On a theorem of Löwner and its connections with resolvents of selfadjoint transformations.

By A. KORÁNYI in Szeged.

In his paper [4] LÖWNER characterized the class C_M of monotone matrix functions of arbitrarily high finite order, i. e. functions for wich $A \geqq B$ implies $f(A) \geqq f(B)$ for any two finite hermitian matrices A, B of the same order and with spectra in $(-1, 1)$.[1]). These functions are at the same time monotone operator functions in Hilbert space, i. e. $f(A) \geqq f(B)$ is implied by $A \geqq B$ for any two selfadjoint operators A, B with spectra in $(-1, 1)$ (see [2]).

LÖWNER first proved by some relatively simple ingenious considerations that C_M is identical with the class C_P of real-valued functions $f(x)$, which are continuously derivable in the interval $(-1, 1)$ and satisfy the inequality

$$\sum_{i=1}^{n} \sum_{j=1}^{n} k(x_j, x_i)\, \alpha_i \bar{\alpha}_j \geqq 0 \tag{1}$$

with

$$k(x, x) = f'(x), \quad k(x, y) = \frac{f(x) - f(y)}{x - y} \quad (x \neq y)$$

for any finite system of points $x_1, \ldots, x_n$ in $(-1, 1)$ and complex numbers $\alpha_1, \ldots, \alpha_n$.

The problem of the explicit characterization of the class C_P has also been solved by LÖWNER. Using his deep results on interpolation by monotone matrix functions he proved that C_P is identical with the class C_A of functions analytic in $(-1, 1)$, which can be continued analytically onto the entire upper half-plane and have there a non-negative imaginary part.

This problem has also been solved by BENDAT and SHERMAN [2] in an other, more direct way. Departing from a transformation of condition (1) and making use of a theorem of S. BERNSTEIN and the Hamburger moment prob-

---

1) Throughout this paper the restriction to the interval $(-1, 1)$ is unessential. All results can easily be transformed to the case of an arbitrary open interval (a, b).

lem they proved that C_P is identical with the class C_I of functions representable in the integral form

$$f(x) = f(0) + \int_{-1}^{1} \frac{x}{1-tx} d\mu(t) \tag{2}$$

with a bounded non-decreasing $\mu(t)$. It is known, however, that $C_I = C_A$.[2])

It was also essentially the same problem that has been solved by WIGNER and NEUMANN [7] in connection with the quantum theory of collisions. Their method rests on function-theoretic arguments and continued fraction expansions.

In section 1 of this paper we shall give a new, perhaps simpler proof of $C_P = C_I$, based on considerations of geometrical nature in Hilbert space. Our method parts from an idea applied by GELFAND and RAIKOV, and GODEMENT in the representation theory of locally compact abelian groups and developed in an abstract form by ARONSZAJN [1]. The result $C_P \subseteq C_A$ will be obtained in a quite elementary way, and a simple use of the spectral theorem will furnish the relation $C_P = C_I$.

In section 2 we show that our method of proof yields also a more general result on operators in Hilbert space. Namely, we shall obtain a necessary and sufficient condition that a bounded symmetric operator T_x depending on a real parameter x be the projection of the resolvent of a self-adjoint operator defined in a wider Hilbert space.

I wish to express my sincere gratitude to professor BÉLA SZ.-NAGY for his kind interest in this work and for his valuable suggestions.

1. Theorem 1. *Let $f(x)$ be continuously derivable in $(-1, 1)$, and suppose that* (1) *holds for any system $x_1, \ldots, x_n \in (-1, 1)$ and any $\alpha_1, \ldots, \alpha_n$. Then $f(x)$ admits of a representation* (2) *with a bounded non-decreasing $\mu(t)$. Conversely, every function of the form* (2) *satisfies condition* (1).

Proof. Consider the set $\mathfrak{H}_1$ of functions representable in the form $\varphi(x) = \sum_{i=1}^{n} \alpha_i k(x, y_i)$ $(y_i \in (-1, 1))$. $\mathfrak{H}_1$ is evidently a linear set. Now define the inner product of the elements $\varphi(x)$ and $\psi(x) = \sum_{j=1}^{m} \beta_j k(x, z_j)$ by

$$(\varphi(x), \psi(x)) = \sum_{i=1}^{n} \sum_{j=1}^{m} k(z_j, y_i) \alpha_i \bar{\beta}_j = \sum_{j=1}^{m} \varphi(z_j) \bar{\beta}_j = \sum_{j=1}^{n} \overline{\psi(y_i)} \alpha_i . \tag{3}$$

[2]) For a simple proof of this fact see [3].

