline{E-f^{-1}(x)}}=\mathfrak{m}$. Let $y\in E-f^{-1}(x)-f(x)$. Obviously the set $\{x,y\}$ is free.

T h e o r e m 11. *If* $\mathfrak{m}$ *is regular,* $\mathfrak{m}\geqq\aleph_0$ *and* $\overline{\overline{f(x)}}\leqq\mathfrak{m}$ *for every* $x\in E$ *then* (C) *does not imply the existence of a free subset of power greater than* 2.

P r o o f. Let E_1 and E_2 two mutually disjoint subsets of E, of power $\mathfrak{m}$, such that $E=E_1\cup E_2$. Let $\{x_\eta^1\}_{\eta<\varphi_\mathfrak{m}}$ and $\{x_\eta^2\}_{\eta<\varphi_\mathfrak{m}}$ be wellorderings of E_1 and E_2, respectively. If $x=x_\eta^1\in E_1$, then let

$$f(x)=\{x_\xi^1\}_{\xi<\eta}\cup\{x_\eta^2\}$$

and if $x=x_\eta^2\in E_2$, then let

$$f(x)=\{x_\xi^2\}_{\xi<\eta}\cup\{x_\eta^1\}.$$

It is easy to see that the sets $f(x)$ satisfy the condition (C) and there does not exist a free subset of power greater than 2.

T h e o r e m 12. *If* $\mathfrak{m}$ *is singular and* $\overline{\overline{f(x)}}<\mathfrak{m}$ *for every* $x\in E$ *,then* (C) *does not imply the existence of an independent pair.*

P r o o f. Let $E=\{\nu:\nu<\varphi_\mathfrak{m}\}$ and for every ordinal number $\nu<\varphi_\mathfrak{m}$, $h_\nu=\{\beta_\xi^{(\nu)}\}_{\xi<\varphi_{\mathfrak{m}^*}}$ a subset of type $\varphi_{\mathfrak{m}^*}$ such that $\lim\limits_{\xi<\varphi_{\mathfrak{m}^*}}\beta_\xi^\nu=\varphi_\mathfrak{m}$ and $h_\mu\cap h_\nu=0$ for $\mu\neq\nu$. Let now $f(\nu)=E_1^{(\nu)}\cup E_2^{(\nu)}$ where $E_1^{(\nu)}=h_\nu$ and $E_2^{(\nu)}=\{\gamma:\gamma\leqq\nu\}$. Obviously the sets $f(x)$ satisfy the condition (C) and does nót exist an independent pair.

III.

We assume in this section that on the sets $f(x)$ the condition (D) holds.

T h e o r e m 13. (D) *implies the existence of a subset with the property* $T(\mathfrak{m}^*,1)$ *i. e. there is a subset* M *of power* $\mathfrak{m}^*$ *such that* (W) *if* $x,y\in M$ *and* $x\neq y$, *then* $f(x)\cap f(y)=0$.

P r o o f. Suppose the contrary. Then the power of a set with the property (W) is less than $\mathfrak{m}^*$. Let N be a maximal set with respect to the property (W), i. e. if $x\notin N$, then there exists an element $y\in N$ such that $f(x)\cap f(y)\neq 0$. We define the sets N_a $(a\in N)$ as follows: Let the element y of $E-N$ be an element of N_a, if $f(y)\cap f(a)\neq 0$. Since $\overline{\overline{N}}<\mathfrak{m}^*$ there is an element $b\in N$ for which $\overline{\overline{N_b}}=\mathfrak{m}$, which contradicts (D).

T h e o r e m 14. *If* $\mathfrak{m}$ *is singular and* $n=3$ *then* (D) *does not imply the existence of a subset with the property* $T(\mathfrak{m},1)$.

