# The von Neumann coordinatization theorem for complemented modular lattices.

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## 1. Introduction.

**1. 1.** In all of what follows $n$ will denote a fixed positive integer, $V$ will denote the set of all vectors $u=(\alpha^1,\ldots,\alpha^n)$ of length $n$, and left modules will always mean *non-empty* left modules of $V$. The coordinates $\alpha^i$ will be arbitrary elements in a ring $\mathfrak{R}$.

If $\mathfrak{R}$ is a division ring it is well known that the set of all left modules of $V$ form a complemented modular lattice. If $\mathfrak{R}$, more generally, is a regular ring with unit element, then, as discovered by JOHN VON NEUMANN, a complemented modular lattice is formed by all left modules *of finite span* (a left module is of finite span if it is spanned by a finite number of vectors). In the case that $\mathfrak{R}$ is a division ring *every* left module is of finite span.

A deep converse to the previous statements was discovered by VON NEUMANN [7, vol. 23, page 18; 8, vol. II, Theorem 14.1, page 141]. Let $L$ be a complemented modular lattice possessing a finite homogeneous basis $a_1,\ldots,a_n$ of order $n$ and let $L_{ij}$ denote the set of inverses of $a_j$ with respect to $a_i+a_j$. VON NEUMANN showed that if $n\geqq 4$ the following theorem holds:

The von Neumann coordinatization theorem. *For every $i\neq j$, addition and multiplication can be defined for the elements of $L_{ij}$ in such a way that:*

(i) *the $L_{ij}$ become regular rings with unit, isomorphic to a common regular ring $\mathfrak{R}$,*

(ii) *all sub-lattices $L(a_i)$ ($L(a_i)$ consists of all $x\leqq a_i$) are isomorphic to the lattice of all left principal ideals of $\mathfrak{R}$,*

(iii) *$L$ is isomorphic to (coordinatized by) the lattice of all left modules of finite span in the space $V$ of vectors $(\alpha^1,\ldots,\alpha^n)$ with all $\alpha^i$ in $\mathfrak{R}$.*

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This generalizes the classical theorem that a projective geometry can be coordinatized (with coordinates in a suitable division ring) if the geometry has dimension $\geqq 3$ (that is, has order $\geqq 4$). But the classical theorem also asserts that a projective geometry of dimension 2 (i. e. a plane geometry) can be so coordinatized if and only if DESARGUES's theorem holds [3, Kap. V; 6, Theorems 10, 11, ex. 19, page 204] and this result is not covered by VON NEUMANN's theorem as formulated hitherto.

In this paper we will give a presentation of VON NEUMANN's coordinatization theorem which further simplifies our previous treatment [1, 2] and which includes the case of plane projective geometry. Our discussion will apply to any complemented modular lattice $L$ possessing a homogeneous basis of order $\geqq 3$; for the case $n=3$, we postulate the additional restrictions (4. 3. 3), (4. 3. 4) and (4. 10. 3). When $L$ is a plane projective geometry these restrictions reduce to the so-called fundamental theorem on quadrangular sets [6, p. 47], which is, in turn, equivalent to DESARGUES's theorem.

Since detailed discussions of the von Neumann coordinatization theorem which have appeared previously [8, vol. II; 4; 1, 2] are not readily accessible, we find it desirable to give here a complete exposition.

**1. 2. Contents of this paper.** This paper does not assume previous knowledge of either VON NEUMANN's theory or general lattice theory. Sections 2, 3 and part of 4 are a simplified exposition of parts of [8, vols I, II].

In section 2 definitions are given for: lattice with zero element, modular lattice, relatively complemented lattice, complemented lattice and independence of a collection of lattice elements, together with some properties which are required later and are easily verified.

In section 3 regular semi-groups and regular rings are defined and some of their properties obtained. With $V$ denoting the module of all vectors of length $n$ with coordinates in a regular ring it is shown that a left module (i. e. sub-module of $V$) of finite span is always spanned by $n$ vectors $(\alpha^{j1},\ldots,\alpha^{jn})$, $j=1,\ldots,n$, with the properties: for each $j$, $\alpha^{jj}$ is idempotent, $=e^j$, say; for all $i>j$, $\alpha^{ji}=0$; for all $i<j$, $e^j\alpha^{ji}=\alpha^{ji}$ and $\alpha^{ji}e^i=0$. Such a set of $n$ vectors will be called a *canonical basis* for the left module. It is shown that the left modules of finite span form a relatively complemented modular lattice; if the regular ring $\mathfrak{R}$ has a unit then this lattice is complemented.

In section 4 a ring of coordinates is constructed for a given complemented modular lattice $L$. In § 4. 1 homogeneous bases and normalized frames for $L$ are defined. Addition and multiplication are defined in §§ 4. 2 and 4. 9 respectively, for elements in a fixed $L_{ij}$ (this is a lattice generalization of familiar constructions in projective geometry). In §§ 4. 2 to 4. 14

it is shown that the $L_{ij}$ then become isomorphic regular rings with unit if $L$ possesses a homogeneous basis of order $n \geqq 3$ and satisfies the additional restrictions (4. 3. 3), (4. 3. 4) and (4. 10. 3). Assuming these conditions on $L$, Parts (i) and (ii) of the von Neumann coordinatization theorem are established (Theorem (4. 14. 6)).

In § 4. 15 certain collections of lattice elements $(x_{ij};\ i \neq j)$, $(x_{ij};\ i > j)$ with all $x_{ij}$ in $L_{ij}$, are called $L$-numbers and upper semi-$L$-numbers respectively. These numbers form rings $\mathfrak{R}'$, $\mathfrak{R}$, respectively, and $\mathfrak{R}'$ is identified in a natural way with a subring of $\mathfrak{R}$ (actually $\mathfrak{R}'$ coincides with $\mathfrak{R}$; this is shown directly if $n \geqq 4$ but for $n = 3$ is obtained only as a *consequence* of Part (iii) of the coordinatization theorem). $\mathfrak{R}$ is called an auxiliary ring for $L$. In (4. 15. 5) $\mathfrak{R}$ is shown to be ring-isomorphic to every $L_{ij}$. The proof of Part (iii) of the coordinatization theorem (to be given in sections 5, 6) is in terms of the space $V$ of vectors $(\alpha^1, \ldots, \alpha^n)$ with $\alpha^i$ in the auxiliary ring $\mathfrak{R}$.

In section 6 we give a rule which assigns to each $x$ in $L$ a family of modules of $V$. It is shown that all left modules assigned by this rule to the same $x$ coincide (Theorem (6. 2. 5)) and that the rule sets up a (1, 1) order preserving correspondence (i. e., lattice isomorphism) between $L$ and the set of all left modules of finite span (Theorems (6. 2. 1), (6. 2. 6) and (6. 2. 7)). This establishes Part (iii) of the coordinatization theorem.

The rule which assigns left modules to an element $x$ is as follows. First we consider special elements $y$ which satisfy: for some integer $i$, $y \leqq a_1 + \ldots + a_i$, $y(a_1 + \ldots + a_{i-1}) = 0$ (such an element is called an $i$-element). We show that every $i$-element can be expressed in terms of suitable „projections“ $\beta^i_{ij}$, $j < i$ (each $\beta^i_{ij}$ in $L_{ij}$), together with a suitable „covering“ idempotent $e$ (see (6. 1. 1)). In § 6. 2 we assign to each $i$-element $y$ a vector $u(y)$, not necessarily unique. Then an arbitrary $x$ is expressed as a sum $x_1 + \ldots + x_n$ with each $x_i$ (not necessarily unique) an $i$-element. The module spanned by vectors $u(x_1), \ldots, u(x_n)$ is assigned by our rule to $x$.

Certain relations required in the proofs of section 6 are collected together in the previous section 5. The involved identity (5. 2. 3) is required in the proof of Theorem (6. 2. 3). In § 5. 3 the nullity $\alpha^0 = (\alpha^0_i;\ i = 1, \ldots, n)$ and the reach $\alpha^r = (\alpha^r_i;\ i = 1, \ldots, n)$ with $0 \leqq \alpha^0_i, \alpha^r_i \leqq a_i$, are defined for each $\alpha$ in $\mathfrak{R}$. If $\mathfrak{R}$ is a division ring, each of $\alpha^0_i = 0$, $\alpha^r_i = a_i$ is equivalent to $\alpha \neq 0$; in the general $\mathfrak{R}$, these conditions are equivalent to: $\alpha$ has a right inverse and $\alpha$ has a left inverse, respectively. Theorems (5. 3. 1) to (5. 3. 7) give properties of reach and nullity and are designed to meet complications which arise in section 6 due to the fact that $\mathfrak{R}$ need not be a division ring.

Section 7 specializes the previous discussion to the case of projective geometry with a normalized frame consisting of points. It is shown that the additional restrictions (4. 3. 3), (4. 3. 4) and (4. 10. 3) are then equivalent to a restricted formulation (7. 4. 3) of DESARGUES's theorem.

**1. 3. Notation.** Greek letters $\alpha, \beta, \gamma, \ldots$ (but excluding $\pi$) *without subscripts* will denote elements in a semi-group $S$ or in a ring $\mathfrak{R}$; *e, f, g will be reserved for ring elements which are idempotent.* For fixed $\alpha^1, \alpha^2, \ldots$ in $\mathfrak{R}$, $(\alpha^1, \alpha^2, \ldots)_l$ will denote the left ideal consisting of all finite sums $\beta^1\alpha^1+\beta^2\alpha^2+\ldots$ with arbitrary $\beta^i$ in $\mathfrak{R}$; similarly $(\alpha^1, \alpha^2, \ldots)_r$ will denote the right ideal of elements $\alpha^1\beta^1+\alpha^2\beta^2+\ldots$; if $\alpha$ is in a semi-group $S$, $(\alpha)_l$ will denote the left coset consisting of all $\beta\alpha$ with arbitrary $\beta$ in $S$, $(\alpha)_r$ will denote the right coset consisting of all $\alpha\beta$ with arbitrary $\beta$ in $S$ (if the semi-group $S$ is the multiplicative semi-group of a ring $\mathfrak{R}$, the left coset $(\alpha)_l$ and the left ideal $(\alpha)_l$ coincide as do the right coset and right ideal $(\alpha)_r$). The letters $u, v, \ldots$ will denote vectors of length $n$ with coordinates in $\mathfrak{R}$ and $(u, v, \ldots)_l$ will denote the left module spanned by $u, v, \ldots$ which consists of all finite sums $\alpha u+\beta v+\ldots$ with arbitrary $\alpha, \beta, \ldots$ in $\mathfrak{R}$. The letters $a, b, c, d, \ldots, x, y, z, \ldots, p, q, w, \ldots, A, B, \ldots$ will denote elements in a lattice $L$. The letters $i, j, k, m, s, t$, will denote positive integers. The same symbols 0, 1 will be used to denote ring elements and lattice elements but there will be no ambiguity. The symbols $+$, $\Sigma$ will denote addition for ring elements and lattice join (i. e. supremum) for lattice elements but there will be no ambiguity. Similarly $\alpha\beta$ and $\Pi_j\alpha^j$ will denote ring multiplication whereas $xy$ and $\Pi_j x^j$ will denote lattice meet (i. e. infimum). With each ring element $\alpha$ there will be associated certain lattice elements to be denoted by $\alpha$ with *subscripts* (with or without superscripts) thus $\alpha_{ij}$, $\alpha_i^0$, and $\alpha_j^r$. For certain lattice elements we will define in §§ 4. 3, 4. 10 new operations $x \dotplus y$, $x \times y$ with values which are again lattice elements; *these should not be confused with the lattice operations* $x+y$, $xy$.

## 2. Complemented modular lattices.

**2. 1. Lattices.** A lattice with zero $L$ is a collection of elements $0, a, b, c, \ldots, x, y, z, \ldots$, partially ordered by a relation $a \leqq b$ (also written $b \geqq a$) such that $0 \leqq x$ for every $x$, and for each pair $a, b$ there are elements $a+b$ and $ab$ (necessarily unique) satisfying:

$$a+b \leqq x \quad \text{if and only if} \quad a \leqq x \text{ and } b \leqq x,$$
$$x \leqq ab \quad \text{if and only if} \quad x \leqq a \text{ and } x \leqq b.$$

$L(a)$ will denote the sub-lattice with zero of all $x \leqq a$.

**2. 2. Modular lattices.** $L$ is called a modular lattice if: $a(b+c)=$ $=b+ac$ for all $a, b, c$ with $b\leqq a$. This *modular law* implies the *absorption law*: $ab+c=a(b+c)$ for all $a, b, c$, with $c\leqq a$; the *clipping identity*: $a(b+c)=a[b(a+c)+c]$ for all $a, b, c$; and the *superfluous term identities*: $ab=a(b+c)$ if $c(a+b)=0$ and $b=bd$ if $b\leqq d$. Applications of these identities will be indicated by (ML), (AL), (CI) and (ST), respectively.

**2. 3. Independence.** In a modular lattice with zero, for each $m=1, 2, \ldots,$ elements $x^1, \ldots, x^m$ are called independent if, for each $i\leqq m$, $x^i(x^1+\cdots+$ $+x^{i-1}+x^{i+1}+\cdots+x^m)=0$. If for some ordering of the $x^i$ it is true that $x^j(x^1+\cdots+x^{j-1})=0$ for $2\leqq j\leqq m$ then the $x^i$ are necessarily independent. If the $x^i$ are independent and for each of a finite number of $j$, $I_j$ is a subset of the integers $1, 2, \ldots, m$, then

$$\Pi_j(\Sigma x^i;\ i \text{ in } I_j)=(\Sigma x^i;\ i \text{ in all } I_j);$$

if the $x^i$ are independent and $x^{ij}\leqq x^i$ for each of a finite number of $j$, then

$$\Pi_j\Sigma_i x^{ij}=\Sigma_i\Pi_j x^{ij}.$$

The symbols $\oplus, \Sigma\oplus$ will sometimes be used in place of $+, \Sigma$ to imply independence of the elements involved.

A detailed treatment of this theory of independence was given by VON NEUMANN [7, vol. 23, page 22, footnote 7; 8, vol. I].

**2. 4. Complements and relative complements.** If $x\leqq z$ in a lattice $L$ with zero then a relative complement, or inverse, of $x$ in $z$ is an element $y$ (not necessarily unique) such that $x\oplus y=z$; $[z-x]$ will be used to denote such an inverse of $x$ in $z$. A lattice $L$ with zero is called *relatively complemented* if there exists at least one relative complement of $x$ in $z$ whenever $x\leqq z$.

A lattice $L$ is said to have a unit 1 (necessarily unique) if $x\leqq 1$ for all $x$ in $L$. If $L$ has zero and unit elements then a relative complement of $x$ in 1 is also called a complement of $x$; $L$ is called *complemented* if each $x$ has at least one complement.

A relatively complemented lattice with unit is obviously complemented; on the other hand, a complemented *modular* lattice is also relatively complemented (indeed, if $x\leqq z$ and $y$ is a complement of $x$ then $yz$ is a relative complement of $x$ in $z$).

The modular law implies the *indivisibility of inverses*, which asserts: *whenever $y_1$ and $y_2$ are both inverses of $a$ in $b$ and $y_1\leqq y_2$, then $y_1=y_2$* (for $y_2=y_2b=y_2(y_1+a)=y_1+y_2a=y_1$). Because of this indivisibility of inverses it is possible to replace „points“ as used in certain constructions in the classical theory of projective geometry, by „inverses". We shall use the phrase *general indivisibility of inverses* to refer to the more general theorem (also a consequence of the modular law): *if* $y_1 + a = y_2 + a$ *and* $y_1 a = y_2 a$ *for some* $a$, *and* $y_1 \leqq y_2$, *then* $y_1 = y_2$.

**2. 5. Perspectivities.** Elements $x^1$ and $x^2$ in a lattice with zero are called perspective if they possess a common inverse in $x^1 + x^2$. Any such common inverse $b$ is called an axis of perspectivity and, if the lattice is modular, sets up a (1, 1) order preserving mapping (called a perspective mapping) of $L(x^1)$ onto $L(x^2)$:

$$\text{if } z^1 \leqq x^1, \quad \text{then} \quad z^1 \to (z^1 + b)x^2,$$
$$\text{if } z^2 \leqq x^2, \quad \text{then} \quad z^2 \to (z^2 + b)x^1.$$

If $z^1$ and $z^2$ correspond under this mapping then $z^1 + b = z^2 + b$.

## 3. Regular rings.

**3. 1. Definition of regular semi-group and regular ring.** A non-empty system $S$ of elements $\alpha, \beta, \ldots$ is called a *semi-group* if an associative multiplication is defined on $S$, i. e. $\alpha\beta$ is defined and is in $S$ whenever $\alpha, \beta$ are in $S$ and $\alpha(\beta\gamma) = (\alpha\beta)\gamma$. The multiplication is called a *regular multiplication* and $S$ is called a *regular semi-group* if, for each $\alpha$ in $S$, $\alpha\beta\alpha = \alpha$ for at least one $\beta$ in $S$ [7, vol. 22, page 708].

It is easy to see that a semi-group $S$ is regular if and only if for each $\alpha$ there exists an idempotent $e$ (that is $ee = e$) such that $e\alpha = \alpha$ and $\alpha\beta = e$ for some $\beta$ (if $\alpha\beta\alpha = \alpha$, then choose $e = \alpha\beta$); similarly a semi-group $S$ is regular if and only if for each $\alpha$ there exists an idempotent $f$ such that $\alpha f = \alpha$ and $\beta\alpha = f$ for some $\beta$ (if $\alpha\beta\alpha = \alpha$, then choose $f = \beta\alpha$).

It is also easy to see that a semi-group $S$ is regular if and only if each left coset $(\alpha)_l$ contains $\alpha$ and is identical with $(e)_l$ for some idempotent $e$ and if and only if each right coset $(\alpha)_r$ contains $\alpha$ and is identical with $(f)_r$ for some idempotent $f$.

A ring $\mathfrak{R}$ (a unit is *not* assumed) is called a *regular ring* if its multiplication is regular; that is, for each $\alpha, \alpha\beta\alpha = \alpha$ for some $\beta$ in $\mathfrak{R}$.

**3. 2. Principal left ideals.** (Throughout this paper, right and left may obviously be interchanged). In a regular ring the principal left ideals form, as we shall show, a relatively complemented modular lattice with zero (complemented, if $\mathfrak{R}$ has a unit) when partially ordered by inclusion; the zero (left principal ideal) of this lattice consists of the zero element of $\mathfrak{R}$ only.

This is easily verified since, if $e, f$ are idempotents [2]):

(i) the smallest left ideal containing $(e)_l$ and $(f)_l$ is precisely $(e+g)_l$ where $g$ is any idempotent with $(g)_l = (f-fe)_l$;

(ii) the left ideal of all ring elements common to $(e)_l$ and $(f)_l$ is precisely $(f-gf)_l$ where $g$ is any idempotent with $(g)_r = (f-fe)_r$;

(iii) $(f-fe)_l$ is a relative complement of $(e)_l$ in $(f)_l$ whenever $(e)_l$ is contained in $(f)_l$;

(iv) if $\mathfrak{R}$ has a unit 1 then $(1)_l \geqq (e)_l$.

It is now easy to prove that: a ring $\mathfrak{R}$ is regular if and only if its principal left ideals form a relatively complemented modular lattice such that every principal left ideal $(\alpha)_l$ contains $\alpha$ and is *contained* in some principal left ideal $(e)_l$ with $e$ idempotent (possibly depending on $\alpha$); and a ring $\mathfrak{R}$ with unit is regular if and only if its principal left ideals form a complemented modular lattice.

**3. 3. Ring conditions on $\alpha$.** If $g$ is an idempotent in a regular ring $\mathfrak{R}$ and $\beta^i, \gamma^i$ $(i = 1, \ldots, m)$ are in $\mathfrak{R}$, then, as we shall now prove, the conditions on $\alpha$: $\alpha$ is in $(g)_l$ and $\alpha\beta^i$ is in $(\gamma^i)_l$ for each $i$, are equivalent to: $\alpha$ is in $(e)_l$ for a suitable idempotent $e = e(g, \beta^1, \ldots, \gamma^1, \ldots)$.