From the evident relations $k(x,y)=k(y,x)$ and (3) it is seen that the inner product does not depend on the special representation of $\varphi(x)$ and $\psi(x)$, and it is easily verified that this definition has the usual properties of the inner product. So we have made $\mathfrak{H}_1$ a (not necessarily complete) Hilbert space. Let $\mathfrak{H}$ be the closure of $\mathfrak{H}_1$.

Now define an operator A_1 in $\mathfrak{H}$, first for the elements of the form $k(x,y)$ with $y\neq 0$ (as a function of x) by

$$A_1 k(x,y)=\frac{1}{y}[k(x,y)-k(x,0)].$$

For any two elements $k(x,y)$, $k(x,z)$ $(y,z\neq 0)$ we have

$$(4)\quad (A_1 k(x,y),k(x,z))=\left(\frac{1}{y}[k(x,y)-k(x,0)],k(x,z)\right)=\frac{1}{y}[k(z,y)-k(z,0)]=$$

$$=\frac{1}{y}\left[\frac{f(y)-f(z)}{y-z}-\frac{f(z)-f(0)}{z}\right]=\frac{zf(y)-yf(z)-(y-z)f(0)}{y(y-z)z}=$$

$$=\frac{1}{z}\left[\frac{f(y)-f(z)}{y-z}-\frac{f(y)-f(0)}{y}\right]=\frac{1}{z}[k(z,y)-k(0,y)]=$$

$$=\left(k(x,y),\frac{1}{z}[k(x,z)-k(x,0)]\right)=(k(x,y),A_1 k(x,z)).$$

(In this calculation we assumed $y\neq z$. In the case $y=z$, however, our result is trivial.) A_1 can now be defined for all linear combinations of the elements $k(x,y)$ $(y\neq 0)$ by linearity. A_1 is densely defined; to see this it suffices to notice that $k(x,0)$ can be represented as the limit of other $k(x,y)$. However, this is evident since

$$\lim_{y\to 0}\|k(x,y)-k(x,0)\|^2=\lim_{y\to 0}\left(f'(y)-2\frac{f(y)-f(0)}{y}+f'(0)\right)=0$$

because of the continuity of $f'(x)$. We have still to show that A_1 is uniquely determined, i. e. that $\sum_i \alpha_i k(x,y_i)=0$ implies $\sum_i \alpha_i A_1 k(x,y_i)=0$. Now this follows from the equality

$$\left(\sum_i \alpha_i A_1 k(x,y_i),k(x,y)\right)=\sum_i \alpha_i (A_1 k(x,y_i),k(x,y))=$$

$$=\sum_i \alpha_i\,(k(x,y_i),A_1 k(x,y))=\left(\sum_i \alpha_i k(x,y_i),A_1 k(x,y)\right)=0$$

whence $\sum_i \alpha_i A_1 k(x,y_i)=0$ owing to the fact that the elements $k(x,y)$ $(y\neq 0)$ span $\mathfrak{H}$.

So A_1 is a densely defined symmetric linear operator in $\mathfrak{H}$. Evidently, A_1 is real with respect to the conjugation determined by

$$J\left(\sum_i \alpha_i k(x, y_i)\right) = \sum_i \bar{\alpha}_i k(x, y_i),$$

so it has a selfadjoint extension A.

We show that $(I-yA)^{-1}$ exists for all y with $|y| < 1$. Since $I-yA$ is selfadjoint, we have to show only that the range of $I-yA$ is dense in $\mathfrak{H}$. Now, by

$$(I-yA)\, k(x, z) = k(x, z) - \frac{y}{z}[k(x, z) - k(x, 0)] = \left(1-\frac{y}{z}\right) k(x, z) + \frac{y}{z} k(x, 0)$$

$(z \neq 0)$, it follows that

$$k(x, 0) = (I-yA)\, k(x, y)$$

and

$$k(x, z) = \frac{1}{1-\dfrac{y}{z}}\left[(I-yA)\, k(x, z) - \frac{y}{z} k(x, 0)\right]$$

for $z \neq y$, thus $k(x, z)$ is in the range of $I-yA$ for all $z \neq y$. Now,

$$k(x, y) = \lim_{z \to y} k(x, z)$$

because of the continuity of $f'(x)$ and so our assertion is proved. Specially we have for $y \neq 0$

$$\frac{f(y)-f(0)}{y} = k(0, y) = (k(x, y), k(x, 0)) =$$

$$= \left((I-yA)^{-1}(I-yA)\, k(x, y), k(x, 0)\right) = \left((I-yA)^{-1} k(x, 0), k(x, 0)\right),$$

or, for any $y \in (-1, 1)$

(5) $$f(y) = f(0) + (y(I-yA)^{-1} k(x, 0), k(x, 0)).$$

From the spectral theorem of selfadjoint operators

$$f(y) = f(0) + \int_{-\infty}^{\infty} \frac{y}{1-ty}\, d(E_t k(x, 0), k(x, 0)) = f(0) + \int_{-\infty}^{\infty} \frac{y}{1-ty}\, d\mu(t).$$

