On Artinian rings.

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§ 1. Introduction.

By an Artinian ring we mean a ring ($\neq 0$) whose left ideals satisfy the minimum condition. Two important classes of Artinian rings have a satisfactory description: 1. those without nonzero nilpotent left ideals (i. e. the semisimple rings) which are — in view of the classical Wedderburn—Artin structure theorems — characterized by a finite set of skew fields and natural integers, 2. the nilpotent Artinian rings whose structure is reduced to the finite nilpotent rings [6].[2]) Here we give a characterization of a third class of Artinian rings containing all the semisimple rings, namely, the class of those rings which are *completely reducible from the left* in the sense that they are direct sums of a finite number of minimal left ideals. The structure of these rings may be described again by a finite number of skew fields and natural integers (Theorem 1).

Turning our attention to general Artinian rings, we first consider the additive structure of Artinian rings. Since this problem has been discussed in full details in [3], here we only mention the principal result according to which the additive group A^+ of an Artinian ring A has a direct decomposition into rational groups $\mathfrak{R}$, a finite number of groups $\mathfrak{C}(p^\infty)$, and cyclic groups of bounded order.[3])

Our next considerations are concerned with Artinian rings containing no subgroup of type p^∞. We shall show that such rings have an important ring-theoretic direct decomposition, namely into the direct sum of a torsion

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1) All the problems discussed in this paper were stated by T. SZELE to whom are due Theorems 1—4 and the necessity parts of Theorems 5—6. After his death his notes on Artinian rings were obtained by the second-named author who made this paper ready for publication.

2) Numbers in brackets refer to the Bibliography given at the end of this paper.

3) By a rational group we mean a group $\mathfrak{R}$ isomorphic to the additive group of all rational numbers. $\mathfrak{C}(p^\infty)$ will denote PRÜFER's group of type p^∞ and $\mathfrak{C}(n)$ the cyclic group of order n.

free Artinian ring and a finite number of Artinian p-rings. Thus the theory of Artinian rings without subgroups of type p^∞ may be reduced to that of Artinian torsion free and p-rings. Moreover, it will turn out that these rings form a very important class of Artinian rings. In fact, this assertion is justified in view of the following two results: 1. an Artinian ring can be imbedded in an Artinian ring with unity element if and only if it contains no subgroup of type p^∞; 2. the left ideals of an Artinian ring A satisfy also the maximum condition if and only if no subgroup of type p^∞ is contained in A. This second statement is an improvement of CH. HOPKINS' well-known result [4] stating that in an Artinian ring with a left or a right unity element the left ideals satisfy the maximum condition too.[4]) The mentioned results also show that the ring structure may depend to a great extent on the additive structure.

Our final result gives a necessary and sufficient condition for the radical of an Artinian ring to be again an Artinian ring.

Some observations concerning terminology may be inserted here. We call a ring R torsion free, torsion or p-ring according as its additive group R^+ is a torsion free, a torsion or a p-group. The elements of the torsion subgroup of R^+ as well as the elements of the maximal algebraically closed[5]) subgroup of R^+ form an ideal of R. If L is a left ideal of R, the elements of the form nx with $x \in L$ and n a fixed natural integer form again a left ideal of R, denoted by nL. The signs $+$ and $\dotplus$ will be used to denote direct sums in the group-theoretic resp. in the ring-theoretic sense.

§ 2. Rings which are completely reducible from the left.

By a ring in the title we shall understand a ring R which may be decomposed into the direct sum of a finite number of minimal left ideals. Clearly, if

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[4]) It follows readily that the presence of a onesided unity element excludes the existence of a subgroup of type p^∞ in Artinian rings. — Let us remark at this stage that AKIZUKI has proved a similar result: *if a commutative ring R with the weakened minimum condition contains at least one element which is no divisor of zero, then in R also the maximum condition holds* [1]. This will also follow from Theorem 6 (moreover, even for the non-commutative case), if one takes into account (see Satz 10 in [3]) that a ring with weakened minimum condition for left ideals and with not torsion free additive group has the same additive structure as the Artinian rings (see Theorem 2), and hence the absence of divisors of zero implies the failure of subgroups of type p^∞ (cf. Corollary 4 in § 5).