We shall prove this for the case $m = 1$ (the general case will then follow at once from § 3. 2 (ii)). We write $\beta$ for $\beta^1$ and $\gamma$ for $\gamma^1$ and we may clearly suppose that $\gamma$ is idempotent. Then the conditions on $\alpha$ are equivalent to: $\alpha = \alpha g$ and $\alpha(\beta - \beta\gamma) = 0$, that is, to the conditions: $\alpha = \alpha g, \alpha f = 0$ where $f$ is an idempotent with $(f)_r = (\beta - \beta\gamma)_r$, that is, to the condition: $\alpha$ is in $(g - hg)_l$ where $h$ is any idempotent with $(h)_r = (gf)_r$.

**3. 4. Canonical basis.** If $M$ is a left module of finite span (of vectors of length $n$ with coordinates in a ring $\mathfrak{R}$) then $M$ is certainly spanned by a finite number of vectors $v^j = (\alpha^{j1}, \ldots, \alpha^{jn})$. If $\mathfrak{R}$ is regular, then $M$ is always spanned by a canonical basis (see § 1. 2), as we shall now verify.

Starting from the given $v^j$ which span $M$, there is an idempotent $e^n$ with $(e^n)_l = (\alpha^{1n}, \alpha^{2n}, \ldots)_l$ (this implies $\alpha^{jn} e^n = \alpha^{jn}$ for all $j$ and $\Sigma_j \beta^j \alpha^{jn} = e^n$

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[2]) In (i), $(e, f)_l \leqq (e+g)_l$ since: $f - fe = (f-fe)g, g = u(f-fe)$, hence $ge = 0$, $e = (e+g) - g(e+g), f = fe + (fg - feg)(e+g)$. Also $e+g)_l \leqq (e, f)_l$ since: $e + g = = e + u(f-fe) = (e - uf)e + uf$. This implies that $(e+g)_l$ is the smallest left ideal containing $(e)_l$ and $(f)_l$.

In (ii), $(f-gf)_l \leqq (e)_l (f)_l$ since: $f - gf = (f-g)f$ and $f - fe = g(f-fe)$, hence $f - gf = (f-gf)e$. Also $(e)_l (f)_l \leqq (f-gf)_l$ since: $g = (f-fe)u$, hence if $x = xe = xf$, then $x(f-gf) = x - x(f-fe)uf = x - (x-x)uf = x$.

In (iii), $(e, f-fe)_l = (f)_l$. Also $(e)_l (f-fe)_l = 0$ since: $u = ue = u(f-fe)$ implies $u = u(f-fe)e = 0$.

for suitable $\beta^j$). Let $w^1 = e^n(\Sigma_k \beta^k v^k)$, and for $j \geqq 1$, $w^{j+1} = v^j - \alpha^{jn} w^1$. This new finite set of vectors (which we shall denote again as $v^j$) span $M$ and have the additional properties: $\alpha^{1n} = e^n$ (idempotent), $e^n \alpha^{1i} = \alpha^{1i}$ for all $i$, and $\alpha^{jn} = 0$ for all $j > 1$.

Now apply the procedure of the preceding paragraph to the vectors $v^j$ $(j \geqq 2)$ to obtain an idempotent $e^{n-1}$ so that the vectors which span $M$ may be supposed to have the additional properties: $\alpha^{2,n-1} = e^{n-1}$, $e^{n-1}\alpha^{2,i} = \alpha^{2,i}$ for all $i$, and $\alpha^{j,n-1} = 0$ for $j > 2$. Successive repetitions of this procedure show that: $M$ can be spanned by vectors $v^j$ (now necessarily $n$ in number) with $\alpha^{j,n+1-j} = e^{n+1-j}$ (idempotent), $e^{n+1-j}\alpha^{ji} = \alpha^{ji}$ for all $i$, and $\alpha^{ji} = 0$ for $i > n+1-j$.

Now replace $v^1$ by $v^1 - \alpha^{1,n-1}v^2$ obtaining the additional property: $\alpha^{1,n-1}e^{n-1} = 0$. By repetition of this procedure, obtain: $\alpha^{1,i}e^i = 0$ for all $i < n$. Similarly, obtain: $\alpha^{ji}e^i = 0$ for all $i < n+1-j$.

If $u^j$ is now defined to be $v^{n+1-j}$, the $u^j$ are a canonical basis for $M$.

**3.5. Vector conditions on $\alpha$.** Suppose $g$ is an idempotent in a regular ring $\Re$ and for each $i = 1, \ldots, m$ suppose $M^i$ is a left module of finite span and $v^i$ is a given vector. We shall now show that the conditions on $\alpha$: $\alpha$ is in $(g)_l$ and $\alpha v^i$ is in $M^i$ for each $i$, are equivalent to: $\alpha$ is in $(e)_l$ for a suitable idempotent $e = e(g, v^1, \ldots, M^1, \ldots)$.

We shall prove this for the case $m = 1$ (the general case will then follow at once from § 3.2 (ii)). We write $v^1 = v = (\alpha^1, \ldots, \alpha^n)$ and we may suppose that $M^1$ has a canonical basis $u^j = (\alpha^{j1}, \ldots, \alpha^{jn})$, $j = 1, \ldots, n$. Then the conditions on $\alpha$ are equivalent to: (i) $\alpha$ is in $(g)_l$ and (ii) $\alpha v = \Sigma_k \beta^k u^k$ for suitable $\beta^k$. But if such $\beta^k$ exist then $\alpha\alpha^j\alpha^{jj} = \beta^j\alpha^{jj}$ for all $j$. Hence condition (ii) on $\alpha$ may be written: $\alpha v \overset{\circ}{=} \Sigma_j \alpha\alpha^j u^j$ and is equivalent to the $n$ conditions: $\alpha(\alpha^k - \Sigma_j \alpha^j \alpha^{jk}) = 0$, $k = 1, \ldots, n$. It is now sufficient to apply the result of § 3.3.

**3.6. The lattice of left modules of finite span.** If $\Re$ is a regular ring then, as we shall prove below, the non-empty left modules of finite span form a relatively complemented modular lattice $L$ when partially ordered by inclusion; if the regular ring $\Re$ has a unit then $L$ has a unit and hence is complemented (note that the vector $u = (\alpha^1, \ldots, \alpha^n)$ is always in $(u)_l$ if $\Re$ is regular, for $eu = u$ with $e$ any idempotent such that $(e)_r = (\alpha^1, \ldots, \alpha^n)_r$). This is now easily verified, using the following statements:

(i) $L$ has a zero (left module of finite span) consisting of the zero vector $(0, \ldots, 0)$ only.

(ii) If $M^1$ is a left module spanned by vectors $u^{11}, \ldots, u^{1n}$ and $M^2$ is a left module spanned by vectors $u^{21}, \ldots, u^{2n}$, then the smallest left module containing $M^1$ and $M^2$ is spanned by $u^{11}, \ldots, u^{1n}$, $u^{21}, \ldots, u^{2n}$.

(iii) If $M^1$ and $M^2$ are left modules with canonical bases

$u^{1j}=(\alpha^{1j1},\ldots,\alpha^{1jn})$, $j=1,\ldots,n$, and $u^{2j}=(\alpha^{2j1},\ldots,\alpha^{2jn})$, $j=1,\ldots,n$,

respectively, then $M^0$, the set of all vectors common to $M^1$ and $M^2$ (clearly a left module) is a left module *of finite span*. We shall prove this now by induction on $n$ (for $n=1$ this is implied by (ii) of § 3.2).

Consider the $n$-th coordinate $\alpha^n$ of a vector $(\alpha^1,\ldots,\alpha^n)$ in $M^0$. For any such $\alpha^n$ it is clear that $\alpha^n\alpha^{1nn}=\alpha^n\alpha^{2nn}=\alpha^n$ so that, without changing the set of vectors in $M^0$, $u^{1n}$ and $u^{2n}$ may be replaced by $e^n u^{1n}$ and $e^n u^{2n}$ respectively where $e^n$ is any idempotent with $(e^n)_l=(\alpha^{1nn})_l(\alpha^{2nn})_l$. Thus we may suppose that $\alpha^{1nn}=\alpha^{2nn}=e^n$. Then necessary and sufficient conditions that $\alpha$ be the $n$-th coordinate of a vector in $M^0$ are: (i) $\alpha$ is in $(e)_l$ and (ii) for some $\alpha^i$, $\beta^i$, $\gamma^i$ $(i=1,\ldots,n-1)$,

$$(\alpha^1,\ldots,\alpha^{n-1})=\sum_{j=1}^{n-1}\beta^j(\alpha^{1j1},\ldots,\alpha^{1j(n-1)})+\alpha(\alpha^{1n1},\ldots,\alpha^{1n(n-1)})=$$
$$=\sum_{j=1}^{n-1}\gamma^j(\alpha^{2ji},\ldots,\alpha^{2j(n-1)})+\alpha(\alpha^{2n1},\ldots,\alpha^{2n(n-1)}).$$

The condition (ii), which involves vectors of length $n-1$, is equivalent to (the $\alpha^i$ may be ignored): $\alpha v$ is in $M$ where $v$ is the vector $(\alpha^{1ni}-\alpha^{2ni};\ i=1,\ldots,n-1)$ and $M$ is the left module spanned by $2n-2$ vectors of length $n-1$: $(\alpha^{1ji};\ i=1,\ldots,n-1)$, $(\alpha^{2ji};\ i=1,\ldots,n-1)$, $j=1,\ldots,n-1$.

It is now sufficient to apply the result of § 3.5 to see that these coordinates $\alpha$ form precisely a left principal ideal $(e)_l$, say.

Let $u$ be a vector in $M^0$ with $n$-th component $e$. Then a vector is in $M^0$ if and only if it differs by a multiple of $u$ from a vector common to $(M^1)'$ and $(M^2)'$, where $(M^1)'$ and $(M^2)'$ are spanned by $u^{1j}$, $j=1,\ldots,n-1$, and $u^{2j}$, $j=1,\ldots,n-1$, respectively.

It follows, by the induction, that $M^0$ is of finite span.

(iv) Suppose $M^1$ and $M^2$ are left modules with canonical bases $u^{1j}=(\alpha^{1j1},\ldots,\alpha^{1jn})$, $j=1,\ldots,n$, and $u^{2j}=(\alpha^{2j1},\ldots,\alpha^{2jn})$, $j=1,\ldots,n$, respectively and suppose $M^1$ is contained in $M^2$.

Then for each $j$, $(\alpha^{1jj})_l$ is contained in $(\alpha^{2jj})_l$. A relative complement of $M^1$ in $M^2$ may be obtained as $M$, the left module spanned by $u^1,\ldots,u^n$ with $u^j=(\alpha^{2jj}-\alpha^{2jj}\alpha^{1jj})u^{2j}$. For clearly this $M$ is a left module of finite span and is contained in $M^2$. Next, $M$ and $M^1$ have only the zero vector in common; for if

$$w=\sum_{j=1}^{n}\beta^j u^j=\sum_{j=1}^{n}\gamma^j u^{1j}$$

then, equating the $n$-th coordinates, we obtain $\beta^n(\alpha^{2nn}-\alpha^{2nn}\alpha^{1nn})=\gamma^n\alpha^{1nn}$;

multiplying on the right with the idempotent $\alpha^{1nn}$ shows that both sides of this equality are zero and hence:

$$w = \sum_{j=1}^{n-1} \beta^j u^j = \sum_{j=1}^{n-1} \gamma^j u^{1j}.$$

Successive reductions show that $w=0$, as stated. Finally $M^2$ is contained in $M \oplus M^1$ (and hence $M^2 = M \oplus M^1$): for the identity:

$$u^{2j} = u^j + \alpha^{2jj} u^{1j} + (\alpha^{2jj} \alpha^{1jj} u^{2j} - \alpha^{2jj} u^{1j})$$

shows that $u^{2j}$ = vector in $M$ + vector in $M^1 + r$ where $r$ is a vector in $M^2$ with $i$-th coordinate zero for all $i \geqq j$. Thus by induction on $k$, every vector in $M^2$ with at most the first $k$ coordinates different from zero, is contained in $M \oplus M^1$; when $k$ takes the value $n$, we obtain: $M^2$ is contained in $M \oplus M^1$, as stated.

(v) If $R$ has a unit 1, then $L$ clearly has as unit (left module of finite span) the left module spanned by $u^1, \ldots, u^n$ with $u^j = (\alpha^{j1}, \ldots, \alpha^{jn})$, $\alpha^{ji} = 0$ if $j \neq i$ and $\alpha^{ji} = 1$ if $j = i$.

## 4. Construction of the auxiliary ring.

**4.1. Homogeneous basis and normalized frame.** Let $L$ be a complemented modular lattice. Then $a_1, \ldots, a_n$ will be called a *homogeneous basis of order n* for $L$ if $a_1 \oplus \cdots \oplus a_n = 1$ and $a_i$ is perspective to $a_j$ for all $i, j$. We shall adopt the notation:

$$A^0 = 0; \quad A^i = a_1 + \cdots + a_i \qquad (i = 1, \ldots, n);$$
$$A^i_j = a_1 + \cdots + a_{j-1} + a_{j+1} + \cdots + a_i \qquad (1 \leqq j \leqq i \leqq n).$$

Suppose that for such a homogeneous basis, $a_1$ is perspective to $a_i$ with axis $x_i$ for $i = 1, \ldots, n$ (clearly $x_1 = 0$): set

$$c_{ij} = (x_i + x_j)\ (a_i + a_j)$$

for all $i, j$. Then as the reader may easily verify, the $c_{ij}$ $(i, j = 1, \ldots, n)$ have the properties: for all $i$, $j$, $k$,

$$c_{ij} = c_{ji}; \quad c_{ii} = 0; \qquad (c_{ij} + c_{jk})(a_i + a_k) = c_{ik}; \tag{4.1.1}$$
$$a_i \oplus c_{ij} = a_j \oplus c_{ij}.$$

A homogeneous basis $a_1, \ldots, a_n$ together with a set of $c_{ij}$ with the properties (4.1.1) will be called a *normalized frame* for $L$.

If $i, j, k$ are all different, $P_{kj:ij} \equiv P_{jk:ji}$ (to be written as $P_{k:i}$ if $j$ is unambiguous) will denote the perspective mapping of $L(a_i + a_j)$ onto

$L(a_k + a_j)$ determined by the axis $c_{ik}$. The perspective mapping $P$ will be called *non-crossing* if both $i > j$ and $k > j$ or both $i < j$ and $k < j$.

The collection of all inverses of $a_j$ in $a_i + a_j$ will be denoted by $L_{ij}$ (these concepts, basic for the coordinatization theorem, are due to VON NEUMANN [7, vol. 23, page 20; 8, vol. II, pages 30, 32, 53]).

Throughout the rest of this paper we shall assume $n \geqq 3$. We shall develop definitions for addition and multiplication to apply to the elements in $L_{ij}$ for arbitrary, but fixed $i, j$ with $i \neq j$, such that $L_{ij}$ becomes a regular ring with unit provided that the normalized frame satisfies the three conditions (4. 3. 3), (4. 3. 4) and (4. 10. 3) below (these conditions are equivalent to Desargues's theorem in the case of projective geometry). These three conditions need to be postulated only for the case $n = 3$ since, as we shall verify, they hold necessarily whenever $n \geqq 4$.

**4. 2. The addition construction for inverses.** An important construction, which for fixed $i, j$ applies to two elements $x, y$ in $L_{ij}$ and yields an element $z$ is the following: Choose any $A$, $B$ satisfying one or more of the properties:

(4. 2. 1) $$a_i + A + B \geqq x,$$

(4. 2. 2) $$a_i(A + B + a_j) = B(a_i + a_j) = 0,$$

(4. 2. 3) $$Aa_j = 0.$$

Then define[3])

(4. 2. 4) $$z = [\{(x + A)(a_i + B) + a_j\}(y + B) + A](a_i + a_j).$$

We shall verify:

(i) (4. 2. 1) *implies* $z + a_j = a_i + a_j$,

(ii) (4. 2. 2) *implies* $za_j = Aa_j$,

so that (4. 2. 1), (4. 2. 2) and (4. 2. 3) together imply that $z$ is in $L_{ij}$.

Proof of (i):

$$\begin{aligned} z + a_j &= [\{(x + A)(a_i + B) + a_j\}(y + B + a_j) + A](a_i + a_j) && \text{(AL)} \\ &= [(x + A)(a_i + B) + a_j + A](a_i + a_j) && \text{(ST)} \\ &= [(x + A)(a_i + B + A) + a_j](a_i + a_j) && \text{(AL)} \\ &= (x + A + a_j)(a_i + a_j) && \text{using (4. 2. 1)} \\ &= a_i + a_j. \end{aligned}$$

---

3) Suppose, in the usual (Cartesian) $u$, $v$ plane, that $x$ is $(u_1, 0)$, $y$ is $(u_2, 0)$, $a_i$ is the origin, $a_j$ is the point at infinity on the $u$ axis, $A$ is the point at infinity on the $v$ axis, and $B$ is the point at infinity on the line $u = v$: then $(u_1 + u_2, 0)$ coincides with the $z$ of (4. 2. 4).

Proof of (ii):

$$\begin{aligned} za_j &= [\{(x+A)(a_i+B)+a_j\}(y+B)+A]a_j \\ &= [\{(x+A)(A+a_j)(B+a_i)+a_j\}(y+B)+A]a_j && \text{(CI)} \\ &= [\{(x+A)(A+a_j)B+a_j\}(y+B)+A]a_j && \text{using (CI) and (4. 2. 2)} \\ &= [(x+A)(A+a_j)B+a_j(y+B)+A]a_j && \text{(ML)} \\ &= [(x+A)(A+a_j)B+A]a_j && \text{using (CI) and (4. 2. 2)} \\ &= (x+A)(A+B)a_j && \text{(AL)} \\ &= [A+x(A+B)(a_i+a_j)]a_j && \text{(ML)} \\ &= [A+x(A+B)a_j]a_j && \text{using (CI) and (4. 2. 2)} \\ &= Aa_j. \end{aligned}$$

We shall now show:

(4. 2. 5) *The z of* (4. 2. 4) $\geqq$ *some element E in* $L_{ij}$ *if* (4. 2. 1) *and* (4. 2. 2) *hold*[4]).

(4. 2. 6) *The z of* (4. 2. 4) $\leqq$ *some element F in* $L_{ij}$ *if* (4. 2. 2) *and* (4. 2. 3) *hold.*

Indeed, (4. 2. 5) holds with $E=[z-za_j]$ since this $E$ is in $L_{ij}$ ($E\oplus a_j = z+a_j=a_i+a_j$, assuming (4. 2. 1)).

Again (4. 2. 6) holds with $F=z+[(a_i+a_j)-(z+a_j)]$ since this $F$ is in $L_{ij}$ ($F+a_j=a_i+a_j$ and $a_jF=a_j(z+a_j)F=a_jz=0$, assuming (4. 2. 2) and (4. 2. 3)).

Of course, if (4. 2. 1), (4. 2. 2) and (4. 2. 3) all hold, then $E\leqq z\leqq F$ and the indivisibility of inverses shows that $E$ and $F$ coincide and coincide with $z$.

**4. 3. Uniqueness of the addition construction.** We shall now show that for $x, y$ fixed, the $E$ of (4. 2. 5) and the $F$ of (4. 2. 6) may be chosen independent of the $A$, $B$ at least to this extent. Suppose $A_0$, $B_0$ are fixed elements which satisfy (4. 2. 1), (4. 2. 2) and (4. 2. 3) hence determine some fixed $z_0$ in $L_{ij}$: *if we now restrict A, B by the additional condition:*

(4. 3. 1) $$(A_0+B_0+a_i+a_j)(A+B+a_i+a_j)=a_i+a_j,$$

*then* (4. 2. 5) *and* (4. 2. 6) *hold with this fixed* $z_0$ *for E and F.* In particular, if $A$, $B$ satisfy (4. 3. 1) and all of (4. 2. 1), (4. 2. 2) and (4. 2. 3), then the $z$ they determine coincides with this fixed $z_0$.

To prove this, we first make the following observations (i) to (iv):

(i) $(A+A_0)a_j=Aa_j$ if $A$, $B$ satisfy (4. 3. 1).

Indeed,

$$\begin{aligned} (A+A_0)a_j &= [A_0(A+a_j)(a_i+a_j)+A]a_j && \text{using (CI) and (4. 3. 1)} \\ &= [A_0\{a_i(A_0+a_j)+a_j\}(A+a_j)+A]a_j && \text{(CI)} \\ &= [A_0a_j(A+a_j)+A]a_j && \text{using (4. 2. 2)} \\ &= Aa_j && \text{using (4. 2. 3).} \end{aligned}$$

[4]) (4. 2. 2) is used only to prove uniqueness of $E$ in § 4. 3.