To finish the proof we have only to show that $\mu(t)$ is constant outside $[-1, 1]$. For $|y| < 1$ we have

$$\int_{-\infty}^{\infty} \frac{1}{(1-ty)^2}\, d\mu(t) = \|(I-yA)^{-1} k(x, 0)\|^2 = \|k(x, y)\|^2 = k(y, y) = f'(y),$$

and so for $|x|>1$

$$\int_{-\infty}^{\infty} \frac{1}{(x-t)^2}\, d\mu(t) = \frac{1}{x^2} f'\left(\frac{1}{x}\right).$$

Whence, for any $|x| \geqq \omega\,(>1)$,

$$\int_{-\infty}^{\infty} \frac{1}{(x-t)^2}\, d\mu(t) \leqq \max_{|y| \leqq \frac{1}{\omega}} f'(y) = M_\omega .$$

Let $\omega \leqq a < b$, $x = \dfrac{a+b}{2}$. Then

$$M_\omega \geqq \int_a^b \frac{1}{(x-t)^2}\, d\mu(t) \geqq \frac{1}{\left(\frac{b-a}{2}\right)^2} \int_a^b d\mu(t) = \frac{4}{(b-a)^2}\,[\mu(t)-\mu(a)],$$

and

$$\frac{\mu(b)-\mu(a)}{b-a} \leqq \frac{b-a}{n} M_\omega .$$

It follows that $\mu'(t)$ exists for $t > \omega$ and $\mu'(t) = 0$. So $\mu(t)$ is constant for $t > \omega$, and since ω was arbitrary, $\omega > 1$, $\mu(t)$ is constant for $t > 1$. The same argument shows that $\mu(t)$ is constant also for $t < -1$.

The prove the converse of the theorem we notice that $f(x)$ being of the form (2) we have

$$k(x, y) = \int_{-1}^{1} \frac{1}{(1-tx)(1-ty)}\, d\mu(t), \tag{6}$$

and so for any $x_1, \ldots, x_n;\ \alpha_1, \ldots, \alpha_n$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} k(x_j, x_i)\,\alpha_i \bar{\alpha}_j = \sum_{i=1}^{n}\sum_{i=1}^{n}\int_{-1}^{1} \frac{1}{(1-tx_j)(1-tx_i)}\, d\mu(t)\,\alpha_i\bar{\alpha}_j =$$

$$= \int_{-1}^{1} \left(\sum_{i=1}^{n} \frac{\alpha_i}{1-tx_i}\right)^2 d\mu(t) \geqq 0.$$

Remark. The assertion $C_P \subseteq C_A$ can be read immediately from the proof, without using the spectral theorem. The representation (5) shows that $f(y)$ has an analytic continuation $f(z)$ onto the upper half-plane, and putting $(I-zA)^{-1} k(x, 0) = v$ we have

$$\operatorname{Im} f(z) = \operatorname{Im}\,(v, (I-zA)v) = \operatorname{Im} z\,[(v, v) - \bar{z}(Av, v)] = \operatorname{Im} z\, \|v\|^2,$$

and so $\operatorname{Im} f(z) \geqq 0$ if $\operatorname{Im} z \geqq 0$.

2. In this section we wish to characterize the functions T_x, the values of which are bounded operators in a Hilbert space $\mathfrak{H}$, which can be represented as projections $\tilde{P}\tilde{R}_x$ onto $\mathfrak{H}$ of the resolvent of a bounded selfadjoint operator defined in a wider Hilbert space $\tilde{\mathfrak{H}}$.[3]) It will be seen that theorem 1 is a special case of theorem 2. For the sake of convenience we shall consider the "resolvent" of A as defined by $Q_x = x(I - xA)^{-1}$ instead of the usually defined resolvent $R_x = (A - xI)^{-1}$. We have evidently the simple relation $Q_x = -R_{\frac{1}{x}}$.