[5]) By an algebraically closed group G (for this terminology see [5]) is meant an abelian group with $nG = G$ for all natural integers n. Such groups are direct sums of rational groups and/or groups of type p^∞, and are, by a well-known result of BAER [2], direct summands of every containing abelian group.

R is assumed to have no nonzero nilpotent left ideals or to contain a unit element, then the notions „completely reducible from the left" and „semisimple" coincide. Our aim is to get a structure theorem for the rings in the title.

In this §, let R denote a ring completely reducible from the left and L a minimal left ideal[6]) of R. Since $RL \subseteq L$, we have either $RL=0$ or $RL=L$. It is further known that a minimal left ideal is either nilpotent or idempotent (and in the latter case it may be generated by an idempotent element).[7]) Hence

$$R = A_1 + \cdots + A_r + B_1 + \cdots + B_s + C_1 + \cdots + C_t \tag{1}$$

where the minimal left ideals A_i, B_j, C_k satisfy: $RA_i=0$; $RB_j=B_j$, $B_j^2=0$; $RC_k=C_k$, $C_k^2=C_k$. Evidently, for the radical[8]) N of R we have

$$N = A_1 + \cdots + A_r + B_1 + \cdots + B_s$$

(for N contains all nilpotent left ideals A_i and B_j, but can contain no element of $C_1+\cdots+C_t$), while

$$A = A_1 + \cdots + A_r$$

is the right annihilator ideal[9]) of R.

Let us now consider the structures of A_i, B_j, C_k.

$C=C_1+\cdots+C_t$ is a left ideal of R and is — as a ring — clearly semisimple. Thus the structure of C is completely known in view of the Wedderburn—Artin structure theorems.

From $RA_i=0$ it follows that each subgroup of A_i is at the same time a left ideal of R, and thus, by the minimality of A_i, we conclude that the additive group of A_i is $\mathfrak{C}(p)$ for some prime p.[10])

Next we intend to prove that A is also a left annihilator of R. It is plainly sufficient to show that for each i, j, k we have $A_i B_j=0$ and $A_i C_k=0$. For this purpose we establish the equality $A_i L=0$ for each minimal left ideal L of R with $RL=L$. Indeed, $A_i L=L$ would imply $L=RL= = R(A_i L)=(RA_i)L=0\cdot L=0$. — The left annihilator ideal of R in general properly contains A; namely, it coincides with the radical N of R. To see this, we verify that $B_j B_{j'}=0$ and $B_j C_k=0$ for all j, j', k. In the contrary case $B_j L=L$ we should have $L=B_j L=B_j(B_j L)=B_j^2\cdot L=0\cdot L=0$. On the

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[6]) A minimal left ideal L is different from 0 and contains no left ideal $\neq 0$ properly.

[7]) See e. g. VAN DER WAERDEN [8], p. 145.

[8]) The notion of radical may be taken in any sense usual in the literature, because all usual definitions coincide under the assumption of the minimum condition. However, for the sake of definiteness, here let the radical be defined as the union of all nilpotent left ideals of the ring.

[9]) I. e. the set of all $y \in R$ with $Ry=0$.

[10]) Every A_i is a zeroring, i. e. any two elements annihilate each other.

other hand, since R/N is semisimple, the left annihilator of R can not be greater than N, q. e. d. Thus $NR=0$ and $N^2=0$.

Consider the product CB_j. We have $CB_j=NB_j+CB_j=RB_j=B_j$, and hence B_j is a minimal C-module, C a semisimple ring. But then B_j is C-isomorphic to some minimal left ideal C_k of C. Hence B_j is a zeroring whose additive group is isomorphic to some C_k.

We have thus proved:

Theorem 1. *A ring R completely reducible from the left has the following structure:*

$$R=(A_1+\cdots+A_r)+(B_1+\cdots+B_s)+(C_1+\cdots+C_t)=A+B+C$$

where the minimal left ideals A_i, B_j, C_k satisfy:

(i) $RA_i=A_iR=0$; *the A_i are zerorings with an additive group $\mathfrak{C}(p)$;*

(ii) $AB_j=BB_j=B_jR=0, CB_j=B_j$; *each B_j is a zeroring whose additive group is C-isomorphic to some C_k;*

(iii) $AC_k=BC_k=C_kA=0, CC_k=C_k$; *$C$ is a semisimple ring.*

Since a semisimple ring C may be characterized by a finite number of skew fields and natural integers, the same holds for B too, consequently, we obtain

Corollary 1. *Any ring completely reducible from the left may be characterized by a finite set of skew fields and natural numbers.*

Let us observe that the left complete reducibility of a ring does not necessarily imply the same for the right. In fact, if R is completely reducible from both sides, then its radical N is the twosided annihilator ideal of R, and therefore in case $s\geqq 1$, i. e. if the set of the B_j is not void, R can not be completely reducible from the right. Moreover, it may happen that the right ideals of R do not satisfy the minimum condition.