(ii) $A_0+A$ and $B_0+B$ satisfy (4. 2. 1) since $A_0$ and $B_0$ do.

(iii) $A_0+A$ and $B_0+B$ satisfy (4. 2. 2) if $A$ and $B$ satisfy (4. 2. 2) and (4. 3. 1).

Indeed,

$$\begin{aligned} a_i(A_0+A+B_0+B+a_j) &= a_i[(A+B)(A_0+B_0+a_i+a_j)+A_0+B_0+a_j] && \text{(CI)} \\ &= a_i[(A+B)(a_i+a_j)+A_0+B_0+a_j] && \text{using (4.3.1)} \\ &= a_i[(A+B+a_j)a_i+A_0+B_0+a_j] && \text{(AL), (ML)} \\ &= a_i(A_0+B_0+a_j) && \text{using (4.2.2)} \\ &= 0 && \text{using (4.2.2);} \end{aligned}$$

$$\begin{aligned} (B_0+B)(a_i+a_j) &= [B_0(B+a_i+a_j)+B](a_i+a_j) && \text{(CI)} \\ &= [B_0(a_i+a_j)+B](a_i+a_j) && \text{using (4.3.1)} \\ &= B(a_i+a_j) && \text{using (4.2.2)} \\ &= 0 && \text{using (4.2.2).} \end{aligned}$$

(iv) $A_0+A$ and $B_0+B$ satisfy (4. 2. 3) if $A$ and $B$ satisfy (4. 2. 3) and (4. 3. 1). This follows immediately from (i) above if $A$ and $B$ satisfy (4. 2. 3).

Let $z_1$ be the $z$ determined by $A_0+A$ and $B_0+B$ and let $z_2$ be the $z$ determined by $A$ and $B$.

Now if $A, B$ satisfy (4. 3. 1), (4. 2. 1) and (4. 2. 2) then $z_1 \geqq z_2$, $z_1+a_j=z_2+a_j$, $z_1a_j=z_2a_j$, hence by the general indivisibility of inverses, $z_1=z_2$. Since clearly $z_1 \geqq z_0$, it follows that $z_2 \leqq z_0$ as required.

Next, if $A, B$ satisfy (4. 3. 1), (4. 2. 2) and (4. 2. 3) then $z_1 \geqq z_2$, $z_1 \geqq z_0$, and $z_1$ is in $L_{ij}$ since $A_0+A$ and $B_0+B$ satisfy all of (4. 2. 1), (4. 2. 2) and (4. 2. 3). From the indivisibility of inverses it follows that $z_0=z_1 \geqq z_2$ as required. This completes the proof of the statement at the beginning of this section.

Thus the following theorem (4. 3. 2) holds if $n \geqq 4$ (as we show below):

(4. 3. 2) *For all* $A, B \leqq a_i+a_j+a_k$ *for some* $k$ *and satisfying all of* (4. 2. 1), (4. 2. 2), (4. 2. 3), *the* $z$ *of* (4. 2. 4) *has the same value (necessarily in* $L_{ij}$*).*

More generally, the following theorems (4. 3. 3), (4. 3. 4) hold if $n \geqq 4$ (as we show below):

(4. 3. 3) *There exists a fixed* $z_0$ *(necessarily in* $L_{ij}$*) such that: for all* $A, B \leqq a_i+a_j+a_k$ *for some* $k$ *and satisfying* (4. 2. 1) *and* (4. 2. 2), *the* $z$ *of* (4. 2. 4) $\geqq z_0$.

(4. 3. 4) *There exists a fixed* $z_0$ *(necessarily in* $L_{ij}$*) such that: for all* $A, B \leqq a_i+a_j+a_k$ *for some* $k$ *and satisfying* (4. 2. 2) *and* (4. 2. 3), *the* $z$ *of* (4. 2. 4) $\leqq z_0$.

*If* (4. 3. 2) *holds, in particular if* $n \geqq 4$, *we shall define* $x \dotplus y$ *to be the common* $z$ *of* (4. 3. 2) *(necessarily uniquely determined by* $x, y$*).*

We note that $A, B \leqq a_i + a_j + a_k$ for some $k$ and satisfying all of (4. 2. 1), (4. 2. 2), (4. 2. 3) do exist; for example, $A = c_{kj}, B = a_k$ with $k \neq i$. Indeed, with this choice for $A, B$,

$$\begin{aligned} &a_i + A + B = a_i + a_j + a_k \geqq x, \\ &a_i(A + B + a_j) = a_i(a_k + a_j) = 0, \\ &B(a_i + a_j) = a_k(a_i + a_j) = 0, \\ &A a_j = c_{kj} a_j = 0. \end{aligned}$$

Substituting $A = c_{kj}$, $B = a_k$ in (4. 2. 4) we obtain:

(4. 3. 5) $$x \dotplus y = [\{(x + c_{jk})(a_i + a_k) + a_j\}(y + a_k) + c_{jk}](a_i + a_j).$$

Hence all of (4. 3. 2), (4. 3. 3), (4. 3. 4) are non-vacuous; either of (4. 3. 3), (4. 3. 4) implies (4. 3. 2) with $z_0$ necessarily identical with the common $z$ of (4. 3. 2).

Easy calculation shows that $a_i$ is a zero for the addition $\dotplus$; that is, $a_i \dotplus x = x \dotplus a_i = x$ for all $x$ in $L_{ij}$.

P r o o f o f (4. 3. 3) a n d (4. 3. 4).

To prove (4. 3. 3) and (4. 3. 4) assuming $n \geqq 4$, let $z_0$ be taken as the $z$ determined by $A_0 = c_{mj}$, $B_0 = a_m$, for any $m \neq i, j$; this $z_0$ will be independent of the choice of $m$. For any $k \neq i, j$, there will be an $m \neq i, j, k$ since $n \geqq 4$.

Throughout the rest of this paper we shall assume *without explicit statement* that (4. 3. 2) does hold, so that $x \dotplus y$ is defined for $x, y$ in $L_{ij}$. Where (4. 3. 3) or (4. 3. 4) is required, an explicit assumption will be made[5]).

In section 7, we show that (4. 3. 2) is equivalent to the apparently stronger (4. 3. 3) and (4. 3. 4) in the case that the elements of $L_{ij}$ are atoms (this occurs when the elements of $L$ are the linear subspaces of a projective geometry).

**4. 4. The symmetric form for the addition construction.** Suppose now that $p, q$ are elements $\leqq a_i + a_j + a_k$ for some $k$ and that $A$ and $B$ are defined in terms of $p, q$ by the relations:

$$A = (p + x)(q + a_j), \quad B = q.$$

Then, as we shall show below, each of (4. 2. 1), (4. 2. 2), (4. 2. 3) is implied

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[5]) (4. 3. 3) and (4. 3. 4), a fortiori (4. 3. 2), hold necessarily, even if $n = 3$, if $x = a_i$ or $y = a_i$. Indeed, if $x = a_i$ and (4. 2. 2) holds, then the $z$ of (4. 2. 4) reduces to $y + A a_j$ which shows that (4. 3. 3) and (4. 3. 4) hold with $y$ for $z_0$; if $y = a_i$ and (4. 2. 2) holds, then the $z$ of (4. 2. 4) reduces to $x(A + B + a_i) + A a_j$ which shows that (4. 3. 3) and (4. 3. 4) hold with $x$ for $z_0$.

by a corresponding condition[6]):

(4.4.1) $$p + q = q + a_i,$$

(4.4.2) $$q(a_i + a_j) = 0, \quad p \leqq a_i + q,$$

(4.4.3) $$p(a_i + a_j) \leqq x.$$

The conditions (4.4.1), (4.4.2) and (4.4.3), for both $p, q$ and $q, p$, are together equivalent to:

(4.4.4) $$\begin{cases} p + a_i = q + a_i = p + q; \\ q(a_i + a_j) = p(a_i + a_j) = 0; \\ p + q \leqq a_i + a_j + a_k. \end{cases}$$

Finally $p = c_{ik}$, $q = a_k$ do satisfy (4.4.4).

That each of (4.4.1), (4.4.2), (4.4.3) implies the corresponding relation (4.2.1), (4.2.2), (4.2.3) respectively is shown as follows:

$$\begin{aligned} a_i + A + B &= a_i + q + (p + x)(q + a_j) \\ &= p + (p + x)(q + a_j) = (p + x)(p + q + a_j) && \text{using (4.4.1)} \\ &\geqq x(a_i + a_j) = x. \\ a_i(A + B + a_j) &= a_i(q + a_j) = 0 && \text{using (CI) and (4.4.2).} \\ A a_j &= (p + x) a_j \\ &= [p(a_i + a_j) + x] a_j && \text{(CI)} \\ &= x a_j && \text{using (4.4.3)} \\ &= 0. \end{aligned}$$

Now substitution for $A, B$ in (4.2.4) gives:

(4.4.5) $$\begin{aligned} z &= [\{(p + x)(q + a_j) + a_i\}(y + q) + (p + x)(q + a_j)](a_i + a_j) \\ &= [(p + a_j)(y + q) + (p + x)(q + a_j)](a_i + a_j) \end{aligned}$$

if (4.4.1) holds (using (ML)).

We can now derive relations between $x \dotplus y$ and $p$ and $q$. First, (4.3.2) shows at once that:

(4.4.6) $$x \dotplus y = [(p + a_j)(y + q) + (p + x)(q + a_j)](a_i + a_j)$$

if $p, q$ satisfy all of (4.4.1), (4.4.2), (4.4.3), in particular if, $p, q$ satisfy (4.4.4); in this case we shall write $(x \dotplus y)_{p,q}$ to denote the formal expression on the right side of (4.4.6) (its value is, of course, $x \dotplus y$). In particular, using $p = c_{ik}, q = a_k$, we obtain

(4.4.7) $$x \dotplus y = [(x + c_{ik})(a_k + a_j) + (y + a_k)(c_{ik} + a_j)](a_i + a_j).$$

Next, as we shall prove below:

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[6]) To derive (4.2.2) we use only $q(a_i + a_j) = 0$ but for a subsequent calculation it is advantageous to restrict (4.4.2) by the condition $p \leqq a_i + q$.

(4.4.8) *If* (4.3.3) *holds, then*:

$$x \dotplus y \leqq [(p+x)(q+a_j)+(q+y)(p+a_j)](a_i+a_j),$$

*provided that* (4.4.1), (4.4.2) *hold.*

(4.4.9) *If* (4.3.4) *holds, then:*

$$x \dotplus y \geqq [(p+x)(q+a_j)+(q+y)(p+a_j)](a_i+a_j),$$

*provided that* (4.4.2), (4.4.3) *hold.*

To prove (4.4.8) it need only be noted that if (4.4.1) and (4.4.2) hold, then (4.4.5) gives:

$$z = [(p+a_j)(y+q)+(p+x)(q+a_j)](a_i+a_j)$$
$$\geqq x \dotplus y$$

since (4.2.1) and (4.2.2) hold.

To prove (4.4.9) we note that if (4.4.2) and (4.4.3) hold, then $p \leqq q+a_i$ and (4.2.2), (4.2.3) hold; hence from (4.4.5):

$$x \dotplus y \geqq z \geqq [\{p+x(q+a_i)+a_j\}(y+q)+(p+x)(q+a_j)](a_i+a_j) \quad \text{(ML)}$$
$$\geqq [(p+a_j)(y+q)+(p+x)(q+a_j)](a_i+a_j).$$

**4.5. Commutativity of the addition construction.** Since $(x \dotplus y)_{p,q}$ is identical with $(y \dotplus x)_{q,p}$ it follows, using any $p, q$ which statisfy (4.4.4), that $x \dotplus y = y \dotplus x$.

**4.6. Associativity of the addition construction.** For fixed $y$ in $L_{ij}$ and $p, q$ satisfying (4.4.4) we define:

$$p' = (q+y)(p+a_j), \quad q' = (a_i+p')(q+a_j).$$

Then $p', q'$ also satisfy (4.4.4). To prove this we note the identities:

$$a_j+p' = a_j+p, \quad a_j+q' = a_j+q,$$
$$y+p' = y+q, \qquad p'q' = pq.$$

Now

$$a_i+q' = (a_i+p')(q+a_j+a_i) = a_i+p',$$
$$p'+q' = (a_i+p')(q+a_j+p') = a_i+p'.$$

Hence

$$p'+a_i = q'+a_i = p'+q'.$$

Also

$$p'(a_i+a_j) = a_j(q+y) = ya_j = 0,$$
$$q'(a_i+a_j) = a_j(a_i+p') = 0,$$
$$p'+q' \leqq p+q+a_j \leqq a_i+a_j+a_k.$$

Now if $x, y, w$ are all in $L_{ij}$, then, as we shall now prove:

(4.6.1) $$[(x \dotplus y)_{p,q} \dotplus w]_{p',q'} = [x \dotplus (y \dotplus w)_{p',q'}]_{p,q}.$$

Indeed,

left side of (4. 6. 1)

$$=[(p'+a_j)(w+q')+[p'+\{p'+(p+x)(q+a_j)\}(a_i+a_j)](q'+a_j)](a_i+a_j)$$
$$=[(p+a_j)(w+q')+\{(p+x)(q+a_j)+p'\}(q+a_j)](a_i+a_j)$$
$$=[(p+a_j)(w+q')+(p+x)(q+a_j)](a_i+a_j);$$

right side of (4. 6. 1)

$$=[(p+a_j)((y\dotplus w)_{p',q'}+q)+(p+x)(q+a_j)](a_i+a_j)$$
$$=[(p+a_j)[\{(p+a_j)(w+q')+q\}(a_i+a_j)+q]+(p+x)(q+a_j)](a_i+a_j)$$
$$=[(p+a_j)(w+q')+(p+x)(q+a_j)](a_i+a_j)$$
$$=\text{left side of (4. 6. 1)}.$$

**4. 7. Subtraction as the inverse of the addition construction.** We shall now verify that for given $x$ and $y$ in $L_{ij}$ the equation $w\dotplus y=x$ has a solution $w$ (the uniqueness of this $w$, to be denoted $x-y$, necessarily follows from the associativity and commutativity of $\dotplus$): for this purpose we set

(4. 7. 1) $$w=[\{x+(y+a_k)(c_{ik}+a_j)\}(a_k+a_j)+c_{ik}](a_i+a_j).$$

First we verify that this $w$ is in $L_{ij}$:

$$wa_j=[\{x+(y+a_k)(c_{ik}+a_j)\}(a_k+a_j)+c_{ik}]a_j$$
$$=[x+(y+a_k)(c_{ik}+a_j)]a_j \qquad \text{(CI), (ST)}$$
$$=[x+(y+a_k)(a_i+a_j)(c_{ik}+a_j)]a_j \qquad \text{(CI)}$$
$$=[x+ya_j]a_j \qquad \text{(ML)}$$
$$=xa_j=0;$$
$$w+a_j=[\{x+(y+a_j+a_k)(c_{ik}+a_j)\}(a_k+a_j)+c_{ik}](a_i+a_j) \qquad \text{(AL)}$$
$$=[(x+c_{ik}+a_j)(a_k+a_j)+c_{ik}](a_i+a_j) \qquad \text{(ST)}$$
$$=(a_k+a_j+c_{ik})(a_i+a_j) \qquad \text{(ST)}$$
$$=a_i+a_j, \qquad \text{(ST)}.$$

Next we verify that $w\dotplus y=x$; indeed from (4. 4. 7) we obtain:

$$w\dotplus y=[(w+c_{ik})(a_k+a_j)+(y+a_k)(c_{ik}+a_j)](a_i+a_j)$$
$$=[\{x+(y+a_k)(c_{ik}+a_j)\}(a_k+a_j)+(y+a_k)(c_{ik}+a_j)](a_i+a_j)$$
$$\text{(AL), (ST), (ML)}$$
$$=[x+(y+a_k)(c_{ik}+a_j)][a_k+a_j+(y+a_k)(c_{ik}+a_j)](a_i+a_j) \qquad \text{(AL)}$$
$$=[x+(y+a_k)(c_{ik}+a_j)](a_i+a_j) \qquad \text{(AL), (ST)}$$
$$=x+(y+a_k)(a_i+a_j)(c_{ik}+a_j) \qquad \text{(ML)}$$
$$=x+ya_j \qquad \text{(ML)}$$
$$=x,$$

as required.

Thus the elements of $L_{ij}$ form an abelian group under the addition $x \dotplus y$.

**4.8. Invariance of $\dotplus$ under the perspectivities $P$.** Suppose $x, y$ are in $L_{ij}$. We shall show:

(4.8.1) $$P(x \dotplus y) = (Px) \dotplus (Py) \quad \text{with} \quad P = P_{ik:ij} \qquad (\text{see § 4.1});$$

(4.8.2) $$P(x \dotplus y) = (Px) \dotplus (Py) \quad \text{with} \quad P = P_{kj:ij}.$$

Proof of (4.8.1). From (4.3.5), using the commutativity of $\dotplus$:

$$P_{ik:ij}(x \dotplus y) = [\{(y + c_{jk})(a_i + a_k) + a_j\}(x + a_k) + c_{jk}](a_i + a_k).$$

We obtain an expression for $(Px) \dotplus (Py)$ using (4.3.5) with $k$ and $j$ interchanged:

$$(P_{ik:ij}x) \dotplus (P_{ik:ij}y)$$
$$= \{[\{(x + c_{jk})(a_i + a_k) + c_{kj}\}(a_i + a_j) + a_k][(y + c_{jk})(a_i + a_k) + a_j] + c_{kj}\}(a_i + a_k)$$
$$= [(x + a_k)\{(y + c_{jk})(a_i + a_k) + a_j\} + c_{kj}](a_i + a_k) \qquad \text{(AL), (ST), (ML)}$$
$$= P_{ik:ij}(x \dotplus y).$$

Proof of (4.8.2). From (4.4.7)

$$x \dotplus y = [(x + c_{ik})(a_k + a_j) + (y + a_k)(c_{ik} + a_j)](a_i + a_j).$$

We may use $A = x \dotplus y$, $B = y$ in (4.2.4) to calculate right side of (4.8.2) for, as we now show, these $A, B$ satisfy the relevant conditions (4.2.1), (4.2.2) and (4.2.3) (with $i$ and $k$ interchanged). Indeed $A, B$ are both in $L_{ij}$, and

$$A + B = [(x + c_{ik})(a_k + a_j) + y + a_k](a_i + a_j), \qquad \text{(AL), (ST)}.$$

Hence:

$$a_k + A + B \geqq (x + c_{ik})(a_k + a_j); \quad a_k(A + B + a_j) = 0; \quad B(a_k + a_j) = 0; \quad Aa_j = 0.$$

With these $A, B$, (4.2.4) (with $i$ and $k$ interchanged) gives:

right side of (4.8.2)

$$= \{[\{(x + c_{ik})(a_k + a_j) + A\}(a_k + y) + a_j][(y + c_{ik})(a_k + a_j) + y] + A\}(a_k + a_j).$$

Since

$$(y + c_{ik})(a_k + a_j) + y = (y + c_{ik})(a_i + a_j + a_k) \geqq c_{ik}, \qquad \text{(AL)}$$

and

$$(x + c_{ik})(a_k + a_j) + A \geqq (y + a_k)(c_{ik} + a_j), \qquad \text{(AL), (ST)},$$

therefore

$$\text{right side of (4.8.2)} \geqq [\{(y + a_k)(c_{ik} + a_j) + a_j\}c_{ik} + A](a_k + a_j)$$
$$= (c_{ik} + A)(a_k + a_j) \qquad \text{(AL), (ST)}$$
$$= \text{left side of (4.8.2)}.$$

Since both sides of (4.8.2) are in $L_{ij}$, the indivisibility of inverses shows that $\geqq$ in (4.8.2) implies $=$ in (4.8.2).

(4. 8. 1) and (4. 8. 2) show that the abelian group $L_{ij}$ (under $\dotplus$) is mapped group-isomorphically on the group $L_{ik}$ by $P_{ik:ij}$ and is mapped group-isomorphically on the group $L_{kj}$ by $P_{kj:ij}$ (in particular, $x-y$, the subtraction of inverses, is invariant under the mappings $P_{ik:ij}, P_{kj:ij}$).