Theorem 2. *In order that the bounded symmetric operator T_x defined in $\mathfrak{H}$ for $-1 < x < 1$ be the projection $\tilde{P}\tilde{Q}_x$ of the resolvent of a selfadjoint operator $\tilde{A}$, $\|\tilde{A}\| \leqq 1$, in a wider Hilbert space $\tilde{\mathfrak{H}} \supseteqq \mathfrak{H}$, the following conditions are necessary and sufficient:*

a) T_x admits of a weakly continuous weak derivative T'_x,[4])

b) $T_0 = O$, $T'_0 = I$,

c) for any finite system of points $x_i \in (-1, 1)$ and elements $f_i \in \mathfrak{H}$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} (K(x_j, x_i) f_i, f_j) \geqq 0,$$

with $K(x, x) = T'_x$, $K(x, y) = \dfrac{T_x - T_y}{x - y}$ $(x \neq y)$.

Proof. In the proof of the necessity we use the relation

$$\tilde{Q}_x - \tilde{Q}_y = (x - y)(\tilde{I} - x\tilde{A})^{-1}(\tilde{I} - y\tilde{A})^{-1},$$

which is essentially the Hilbert functional equation of the resolvent. The necessity of conditions a), b) can easily be seen from this (and is also well known). To see the necessity of c) let $\tilde{K}(x, y)$ be the kernel operator built from $\tilde{Q}_x$ instead of T_x; then it follows from $T_x = \tilde{P}\tilde{Q}_x$ that

$$\sum_i \sum_j (K(x_j, x_i) f_i, f_j) = \sum_i \sum_j (\tilde{P}\tilde{K}(x_j, x_i) f_i, f_j) = \sum_i \sum_j (\tilde{K}(x_j, x_i) f_i, f_j) =$$
$$= \sum_i \sum_j ((\tilde{I} - x_j\tilde{A})^{-1} (\tilde{I} - x_i\tilde{A})^{-1} f_i, f_j) = \|\sum_i (\tilde{I} - x_i\tilde{A})^{-1} f_i\|^2 \geqq 0.$$

For the proof of the sufficiency we construct the space $\tilde{\mathfrak{H}}$. We consider the set $\tilde{\mathfrak{H}}_1$ of functions defined for $-1 < x < 1$ and taking their values in $\mathfrak{H}$, representable in the form $f(x) = \sum_{i=1}^{n} K(x, y_i) f_i$. The inner product of $f(x)$ and

---

[3]) This terminology has been introduced in [6]. In this case it means simply that $T_x f = \tilde{P}\tilde{Q}_x f$ for any $f \in \mathfrak{H}$.

[4]) I. e. $(T_x f, g)$ has a continuous derivative with respect to x for any fixed f, g.

$g(x)=\sum_{j=1}^{m} K(x,z_j)g_j$ will be

$$\langle f(x), g(x)\rangle = \sum_{j=1}^{n}\sum_{j=1}^{m}(K(z_j, y_i)f_i, g_j).$$

It is seen in the same way as above, for (3), that this is a legitimate definition, and the set $\tilde{\mathfrak{H}}_1$ can be completed to a complete Hilbert space $\tilde{\mathfrak{H}}$.

Owing to the equalities

$$cK(x,0)f=K(x,0)cf,\quad K(x,0)(f_1+f_2)=K(x,0)f_1+K(x,0)f_2,$$
$$\langle K(x,0)f, K(x,0)g\rangle=(K(0,0)f,g)=(f,g)$$

$\mathfrak{H}$ can be imbedded into $\tilde{\mathfrak{H}}$ with the identification $f\sim K(x,0)f$.[5])

We shall need the projection $\tilde{P}$ of an element $K(x,y)f$ onto $\mathfrak{H}$. For every $h\in\mathfrak{H}$ we have

$$\langle \tilde{P}K(x,y)f, h\rangle = \langle K(x,y)f, K(x,0)h\rangle = (K(0,y)f,h),$$

therefore

$$\tilde{P}K(x,y)f=K(0,y)f.$$

We define the operator $\tilde{A}_1$ in $\tilde{\mathfrak{H}}$ for the elements $K(x,y)f$ $(y\neq 0)$ by

$$\tilde{A}_1 K(x,y)f=\frac{1}{y}[K(x,y)f-K(x,0)f].$$

The property

$$\langle \tilde{A}_1 K(x,y)f,\ K(x,z)g\rangle = \langle K(x,y)f,\ \tilde{A}_1 K(x,z)g\rangle$$

