Remarks to the preceding paper of A. Korányi.*)

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1. Theorem 1 of the cited paper may be generalized as follows:

Theorem A. *Let $f(x)$ be a real-valued, continuous function on $(-1, 1)$, derivable on a subset S of full measure, including the point $x=0$. Suppose that the function $k(x, y)$, defined on $S \times S$ by the formulas*

$$k(x, y)=\frac{f(y)-f(x)}{y-x} \quad \text{if} \quad y \neq x, \quad \text{and} \quad k(x, x)=f'(x),$$

be positively definite, i.e.

$$\sum_i \sum_j k(x_j, x_i)\, \alpha_i \bar{\alpha}_j \geqq 0 \tag{1}$$

holds for any finite system of points $x_1, \ldots, x_n \in S$, and any complex $\alpha_1, \ldots, \alpha_n$. Then $f(x)$ may be represented in the form

$$f(x)=f(0)+\int_{-1-0}^{1} \frac{x}{1-xt}\, dm(t) \tag{2}$$

with a bounded non-decreasing, right-continuous function $m(t)$.

The cited theorem settles the same fact under the more restringent condition that $f(x)$ is continuously derivable throughout $(-1, 1)$. We may reduce, however, the above more general case to this particular one.

Let $\varepsilon>0$. If $x_1, \ldots, x_n$ are any given points in $(-1, 1)$, the points

$$x_i(t)=(1+\varepsilon)^{-1}(x_i+t) \qquad (i=1, \ldots, n)$$

belong to S for almost every value of the parameter t with $|t|<\varepsilon$. This follows readily from the fact that S is of full measure. Thus, for almost all t in $(-\varepsilon, \varepsilon)$, and any complex α_i, we have

$$\sum_i \sum_j k(x_j(t), x_i(t))\, \alpha_i \bar{\alpha}_j \geqq 0,$$

*) A. Korányi, On a theorem of Löwner and its connections with resolvents of selfadjoint transformations, *these Acta*, **17** (1956), 63—70.

and consequently

$$\sum_i \sum_j \int_{-\varepsilon}^{\varepsilon} k(x_j(t), x_i(t))\, dt \cdot \alpha_i \bar{\alpha}_j \geqq 0. \tag{3}$$

Put

$$f_\varepsilon(x) = \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} f\left(\frac{x+t}{1+\varepsilon}\right) dt = \frac{1+\varepsilon}{2\varepsilon} \int_{\frac{x-\varepsilon}{1+\varepsilon}}^{\frac{x+\varepsilon}{1+\varepsilon}} f(\tau)\, d\tau \qquad (-1 < x < 1);$$

the corresponding kernel function $k_\varepsilon(x, y)$ is obviously equal to

$$\frac{1}{1+\varepsilon} \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} k\left(\frac{x+t}{1+\varepsilon}, \frac{y+t}{1+\varepsilon}\right) dt;$$

thus, by (3)

$$\sum_i \sum_j k_\varepsilon(x_j, x_i)\, \alpha_i \bar{\alpha}_j \geqq 0.$$

Now, the function $f_\varepsilon(x)$ is continuously derivable throughout $(-1, 1)$:

$$f'_\varepsilon(x) = \frac{1}{2\varepsilon}\left[f\left(\frac{x+\varepsilon}{1+\varepsilon}\right) - f\left(\frac{x-\varepsilon}{1+\varepsilon}\right)\right], \tag{4}$$

and we are in the particular case considered by KORÁNYI. Thus $f_\varepsilon(x)$ may be represented in the form

$$f_\varepsilon(x) = f_\varepsilon(0) + \int_{-1-0}^{1} \frac{x}{1-xt}\, dm_\varepsilon(t) \tag{5}$$

with a non-decreasing bounded, right-continuous $m_\varepsilon(t)$ with $m_\varepsilon(-1-0) = 0$. By (5) we have

$$f'_\varepsilon(0) = \lim_{x \to 0} \frac{f_\varepsilon(x) - f_\varepsilon(0)}{x} = \lim_{x \to 0} \int_{-1-0}^{1} \frac{1}{1-xt}\, dm_\varepsilon(t) = \int_{-1-0}^{1} dm_\varepsilon(t) = m_\varepsilon(1),$$

and by (4)

$$f'_\varepsilon(0) \to f'(0) \quad \text{for} \quad \varepsilon \to 0.$$

Thus $m_\varepsilon(1) \to f'(0)$ as $\varepsilon \to 0$, and so we may apply the theorem of HELLY: there exists a non-decreasing, right-continuous function $m(t)$ with $m(-1-0) = 0$, $m(1) = f'(0)$, such that

$$\int_{-1-0}^{1} g(t)\, dm_{\varepsilon_n}(t) \to \int_{-1-0}^{1} g(t)\, dm(t)$$

for a conveniently chosen sequence $\varepsilon_n \to 0$ and for all continuous $g(t)$, in particular for

$$g(t) = g_x(t) = \frac{x}{1-xt}, \qquad |x| < 1.$$

On the other hand, we have $\lim_{\varepsilon \to 0} f_\varepsilon(x) = f(x)$ by the continuity of $f(x)$, and so the relation (2) results from (5) for $\varepsilon = \varepsilon_n \to 0$.

2. Our second remark concerns theorem 2 of the cited paper, on operator valued functions. We generalize it in two directions: we drop, as in the preceding theorem, the hypothesis that the derivative be continuous, and we replace the hypothesis that the kernel function be positive definite by a weaker one ("weak positive definiteness").