**4. 9. The multiplication construction for inverses.** A second construction which, for fixed $i, j$, applies to two elements $x, y$ in $L_{ij}$ and yields an element $z$ is the following: Choose any $A, B$ satisfying one or more of the properties[7]):

(4. 9. 1) $$x+A+B \geqq a_i.$$

(4. 9. 2) $$a_i(A+B+a_j)=B(a_i+a_j)=0.$$

(4. 9. 3) $$Aa_j=0.$$

Then define:

(4. 9. 4) $$z=[(x+A)(a_i+B)+\{(c_{ij}+A)(a_i+B)+y\}(a_j+B)](a_i+a_j).$$

We shall verify:

(i) (4. 9. 1) *implies* $z+a_j=a_i+a_j$,

(ii) (4. 9. 2) *implies* $z\,a_j=[(c_{ij}+A\,a_j)a_i+y]a_j$,

so that (4. 9. 1), (4. 9. 2), and (4. 9. 3) together imply that $z$ is in $L_{ij}$.

Proof of (i):

$$\begin{aligned} z+a_j &= [(x+A)(a_i+B)+\{(c_{ij}+A)(a_i+B)+y+a_j\}(a_j+B)](a_i+a_j) && \text{(AL)}\\ &= [(x+A)(a_i+B)+(a_i+a_j+A)(a_i+a_j+B)(a_j+B)](a_i+a_j) && \text{(AL)}\\ &= [(x+A)(a_i+B)+(a_i+a_j+A)(a_j+B)](a_i+a_j) && \text{(ST)}\\ &= (a_i+a_j+A)[a_j+B+(x+A)(a_i+B)](a_i+a_j) && \text{(AL)}\\ &= a_j+(x+A+B)(a_i+B)(a_i+a_j) && \text{(ST), (ML), (AL)}\\ &= a_j+a_i && \text{from (4. 9. 1), (ST), (ML) and (4. 9. 2).} \end{aligned}$$

Proof of (ii):

$$\begin{aligned} z\,a_j &= [(x+A)(a_i+B)(a_j+B)+\{(c_{ij}+A)(a_i+B)+y\}(a_j+B)]a_j && \text{(CI)}\\ &= [(x+A)B+\{(c_{ij}+A)(a_i+B)+y\}(a_j+B)]a_j && \text{(ML) and (4. 9. 2)}\\ &= [AB+\{(c_{ij}+A)(a_i+B)+y\}(a_j+B)]a_j && \text{(CI), (ST) and (4. 9. 2)}\\ &= [(c_{ij}+A)(a_i+B)+y]a_j && \text{(ST)}\\ &= [(c_{ij}+A)(a_i+B)(a_i+a_j)+y]a_j && \text{(CI)}\\ &= [\{c_{ij}+A(a_i+a_j)\}a_i+y]a_j && \text{(ML) and (4. 9. 2)}\\ &= [(c_{ij}+Aa_j)a_i+y]a_j && \text{(CI) and (4. 9. 2).} \end{aligned}$$

---

[7]) (4. 9. 2) is identical with (4. 2. 2), (4. 9. 3) with (4. 2. 3); in the presence of (4. 9. 2), (4. 9. 1) and (4. 2. 1) are equivalent and equivalent to $a_i+A+B=x+A+B$ (this follows from the general indivisibility of inverses since $a_i+A+B+a_j=x+A+B+a_j$ and $(a_i+A+B)a_j=(A+B)a_j=(x+A+B)a_j$).

It now follows, as in the proof of (4.2.5), (4.2.6) that:

(4.9.5) *The z of* (4.9.4) $\geqq$ *some element E in* $L_{ij}$ *if* (4.9.1) *and* (4.9.2) *hold*[8].
(4.9.6) *The z of* (4.9.4) $\leqq$ *some element F in* $L_{ij}$ *if* (4.9.2) *and* (4.9.3) *hold*.

Of course, if (4.9.1), (4.9.2) and (4.9.3) all hold, then $E \leqq z \leqq F$ and the indivisibility of inverses shows that $E$ and $F$ coincide and coincide with $z$.

**4.10. Uniqueness of the multiplication construction.** The argument of § 4.3 shows: for $x, y$ fixed, the $E$ of (4.9.5) and the $F$ of (4.9.6) may be chosen independent of the $A$, $B$ at least to this extent. Suppose $A_0$, $B_0$ are fixed elements which satisfy all of (4.9.1), (4.9.2) and (4.9.3) and hence determine some fixed $z_0$ in $L_{ij}$; *if we now restrict A, B by the additional condition:*

(4.10.1) $$(A_0+B_0+a_i+a_j)(A+B+a_i+a_j)=a_i+a_j,$$

*then* (4.9.5) *and* (4.9.6) *hold with this fixed* $z_0$ *for E and F.* In particular, if $A$, $B$ satisfy (4.10.1) and all of (4.9.1), (4.9.2) and (4.9.3) then the $z$ they determine coincides with this fixed $z_0$.

Thus, with proofs as in § 4.3, if $n \geqq 4$:

(4.10.2) *For all* $A, B \leqq a_i+a_j+a_k$ *for some k and satisfying all of* (4.9.1), (4.9.2), (4.9.3), *the z of* (4.9.4) *has the same value (necessarily in* $L_{ij}$).

More generally, the following theorems (4.10.3), (4.10.4) hold if $n \geqq 4$:

(4.10.3) *There exists a fixed* $z_0$ *(necessarily in* $L_{ij}$*) such that: for all* $A, B \leqq a_i+a_j+a_k$ *for some k and satisfying* (4.9.1) *and* (4.9.2), *the z of* (4.9.4) $\geqq z_0$.

(4,10.4) *There exists a fixed* $z_0$ *(necessarily in* $L_{ij}$*) such that: for all* $A, B \leqq a_i+a_j+a_k$ *for some k and satisfying* (4.9.2) *and* (4.9.3), *the z of* (4.9.4) $\leqq z_0$.

*If* (4.10.2) *holds, in particular if* $n \geqq 4$, *we shall define* $x \times y$ *to be the common z of* (4.3.2) *(necessarily uniquely determined by x, y).*

---

[8]) We actually use only (4.9.1) here but the uniqueness of $E$ as established in the next section applies only if the additional condition (4.9.2) holds. We note here that in the presence of condition (4.2.2), that is, (4.9.2), the discussions of both §§ 4.2, 4.3 for $\dotplus$ and §§ 4.9, 4.10 for $\times$, could be included (as special cases) in a single discussion of a general construction, for $x, y, w$ in $L_{ij}$:

$$z' = [\{(x+A)(a_i+B)+a_j\}(y+B)+\{(A+c_{ij})(a_i+B)+w\}(a_j+B)](a_i+a_j).$$

This $z'$ reduces to (4.2.4) if $c_{ij}$ is chosen for $w$; on the other hand, if $a_i$ is chosen for $y$ and then $w$ replaced by $y$, $z'$ reduces to (4.9.4). This $z'$ actually expresses $(x \times w) \dotplus y$ (see (5.2.2)).

We note that $A, B \leqq a_i + a_j + a_k$ for some $k$ and satisfying all of (4.9.1), (4.9.2), (4.9.3) do exist, for example $A = c_{kj}$, $B = a_k$ with $k \neq i, j$. Substituting $A = c_{kj}$, $B = a_k$ in (4.9.4) we obtain:

$$(4.10.5) \qquad x \times y = [(x + c_{jk})(a_i + a_k) + (y + c_{ik})(a_j + a_k)](a_i + a_j)$$
$$= (P_{ik:ij}x + P_{kj:ij}y)(a_i + a_j).$$

Hence if any of (4.10.2), (4.10.3), (4.10.4) hold, they are non-vacuous and either of (4.10.3), (4.10.4) implies (4.10.2) with $z_0$ necessarily identical with the common $z$ of (4.3.2).

Throughout the rest of this paper we shall assume *without explicit statement* that (4.10.2) does hold, so that $x \times y$ is defined for $x, y$ in $L_{ij}$. Where (4.10.3) is required, an explicit assumption will be made[9,10]).

We shall show in section 7 that (4.10.2) is equivalent to the apparently stronger (4.10.3) and (4.10.4) in the case that the elements of $L_{ij}$ are atoms.

Easy calculation shows that $a_i$ is a two-sided zero and $c_{ij}$ is a two-sided unit for this multiplication; that is,

$$a_i \times x = x \times a_i = a_i, \quad c_{ij} \times x = x \times c_{ij} = x,$$

for all $x$ in $L_{ij}$.

**4.11. Associativity of multiplication.** We will now verify that if $w, x, y$ are in $L_{ij}$, then:

$$(w \times x) \times y = w \times (x \times y).$$

We note that if $u = v \times x$ where $v$ is an arbitrary inverse, then $A = (x + c_{ik})(a_k + a_j)$, $B = a_k$ may be used in (4.9.4) to obtain $u \times y$. For as we shall show, these $A$, $B$ satisfy the relevant conditions (4.9.1), (4.9.2) and (4.9.3):

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[9]) (4.10.4) has been given for completeness but is never actually assumed in our present deduction of the coordinatization theorem (see footnote 12) so that, as follows from this theorem, (4.3.3), (4.3.4) and (4.10.3) together imply (4.10.4). The existence of non-Desarguesian harmonic-point projective plane geometries shows that (4.3.3) and (4.3.4) do not necessarily imply (4.10.2) (see footnote 19, p. 245).

[10]) (4.10.3) and (4.10.4), a fortiori (4.10.2), hold necessarily, even in the case $n=3$, if $x = a_i$, or $x = c_{ij}$, or $y = a_i$ or $y = c_{ij}$. Indeed if $x = a_i$ and (4.9.2) holds, then the $z$ of (4.9.4) reduces to $a_i + [y + (c_{ij} + Aa_j)a_i]a_j$ which shows that (4.10.3) and (4.10.4) hold with $a_i$ for $z_0$; if $x = c_{ij}$ and (4.9.2) holds, then the $z$ of (4.9.4) reduces to $(Aa_j + c_{ij})a_i + y[a_j + (A + B \quad c_{ij})a_i]$ which shows that (4.10.3) and (4.10.4) hold with $y$ for $z_0$. If $y = a_i$ and (4.9.2) holds, then the $z$ of (4.9.4) reduces to $(x + A + B)a_i$ which shows that (4.10.3) and (4.10.4) hold with $a_i$ for $z_0$; if $y = c_{ij}$ and (4.9.2) holds, then the $z$ of (4.9.4) reduces to $Aa_j + x(a_i + A + B)$ which, with the help of footnote 7, shows that (4.10.3) and (4.10.4) hold with $x$ for $z_0$.

indeed, using (4. 10. 5) to express $v \times x$, yields:

(4. 11. 1)
$$(v \times x)+A=[(v+c_{jk})(a_i+a_k)+A](a_i+a_j)+A$$
$$=(v+c_{jk})(a_i+a_k)+A, \qquad \text{(AL), (ST)};$$
$$(v \times x)+A+B=a_i+a_k+A \geqq a_i, \qquad \text{(AL), (ST)};$$
$$a_i(A+B+a_j)=a_i(a_k+a_j)=0; \quad B(a_i+a_j)=a_k(a_i+a_j)=0;$$
$$Aa_j=(x+c_{ik})a_j=xa_j=0 \qquad \text{(CI)}.$$

Now use these $A$, $B$ in (4. 9. 4) to obtain $x \times y$ and also $(w \times x) \times y$. Then:

$$x \times y=[(x+A)(a_i+a_k)+\{(c_{ij}+A)(a_i+a_k)+y\}(a_j+a_k)](a_i+a_j)$$
$$=[c_{ik}+xa_i+\{(c_{ij}+A)(a_i+a_k)+y\}(a_j+a_k)](a_i+a_j) \qquad \text{(AL), (ST), (ML)}$$
$$\geqq [c_{ik}+\{(c_{ij}+A)(a_i+a_k)+y\}(a_j+a_k)](a_i+a_j);$$

this last expression is in $L_{ij}$ along with $x \times y$, and hence $= x \times y$ by the indivisibility of inverses. Hence:

$$[(x \times y)+c_{ik}](a_j+a_k)=[(c_{ij}+A)(a_i+a_k)+y](a_i+a_k) \qquad \text{(AL), (ST), (ML)},$$

and so, using (4. 10. 5),

$$w \times (x \times y)=[(w+c_{jk})(a_i+a_k)+((x \times y)+c_{ij})(a_j+a_k)](a_i+a_j)$$
$$=[(w+c_{jk})(a_i+a_k)+\{(c_{ij}+A)(a_i+a_k)+y\}(a_j+a_k)](a_i+a_j).$$

Again,

$$(w \times x) \times y$$
$$=[((w \times x)+A)(a_i+a_k)+\{(c_{ij}+A)(a_i+a_k)+y\}(a_j+a_k)](a_i+a_j),$$

and, using (4. 11. 1) to express $w \times x$,

$$=[\{(w+c_{jk})(a_i+a_k)+A\}(a_i+a_k)+\{(c_{ij}+A)(a_i+a_k)+y\}(a_j+a_k)](a_i+a_j)$$
$$=[(w+c_{jk})(a_i+a_k)+Aa_k+\{(c_{ij}+A)(a_i+a_k)+y\}(a_j+a_k)](a_i+a_j) \qquad \text{(ML)}$$
$$=[(w+c_{jk})(a_i+a_k)+\{(c_{ij}+A)(a_i+a_k)+y\}(a_j+a_k)](a_i+a_j) \qquad \text{(ST)}$$
$$=w \times (x \times y),$$

which establishes the associativity of multiplication.

**4. 12. The regularity of multiplication for inverses.** From §§ 4. 9, 4. 10, 4. 11 it follows that the elements of $L_{ij}$ under the multiplication $x \times y$ form a semi-group with unit.

We shall now show that the multiplication is regular (see § 3. 1 for definition of regularity). For this purpose we associate with each $x$ in $L_{ij}$ a lattice element $x^r$ which we shall call the *reach of x*, defined as

$$x^r=(x+a_i)a_j.$$

We shall prove below:

(4. 12. 1) *For every* $b \leqq a_j$ *and* $d = [a_j - b]$ *there is an* $e$ *in* $L_{ij}$ *with* $e \times e = e$, $e^{r} = b$, $(c_{ij} - e)^{r} = d$.

(4. 12. 2) *For* $x, y$ *in* $L_{ij}$ *there is a* $w$ *with* $w \times x = y$ *if and only if* $x^r \geqq y^r$.

From (4. 12. 1) it will follow that there is an idempotent $e$ with $e^r = x^r$ and hence from (4. 12. 2), for suitable $y$ and $w$ in $L_{ij}$, $y \times e = x$ and $w \times x = e$. From the associativity of multiplication for inverses, this implies

$$x \times w \times x = x \times e = (y \times e) \times e = y \times e = x,$$

which shows that the multiplication is regular.

We note that this will also show:

(4. 12. 3) *The correspondence* $(x)_l \rightarrow x^r$ *sets up a* $(1, 1)$ *order preserving mapping between the set of all left cosets of* $L_{ij}$ *and the lattice* $L(a_j)$.

(4. 12. 3) implies that the cosets $(x)_l$ form a complemented modular lattice under the relation of inclusion.

Proof of (4. 12. 1): Set $e = (b + a_i)(d + c_{ij})$. Then $e$ can be expressed as $e = ea_i + ec_{ij}$. This $e$ is in $L_{ij}$ since:

$$\begin{aligned} ea_j &= (b + a_i)(d + c_{ij})a_j = (b + a_i a_j)(d + c_{ij} a_j), \quad \text{using } b, d \leqq a_j, \text{(ML)} \\ &= bd = 0, \end{aligned}$$

$$\begin{aligned} e + a_j &= (b + a_i)(d + c_{ij}) + b + d \\ &= (b + a_i + d)(b + d + c_{ij}) && \text{(AL)} \\ &= (a_i + a_j)(a_i + a_j) = a_i + a_j. \end{aligned}$$

This $e$ satisfies the requirements of (4. 12. 1), for using (4. 10. 5) and (ML),

$$\begin{aligned} e \times e &= ea_i + [(ec_{ij} + c_{jk})(a_i + a_k) + (e + c_{ik})(a_k + a_j)](a_i + a_j) \\ &= ea_i + [(ec_{ij} + c_{jk})c_{ik} + (e + c_{ik})(a_k + a_j)](a_i + a_j) && \text{(ST)} \\ &= ea_i + (e + c_{ik})[(ec_{ij} + c_{jk})c_{ik} + a_k + a_j](a_i + a_j) && \text{(AL)} \\ &= ea_i + e[(ec_{ij} + c_{jk})(a_i + a_k) + a_k + a_j] && \text{(ML)} \\ &= ea_i + e(ec_{ij} + c_{jk} + a_k)(a_i + a_j + a_k) && \text{(AL)} \\ &= ea_i + ec_{ij} = e && \text{(ST), (ML).} \end{aligned}$$

Next,

$$e_j^{r} = [(b + a_i)(d + c_{ij}) + a_i]a_j = (b + a_i)(a_i + a_j)a_j = b.$$

Finally, since (4. 7. 1) implies that $(x - y)^{r} = (x + y)a_j$ for all $x, y$ in $L_{ij}$,

$$(c_{ij} - e)^{r} = (c_{ij} + e)a_j = [c_{ij} + (b + a_i)(d + c_{ij})]a_j = d.$$

This completes the proof of (4. 12. 1).

Proof of (4. 12. 2): Because of the indivisibility of inverses, $w \times x = y$ is equivalent to $w \times x \leqq y$, and this in turn is equivalent to:

$$(4.12.4) \qquad (w \times x) + (x + c_{ik})(a_j + a_k) \leqq y + (x + c_{ik})(a_j + a_k)$$

(clip to both sides of (4. 12. 4) by $(a_i + a_j)$ to derive $w \times x \leqq y$). Now (4. 12. 4) is equivalent (use (4. 10. 5) to express $w \times x$) to:

$$(w + c_{jk})(a_i + a_k) + (x + c_{ik})(a_j + a_k) \leqq y + (x + c_{ik})(a_j + a_k) \qquad \text{(AL), (ST)}$$

hence to:

$$(w + c_{jk})(a_i + a_k) \leqq y + (x + c_{ik})(a_j + a_k);$$

hence to:

$$(4.12.5) \qquad (w + c_{jk})(a_i + a_k) \leqq [y + (x + c_{ik})(a_j + a_k)](a_i + a_k);$$

hence to

$$(4.12.6) \quad (w + c_{jk})(a_i + a_k) + c_{jk} \leqq [y + (x + c_{ik})(a_j + a_k)](a_i + a_k) + c_{jk}$$

(clip both sides of (4. 12. 6) by $a_i + a_k$ to recover (4. 12. 5)); hence to:

$$(4.12.7) \qquad w \leqq [y + (x + c_{ik})(a_j + a_k)](a_i + a_k) + c_{jk}.$$

Now the right side of (4. 12. 7) $\geqq$ some $w$ in $L_{ij}$ if and only if:

$$(4.12.8) \qquad (\text{right side of } (4.12.7)) + a_j \geqq a_i + a_j$$

(if (4. 12. 8) holds, $w$ may be chosen as

$$w = [(\text{right side of } (4.12.7))(a_i + a_j) - \text{right side of } (4.12.7))a_j]).$$

Thus $w \times x = y$ for some $w$ in $L_{ij}$ if and only if:

$$(y + (x + c_{ik})(a_j + a_k)](a_i + a_k) + c_{jk} + a_j \geqq a_i + a_j.$$

This last condition is equivalent to each of the following:

$$(4.12.9) \qquad y + (x + a_i + a_k)(a_j + a_k) \geqq a_i,$$

$$(4.12.10) \qquad y + (x + a_i + a_k)a_j \geqq a_i,$$

$$(4.12.11) \qquad y + (x + a_i)a_j \geqq y + a_i,$$

$$(4.12.12) \qquad (x + a_i)a_j \geqq (y + a_i)a_j,$$

(clip both sides of (4. 12. 11) by $a_j$ to obtain (4. 12. 12); add $y$ to both sides of (4. 12. 12) to recover (4. 12. 11)). This completes the proof of (4. 12. 2) and establishes the regularity of multiplication.

We note that $(a_i)^r = (a_i + a_j)a_j = 0$ and hence using (4. 12. 2), $x^r = 0$ if and only if $x = a_i$.