Proof. Let $\{E_\xi\}_{\xi<\varphi_{\mathfrak{m}^*}}$ be a sequence of type $\varphi_{\mathfrak{m}^*}$, of mutually disjoint subsets of E such that

$$E = \bigcup_{\xi<\varphi_{\mathfrak{m}^*}} E_\xi,$$

$\overline{\overline{E_\xi}} = \mathfrak{m}_\xi < \mathfrak{m}$ and $\mathfrak{m}_\eta < \mathfrak{m}_\nu$ for $\eta < \nu$. Let $\{x^\xi_\eta\}_{\eta<\varphi_{\mathfrak{m}_\xi}}$ be any wellordering of the type $\varphi_{\mathfrak{m}_\xi}$, of E_ξ. We define the sets $f(x)$ as follows: if $x^\xi_\eta \in E_\xi$, then let $f(x) = \{x^\xi_0, x^\xi_\eta\}$. Obviously the sets $f(x)$ satisfy the condition (D), and does not exist a subset of E with the property $T(\mathfrak{m}, 1)$.

Theorem 15. (D) *implies the existence of a free subset of E, of power* $\mathfrak{m}^*$.

Proof. We consider two cases: a) E has a subset F of power $\mathfrak{m}$ such that, if $x \in E_1$, then $\overline{\overline{f(x)}} = \mathfrak{m}$, b) there is no such a subset of power $\mathfrak{m}$.

In the case a) we prove the following

Lemma 6. *If* $M \subset E$ *and* $\overline{\overline{M}} < \mathfrak{m}^*$, *then* $\overline{\overline{E - \bigcup_{x\in M} f(x)}} = \mathfrak{m}$.

Proof. Suppose the contrary, i.e. E has a subset M such that $\overline{\overline{M}} < \mathfrak{m}^*$ and $\overline{\overline{E - \bigcup_{x\in M} f(x)}} < \mathfrak{m}$. Then there is an element y of M such that $\overline{\overline{f(y)}} = \mathfrak{m}$ and $f(y)$ has a subset $F(y)$ of power $\mathfrak{m}$ such that, if $z \in F(y)$, then $\overline{\overline{f(z)}} = \mathfrak{m}$. Since $\overline{\overline{M}} < \mathfrak{m}^*$, it follows from (D) that the set $F(y)$ has an element z_0 for which $f(z_0) \cap f(z) = 0$ for every $z \in M$. Thus $f(z_0) \subset E - \bigcup_{x\in M} f(x)$ which is impossible because $\overline{\overline{f(z_0)}} = \mathfrak{m}$.

Let $E_1 = \{y : \overline{\overline{f(y)}} < \mathfrak{m}\}$. Further let $V = E$ in the case a), $V = E_1$ in the case b) and $\{x_\nu\}_{\nu<\varphi_\mathfrak{m}}$ any wellordering of the type $\varphi_\mathfrak{m}$, of V. We define the sequence $\{y_\nu\}_{\nu<\varphi_{\mathfrak{m}^*}}$ as follows: Put $y_0 = x_0$. Let now β be an ordinal number, $1 \leqq \beta < \varphi_{\mathfrak{m}^*}$, and suppose that all elements y_ξ, where $0 \leqq \xi < \beta$, have been already defined. Let $F_\beta = \{x_\nu\}_{\nu<\varphi_\mathfrak{m}} - (\{y_\nu\}_{\nu<\beta} \cup (\bigcup_{\nu<\beta} f(y_\nu))$.

We now prove $\overline{\overline{F_\beta}} = \mathfrak{m}$. In case b) this is clear and in case a) it follows from lemma 6 $(M = \{y_\nu\}_{\nu<\beta})$.

Let D_β be the set of elements $y \in F_\beta$ for which there is a $\nu < \beta$ such that $y_\nu \in f(y)$. Since $\overline{\overline{\beta}} < \mathfrak{m}^*$, by (D), $\overline{\overline{D_\beta}} < \mathfrak{m}$. It follows that $\overline{\overline{F_\beta - D_\beta}} = \mathfrak{m}$. Let y_β be the first element of $F_\beta - D_\beta$. Thus the set $\{y_\nu\}_{\nu<\varphi_{\mathfrak{m}^*}}$ is defined. Put $E' = \{y_\nu\}_{\nu<\varphi_{\mathfrak{m}^*}}$. Clearly the set E' is free and $\overline{\overline{E'}} = \mathfrak{m}^*$.