As a simple consequence of our result we mention:

Corollary 2. *A ring completely reducible from the left is semisimple if and only if it contains no nonzero left annihilator.*

§ 3. Decompositions of Artinian rings.

Let A be an Artinian ring. The additive structure of A is completely described by

Theorem 2. *The additive group A^+ of an Artinian ring A is of the form*

(2) $$A^+=\sum\mathfrak{R}+\sum\mathfrak{C}(p^\infty)+\sum_{p^k|m}\mathfrak{C}(p^k) \qquad (m \text{ fixed})$$

where the cardinal number of the components in the first and third summand is arbitrary, while that in the second summand is finite.

For the proof of this theorem we refer to [3] where it is also shown that to any given group A^+ of the form (2) there exists an Artinian ring A whose additive group is A^+.

Let us here observe that the aditive group R^+ of a ring R completely reducible from the left has the form

$$R^+ = \sum \mathfrak{R} + \sum \mathfrak{C}(p_1) + \cdots + \sum \mathfrak{C}(p_n) \qquad (p_i \text{ fixed})$$

where the cardinal number of the components in each direct summand is arbitrary. In fact, this follows at once from the structure theorem (Theorem 1), if we take into account that any complete matrix ring over a skew field has the additive structure $\sum \mathfrak{R}$ or $\sum \mathfrak{C}(p)$ according as the characteristic of the skew field is 0 or p.

While Theorem 2 establishes a direct decomposition in the group-theoretic sense, the next result shows that in the important case of the absence of subgroups of type p^∞ the Artinian rings admit a direct decomposition in the ring-theoretic sense.

Theorem 3. *An Artinian ring A without subgroups of type p^∞ is the ring-theoretic direct sum of a torsion free Artinian ring B and a finite number of Artinian p-rings C_i, belonging to different primes p_i,*

$$A = B \oplus C_1 \oplus C_2 \oplus \cdots \oplus C_r. \tag{3}$$

The components $B, C_1, \cdots, C_r$ are uniquely determined by A.

If the Artinian ring A contains no subgroup of type p^∞, then its maximal algebraically closed ideal B and its torsion subideal C have no nonzero element in common. Since, by Theorem 2, B and C together generate A, we have $A = B \oplus C$. If we decompose C into its p-components, we arrive at (3). Evidently, the ideals B and C_i are Artinian rings and are uniquely determined as the maximal algebraically closed ideal resp. the maximal p-subrings of A.

Theorem 3 reduces the theory of Artinian rings with no subgroup of type p^∞ to the theory of torsion free Artinian rings and to that of Artinian p-rings whose elements are of bounded order.[11])

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[11]) We have not succeeded in deciding whether or not Theorem 3 holds in general. In case subgroups of type p^∞ are present, the difficulty arises from the possibility that the product of two elements of B (B is now defined as the torsion free component of the algebraically closed subideal D) belongs to D, but not necessarily to B. It is not hard to see that (3) is not true in general for every choice of B, but it is an open question whether B can always be chosen appropriately so as to satisfy (3).

§ 4. Artinian rings with unity element.

Let the Artinian ring A contain a (left, right or twosided) unity element. Then A contains no subgroup $\mathfrak{C}(p^\infty)$. Indeed, assume a is an element of order p and of infinite height[12]) in A. For a left[13]) unity element e of A there exists no $x \in A$ with $p^{n+1}x = p^n e$ where n is any non-negative integer, since in the contrary case we should have

$$a = ea = e(p^n y) = (p^n e)y = (p^{n+1}x)y = x(p^{n+1}y) = x(pa) = x\cdot 0 = 0,$$

a contradiction. Therefore $p^n e$ does not belong to $p^{n+1}A$, i. e. $p^{n+1}A$ is a proper subideal in $p^n A$. Thus

$$A \supset pA \supset \cdots \supset p^n A \supset p^{n+1} A \supset \cdots$$

is an infinite descending chain of ideals of A, contradicting the minimal condition for left ideals. Hence, by Theorem 3, we get