We shall now prove that the idempotent $e$ of (4. 12. 1) is uniquely determined by $b$ and $d$. First we prove:

(4.12.13) *If* $b \oplus d = a_j$, *then* $(b+a_i)(d+c_{ij})$ *is an idempotent* $e$ *in* $L_{ij}$ *such that* $e^r = b$, $(c_{ij}-e)^r = d$; *conversely, if* $e$ *is an idempotent, and if* $b = e_r$, $d = (c_{ij}-e)^r$, *then* $b \oplus d = a_j$ *and* $e = (b+a_i)(d+c_{ij}) = = ea_i + ec_{ij}$.[11])

Proof of (4.12.13). The "if" part was shown in the proof of (4.12.1). To establish the "only if" part we assume $e$ to be idempotent; that is, using (4.10.5):

$$e = [(e+c_{ik})(a_k+a_j)+(e\dot{+}c_{kj})(a_i+a_k)](a_i+a_j).$$

Now

$$e \geqq c_{ij}(e+a_i). \tag{4.12.16}$$

For

$$\begin{aligned} e &= e+e \qquad \text{(lattice union, not to be confused with } e\dot{+}e) \\ &= [e+c_{ik}+(e+c_{kj})(a_i+a_k)](a_i+a_j) \qquad \text{(AL), (ST)} \\ &\geqq [c_{ik}+c_{kj}(e+a_i+a_k)](a_i+a_j) \geqq c_{ij}(e+a_i). \end{aligned}$$

Since $(x-y)^r = (x+y)a_j$, therefore $b = (e+a_i)a_j$ and $d = (c_{ij}+e)a_j$. Hence:

$$b+d = (e+a_i)a_j+(c_{ij}+e)a_j = [e+a_i+(c_{ij}+e)a_j]a_j = (e+a_i+c_{ij})a_j = a_j;$$

$$bd = (e+a_i)(c_{ij}+e)a_j = [e+c_{ij}(e+a_i)]a_j = 0, \qquad \text{using (4.12.16);}$$

$$(b+a_i)(d+c_{ij}) = (e+a_i)(e+c_{ij}) = e+c_{ij}(e+a_i) = e, \qquad \text{using (4.12.16).}$$

This completes the proof of (4.12.13). The uniqueness of the idempotent $e$ in (4.12.1) follows since (4.12.13) shows that an idempotent $e$ is determined by $e^r$ and $(c_{ij}-e)^r$.

**4.13. Invariance of $\times$ under the perspectivities $P$.** Suppose $x, y$ are in $L_{ij}$. We shall show, assuming (4.10.3)[12]):

(4.13.1) $\quad P(x\times y) = (Px)\times(Py)$ with $P = P_{ik;ij}$ (see § 4.1);

(4.13.2) $\quad P(x\times y) = (Px)\times(Py)$ with $P = P_{kj;ij}$.

[11]) If $a_i$ is an atom, clearly the only idempotents in $L_{ij}$ are the zero $a_i$ and the unit $c_{ij}$.

[12]) Note that the abelian group character of the $L_{ij}$ under $\dot{+}$ and their group-isomorphism under the perspectivities $P$ follow from the assumption (4.3.2) alone (in this connection see footnote 19); the semi-group character of the $L_{ij}$ under $\times$, the regularity of $\times$, and (4.13.2) follow from the assumption (4.10.2) alone. However our proof of (4.13.1) requires the additional assumption (4.10.3) and our proofs of distributivity (4.14.1) and (4.14.2) require the additional assumptions (4.3.4) and (4.3.3) respectively; whether some or all of these additional conditions (4.3.3), (4.3.4) and (4.10.3) are actually implied by (4.3.2) and (4.10.2) is not known but it is not difficult to verify, assuming only (4.3.2) and (4.10.2), that: the distributivity (4.14.1) holds in the case that $w$ is an idempotent or $wa_i = 0$, and the distributivity (4.14.2) holds in the case that $w$ is an idempotent or $w+a_i = a_i+a_j$.

Proof of (4.13.1). We may obtain an element $\geqq x \times y$ from (4.10.3) with $A = (P_{k:j}y + c_{ij})(a_k + a_j)$ and $B = a_k$ for these $A, B$ satisfy the conditions (4.9.1) and (4.9.2) (this follows from the fact that $A$ is in $L_{jk}$ and $A + B = a_j + a_k$). We obtain:

$$
\begin{aligned}
x \times y &\leqq [(x+A)(a_i + a_k) + \{(c_{ij} + A)(a_i + a_k) + y\}(a_j + a_k)](a_i + a_j) \\
&= [(x+A)(a_i + a_k) + (P_{k:j}y + y)(a_j + a_k)](a_i + a_j) && \text{(AL), ST)} \\
&= [(x+A)(a_i + a_k) + (y + c_{jk})(y + a_i + a_k)(a_j + a_k)](a_i + a_j) && \text{(AL)} \\
&= [(x+A)(a_i + a_k) + c_{jk}(y + a_i + a_k)(a_j + a_k)](a_i + a_j) && \text{(ML)} \\
&\leqq [(x+A)(a_i + a_k) + c_{jk}](a_i + a_j).
\end{aligned}
$$

Then

$$
\begin{aligned}
P_{k:i}(x \times y) &\leqq (x + A)(a_i + a_k) \\
&= [x + (P_{k:j}y + c_{ij})(a_k + a_j)](a_i + a_k) \\
&= P_{k:j}x \times P_{k:j}y
\end{aligned}
$$

using (4.10.5) with $j$ and $k$ interchanged to express $P_{k:j}x \times P_{k:j}y$. The indivisibility of inverses now shows that equality holds in (4.13.1).

Proof of (4.13.2). We may express $x \times y$ by (4.9.4) with $A = P_{k:i}x$ and $B = a_k$ for these $A$, $B$ satisfy the conditions (4.9.1), (4.9.2), (4.9.3) (this follows from the fact that $A$ and $B$ are both in $L_{kj}$ and

$$x + A + B = x + c_{ik} + a_k = x + a_i + a_k \geqq a_i).$$

We obtain:

$$
\begin{aligned}
x \times y &= [(x + c_{ik})(a_i + a_k) + \{(c_{ij} + P_{k:i}x)(a_i + a_k) + y\}(a_j + a_k)](a_i + a_j) \\
&\geqq [c_{ik} + \{(c_{ij} + P_{k:i}x)(a_i + a_k) + y\}(a_j + a_k)](a_i + a_j).
\end{aligned}
$$

$$P_{k:i}(x \times y) \geqq [(c_{ij} + P_{k:i}x)(a_i + a_k) + y](a_k + a_j) = P_{k:i}x \times P_{k:i}y$$

using (4.10.5) with $i$ and $k$ interchanged to express $P_{k:i}x \times P_{k:i}y$. The indivisibility of inverses now shows that equality holds in (4.13.2).

**4.14. The distributivity of $\times$ with respect to $\dotplus$.** Suppose $w, x, y$ are in $L_{ij}$. We shall show, assuming (4.3.3) and (4.3.4)[12]):

(4.14.1) $$w \times (x \dotplus y) = (w \times x) \dotplus (w \times y);$$

(4.14.2) $$(x \dotplus y) \times w = (x \times w) \dotplus (y \times w).$$

Proof of (4.14.1). We may obtain an element $\leqq P_{k:i}x \dotplus P_{k:i}y$ from (4.4.9) (which in turn is derived from (4.3.4)) with $i$ and $k$ interchanged, using $q = w \times y$ and $p = (q + a_k)(P_{k:j}w + a_j)$, for these $p, q$ satisfy the relevant conditions (4.4.2) and (4.4.3) with $i$ and $k$ interchanged; this follows from the fact that $q$ is in $L_{ij}$ and

$$p(a_k + a_j) = a_k(P_{k:j}w + a_j) = (w + c_{jk})a_k = 0 \leqq P_{k:i}x.$$

We obtain:

(4. 14. 13) $P_{k:i}x \dotplus P_{k:i}y \geqq [(p+P_{k:i}x)(a_i+a_j)+(q+P_{k:i}y)\,P_{k:j}w+a_j)]\,(a_k+a_j)$.

Now, using (4. 10. 5) with $w$ in place of $x$,

$$q+P_{k:i}y=(w\times y)+P_{k:i}y=(P_{k:j}w+P_{k:i}y)\,(a_i+a_j)+P_{k:i}y=P_{k:j}w+P_{k:j}y.$$

Now from (4. 14. 3), since $P(x\dotplus y)=Px\dotplus Py$ (from (4. 8. 2)),

$$P_{k:i}(x\dotplus y)\geqq[(p+P_{k:i}x)\,(a_i+a_j)+P_{k:j}w]\,(a_k+a_j).$$

Hence, using (4. 10. 5) with $w$ in place of $x$ and $x\dotplus y$ in place of $y$,

$$\begin{aligned} w\times(x\dotplus y)&=[P_{k:j}w+P_{k:i}(x\dotplus y)]\,(a_i+a_j)\\ &\geqq[P_{k:j}w+\{(p+P_{k:i}x)\,(a_i+a_j)+P_{k:j}w\}\,(a_k+a_j)]\,(a_i+a_j)\\ &=[(p+P_{k:i}x)\,(a_i+a_j)+P_{k:j}w]\,(a_i+a_j) \qquad \text{(AL), (ST)}\\ &\geqq(p+P_{k:i}x)\,(a_i+a_j)\\ &=[((w\times y)+a_k)P_{k:j}w+a_j)+P_{k:i}x]\,(a_i+a_j). \end{aligned}$$

Now we may calculate $(w\times x)\dotplus(w\times y)$ from (4. 4. 6) with $(w\times x)$ in place of $x$ and $(w\times y)$ in place of $y$, using $p=P_{k:j}w$ and $q=a_k$, for these $p, q$ satisfy the relevant conditions (4. 4. 1), (4. 4. 2), and (4. 4. 3); indeed, $p$ is in $L_{ik}$ and

$$p(a_i+a_j)\leqq(P_{k:j}w+P_{k:i}x)\,(a_i+a_j)=w\times x.$$

We obtain:

$$\begin{aligned} &(w\times x)\dotplus(w\times y)\\ &\quad=[(P_{k:j}w+a_j)\,((w\times y)+a_k)+(P_{k:j}w+(w\times x))\,(a_k+a_j)]\,(a_i+a_j). \end{aligned}$$

Now, using (4. 10. 5) with $w$ in place of $x$, $x$ in place of $y$,

$$P_{k:j}w+(w\times x)\leqq P_{k:j}w+P_{k:i}x;$$

$$(P_{k:j}w+(w\times x))\,(a_k+a_j)\leqq P_{k:i}x+P_{k:j}w(a_k+a_j)=P_{k:i}x.$$

Then

$$\begin{aligned} (w\times x)\dotplus(w\times y)&\leqq[(P_{k:j}w+a_j)\,((w\times y)+a_k)+P_{k:i}x]\,(a_i+a_j)\\ &\leqq w\times(x\dotplus y). \end{aligned}$$

Now the indivisibility of inverses shows that equality holds in (4. 14. 1).

Proof of (4. 14. 2). We may obtain an element $\geqq P_{k:j}x\dotplus P_{k:j}y$ from (4. 3. 3) with $j$ and $k$ interchanged, using $A=P_{k:i}w$ and $B=a_j$; for these $A$, $B$ satisfy the relevant conditions (4. 2. 1) and (4. 2. 2) with $j$ and $k$ interchanged ($A$ is in $L_{kj}$). We obtain:

$$\begin{aligned} \text{(4. 14. 4)}\quad P_{k:j}(x\dotplus y)&=P_{k:j}x\dotplus P_{k:j}y\\ &\leqq[(P_{k:j}x+P_{k:i}w)\,(a_i+a_j)+a_k]\,(P_{k:j}y+a_j)+P_{k:i}w\\ &\leqq((x\times w)+a_k)\,(P_{k:j}y+a_j)+P_{k:i}w. \end{aligned}$$

Hence, using (4. 10. 5) with $x \dot{+} y$ in place of $x$ and $w$ in place of $y$,

$$(x \dot{+} y) \times w = [P_{k:j}(x \dot{+} y) + P_{k:i} w]\,(a_i + a_j)$$
$$\leqq [((x \times w) + a_k)\,(P_{k:j} y + a_j) + P_{k:i} w]\,(a_i + a_j).$$

The inequality implies equality because of the indivisibility of inverses (the last member is easily shown to be in $L_{ij}$). Therefore

(4. 14. 5) $\quad (x \dot{+} y) \times w = [((x \times w) + a_k)\,(P_{k:j} y + a_j) + P_{k:i} w]\,(a_i + a_j).$

Now we may calculate $(y \times w) \dot{+} (x \times w)$ from (4. 4. 6) with $y \times w$ in place of $x$ and $x \times w$ in place of $y$, using $p = P_{k:j} y$ and $q = a_k$; for these $p, q$ satisfy the relevant conditions (4. 4. 1), (4. 4. 2) and (4. 4. 3); indeed, $p$ is in $L_{jk}$ and

$$p(a_i + a_j) \leqq (P_{k:j} y + P_{k:i} w)\,(a_i + a_j) = y \times w.$$

We obtain:

$$(y \times w) \dot{+} (x \times w)$$
$$= [(P_{k:j} y + a_j)\,((x \times w) + a_k) + (P_{k:j} y + (y \times w))\,(a_k + a_j)]\,(a_i + a_j).$$

Now using (4. 10. 5) with $y$ in place of $x$ and $w$ in place of $y$,

$$P_{k:j} y + (y \times w) \leqq P_{k:j} y + P_{k:i} w;$$
$$(P_{k:j} y + (y \times w))(a_k + a_j) \leqq P_{k:i} w + P_{k:j} y(a_k + a_j) = P_{k:i} w.$$

Then

$$(y \times w) \dot{+} (x \times w) \leqq [(P_{k:j} y + a_j)((x \times w) + a_k) + P_{k:i} w](a_i + a_j).$$

Now (4. 14. 5), the indivisibility of inverses and the commutativity of $\dot{+}$ show that equality holds in (4. 14. 2).

This completes the proof that $\times$ is distributive with respect to $\dot{+}$.

Thus, under the operations $\dot{+}$ and $\times$ the $L_{km}$ become regular rings with unit if the two conditions (4. 3. 3), (4. 3. 4) hold for all pairs $i, j$. If in addition (4. 10. 3) holds, §§ 4. 8, 4. 13 show that the mappings $P_{kj:ij}$, $P_{ik:ij}$ yield ring isomorphisms of $L_{ij}$ onto $L_{kj}$, $L_{ik}$ respectively, so that as regular rings, the $L_{km}$ are all isomorphic.

This, together with (4. 12. 3), establishes:

(4. 14. 6) *Parts* (i) *and* (ii) *of the von Neumann coordinatization theorem hold if* $n > 3$ *or if* $n = 3$ *and* $L$ *possesses a normalized frame satisfying* (4. 3. 3), (4. 3. 4) *and* (4. 10. 3).

**4. 15. Fraternal systems, *L*-numbers, upper semi-fraternal systems and upper semi-*L*-numbers.** A *fraternal system* is defined to be a set of lattice elements $\langle b \rangle = \langle b_{ij} \rangle = \langle b_{ij};\ i, j = 1, \ldots, n,\ i \neq j \rangle$ satisfying (i) $b_{ij} \leqq a_i + a_j$, (ii) $P_{kj:ij} b_{ij} = b_{kj}$ and (iii) $P_{ik:ij} b_{ij} = b_{ik}$ for all $i, j, k$. A lattice element $x$ is called an *(i, j) fraternal element* if there is a fraternal system $\langle b\rangle$ with $b_{ij}=x$ (if such a $b$ exists it is clearly uniquely determined by $x$).

An *upper semi-fraternal system* is defined to be a set of lattice elements $\langle b\rangle=\langle b_{ij}\rangle=\langle b_{ij};\ i,\ j=1,\dots,n, i>j\rangle$ satisfying (i) $b_{ij}\leqq a_i+a_j$, (ii) $P_{kj:ij}b_{ij}=b_{kj}$ with both $i>j, k>j$, (iii) $P_{ik:ij}b_{ij}=b_{ik}$ with both $i>j,\ i>k$ (the mappings $P$ in (ii) and (iii) are non-crossing according to the definition in § 4. 1).

Suppose $\langle b\rangle$ is a fraternal system or an upper semi-fraternal system; then: if $b_{ij}\leqq a_i$ for some $i, j$ it follows from the definition of $P_{ik:ij}$ that this holds for all $i, j$ and $b_{ij}$ is independent of $j$; similarly if $b_{ij}\leqq a_j$ for some $i, j$ then it follows from the definition of $P_{kj:ij}$ that this holds for all $i, j$ and $b_{ij}$ is independent of $i$; finally, if $b_{ij}$ is in $L_{ij}$ for some $i, j$ it is clear that this holds for all $i, j$.

Fraternal systems $\langle b\rangle$ with $b_{ij}$ in $L_{ij}$ will be called *L-numbers*; upper semi-fraternal systems $\langle b\rangle$ with $b_{ij}$ in $L_{ij}$ will be called *upper semi-L-numbers*. If $\beta$ denotes an $L$-number or an upper semi-$L$-number $\langle b\rangle$ we shall sometimes write $\beta_{ij}$ to mean $b_{ij}$.

If $\alpha$ and $\beta$ are both $L$-numbers we define $\alpha+\beta$ to be the system $\langle b\rangle$ with $b_{ij}=\alpha_{ij}\dotplus\beta_{ij}$ for all $i\neq j$ and $\alpha\beta$ to be the system $\langle b\rangle$ with $b_{ij}=\alpha_{ij}\times\beta_{ij}$ for all $i\neq j$. It is clear from §§ 4. 8, 4. 13 that $\alpha+\beta$ is an $L$-number and if (4. 10. 3) holds, then $\alpha\beta$ is also an $L$-number. Subtraction is defined for $L$-numbers with $\alpha-\beta=\langle\alpha_{ij}-\beta_{ij};\ i,\ j=1,\dots,n,\ i\neq j\rangle$ (the last paragraph of § 4. 8 shows that this system is an $L$-number since subtraction of inverses is invariant under the perspectivities $P$). Finally the $L$-numbers form a ring $\mathfrak{R}'$ with two-sided unit $1=\langle b; b_{ij}=c_{ij}$ for all $i\neq j\rangle$ if (4. 3. 3), (4. 3. 4) and (4. 10. 3) hold.

Similarly if (4. 3. 3), (4. 3. 4) and (4. 10. 3) hold the upper semi-$L$-numbers form a ring $\mathfrak{R}$ with two-sided unit; that $\mathfrak{R}$ is regular and ring-isomorphic to every $L_{ij}$ will be shown in (4. 15. 5) below.

*The regular ring* $\mathfrak{R}\equiv\mathfrak{R}(a_i, c_{ij}; i, j=1,\dots,n,\ i\neq j)$ *will be called an auxiliary ring for the lattice* $L$[13]).

---

[13]) It follows easily from the definitions of $\dotplus$ and $\times$ that $\mathfrak{R}$ (the *abstract* ring) is completely determined by any *three* elements of the homogeneous basis, thus $\mathfrak{R}=\mathfrak{R}(a_1, a_2, a_3)$. To what extent $\mathfrak{R}$ is completely determined by $L$ (of order $n$) is not yet known. However it was shown by von Neumann [7, vol. 23, page 20, line 38; 8, vol. II, Theorem 4. 2] that $\mathfrak{R}_n$, the regular ring of all $n\times n$ matrices with elements in $\mathfrak{R}$, is uniquely determined by $L$. It is not difficult to show that if $L$, of order $n$, has an auxiliary ring which is a *field* (footnote 11 implies that this occurs if $L$ is a projective geometry and the $a_i$ are points) then the auxiliary ring (corresponding to this order $n$) is uniquely determined.

If $\langle b_{km};\ k \neq m\rangle$ is a fraternal system and $\langle d_{km};\ k > m\rangle$ is an upper semi-fraternal system and if $b_{ij} = d_{ij}$ for some $i > j$, then clearly $b_{ij} = d_{ij}$ for all $i > j$. Also every fraternal system $\langle b_{ij}\rangle$ when restricted to $i > j$ clearly gives a semi-fraternal system and we shall identify these when there is no possibility of confusion. Then $\mathfrak{R}'$ becomes a subring of $\mathfrak{R}$. Actually it will follow as a consequence of the coordinatization theorem (§ 1. 1) that $\mathfrak{R}' = \mathfrak{R}$ (in the case $n \geqq 4$ the equality $\mathfrak{R}' = \mathfrak{R}$ follows directly from the lemma of von Neumann, (4. 15. 2) below).