Theorem 16. *If* $\mathfrak{m}$ *is singular, then the condition* (D) *on the sets* $f(x)$ *does not imply the existence of a free subset of E, of power* $\mathfrak{m}$.

Proof. Let $\{E_\xi\}_{\xi<\varphi_{\mathfrak{m}^*}}$ be a sequence of type $\varphi_{\mathfrak{m}^*}$, of mutually disjoint subsets of E such that

$$E = \bigcup_{\xi<\varphi_{\mathfrak{m}^*}} E_\xi,$$

$\overline{\overline{E}}_\xi = \mathfrak{m}_\xi < \mathfrak{m}$ and $\mathfrak{m}_\eta < \mathfrak{m}_\nu$ for $\eta < \nu$. Let $\{x_\eta^\xi\}_{\eta<\varphi_{\mathfrak{m}_\xi}}$ be any wellordering of the type $\varphi_{\mathfrak{m}_\xi}$ of E_ξ. We define the sets $f(x)$ as follows: if $x = x_\eta^\xi \in E$, then let $f(x) = \{x_\zeta^\xi\}_{\zeta<\eta}$. It is obvious that the sets $f(x)$ satisfy (D) and there does not exist a free subset of E of power $\mathfrak{m}$.

IV.

We assume in this section that the sets $f(x)$ satisfy (A), and we give the solutions of questions a) and b).

Lemma 7. *If* $\mathfrak{m} > \aleph_1$, *and there is a regular cardinal number* $\mathfrak{r}$ *for which* $\aleph_0 < \mathfrak{n} \leqq \mathfrak{r} < \mathfrak{m}$, *then to every element* x *of* E *there corresponds a closed subset* $g(x)$ *of* E *such that* $x \in g(x)$ *and* $\overline{\overline{g(x)}} < \mathfrak{r}$.

Proof. Let x be a given element of E and

$$\{x\} \cup f(x) = E_1,\ f(E_1) = E_2, \ldots, f(E_{k-1}) = E_k, \ldots$$

It is easy to see that $\overline{\overline{E}}_k < \mathfrak{r}$ $(k = 1, 2, \ldots)$. Put $g(x) = \bigcup_{k=1}^{\infty} E_k$.

Theorem 17. *If there exists a regular cardinal number* $\mathfrak{r}$ *such that* $\aleph_0 < \mathfrak{n} \leqq \mathfrak{r} < \mathfrak{m}$, *then* (A) *implies the existence of a closed proper subset of* E, *of power* $\mathfrak{m}$.

Proof. By lemma 7 to every $x \in E$ there corresponds a closed subset $g(x)$ of E such that $\overline{\overline{g(x)}} < \mathfrak{r}$. By a lemma of [3] (see p. 55) there is a subset F of E for which $\overline{\overline{F}} = \mathfrak{m}$ and

$$E - \bigcup_{x\in F} g(x) \neq 0.$$

Since $\bigcup_{x\in F} g(x)$ is obviously closed, the theorem is proved.

Theorem 18. *If* $\mathfrak{m} > \aleph_0$, $\mathfrak{m}^-$ *is singular and* $\mathfrak{n} = \mathfrak{m}^-$, *then* (A) *does not imply the existence of a closed proper subset of* E, *of power* $\mathfrak{m}$.