Theorem 4. *An Artinian ring A with a left, right or twosided unity element contains no subgroup of type p^∞ and is the (ring-theoretic) direct sum of a torsion-free Artinian ring and a finite number of Artinian p-rings, all with the same sided unity element.*

It is known that every ring may be imbedded in a ring with unity element. If we perform the usual construction of imbedding for an Artinian ring, we do not get, in general, an Artinian ring again. Hence the problem arises: under what conditions may an Artinian ring be imbedded in an Artinian ring with unity element?[14]) This question is completely answered by

Theorem 5. *An Artinian ring A can be imbedded in an Artinian ring R with unity element if and only if A contains no subgroup of type p^∞.*

That the failure of subgroups $\mathfrak{C}(p^\infty)$ is a necessary condition follows immediately from Theorem 4. Now suppose, conversely, that the Artinian ring A contains no subgroup of type p^∞. Then we have (3) and it is clearly sufficient to show that all of B and C_i may be imbedded in Artinian rings with unity elements.

First let us consider the torsion free Artinian ring B. We define an overring U of B as follows: $U^+ = B^+ + \mathfrak{R}$ and define the multiplication for

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[12]) I. e. the equation $p^n y = a$ is solvable for some y, for each natural integer n.

[13]) The same inference can be applied if e is a right unity element.

[14]) A problem of the kind „a ring of property P is to be imbedded in a ring with unity element and again of property P" has been discussed by J. SZENDREI [7]; he has proved that every ring without divisors of zero can be imbedded in a ring with unity element and without divisors of zero.

the elements (a,ϱ) of U^+ by

(4) $$(a,\varrho)\cdot(b,\sigma)=(ab+\varrho b+\sigma a,\varrho\sigma) \qquad (a,b\in B;\varrho,\sigma\in\mathfrak{R})$$

where ϱb (the product of an element B by a rational number) is, owing to the torsion free character of B, a uniquely determined element of B (by Theorem 2, B^+ is algebraically closed!). It follows by an easy calculation that U is a ring with the unity element $(0, 1)$. In order to show that U is an Artinian ring, let $L_1\supseteq L_2\supseteq\cdots$ be a descending chain of left ideals L_n of U. Put $K_n=L_n\cap B$; then $K_1\supseteq K_2\supseteq\cdots$ is a descending chain of left ideals of B, hence it contains but a finite number of different left ideals. We have thus to prove that there is no infinite properly descending chain $L_1\supset L_2\supset\cdots$ of left ideals of U such that $L_1\cap B=L_2\cap B=\cdots=K$.

The left ideal $K=L_n\cap B$ of B consists of all $(a,\varrho)\in L_n$ with $\varrho=0$. If $(a,\varrho),(b,\sigma)\in L_n$ and $\varrho\neq 0$, then

$$(b,\sigma)-\left(0,\frac{\sigma}{\varrho}\right)(a,\varrho)=\left(b-\frac{\sigma}{\varrho}a,0\right)\in K,$$

whence we conclude that each element of L_n lies in the subgroup K^++P where P denotes the rational subgroup of U^+ containing (a,ϱ). We have

$$K^+\subseteq L_n^+\subseteq K^++P.$$

But any left ideal of an algebraically closed ring U with unity element is again algebraically closed,[15]) hence either $L_n=K$ or $L_n=K+P$. This implies that in the descending chain in question at most two different ideals may exist (whose meets with B coincide). Therefore, U is an Artinian ring.

Now we proceed to the case of an Artinian p-ring C whose elements are of bounded order, say, with the bound p^k. We construct a ring V with the additive group $V^+=C^++M^+$ where M is the ring of the residue classes of the rational integers modulo p^k. Let the multiplication of the elements (a,ϱ) $(a\in V,\varrho\in M)$ be defined by the rule (4). As before it follows that the only thing we must verify is that there exists no infinite properly descending chain $L_1\supset L_2\supset\ldots$ of left ideals L_n of V such that $L_n\cap C$ is the same left ideal K of C. Let ϱ be the least natural integer with $(a,\varrho)\in L_n$. Then for any $(b,\sigma)\in L_n$ there is a $\tau\in M$ with $\sigma=\tau\varrho$. Now

$$(b,\sigma)-(0,\tau)(a,\varrho)=(b-\tau a,0)\in K$$

implies that L_n^+/K^+ is isomorphic to some subgroup of $\mathfrak{C}(p^k)$. Consequently, the chain $L_1\supset L_2\supset\ldots$ in question may contain at most $k+1$ different terms, i. e., V is an Artinian ring.