We shall show below:

(4. 15. 1) *Let $\varphi(x_1, \ldots, x_r)$ be a lattice polynomial in $x_1, \ldots, x_r$ and let $\varphi(y_{1,ij}, \ldots, y_{r,ij}) = y_{ij}$. If the $\langle y_{m,ij}; i > j\rangle$ are all upper semi-fraternal systems, then so is $\langle y_{ij};\ i > j\rangle$; if the $\langle y_{m,ij};\ i \neq j\rangle$ are all fraternal systems, then so is $\langle y_{ij}; i \neq j\rangle$.*

(4. 15. 2) *Lemma of von Neumann* [7, vol. II, lemma 6. 1]: *If $n \geqq 4$, then for each $x \leqq a_i + a_j$ for some $i \neq j$, there is one and only one fraternal system $\langle b\rangle$ with $b_{ij} = x$ (that is $x$ is an $(i, j)$ fraternal element).*

(4. 15. 3) *If $x \leqq a_i + a_j$ for some $i > j$ then there is one and only one upper semi-fraternal system $\langle b\rangle$ with $b_{ij} = x$.*

(4. 15. 4) *Every $x \leqq$ some $a_i$ or some $c_{ij}$ is an $(i, j)$ fraternal element.*

(4. 15. 2) was used by von Neumann as a technical aid for proving the coordinatization theorem for the case $n \geqq 4$. For this case von Neumann showed that the $L$-numbers form a regular ring with unit and he proved the coordinatization theorem using coordinates from this ring.

In the present paper we shall establish the coordinatization theorem for the general complemented modular lattice with $n \geqq 3$ (assuming the Desarguesian-type conditions (4. 3. 3), (4. 3. 4), (4. 10. 3) for the case $n = 3$) by using as a technical aid the apparently weaker lemma (4.15. 3) and using coordinates from the regular ring of all upper semi-$L$-numbers.

P r o o f  o f (4. 15. 1). This holds since lattice union and lattice intersection are preserved under the perspective mappings $P$.

P r o o f  o f (4. 15. 2). We need only prove, assuming $n \geqq 4$: if $T \equiv T_{km:ij}$ $(i \neq j, k \neq m)$ is the product of an ordered sequence of $s$ perspective mappings $P$ such that $T$ maps $L(a_i + a_j)$ onto $L(a_k + a_m)$ and $T(a_i) = a_k$ (such $T$ exist) then $T$ is uniquely determined by $i, j, k, m$ ((4. 15. 2) then follows by setting $b_{km} = T_{km:ij}x$ for all $k \neq m$).

To verify the general uniqueness of such $T$ it is clearly sufficient to confirm that, in the case $i = k$ and $j = m$, $T$ cannot fail to be the identity mapping. If it could, we would choose $s$ to have its least possible value to give such a $T$ different from the identity and derive a contradiction as follows.

Easy calculations, using the modular law, establish the identities:

(i) $P_{kj:hj}P_{hj:ij} = P_{kj:ij}$ if all of $i, h, k \neq j$,

equivalently, $P_{ik:ih}P_{ih:ij} = P_{ik:ij}$ if all $j, h, k \neq i$;

(ii) $P_{km:kj}P_{kj:ij} = P_{km:im}P_{im:ij}$ if $i, j, k, m$ are all different.

It may therefore be supposed (since $s$ has its least value) that the sequence of mappings which defines $T$ begins:

$$\ldots\; P_{mr:mk}P_{mk:hk}P_{hk:hj}P_{hj:ij}$$

(necessarily: $h \neq i$; $k \neq h, j$).

If $k \neq i$ so that $i, j, h, k$ are all different, we can, without changing $T$, replace $P_{hk:hj}P_{hj:ij}$ by $P_{hk:ik}P_{ik:ij}$; then we can replace $P_{mk:hk}P_{hk:ik}$ by $P_{mk:ik}$. This would express $T$ as a product of fewer than $s$ mappings. Therefore we must have $k = i$.

The same argument shows that $m = j$ and that $T$ is defined by mappings beginning:

$$\ldots\; P_{jr:ji}P_{ji:hi}P_{hi:hj}P_{hj:ij}.$$

Since $n \geqq 4$, there is an integer $t$ such that $i, j, h, t$ are all different. Then we may replace $P_{hi:hj}$ by $P_{hi:ht}P_{ht:hj}$; then $P_{ht:hj}$; $P_{hj:ij}$ by $P_{ht:it}P_{it:ij}$; then $P_{ji:hi}P_{hi:ht}$ by $P_{ji:jt}P_{jt:ht}$; then we may replace $P_{jt:ht}P_{ht:it}$ by $P_{jt:it}$; then $P_{jr:ji}P_{ji:jt}$ by $P_{jr:jt}$. $T$ will now be expressed by fewer than $s$ mappings; this contradiction establishes (4. 15. 2).

Proof of (4. 15. 3). We need only prove: if $T = T_{km:ij}$ $(i > j, k > m)$ is the product of an ordered sequence of $s$ *non-crossing* perspective mappings $P$ such that $T$ maps $L(a_i + a_j)$ onto $L(a_k + a_m)$ and $T(a_i) = a_k$, then $T$ is uniquely determined by $i, j, k, m$ ((4. 15. 3) then follows by setting $b_{km} = T_{km:ij}x$ for all $k > m$).

To verify the general uniqueness of such $T$ it is clearly sufficient to confirm that, in the case $k = i$, $m = j$, $T$ cannot fail to be the identity mapping. If it could, we would choose $s$ to have its least possible value to give such a $T$ different from the identity and derive a contradiction as follows.

When $n \geqq 4$, (4. 15. 3) is implied by (4. 15. 2), hence we may assume that $n = 3$, so that there are 3 different indices $i, j, k$. The sequence of *non-crossing* mappings which defines $T$ must begin:

$$\ldots\; P_{kj:ki}P_{ki:kj}P_{kj:ij}.$$

But $P_{kj:ki}P_{ki:kj}$ is the identity and can be omitted, contradicting the minimum character of $s$. This completes the proof of (4. 15. 3).

Proof of (4.15.4). Because of (4.15.2), we may assume $n=3$. Now if $x \leqq a_i$, we need only note that $P_{kj:ij}x \leqq a_k$ and hence:

$$P_{ji:ki}P_{ki:kj}P_{kj:ij}x = P_{ji:ki}(x+c_{ik})(a_k+a_j) = (x+c_{ij})a_j;$$
$$P_{ji:jk}P_{jk:ik}P_{ik:ij}x = P_{jk:ik}x = P_{ji:ki}P_{ki:kj}P_{kj:ij}x.$$

If $x \leqq c_{ij} = c_{ji}$, we need only note that:

$$P_{ji:ki}P_{ki:kj}P_{kj:ij}x = P_{ji:ki}P_{ki:kj}(x+c_{ik})c_{kj}$$
$$= P_{ji:ki}[(x+c_{ik})c_{kj}+c_{ij}+x](a_i+a_k) = P_{ji:jk}P_{jk:ik}P_{ik:ij}x.$$

From (4.15.3) and the definitions of addition and multiplication for upper semi-$L$-numbers, it follows at once that:

(4.15.4) *$\mathfrak{R}$ is a regular ring with unit, isomorphic to every $L_{ij}$.*

## 5. Properties of the auxiliary ring.

**5.1. Addition and multiplication formulae for elements in $\mathfrak{R}$.** For future reference we list the following formulae:

(5.1.1) $(\alpha+\beta)_{ij} = [P_{kj:ij}\alpha_{ij}+(\beta_{ij}+a_k)(c_{ik}+a_j)](a_i+a_j)$, if $i>j, i \neq k \neq j$;
$\qquad = [\alpha_{kj}+(\beta_{ij}+a_k)(c_{ik}+a_j)](a_i+a_j)$, if $i>j, i \neq k>j$.

(5.1.2) $(\alpha+\beta)_{ij} = [(P_{ik:ij}\alpha_{ij}+a_j)(\beta_{ij}+a_k)+c_{jk}](a_i+a_j)$, if $i>j, i \neq k \neq j$;
$\qquad = [(\alpha_{ik}+a_j)(\beta_{ij}+a_k)+c_{jk}](a_i+a_j)$, if $i>j, i>k \neq j$.

(5.1.3) $(\alpha-\beta)_{kj} = [\alpha_{ij}+(a_k+\beta_{ij})(a_j+c_{ik})](a_k+a_j)$, if $i>j, k>j, i \neq k$.

(5.1.4) $(\alpha\beta)_{ij} = (P_{ik:ij}\alpha_{ij}+P_{kj:ij}\beta_{ij})(a_i+a_j)$, if $i>j, i \neq k \neq j$;
$\qquad = (\alpha_{ik}+\beta_{kj})(a_i+a_j)$, if $i>k>j$.

Such formulae were first given by von Neumann [8, vol. II] and follow immediately from (4.4.7), (4.3.5), (4.7.1), (4.10.5).

**5.2. An important identity.** Suppose that $1<j<i \leqq n$ and let $\delta^m$ and $\theta^m$, $m=1,\ldots,j-1$ and $\beta$ be arbitrary elements in $\mathfrak{R}$. We shall now prove that the following identity holds[14, 15]:

(5.2.1)
$$\prod_{m=1}^{j-1}[(\delta^m+\beta\theta^m)_{im}+A_m^{j-1}]$$
$$= \left[(\beta_{ij}+A^{j-1})\prod_{m=1}^{j-1}(\delta_{im}^m+A_m^j)+\prod_{m=1}^{j-1}(\theta_{jm}^m+A_m^{j-1})\right](A^{j-1}+a_i).$$

[14]) (5.2.1) was established in [2, § 5] for the case $n \geqq 4$. The present proof holds for $n \geqq 3$ assuming (4.3.3), (4.3.4) and (4.10.3). The identity (5.2.1) makes possible a simple proof that the module which we shall assign to an $x$ in $L$ (see § 6.2) is uniquely determined by $x$ (Theorem (6.2.5)). This is a critical step in the proof of the coordinatization theorem as given in the present paper.

[15]) As defined in § 4.1,
$$A^0=0;\ A^i=a_1+\cdots+a_i;\ A_j^i=a_1+\cdots+a_{j-1}+a_{j+1}+\cdots+a_i.$$

To prove this identity we shall first establish that for arbitrary indices $1 \leqq m < j < i \leqq n$ and arbitrary $\delta, \beta, \theta$ in $\mathfrak{R}$,

(5.2.2) $$(\delta+\beta\theta)_{im} = [\theta_{jm} + (\delta_{im} + a_j)(\beta_{ij} + a_m)](a_i + a_m).$$

We may calculate $(\beta\theta+\delta)_{im} = (\delta+\beta\theta)_{im}$ from (4.2.4) with $\beta\theta$ in place of $x$, with $\delta$ in place of $y$, with $j$ replaced by $m$ and using $A = \theta_{jm}$, $B = a_j$ (indeed, these $A, B$ satisfy the relevant formulae (4.2.1), (4.2.2), (4.2.3) with $j$ replaced by $m$, since $A$ is in $L_{jm}$ and

$$a_i + A + B = a_i + a_j + \theta_{jm} \geqq (\beta_{ij} + \theta_{jm})(a_i + a_m) = (\beta\theta)_{im}$$

from (5.1.4) with $\beta$ in place of $\alpha, \theta$ in place of $\beta$). We obtain:

$$\begin{aligned}(\beta\theta+\delta)_{im} &= \{[((\beta\theta)_{im} + \theta_{jm})(a_i + a_j) + a_m](a_j + \delta_{im}) + \theta_{jm}\}(a_i + a_m) \\ &= \{[(\beta_{ij} + \theta_{jm})(a_i + a_j) + a_m](a_j + \delta_{im}) + \theta_{jm}\}(a_i + a_m) \\ &= [(\beta_{ij} + \theta_{jm} a_j + a_m)(a_j + \delta_{im}) + \theta_{jm}](a_i + a_m) \\ &= [(\beta_{ij} + a_m)(a_j + \delta_{im}) + \theta_{jm} a_j + \theta_{jm}](a_i + a_m) \\ &= [(\beta_{ij} + a_m)(a_j + \delta_{im}) + \theta_{jm}](a_i + a_m).\end{aligned}$$

This proves (5.2.2). It follows that (in the rest of § 5.2 we shall write $\Pi$ in place of $\prod_{m=1}^{j-1}$):

$$\begin{aligned}\text{left side of (5.2.1)} &= \Pi[\{\theta_{jm}''' + (\delta_{im}''' + a_j)(\beta_{ij} + a_m)\}(a_i + a_m) + A_m^{j-1}] \\ &= (a_i + A^{j-1})\,\Pi[(\theta_{jm}''' + A_m^{j-1}) + (\delta_{im}''' + A_m^j)(\beta_{ij} + A^{j-1})] \\ &\geqq \text{right side of (5.2.1)}.\end{aligned}$$

But each side of (5.2.1) is an inverse of $A^{j-1}$ in $a_i + A^{j-1}$ since, for:

$$\begin{aligned}&A^{j-1}(\text{right side of (5.2.1)}) \leqq A^{j-1}(\text{left side of (5.2.1)}) = 0; \\ &a_i + A^{j-1} \geqq (\text{left side of (5.2.1)}) + A^{j-1} \\ &\qquad \geqq (\text{right side of (5.2.1)}) + A^{j-1} \\ &\qquad = (a_i + A^{j-1})\,[\Pi(\theta_{jm}''' + A^{j-1}) + (\beta_{ij} + A^{j-1})\Pi(\delta_{im}''' + A^j)] \\ &\qquad = a_i + A^{j-i}.\end{aligned}$$

Now the indivisibility of inverses shows that equality holds in (5.2.1).

As a corollary to (5.2.1) we shall derive the following identity which holds for arbitrary $\beta'''$ and $\theta'''$, $m = 1, \ldots, j, j < i \leqq n$, and arbitrary $\gamma$:

(5.2.3) $$[(\beta^j - \gamma)_{ij} + A^{j-1}]\,\Pi[(\beta''' + \gamma\theta''')_{im} + A_m^j] + \Pi(\theta_{jm}''' + A_m^{j-1}) = \\ = (\beta_{ij}^j + A^{j-1})\,\Pi(\beta_{im}''' + A_m^j) + \Pi(\theta_{jm}''' + A_m^{j-1}).$$

The right side of (5.2.3) is precisely the left side of (5.2.3) with 0 in place of $\gamma$. Thus we need only show that the left side of (5.2.3) has the same value for all $\gamma$, equal to its value when $\gamma = \beta^j$, say. It is therefore sufficient to prove:

(5. 2. 4) left side of (5. 2. 3) $= \Pi[(\beta^m + \beta^i \theta^m)_{im} + A_m^{j-1}] + \Pi(\theta_{jm}^m + A_m^{j-1})$.

Now (5. 2. 4) can be obtained by substituting in (5. 2. 1): $\beta^j - \gamma$ for $\beta^r$ and $\beta^m + \gamma\theta^m$ for $\delta^m$, $m = 1, \ldots, j-1$ and adding the term $\Pi(\theta_{jm}^m + A_m^{j-1})$ to both sides.

**5. 3. Reach and nullity.** We shall define below, for each $\alpha$ in $\mathfrak{R}$, two fraternal systems which we shall call the *nullity* of $\alpha$ and *reach of* $\alpha$.

Suppose $\alpha$ is an upper semi-$L$-number and $i, j$ fixed, $i > j$. Lemmas (4. 15. 1) and (4. 15. 4) show that there exists a unique *fraternal* system $\langle b \rangle$ with $b_{ij} = \alpha_{ij} a_i$; since $\langle b \rangle$ is a fraternal system it follows from the definition of upper semi-fraternal systems that, for all $k > m$, $b_{km} = \alpha_{km} a_k$ and does not depend on $m$. Note: $b_{km}$ is defined for *all* $k \neq m$ although $\alpha_{km}$ is defined only for $k > m$. We shall call this fraternal system, to be denoted as $\alpha^0$, the *nullity* of $\alpha$; so that, if $i > 1$, $\alpha_i^0 = \alpha_{ij}\, a_i$ for all $j < i$ and if $\alpha$ is an $L$-number, then $\alpha_i^0 = \alpha_{ij}\, a_i$ for *all* $i \neq j$.

Similarly, lemmas (4. 15. 1) and (4. 15. 4) show that for each $\alpha$ in $\mathfrak{R}$, the system $b_{ij} = (\alpha_{ij} + a_i) a_j$, $i > j$ is an upper semi-fraternal system, which is part of a fraternal system. We shall call this fraternal system, to be denoted as $\alpha^r$, the *reach of* $\alpha$; so that, if $j < n$, $\alpha_j^r = (\alpha_{ij} + a_i) a_j$ for all $i > j$ and if $\alpha$ is an $L$-number, then $\alpha_j^r = (\alpha_{ij} + a_i)\, a_j$ for all $i \neq j$.[16])

We shall prove the following relations:

(5. 3. 1) *Every idempotent $e$ in $\mathfrak{R}$ is an $L$-number and for every decomposition $a = b \oplus d$, for some fixed $j$, there is a unique idempotent $e$ in $\mathfrak{R}$ with $e_j^r = b$ and $(1-e)_j^r = d$.*

(5. 3. 2) *For $\alpha, \beta$ in $\mathfrak{R}$ there is a $\gamma$ satisfying $\gamma\alpha = \beta$ if and only if $\alpha_j^r \geqq \beta_j^r$.*

(5. 3. 3) $\alpha\beta = 0$ *if and only if* $\alpha_j^r \leqq \beta_j^0$.

(5. 3. 4) $e_j^0 = (1-e)_j^r$ *for every idempotent $e$.*

(5. 3. 5) $e_j^0 \oplus (1-e)_j^0 = e_j^0 \oplus e_j^r = a_j$ *for every idempotent $e$.*

(5. 3. 6) $(\alpha - \beta)_i^0 = (\alpha_{ij}\beta_{ij} + a_j)\, a_i$ *if* $i > j$.

(5. 3. 7) $\alpha_{kj} \leqq (\alpha - \beta)_{kj} + \beta_j^r$ *if* $k > j$.

Proof of (5. 3. 1). If $e$ is an idempotent element in $\mathfrak{R}$, then for any *fixed* $i > j$, lemma (4. 12. 13) shows that $e_{ij}$ is of the form $e_{ij}\; a_i + e_{ij}\; c_{ij}$ and

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[16]) The reach of $\alpha$, $\alpha_j^r$, is identical with the $(a)_j$ of VON NEUMANN [8, vol. II, Definition 9. 1]; (5. 3. 1) and (5. 3. 2) were established in [8, vol. II, Theorem 9. 3, Lemma 9. 1].

lemmas (4. 15. 1) and (4. 15. 4) show that $e_{ij}$ is an $(i, j)$ fraternal element and hence $e$ is an $L$-number. The rest of (5. 3. 1) need only be verified for some particular $j$ which can be taken $< n$ and this follows from (4. 12. 1).

Proof of (5. 3. 2). The proof of (5. 3. 2) follows easily from (4. 12. 2).

Proof of (5. 3. 3). We need only prove this assuming some $i > j > k$. Then $\alpha_i\beta = 0$ means

$$a_i = (\alpha_{ij} + \beta_{jk})(a_i + a_k).$$

Because of the indivisibility of inverses, this is equivalent to each of:

$$a_i \leqq (\alpha_{ij} + \beta_{jk})(a_i + a_k)$$

$$a_i \leqq \alpha_{ij} + \beta_{jk}$$

(5. 3. 8) $$a_i + \alpha_{ij} \leqq \alpha_{ij} + \beta_{jk}$$

(5. 3. 9) $$(a_i + \alpha_{ij})(a_j + a_k) \leqq \beta_{jk}$$

(add $\alpha_{ij}$ to both sides of (5. 3. 9) to derive (5. 3. 8)). Thus $\alpha_i\beta = 0$ is equivalent to:

$$\alpha_j^r \leqq a_j \beta_{jk} = \beta_j^0.$$

Proof of (5. 3. 4). We need only prove this assuming some $i > j > k$. That $(1-e)_j^r \leqq e_j^0$ follows from (5. 3. 3). By (5. 3. 1) there exists an idempotent $f$ with $f_j^r = [e_j^0 - (1-e)_j^r]$. Then $f_j^r \leqq e_j^0$ ; by (5. 3. 3) this implies $fe = 0$, $f(1-e) = f$, hence, using (5. 3. 2), $f_j^r = f_j^r(1-e)_j^r = 0$, that is, $e_j^0 = (1-e)_j^r$.

Proof of (5. 3. 5). Because of (5. 3. 1) and (5. 3. 2) the correspondence $(\alpha)_r \leftrightarrow \alpha_j^r$ is a (1, 1) order preserving correspondence between the left principal ideals of $\mathfrak{R}$ and the $x \leqq L(a_j)$; it follows that $e_j^r \oplus (1-e)_j^r = a_j$. Now (5. 3. 5) follows from (5. 3. 4).