can be seen in the same way as on p. 65 and $\tilde{A}_1$ can be continued for all linear combinations of the $K(x,y)f$. The fact that these combinations are dense in $\tilde{\mathfrak{H}}$ follows from

$$\lim_{y\to 0}\|K(x,y)f-K(x,0)f\|^2=\lim_{y\to 0}\left[(T_y'f,f)-2\left(\frac{T_y}{y}f,f\right)+(f,f)\right]=0$$

by conditions a), b). Introducing a conjugation J[6]) in $\mathfrak{H}$, $\tilde{A}_1$ is seen to be a real operator with respect to the conjugation determined by $\tilde{J}\sum K(x,y_i)f_i=$ $=\sum K(x,y_i)Jf_i$, so it has a selfadjoint extension $\tilde{A}$. Analogously to the proof of theorem 1 it can also be seen that $(\tilde{I}-y\tilde{A})^{-1}$ exists if $|y|<1$.

For every $y\neq 0$ we have

$$(\tilde{I}-y\tilde{A})^{-1}K(x,0)f=K(x,y)f, \tag{7}$$

---

[5]) These considerations are analogous to those in [6] for operator functions on $*$-semigroups.

[6]) See [5] p. 40—41. We may define J as follows. Let $\{\varphi_\alpha\}$ be any complete orthonormal set in $\mathfrak{H}$. Then put, for any $f\in\mathfrak{H}$, $Jf=\sum_\alpha(\varphi_\alpha, f)\varphi_\alpha$.

and so

$$K(0, y)f = \tilde{P}K(x, y)f = \tilde{P}(\tilde{I} - y\tilde{A})^{-1}K(x, 0)f = \tilde{P}(\tilde{I} - y\tilde{A})^{-1}f$$

i. e.

$$\frac{T_y}{y}f = \tilde{P}(\tilde{I} - y\tilde{A})^{-1}f$$

for any $f \in \mathfrak{H}$. Multiplying by y we have $T_y = \tilde{P}\tilde{Q}_y f$ for any $f \in \mathfrak{H}$; and this equality holds evidently also for $y = 0$.

It remains to prove that $\tilde{A}$ is bounded, with $\|\tilde{A}\| \leqq 1$. Denote the resolution of the identity belonging to $\tilde{A}$ by $\tilde{E}_t$. For any $f \in \mathfrak{H}$ we have

$$(T_y f, f) = (\tilde{P}y(\tilde{I} - y\tilde{A})^{-1}f, f) = (y(\tilde{I} - y\tilde{A})^{-1}f, f) = \int_{-\infty}^{\infty} \frac{y}{1 - ty}\, d(\tilde{E}_t f, f);$$

and we conclude, in the same way as in the case of Theorem 1 that $(\tilde{E}_t f, f)$ is constant outside $[-1, 1]$. Now, $\tilde{\mathfrak{H}}$ is minimal in the sense that the elements $\tilde{E}_t f$ $(f \in \mathfrak{H})$ span $\tilde{\mathfrak{H}}$, as it can be seen immediately from (7):

$$K(x, y)f = \int_{-\infty}^{\infty} \frac{1}{1 - ty}\, d\tilde{E}_t f \qquad (f \in \mathfrak{H}).$$

By a known argument (see [6], p. 4—5) it follows then that $\tilde{E}_t$ is constant outside $[-1, 1]$, thus proving our assertion.

Bibliography.

[1] N. Aronszajn, La théorie des noyaux reproduisants et ses applications, *Proceedings Cambridge Phil. Soc.*, **39** (1943), 133—153.

[2] J. Bendat and S. Sherman, Monotone and convex operator functions, *Transactions Amer. Math. Soc.*, **79** (1955), 58—71.

[3] A. Korányi, Note on the theory of monotone operator functions, *these Acta*, **16** (1955). 241—245.

[4] K. Löwner, Über monotone Matrixfunktionen, *Math. Zeitschrift*, **38** (1934), 177—216.

[5] B. Sz.-Nagy, *Spektraldarstellung linearer Transformationen des Hilbertschen Raumes* (Berlin, 1942).

[6] B. Sz.-Nagy, *Prolongements des transformations de l'espace de Hilbert qui sortent de cet espace.* Appendice au livre "Leçons d'analyse fonctionnelle" par F. Riesz et B. Sz.-Nagy (Budapest, 1955).

[7] E. P. Wigner and J. v. Neumann, Significance of Löwner's theorem in the quantum theory of collisions, *Annals of Math.*, **59** (1954), 418—433.

(Received May 1, 1956.)