Theorem B. *Let $F(x)$ be a function defined on $(-1, 1)$, whose values are bounded symmetric operators on Hilbert space $\mathfrak{H}$. Suppose that*

a) *$F(x)$ is weakly continuous in x throughout $(-1, 1)$, and weakly derivable on a subset S of $(-1, 1)$ of full measure, including the point $x = 0$; i. e. suppose that, for any fixed $u, v \in \mathfrak{H}$,*

$$(F(y)\,u, v) \to (F(x)\,u, v) \quad \text{for} \quad y \to x,$$

$$\left(\frac{F(y)-F(x)}{y-x}\,u, v\right) \to (F'(x)\,u, v) \quad \text{for} \quad y \to x,\ x \in S,$$

the "derivative" $F'(x)$ being necessarily a bounded symmetric operator;

b) $F(0) = O$, $F'(0) = I$;

c) *putting $K(x,y) = \dfrac{F(y)-F(x)}{y-x}$ if $x \neq y$, and $K(x,x) = F'(x)$, then*

$$\sum_i \sum_j \alpha_i \bar{\alpha}_j K(x_j, x_i) \geqq O \tag{6}$$

for any $x_1, \ldots, x_n \in S$, and any complex $\alpha_1, \ldots, \alpha_n$ ("weak positive definiteness").

Then there exists, in a Hilbert space $\mathfrak{K} \supset \mathfrak{H}$, a self-adjoint operator A with $\|A\| \leqq 1$, such that $F(x)$ be the "projection" of $x(I-xA)^{-1}$ on $\mathfrak{H}$:

$$F(x) = \operatorname{pr} x(I-xA)^{-1}, \qquad |x| < 1.^{2)}$$

Proof. For any fixed $u \in \mathfrak{H}$, $(F(x)\,u, u)$ satisfies the conditions of theorem A, thus we have

$$(F(x)\,u, u) = \int_{-1-0}^{1} \frac{x}{1-xt}\, dm(u; t) \qquad (|x| < 1)$$

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2) For this terminology and notation see B. Sz.-Nagy, *Prolongements des transformations de l'espace de Hilbert qui sortent de cet espace.* Appendice au livre "Leçons d'analyse fonctionnelle" par F. Riesz et B. Sz.-Nagy (Budapest, 1955).

where $m(u;t)$ is a non-decreasing function of t, which is normalized by the condition that it is right-continuous and $m(u;-1-0)=0$. We have, moreover,

$$m(u;1)=(F'(0)\,u,u). \tag{7}$$

Putting, for $u, v \in \mathfrak{H}$,

$$m(u,v;t)=\frac{1}{4}[m(u+v;t)-m(u-v;t)+i\,m(u+iv;t)-i\,m(u-iv;t)],$$

we get a right-continuous function of bounded variation, with $m(u,v;-1-0)=0$, such that

$$(F(x)\,u,v)=\int_{-1-0}^{1}\frac{x}{1-xt}\,dm(u,v;t) \qquad (|x|<1). \tag{8}$$

This relation (8) and the normalization conditions determine the function $m(u,v;t)$ uniquely. As a matter of fact, the function

$$w(z)=\int_{-1-0}^{1}\frac{1}{z-t}\,dm(t)$$

is, for any $m(t)$ of bounded variation, regular in the complex plane cut along the segment $[-1,1]$ of the real axis. In our case, the values of $w(z)$ are given for $z=1/x, |x|<1$, thus $w(z)$ is uniquely determined in its whole domain, and $m(t)$ is determined by $w(z)$ by the well-known inversion formula of Stieltjes.

We have in particular $m(u,u;t)=m(u;t)$, and, since the left-hand side of (8) is, for any fixed t, a (hermitian) symmetric bilinear form in f, g, so is $m(u,v;t)$ necessarily a symmetric bilinear form in f, g, too.

Therefore, there exists a bounded symmetric operator $B(t)$ on $\mathfrak{H}$ such that

$$m(u,v;t)=(B(t)\,u,v);$$

$B(t)$ is a non-decreasing, right-continuous function of t; with $B(-1-0)=O$ $B(1)=F'(0)=I$. In other words, $\{B(t)\}$ is a generalized spectral family. By a well-known theorem of M. Neumark, it may be represented in the form

$$B(t)=\mathrm{pr}\,E(t),$$

where $\{E(t)\}$ is an ordinary spectral family in a convenient larger Hilbert space $\mathfrak{K}$.[3]) It results of (8), then, that

$$F(x)=\mathrm{pr}\,x(I-xA)^{-1}$$

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[3]) See f. i. [2])

where A denotes a self-adjoint operator on $\mathfrak{K}$, namely:

$$A = \int_{-1-0}^{1} t\,dE(t).$$

This finishes the proof.

3. We may generalize the theorem still further, by dropping the condition $F'(0) = I$. Then the following representation holds:

$$F(x) = R \cdot \mathrm{pr}\, x(I - xA)^{-1} \cdot R$$

with a self-adjoint A in $\mathfrak{K} \supset \mathfrak{H}$, $\|A\| \leqq 1$, and with a positive self-adjoint R in $\mathfrak{H}$,

$$R = [F'(0)]^{1/2}.$$

We omit the proof; it goes partially along similar lines that were followed by the author in a previous paper.[4])

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[4]) BÉLA SZ.-NAGY, A moment problem for self-adjoint operators, *Acta Math. Acad. Sci. Hung.*, **3** (1952), 285—293.