Proof of (5. 3. 6). Using (5. 1. 3):

$$(\alpha-\beta)_k^0 = [\alpha_{ij} + (a_k + \beta_{ij})(a_j + c_{ik})]a_k$$

$$= [\alpha_{ij}(a_k + \beta_{ij}) + (a_j + c_{ik})]a_k;$$

$$(\alpha-\beta)_i^0 = [(\alpha-\beta)_k^0 + c_{ik}]a_i = (\alpha_{ij}\beta_{ij} + a_j + c_{ik})a_i$$

$$= (\alpha_{ij}\beta_{ij} + a_j)a_i.$$

Proof of (5. 3. 7). We need only prove this assuming some $k \neq i > j$. Using (5. 1. 3):

$$(\alpha-\beta)_{kj} + \beta_j^r = [\alpha_{ij} + (a_k + \beta_{ij})(a_j + c_{ik}) + (\beta_{ij} + a_i)a_j](a_k + a_j)$$

$$= [\alpha_{ij} + (\beta_{ij} + a_i + a_k)(a_j + c_{ik})](a_k + a_j)$$

$$\geqq (\alpha_{ij} + c_{ik})(a_k + a_j)\alpha_{kj}.$$

The methods used to prove (5. 3. 2) will also show the following theorems (we omit proofs since we do not make use of these theorems).

(5. 3. 10) For $\alpha, \beta$ in $\mathfrak{R}$ there is a $\gamma$ in $\mathfrak{R}$ satisfying $\alpha\gamma=\beta$ if and only if $\alpha_i^0 \leqq \beta_i^0$.

For $\alpha$ in $\mathfrak{R}$ and $\beta$ in $\mathfrak{R}'$:

(5. 3. 11) $\alpha_{ij} \leqq \beta_{ji}$ if and only if $\alpha\beta=1$;

(5. 3. 12) $\alpha_{ij} \geqq \beta_{ji}$ if and only if $\beta\alpha=1$.

## 6. The procedure for assigning coordinates.

**6. 1. The $i$-elements in $L$.** We shall call $x$ an *i-element* if $x \leqq A^i$ and $xA^{i-1}=0$. We shall prove that for arbitrary $\beta^j$ in $\mathfrak{R}$, $j=1,\ldots, i-1$, and arbitrary idempotent $e$ in $\mathfrak{R}$, the formula:

(6. 1. 1) $$y=(e_i^r+A^{i-})\prod_{j=1}^{i-1}(\beta_{ij}^j+A_j^{i-1})$$

(the factor $\Pi$ to be omitted if $i=1$) defines an $i$-element. If $e=1$ then $e_i^r+A^{i-1}=a_i+A^{i-1}$ and may be omitted.

Indeed, $y \leqq A^i$ and

$$yA^{i-1}=A^{i-1}\prod_{j=1}^{i-1}(\beta_{ij}^j+A_j^{i-1})=A^{i-1}\prod_{j=1}^{i-1}(\beta_{ij}^j a_j+A_j^{i-1})=0.$$

Now consider an arbitrary but fixed $i$-element $x$. Define

(6. 1. 2) $$x^j=(x+A_j^{i-1})(a_i+a_j), \qquad j=1,\ldots,i-1.$$

For fixed choice of idempotent $e=e(x)$ with $e_i^r=(x+A^{i-1})a_i$ (such an $e$ exists by (5. 3. 1)) we define:

(6. 1. 3) $$B=B(x)=x+e_i^0; \quad B^j=(B+A_j^{i-1})(a_i+a_j)$$

(*B may not be uniquely determined by x*). The following relations hold:

(6. 1. 4) $e_i^r+a_j=x^j+a_j$; $\quad x^j+A_j^{i-1}=x+A_j^{i-1}$; $\quad e_i^r+A^{i-1}=x+A^{i-1}$.

(6. 1. 5) $B^j-e_i^0+(x+A_j^{i-1})(a_i+a_j)=e_i^0+x^j$;
$B^j a_i=e_i^0+x^j a_i$;
$BA^{i-1}=[x+e_i^0(x+A^{i-1})]A^{i-1}=(x+e_i^0e_i^r)A^{i-1}=0$ by (5. 3. 5);
$B+A^{i-1}=A^{i-1}+x+e_i^0=A^{i-1}+e_i^r+e_i^0=A^i$.

(6. 1. 6) $B^j a_j=(B+A_j^{i-1})a_j=(BA^{i-1}+A_j^{i-1})a_j=0$;
$B^j+a_j=(B+A^{i-1})(a_i+a_j)=a_i+a_j$.

(6. 1. 7) $B(e_i^r+A^{i-1})=B(x+A^{i-1})=x$ (use (6. 1. 3), (6. 1. 5));
$B^j(e_i^r+a_j)=B^j(x^j+a_j)=x^j$ (use (6. 1. 5), (6. 1. 6)).

(6.1.6) shows that $B^j$ is in $L_{ij}$ for each $j<i$. Let $\alpha^j$ be the element in $\Re$ with $\alpha^j_{ij}=B^j$. Then for every $\beta^j$ in $\Re$ with $\beta^j_{ij}\geqq x^j$, as we shall now show:

(6.1.8) $$e\alpha^j=\alpha^j=e\beta^j.$$

To verify (6.1.8): from (6.1.7), $\alpha^j_{ij}=B^j\geqq x^j$; from (6.1.5) and (5.3.4), $(\alpha^j)^0_i=B^j a_i\geqq e^0_i=(1-e)^r_i$; then (5.3.3) implies $(1-e)\alpha^j=0$, that is, $e\alpha^j=\alpha^j$.

Next, since $\alpha^j_{ij}\beta^j_{ij}\geqq x^j$, it follows from (5.3.6) that:

$$(\alpha^j-\beta^j)^0_i\geqq(x^j+a_j)a_i=e^r_i.$$

(5.3.3) now shows that $e(\alpha^j-\beta^j)=0$, hence $e\alpha^j=\alpha^j=e\beta^j$.

In the next section we shall make use of the following remark. Suppose $\bar{e}$ is also a possible choice for $e(x)$ and let $\bar{\alpha}^j$, $j<i$, denote the corresponding $\Re$-elements. By definition, $\bar{e}^r=e^r$; (5.3.2) implies $\bar{e}e=\bar{e}$, $e\bar{e}=e$; and from (6.1.8)

(6.1.9) $$\bar{e}\alpha^j=\bar{\alpha}^j \qquad (1\leqq j<i).$$

Since $BA^{i-1}=0$,

(6.1.10) $$\prod_{j=1}^{i-1}(\alpha^j_{ij}+A^{i-1}_j)=\prod_{j=1}^{i-1}(B+A^{i-1}_j)=B+\prod_{j=1}^{i-1}A^{i-1}_j=B;$$

(6.1.11) $$(e^r_i+A^{i-1})\prod_{j=1}^{i-1}(\alpha^j_{ij}+A^{i-1}_j)=(x+A^{i-1})B=x.$$

We shall now prove that for arbitrary $\beta^j$ with $\beta^j_{ij}\geqq x^j$, $j<i$, the $i$-element $y$ given by (6.1.1) is identical with $x$. Indeed,

$$\begin{aligned}(e^r_i+A^{i-1})(\alpha^j_{ij}+A^{i-1}_j)&=A^{i-1}_j+B^j(e^r_i+A^{i-1})(a_i+a_j)\\&=A^{i-1}_j+B^j(e^r_i+a_j)=A^{i-1}_j+x^j \quad \text{(use (6.1.4), (6.1.5), (6.1.6))}\\&\leqq A^{i-1}_j+\beta^j_{ij}\end{aligned}$$

for each $j<i$; hence $x\leqq y$. It follows from the indivisibity of inverses that $x=y$ $(yA^{i-1}=xA^{i-1}=0$ and $y+A^{i-1}=e^r_i+A^{i-1}=x+A^{i-1})$.

Conversely, as we now show, for arbitrary idempotent $e$ and arbitrary $\beta^j$ in $\Re$, the $i$-element $y$ of (6.1.1) has the properties: $y^j\leqq\beta^j_{ij}$ for $j<i$, and $e$ is a possible choice for $e(y)$. Indeed, $(y+A^{i-1})a_i=e^r_i$ and

$$\begin{aligned}y^j&=(e^r_i+A^{i-1})\left[\prod_{k=1}^{i-1}(\beta^k_{ik}+A^{i-1}_k)+A^{i-1}_j\right](a_i+a_j)\\&\leqq(\beta^j_{ij}+A^{i-1}_j)(a_i+a_j)=\beta^j_{ij},\end{aligned}$$

as stated. Moreover, (6.1.8) shows that $\alpha^j(y)=e\beta^j$.

**6.2. The rule for assigning left modules to elements of $L$.** For each $x$ in $L$ call $x_1, \dots, x_n$ a base-decomposition of $x$ if each $x_i$ is an inverse:

$$x_i = [xA^i - xA^{i-1}] \qquad (i = 1, \dots, n).$$

Clearly each $x_i$ is an $i$-element and $x = x_1 \oplus \cdots \oplus x_n$. For each base-decomposition of $x$ and for any idempotent $e^i$ satisfying:

$$(e^i)_i^r = (x_i + A^{i-1})a_i = (xA^i + A^{i-1})a_i,$$

let $B(x_i)$, $B^j(x_i)$, $\alpha^{ij} = \alpha^j(x_i)$ be determined as in § 6.1, and define the vector

$$u(x_i) = (-\alpha^{i1}, \dots, -\alpha^{i,i-1}, e^i, 0, \dots, 0).$$

Now, for each such $u(x_i)$, $i = 1, \dots, n$, assign to $x$ the left module

$$M(x_1, \dots, x_n) = (u(x_1), \dots, u(x_n))_l.$$

We note that: (i) the $x_i$ may not be uniquely determined by $x$ and for each $x_i$ the idempotent $e^i$ may not be uniquely determined by $x_i$, however it follows from (6.1.9) that $(u(x_i))_l$ is uniquely determined by $x_i$ so that $M(x_1, \dots, x_n)$ is uniquely determined by $x_1, \dots, x_n$; (ii) if $x$ is a $j$-element then the $x_i$ *are* uniquely determined with $x_i = x$ for $i = j$ and $x_i = 0$ for $i \neq j$; (iii) if $x_i$ is an arbitrary $i$-element for each $i = 1, \dots, n$, and $x = x_1 \oplus \cdots \oplus x_n$ then $x_1, \dots, x_n$ is a base-decomposition for $x$.

We shall prove below the following statements (6.2.1)—(6.2.7).

(6.2.1) *Every left module $M$ of finite span is identical with $M(x_1, \dots, x_n)$ for some base-decomposition $x_1, \dots, x_n$ of some $x$ in $L$.*

(6.2.2) *Suppose $x_1, \dots, x_n$ and $\bar{x}_1, \dots, \bar{x}_n$ are base-decompositions for the same $x$. If*

$$u(x_m) = (-\alpha^{m,1}, \dots, -\alpha^{m,m-1}, e^m, 0, \dots, 0)$$

*and*

$$u(\bar{x}_m) = (-\bar{\alpha}^{m,1}, \dots, -\bar{\alpha}^{m,m-1}, \bar{e}^m, 0, \dots, 0)$$

*and the $\alpha^{m,k}$ and the $\bar{\alpha}^{m,k}$ both form canonical matrices*[17]) *with $\bar{e}_m = e^m$, then $M(x_1, \dots, x_n) = M(\bar{x}_1, \dots, \bar{x}_n)$.*

(6.2.3) *Suppose $y$ is an $i$-element, $u(y)$ has $i$-th coordinate $\bar{e}$ and $z$ is a $j$-element with $1 \leq j < i \leq n$. Then* (i) *if $\gamma$ in $\Re$ satisfies $\bar{e}\gamma = \gamma$, the vector $u(y) + \gamma u(z)$ also has $i$-th coordinate equal to $\bar{e}$ and coincides with $u(x)$ for some $i$-element $x$; and* (ii) *if for some $i$-element $x$ there is a relation $u(x) = u(y) + \gamma u(z)$ for some $\gamma$ in $\Re$, then $x + z = y + z$.*

[17]) A matrix whose rows form a canonical basis (see § 1.2) is called a *canonical matrix.*

(6.2.4) *If $x$ has a base-decomposition $x_1, \dots, x_n$ with*

$$u(x_m) = (-\alpha^{m,1}, \dots, -\alpha^{m,m-1}, e^m, 0, \dots, 0),$$

*with the same $e^m$, such that the $\alpha^{m,k}$ form a canonical matrix and* $M(\bar{x}_1, \dots, \bar{x}_n) = M(x_1, \dots, x_n)$, *then*

(6.2.5) *for each $x$ in $L$, all $M(x_1, \dots, x_n)$ assigned to $x$ coincide, so that we may write $M(x)$ for $M(x_1, \dots, x_n)$;*

(6.2.6) *$x \leqq y$ implies that $M(x) \leqq M(y)$;*

(6.2.7) *$M(x) \leqq M(y)$ implies $x \leqq y$.*

The coordinatization theorem follows easily from (6.2.1), (6.2.5), (6.2.6) and (6.2.7).

**6.3. Proof of** (6.2.1). $M$ is spanned by some canonical basis $u^i = (\alpha^{i1}, \dots, \alpha^{in})$, $i = 1, \dots, n$ (§ 3.4). Choose $e^i$, $x(i)$ as follows:

$$e^i = \alpha^{ii},$$

$$x(i) = [(e^i)_i^r + A^{i-1}] \prod_{j=1}^{i-1} [(-\alpha^{ij})_{ij} + A_j^{i-1}].$$

Then § 6.1 shows that each $x(i)$ is an $i$-element and that $e^i$ is a possible choice for $e(x(i))$; with this choice of $e(x(i))$ it follows from the last paragraph of § 6.1 that $\alpha^j(x(i)) = -\alpha^{ij}$ (for a canonical basis $e^i \alpha^{ij} = \alpha^{ij}$), that $u^i$ is a possible choice for $u(x(i))$ and hence $M$ coincides with $M(x_1, \dots, x_n)$.

**6. 4. Proof of** (6.2.2). We shall show that $\bar{x}_m = x_m$ for all $m$ so that the $\alpha^{mk}$ are uniquely determined by $x$ and the $e^m$ (if the $\alpha^{mk}$ are to form a canonical matrix).

Set $U^k = (e^k)_k^0 \leqq a_k$ for each $k < m$. Since (5.3.5) shows that $(e_k)_k^0 \, (e_k)_k^r = 0$, it follows that

(6.4.1) $$U^k(x_k + A^{k-1}) = 0; \quad x_k(U^k + A^{k-1}) = 0.$$

We shall show that

(6.4.2) $$x_m = x \prod_{k=1}^{m-1} (U^k + A_k^m);$$

this will establish the uniqueness of $x_m$ since the $U^k$ are uniquely determined by the $e^k$.

From § 6.2 there is a $B(x_m) \geqq x_m$ for which

$$\alpha_{mk}^{mk} = B(x_m) + A_k^{m-1}) \, (a_m + a_k);$$

now (5.3.3) implies that $(\alpha^{mk})_k^r \leqq (e^k)_k^0$; i. e., $(B(x_m) + A_k^m) a_k \leqq U^k$. Hence $U^k + A_k^m \geqq B(x_m) \geqq x_m$ for each $k$ and so

right side of (6.4.2) $\geqq x_m$.

Equality in (6.4.2) now follows from the indivisibility of inverses; indeed,

(6.4.3) $\quad$ (right side of (6.4.2)) $+A^{m-1} \leqq xA^m + A^{m-1} = x_m + A^{m-1}$

so that equality holds in (6.4.3), and

$$\text{(right side of (6.4.2))}\ A^{m-1} = (x_1 + \cdots + x_{m-1}) \prod_{k=1}^{m-1} (U^k + A_k^m)$$

$$= (x_1 + \cdots + x_{m-1}) \prod_{k=1}^{m-1} (U^k + A_k^{m-1})$$

$$= (x_1 + \cdots + x_{m-1}) (U^{m-1} + A^{m-2}) \prod_{k=1}^{m-2} (U^k + A_k^{m-1})$$

$$= (x_1 + \cdots + x_{m-2}) \prod_{k=1}^{m-2} (U^k + A_k^{m-1}) \qquad \text{(use (6.4.1))}$$

$$= (x_1 + \cdots + x_{m-3}) \prod_{k=1}^{m-3} (U^k + A_k^{m-2}) = \cdots = 0 = x_m A^{m-1}.$$

**6.5. Proof of** (6.2.3). (i) $u(y) + \gamma u(z)$ is of the form

$$(-\bar{e}\beta^1, -\bar{e}\beta^2, \ldots, -\bar{e}\beta^{i-1}, \bar{e}, 0, \ldots, 0)$$

and hence coincides with $u(x)$ for some $i$-element $x$ by the last paragraph of § 6.1.

(ii) We need only prove $x \leqq y+z$; for the relation $u(y) = u(x) + (-\gamma)u(z)$ would imply, in the same way, $y \leqq x+z$, hence $x+z \leqq y+z \leqq x+z$.

To show $x \leqq y+z$ we need only prove the statement: for arbitrary idempotents $\bar{e}$ and $e$ and $\alpha^m$ with $\bar{e}\alpha^m = \alpha^m$ for $m < i$, $\beta^m$ with $\bar{e}\beta^m = \beta^m$ for $m < i$, and $\theta^m$ with $e\theta^m = \theta^m$ for $m < j$, the conditions

$$\text{(6.5.1)} \qquad \begin{cases} \alpha^m = \beta^m & \text{for } j < m \leqq i \\ \alpha^j = \beta^j - \gamma e & \text{for some } \gamma \\ \alpha^m = \beta^m + \gamma\theta^m & \text{for } 1 \leqq m < j \end{cases}$$

imply:

$$\text{(6.5.2)} \qquad (\bar{e}_i^r + A^{i-1}) \prod_{m=1}^{i-1} (\alpha_{im}^m + A_m^{i-1})$$

$$\leqq (\bar{e}_i^r + A^{i-1}) \prod_{m=1}^{i-1} (\beta_{im}^m + A_m^{i-1}) + (e_j^r + A^{j-1}) \prod_{m=1}^{j-1} (\theta_{jm}^m + A_m^{j-1}).$$

It is sufficient to establish in place of (6.5.2):

$$\text{(6.5.3)} \qquad \prod_{m=1}^{j} (\alpha_{im}^m + A_m^j)$$

$$\leqq \prod_{m=1}^{j} (\beta_{im}^m + A_m^j) + (e_j^r + A^{j-1}) \prod_{m=1}^{j-1} (\theta_{jm}^m + A_m^{j-1})$$

for we can then derive (6. 5. 2) by adding $a_{j+1}+\cdots+a_{i-1}$ to both sides of (6. 5. 3) and then clipping both sides by

$$(\bar{e}_i^r+A^{i-1})\prod_{m=j+1}^{i-1}(a_{im}^m+A_m^{i-1}).$$

But (6. 5. 3) can be derived from

$$\text{(6. 5. 4)} \qquad \text{left side of (6. 5. 3)} \leqq \prod_{m=1}^{j}(\beta_{im}^m+A_m^j)+\prod_{m=1}^{j-1}(\theta_{jm}^m+A_m^{j-1})$$

by clipping both sides of (6. 5. 4) by $\beta_{ij}^j+e_j^r+A^{j-1}$: indeed, this clipping does not change the left side of (6. 5. 4) since one of its factors is

$$\begin{aligned} \alpha_{ij}^j+A^{j-1} &= (\beta^j-\gamma e)_{ij}+A^{j-1} \\ &\leqq \beta_{ij}^j+(-\gamma e)_j^r+A^{j-1} && \text{using (5. 3. 7)} \\ &\leqq \beta_{ij}^j+e_j^r+A^{j-1} && \text{using (5. 3. 2);} \end{aligned}$$

on the other hand this clipping changes the right side of (6. 5. 4) to

$$\prod_{m=1}^{j}(\beta_{im}^m+A_m^j)+(\beta_{ij}^j+e_j^r+A^{j-1})\prod_{m=1}^{j-1}(\theta_{jm}^m+A_m^{j-1}) \qquad \text{(ML)}$$
$$=\text{right side of (6. 5. 3)}$$

since the modular law implies $(\beta_{ij}^j+e_j^r+A^{j-1})A^j=e_j^r+A^{j-1}$.

Since $\alpha^m=\beta^m+\gamma e\theta^m$ for $m<j$ and $\alpha^j=\beta^j-\gamma e$, the desired (6. 5. 4) follows immediately from (5. 2. 3), using $\gamma e$ in place of the $\gamma$ in (5. 2. 3).