Proof. Let $\{E_\beta\}_{\beta<\varphi_{\mathfrak{m}}}$ be a sequence of the type $\varphi_{\mathfrak{m}}$, of mutually disjoint subsets of E such that $E = \bigcup_{\beta<\varphi_{\mathfrak{m}}} E_\beta$ and $\overline{\overline{E}}_\beta = \mathfrak{m}^-$ $(\beta < \varphi_{\mathfrak{m}})$. Further let $\{x_\alpha^{(\beta)}\}_{\alpha<\varphi_{\mathfrak{m}}}$ be a wellordering of the type $\varphi_{\mathfrak{m}^-}$ of E_β. We define the sets $f(x)$ as follows: Let $\{\alpha_\nu\}_{\nu<\varphi_{(\mathfrak{m}^-)^*}}$ be a set of type $\varphi_{(\mathfrak{m}^-)^*}$ of ordinal numbers such that $\lim_{\nu<\varphi_{(\mathfrak{m}^-)^*}} \alpha_\nu = \varphi_{\mathfrak{m}^-}$. If $\beta > 0$, then let H_β be a one to one mapping of the set $\{x_\alpha^{(\beta)}\}_{\alpha<\varphi_{\mathfrak{m}^-}}$ onto the set $\{x_\alpha^{(\gamma)}\}_{\alpha<\varphi_{\mathfrak{m}^-}}^{\gamma<\beta}$. Since the powers of both sets are equal to $\mathfrak{m}^-$ there is such a mapping. If $x=x_\alpha^{(\beta)}\in E_\beta$, then let

$$f(x)=E_1^{(x)}\cup E_2^{(x)}\cup E_3^{(x)}$$

where $E_1^{(x)}=\{x_\gamma^{(\beta)}\}_{\gamma<\alpha}$, $E_2^{(x)}=\{x_{\alpha_\nu}^{\beta^-}\}_{\nu<\varphi(\mathfrak{m}^-)^*}$, further $E_3^{(x)}=0$, if $\beta=0$ and $E_3^{(x)}=\{H_\beta(x)\}$ if $\beta>0$.

Obviously $\overline{\overline{f(x)}}<\mathfrak{n}$ for every $x\in E$. If $g(x)=\bigcup\limits_{k=1}^{\infty}E_k$, where $E_1=f(x)$ and $E_k=f(E_{k-1})$ for $k>1$, then by the definition of $f(x)$, for $x=x_\alpha^{(\beta)}$,

$$g(x_\alpha^\beta)=\bigcup_{\varrho\leqq\beta}\{x_\alpha^{(\varrho)}\}_{\alpha<\varphi_{\mathfrak{m}^-}}.$$

It follows that E does not have a closed proper subset of power $\mathfrak{m}$.

Theorem 19. *If there exists a regular cardinal number* $\mathfrak{r}$ *such that* $\aleph_0<\mathfrak{n}\leqq\mathfrak{r}<\mathfrak{m}$, *then* (A) *implies the existence of two almost disjoint closed subsets of* E, *of power* $\mathfrak{m}$. *If* $\mathfrak{m}\,(\neq\aleph_{\alpha+\omega})$ *is the sum of* $\mathfrak{n}$ *cardinal numbers, each of which is smaller than* $\mathfrak{n}$, *we assume the generalised continuum hypothesis.*

Proof. By the lemma 7 to every $x\in E$ there corresponds a closed subset $g(x)$ of E such that $\overline{\overline{g(x)}}<\mathfrak{r}$. By the theorems 1, 6, and 8 of [4], there is a subset Γ of power $\mathfrak{m}$ of E, for which

$$\overline{\overline{\prod_\Gamma^g}}<\mathfrak{m}\quad\text{and}\quad\overline{\overline{\sum_\Gamma^g}}=\mathfrak{m}.$$

Let $\Gamma=\Gamma_1\cup\Gamma_2$ such that $\Gamma_1\cap\Gamma_2=0$ and $\overline{\overline{\Gamma_1}}=\overline{\overline{\Gamma_2}}=\mathfrak{m}$. Let $E_1=\bigcup\limits_{x\in\Gamma_1}g(x)$ and $E_2=\bigcup\limits_{x\in\Gamma_2}g(x)$. Obviously E_1 and E_2 are almost disjoint closed sets of power $\mathfrak{m}$.

Theorem 20. *If* $\mathfrak{m}>\aleph_0$, $\mathfrak{m}^-$ *is singular and* $\mathfrak{n}=\mathfrak{m}^-$, *then* (A) *does not imply the existence of two almost disjoint closed subsets of* E, *of power* $\mathfrak{m}$.

This follows from the proof of Theorem 18.

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