**6. 6. Proof of** (6. 2. 4). By (6. 2. 3), for each $m>1$, the vector $u(x_m)+\alpha^{m,1}u(x_1)$ coincides with $u(\bar{x}_m)$ for some $m$-element $\bar{x}_m$ such that $\bar{x}_m+x_1=x_m+x_1$. Then $x_1, \bar{x}_2, \ldots, \bar{x}_n$ is again a base-decomposition of $x$ with $\bar{\alpha}^{m,1}e^1=0$ for $m>1$. Similarly $\bar{x}_3, \ldots, \bar{x}_n$ can be replaced so that the new $\bar{\alpha}^{i,j}$ satisfy also $\bar{\alpha}^{m,2}e^2=0$ for $m>2$. Successive repetition of this procedure establishes (6. 2. 4).

**6. 7. Proof of** (6. 2. 5). If $M=M(x_1,\ldots,x_n)=(u(x_1),\ldots,u(x_n))_l$ and $e^m$ is the $m$-th coordinate of $u(x_m)$ we may, without changing $M$, replace $u(x_m)$ by $f^m(u(x_m))$ where $f^m$ is *any* idempotent satisfying $(f^m)_m^r=(xA^m+A^{m-1})a_m$. The statement (6. 2. 5) now follows from (6. 2. 4) and (6. 2. 2).

**6. 8. Proof of** (6. 2. 6). If $x\leqq y$, we may choose base-decompositions $x_i, i\leqq n$ and $y_i, i\leqq n$ so that $x_i\leqq y_i$ (for example, choose $y_i=x_i+[yA^i-(yA^{i-1}+x_i)]$. Then $(e(x_i))_i^r\leqq(e(y_i))_i^r$ which implies $e(x_i)e(y_i)=e(x_i)$; we may choose $\beta^j(x_i)$ to coincide with $\beta^j(y_i)$ since

$$\beta_{ij}^j(y_i)\geqq(y_i+A_j^{i-1})(a_i+a_j)\geqq x_i+A_j^{i-1})(a_i+a_j).$$

Now $e(x_i)u(y_i)=u(x_i)$ for $i \leqq n$, hence $M(x_1, \ldots, x_n) \leqq M(y_1, \ldots, y_n)$. Because of (6. 2. 5) it follows that $M(x) \leqq M(y)$.

**6. 9. Proof of** (6. 2. 7). Since $x$ is a union of $i$-elements, it is clearly sufficient to prove (6. 2. 7) with the restriction that $x$ is an $i$-element. Then we suppose that $y$ has a base-decomposition $y_i, i \leqq n$, and that

(6. 9. 1) $$u(x)=\gamma_1 u(y_1)+\cdots+\gamma_n u(y_n).$$

Let $e(x)$ be the $i$-th coordinate of $u(x)$ and let $e(y_m)$ be the $m$-th coordinate of $u(y_m)$. Then in (6. 9. 1), we may suppose that (i) $\gamma_m=0$ for $m>i$, (ii) $e(x)e(y_i)=\gamma_i=e(x)$, (iii) $e(x)\gamma_m=\gamma_m$ for all $m$.

To prove (i), suppose $i<n$. The $n$-th coordinates of $u(x)$ and of $\gamma_m u(y_m)$ for $m<n$ are all 0; hence the $n$-th coordinate of $\gamma_n u(y_n)$ equals 0 and we may suppose $\gamma_n=0$. Successive applications of this argument establish (i).

Now consider $i$-th coordinates; $e(x)=\gamma_i e(y_i)$, hence $e(x)e(y_i)=\gamma_i e(y_i)$ $=e(x)$ and we may replace $\gamma_i$ in (6. 9. 1) by $e(x)$ to obtain (ii).

Since $e(x)u(x)=u(x)$ we may replace $\gamma_m$ in (6. 9. 1) by $e(x)\gamma_m$ to obtain (iii).

We may even assume that $\gamma_i=1$ in (6. 9. 1), for the last paragraph of § 6. 1 shows that $\gamma_i u(y_i)$, (that is, $e(x)u(y_i)$), coincides with $u(\bar{y}_i)$ for some $i$-element $\bar{y}_i$. The last paragraph of § 6. 1 shows that $\bar{y}_i \leqq y_i$ since

$$(e(\bar{y}_i))_i^r=(e(x)e(y_i))_i^r \leqq (e(y_i))_i^r.$$

Hence it is sufficient to prove (6. 2. 7) with $y_i$ replaced by $\bar{y}_i$. Thus we may suppose that in (6. 9. 1), $\gamma_i=1$ and $e(y_i)=e(x)=e$ (say) and $e\gamma_m=\gamma_m$ for all $m$. Now (6. 2. 3) shows that $u(y_i)+\gamma_{i-1}u(y_{i-1})=u(z)$ for some $i$-element $z$ with $z \leqq y_i+y_{i-1}$. Similarly, $u(z)+\gamma_{i-2}u(y_{i-2})=u(\bar{z})$ for some $i$-element $\bar{z} \leqq z+y_{i-2} \leqq y_i+y_{i-1}+y_{i-2}$. Repetition of this argument finally yields $x \leqq y_i+y_{i-1}+\cdots+y_1 \leqq y$, as required.

This completes the proof of all statements from (6. 2. 1) to (6. 2. 7) and establishes the coordinatization theorem.

## 7. The case of projective geometry.

**7. 1.** The previous discussion clearly applies to the case of classical projective geometry with a normalized frame consisting of points, that is, atoms[18]). We shall now investigate the meaning of our conditions (4. 3. 3), (4. 3. 4) and (4. 10. 3) in the case of *plane* projective geometry (the only case in which these conditions need to be postulated).

[18]) In this case $\mathfrak{R}$, the ring of coordinates, is a field (see footnote 13).

**7. 2.** Because of footnotes 5 and 10 and since $x, y, a_i, a_j, c_{ij}$ are all atoms, (4. 3. 3) and (4. 3. 4) need be assumed only for the case of $xa_i = 0$, $ya_i = 0$, and (4. 10. 3) need be assumed only for the case that $xa_i$, $ya_i$, $xc_{ij}$ and $yc_{ij}$ are all 0.

Now in the case of projective geometry, we shall show that (4. 3. 2) implies (4. 3. 3). Indeed, suppose (4. 2. 1) and (4. 2. 2) hold; we need consider only the case that (4. 2. 3) fails. Then $A \geqq a_j$ and the $z$ of (4. 2. 4) reduces to $a_i + a_j$ so that (4. 3. 3) does hold.

Similarly we shall show that (4. 3. 2) implies (4. 3. 4). Indeed, suppose (4. 2. 2) and (4. 2. 3) hold; we need consider only the case that (4. 2. 1) fails. Then $(a_i + A + B)x = 0$ and the $z$ of (4. 2. 4) reduces to 0 so that (4. 3. 4) does hold.

Similarly we shall show that (4. 10. 2) implies (4. 10. 3). Indeed, suppose (4. 9. 1) and (4. 9. 2) hold. If (4. 9. 3) fails, $A \geqq a_j$ and the $z$ of (4. 9. 4) reduces to $a_i + y$. If (4. 9 .3) holds, then

$$(z \text{ of } (4.9.4)) + y = a_i(x + A + c_{ij}) + y \leqq a_i + y$$

so that (4. 10. 3) does hold.

**7. 3.** Thus we need assume only (4. 3. 2) and (4. 10. 2) under the restrictions of § 7. 2. Straight-forward inspection confirms that the assumptions of § 7. 3 do hold if we have the following conditions[19, 20]):

Quadrangle condition ((7. 3. 1)—(7. 3. 4)): Suppose two quadrangles $P_i$, $i = 1, 2, 3, 4$, and $P'_i$, $i = 1, 2, 3, 4$, and a line $W$ are such that:

(7. 3. 1) *no three of the vertices of the same quadrangle lie on a common line;*

(7. 3. 2) *$W$ contains none of the vertices of either quadrangle.*

---

19) To derive (4. 3. 2) from the quadrangle condition, choose $P_1 = A$, $P_2 = B$, $P_3 = (x + A)(a_i + B)$, $P_4 = [(x + A)(a_i + B) + a_j](y + B)$ and $W = a_i + a_j$ (the conditions (4. 2. 1), (4. 2. 2) and (4. 2. 3) together with the restrictions $x \neq a_i$, $y \neq a_i$ imply that $A, B$ are points with $A \neq B$, and that (7. 3. 1), (7. 3. 2) hold). Actually (4. 3. 2) can be derived from the uniqueness of the harmonic point condition; indeed the $z$ of (4. 2. 4) coincides with the harmonic conjugate of $a_j$ with respect to $a_i$ and $b$ where $b$ itself is the harmonic conjugate of $a_j$ with respect to $x$ and $y$. The uniqueness of the harmonic point condition is of course implied by the quadrangle condition but the converse need not hold [5, § III 3]. To derive (4. 10. 2) from the quadrangle condition, choose $P_1 = A$, $P_2 = [y + (c_{ij} + A)(a_i + B)](a_j + B)$, $P_3 = (a_i + B)(c_{ij} + A)$, $P_4 = (x + A)(a_i + B)$ and $W = a_i + a_j$ (the restrictions (7. 3. 2) and (7. 3. 3) follow from (4. 9. 1), (4. 9. 2) and (4. 9. 3) together with the restrictions $x \neq a_i$, $x \neq c_{ij}$, $y \neq a_i$, $y \neq c_{ij}$).

20) (7. 3. 1) is the *fundamental theorem on quadrangular sets* as given by Veblen and Young [6, Theorem 3, page 47]; (7. 4. 3) is the *theorem of Desargues* [6, Theorem 1, page 41], and (7. 4. 2) is the dual (and converse) of the theorem of Desargues [6, Theorem 1, page 41]. However, the hypotheses of our quadrangle condition, and (7. 4. 2), (7. 4. 3) are subject to restrictions which are not specifically included in the corresponding theorems of Veblen and Young.

For $i, j = 1, 2, 3, i \neq j$, let $P_{ij} = (P_i + P_j) W$ ($P_{ij}$ is a point since $P_i + P_j$ and $W$ are different lines). Similarly let $P'_{ij} = (P'_i + P'_j) W$. Now suppose also that:

(7. 3. 3) $P_{ij} = P'_{ij}$ *except possibly for the pair* $(i, j) = (3, 4)$.

(7. 3. 4) *Then* (7. 3. 3) *holds also for the pair* $(i, j) = (3, 4)$.

**7. 4.** It is well known (and easily shown) that in a plane projective geometry all lines contain the same number of points, $N$ (say), and each point lies on precisely $N$ distinct lines. We shall prove:

(7. 4. 1) (i) *If* $N = 3$ *or* 4 *the quadrangle condition necessarily holds.*
(ii) $N$ *is infinite or finite but* $> 4$, *the quadrangle condition*[20]) *can be deduced from the following two triangle conditions:*

(7. 4. 2) *If* $P_i$, $i = 1, 2, 3$, *are points not on a line, if* $P'_i$, $i = 1, 2, 3$, *are points not on a line, if* $W$ *is a line containing none of the* $P_i$, $P'_i$, $i = 1, 2, 3$, *and if for each pair* $i, j = 1, 2, 3$, $i \neq j$: *the lines* $P_i + P_j$ and $P'_i + P'_j$ *(these are lines different from* $W$*) meet* $W$ *in the same point, then a point* $Q$ *exists such that for each* $i = 1, 2, 3$: *the points* $Q, P_i, P'_i$ *are on a line.*

(7. 4. 3) *If* $P_i$, $i = 1, 2, 3$, *are points not on a line, and* $P'_i$, $i = 1, 2, 3$, *are also points not on a line, and* $Q$ *is a point on none of* $P_i + P_j$, $P'_i + P'_j$, $i, j = 1, 2, 3$, $i \neq j$, *and if for each* $i = 1, 2, 3$: $Q + P_i = Q + P'_i$, *then a line* $W$ *exists such that for each pair* $i, j = 1, 2, 3$, $i \neq j$: *the lines* $W, P_i + P_j, P'_i + P'_j$ *contain a common point.*

Proof of (7. 4. 1) (i). From (7. 3. 1), (7. 3. 2) and (7. 3. 3) each of $P_{12}$, $P_{34}$, $P'_{34}$ is different from each of $P_{13}$, $P_{14}$, $P_{23}$, $P_{24}$, and $P_{13} \neq P_{14} \neq P_{24} \neq P_{23} \neq P_{13}$. Suppose if possible that $P_{34} \neq P'_{34}$. Then $N$ must be $> 3$; hence $N = 4$ and $P_{13} = P_{24}$, $P_{14} = P_{23}$, $P_{12} = P_{34}$ or $P'_{34}$ (without loss of generality, assume $P_{12} = P_{34}$). Let $Q$ be a point on $P_1 + P_4$, with $Q$ different from each of $P_1$, $P_4$, $P_{14}$ (such $Q$ exist since $N = 4$). Now each of $Q + P_3$, $Q + P_2$ is a line and must contain the point of $W$ which is different from each of $P_{12}$, $P_{23}$, $P_{13}$. Since $Q$ is different from $P_{14}$, it follows that $Q$ lies on $P_2 + P_3$; since $Q$ also lies on $P_1 + P_4$ and $P_{23} = P_{14}$, this implies $Q = P_{14}$. This contradiction establishes (7. 4. 1) (i).

Proof of (7. 4. 1) (ii): *Case* I. Suppose, as a special case, that:

(7. 4. 4) $P_1 + P_2 \neq P'_1 + P'_2$; $P_1 + P_3 \neq P'_1 + P'_3$; $P_1 + P_4 \neq P'_1 + P'_4$.

We may assume

(7. 4. 5) $$P_3 + P_4 \neq P'_3 + P'_4$$

for equality in (7. 4. 5) would immediately imply $P_{34} = P'_{34}$.

(7.4.2), applied to $P_1$, $P_2$, $P_3$ and $P'_1$, $P'_2$, $P'_3$, shows that a point $Q$ exists such that for each $i=1,2,3$: the points $Q$, $P_i$, $P'_i$ lie on a line. Similarly a point $Q'$ exists such that for each $i=1, 2, 4$: the points $Q'$, $P_i, P'_i$ lie on a line. From (7.4.4) it follows easily that $Q=(P_1+P'_1)(P_2+P'_2)=Q'$ and that $Q$ is different from each of $P_1, P_2, P_3, P_4, P'_1, P'_2, P'_3, P'_4$.

If the line $P_1+P_4$ contained $Q$ it would also contain $P'_1$; then $P_1+P_4=P_1+P_{14}=P'_1+P'_{14}=P'_1+P'_4$, contradicting (7.4.4). Thus $Q$ does *not* lie on $P_1+P_4$. Similarly $Q$ does *not* lie on any of $P_1+P_3, P'_1+P'_4, P'_1+P'_3$.

Next, if $P_3+P_4$ contained $Q$, it would also contain $P'_3+P'_4$, contradicting (7.4.5). Thus $Q$ does *not* lie on $P_3+P_4$, and similarly $Q$ does *not* lie on $P'_3+P'_4$.

The preceding paragraphs show that (7.4.3) applies to $P_1, P_3, P_4$ and $P'_1, P'_3, P'_4$. Hence there exists a line $W'$ such that for each pair $i, j=1, 3, 4$, $i\neq j$: $W'$, $P_i+P_j$, $P'_i+P'_j$ contain a common point.

Now by (7.4.4), $P_1+P_3\neq P_1+P'_3$; hence their intersection $P_{13}$ (which lies on $W$) must lie on $W'$. Similarly, $P_{14}$ (which lies on $W$) must lie on $W'$. Since $P_{13}\neq P_{14}$ it follows that $W=W'$. Hence $P_{34}=(P_3+P_4)\,W=(P'_3+P'_4)\,W=P'_{34}$ as required.

*Case* II. Suppose, as a special case, that:

(7.4.6) $P_1+P_2\neq P'_1+P'_2$; $P_2+P_3\neq P'_2+P'_3$; $P_2+P_4\neq P'_2+P'_4$.

Then the proof for Case I with 1 and 2 interchanged shows that (7.4.1)(ii) holds in this case also.

*General Case.* We shall show the existence of points $P^*_i$, $i=1, 2, 3, 4$, such that:

(7.4.7) the conditions (7.3.1), (7.3.2) and (7.3.3) and also (7.4.4) are satisfied by $P_i$ and $P^*_i$, $i=1,2,3,4$;

(7.4.8) the conditions (7.3.1), (7.3.2) and (7.3.3) and also (7.4.6) are satisfied by $P^*_i$ and $P'_i$, $i=1,2,3,4$.

It will then follow that $P_{34}=P^*_{34}=P'_{34}$, which will complete the proof of (7.4.1) (ii).

To determine such $P^*_i$, choose any line $W'$ through $P_{12}$ different from $W, P_1+P_2, P'_1+P'_2$ (this is possible since $N\geqq 4$). Choose a point $P^*_1$ on $W'$ but different from each of $P_{12}, (P_1+P_3)\,W', (P_1+P_4)\,W'$ (such $P^*_1$ exist since $N\geqq 4$). Choose a point $P^*_2$ on $W'$ but different from each of $P_{12}$, $P^*_1$, $(P'_2+P'_3)\,W', (P'_2+P'_4)\,W'$ (such $P^*_2$ exist since $N>4$).

Now let $P^*_3=(P^*_1+P_{13})(P^*_2+P_{23})$, $P^*_4=(P^*_1+P_{14})(P^*_2+P_{24})$. It is easily verified that $P^*_3$, $P^*_4$ are points and that these $P^*_i$ satisfy the conditions (7.4.7) and (7.4.8).

Next we shall verify:

(7.4.9) *The dual (and converse) triangle conditions* (7.4.2) *and* (7.4.3) *are equivalent.*

Proof of (7.4.9). Assume (7.4.2) and suppose that $P_i, P'_i, i=1,2,3,$ and $Q$ satisfy the conditions of (7.4.3).

If $P_1 = P'_1$, then the line containing $P_1$ and any point of $(P_2+P_3)(P'_2+P'_3)$ will serve as the required $W$; hence we may suppose $P_1 \neq P'_1$. Similarly we may also assume $P_2 \neq P'_2, P_3 \neq P'_3$ and hence $P_1+P'_1, P_2+P'_2, P_3+P'_3$ are different lines.

Let $A_1 = (P_2+P_3)(P'_2+P'_3), A_2 = (P_3+P_1)(P'_3+P'_1), A_3 = (P_1+P_2)(P'_1+P'_2)$. Then the $A_i$ must be different points.

Let $A^*_3 = (A_1+A_2)(P_1+P_2), P^*_1 = (A^*_3+P'_3)(P'_1+P'_2)$. Necessarily, $A^*_3$ is a point different from $A_1$. If $A^*_3$ were on $P'_2+P'_3$, this would imply that $A_1+A^*_3$ coincides with $P'_2+P'_3$ and contains $A_2$, hence also $P'_1$; thus $P'_1, P'_2, P'_3$ would lie on a line, contrary to hypothesis. Thus $A^*_3$ is *not* on $P'_2+P'_3$ (similarly $A^*_3$ is not on $P'_1+P'_3$). Hence $P^*_1$ is a point not on $P'_2+P'_3$.

Now $A^*_3 = A_3$, so that the line $A_1+A_2$ serves as the required $W$; for if $A^*_3 \neq A_3$, then $A^*_3$ does *not* lie on $P'_1+P'_2$ and the conditions of (7.4.2) are satisfied by $P_1, P_2, P_3$ and $P^*_1, P'_2, P'_3$ and the line $A_1+A_2$. Hence there exists a point $Q'$ such that $Q'$, $P_1$, $P^*_1$ lie on a line and $Q'$, $P_i, P'_i$ lie on a line for $i=2,3$. Then $Q'$ must be $Q$ and $P^*_1$ lies on $P_1+P'_1$, hence $P^*_1 = P'_1$, and hence $A^*_3$ lies on $P'_1+P'_3$, a contradiction. This shows that $A^*_3 = A_3$ as stated and (7.4.3) is proved assuming (7.4.2).

The dual of the above proof shows that (7.4.2) holds if (7.4.3) is assumed. This completes the proof of (7.4.9).

Thus our coordinatization theorem proves that a projective geometry can be coordinatized if its dimension is $\geqq 3$ or if its dimension is 2 and DESARGUES's theorem, *as stated in* (7.4.3), holds.

On the other hand, if a projective geometry can be coordinatized, the projective geometry can be embedded in a projective geometry of dimension $\geqq 3$, and it is then easy to verify that DESARGUES's theorem must hold.

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