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[15]) For, together with each element a, all of its rational multiples $\varrho e a$ belong to the same left ideal.

This completes the proof of Theorem 5.

A simple consequence of our last result is

Corollary 3. *A nilpotent Artinian ring A may be imbedded in an Artinian ring with unity element if and only if A is finite.*

In fact, by [6], the additive group of a nilpotent Artinian ring is the direct sum of a finite number of groups $\mathcal{C}(p^k)$ with $1 \leqq k \leqq \infty$. Now, Theorem 5 implies the assertion.

Finally, let us mention the following interesting problem: *characterize all rings which may be imbedded in an Artinian ring.* Evidently, a necessary condition is that the additive group of the ring is a subgroup of (2), i. e. the direct sum of a torsion free group, a finite number of groups of type p^∞ and a torsion group with elements of bounded order. But this condition is not sufficient. The question of finding a necessary and sufficient condition is open.

§ 5. Artinian rings with the maximum condition for left ideals.

Next we turn our attention to the problem of finding a necessary and sufficient condition that the minimum condition for left ideals imply the maximum condition for the same ideals. Our result is contained in

Theorem 6. *The left ideals of an Artinian ring A satisfy the maximum condition if and only if A contains no subgroup of type p^∞.*

The necessity of the condition follows immediately from the observation that in an Artinian ring A each subgroup of a group of type p^∞ is an ideal. In fact, $a \in \mathcal{C}(p^\infty)$ is annihilated by each element $b \in \sum \mathfrak{R} + \sum \mathcal{C}(q^\infty) + \sum_{q \neq p} \mathcal{C}(q^k)$ in (2), for $ba = (p^n x)a = x(p^n a) = 0$ if[16] $O(a) = p^n$ and $x \in A$ satisfies $p^n x = b$, while for $c \in \sum \mathcal{C}(p^k)$ we have $ca = c(p^s y) = (p^s c) y = 0$ if $O(c) = p^s$ and $y \in A$ is chosen so as to satisfy $p^s y = a$. Hence the elements of $\mathcal{C}(p^\infty)$ annihilate the whole ring and therefore each subgroup of $\mathcal{C}(p^\infty)$ is actually an ideal. Since the subgroups of $\mathcal{C}(p^\infty)$ do not satisfy the maximum condition, the necessity of the condition in the theorem is established.

In order to prove the sufficiency, let us assume that A is an Artinian ring with no subgroup of type p^∞. Then, by Theorem 3, we have $A = B \oplus C_1 \oplus \cdots \oplus C_r$ where B is a torsion free Artinian ring and C_i are Artinian p-rings with elements of bounded order. It is plainly enough

[16]) $O(x)$ denotes the order of the group element x.

to verify that in each of B and C_i the left ideals satisfy the maximum condition.

First consider the ring B. We imbed B as shown in § 4 in an Artinian ring U with unity element. Any left ideal L of B is algebraically closed. For, denoting by mL a minimal one among the left ideals nL $(n=1,2,\ldots)$ of B, from the algebraic closure of mL it follows $L=mL+K$ for some $K\subseteq L$. But hence in view of $mL=m(mL+K)=mL+mK$ we get $mK=0$, i.e. $K=0$. Therefore, for any rational number ϱ we have $\varrho L=L$. Using this fact, we may show that L is a left ideal of U too. Indeed, $(a,0)\in L$ and $(b,\varrho)\in U$ imply

$$(b,\varrho)(a,0)=(ba+\varrho a,0)=(ba,0)+\varrho(a,0)\in L,$$

considering that $(ba,0)\in L$ and $\varrho(a,0)\in L$. By Hopkins' result [4], the left ideals of U satisfy the maximum condition, consequently, the same is true for B.

The case of Artinian p-rings C_i is somewhat easier. Constructing the ring V of § 4, we see that, for any left ideal L of C_i $(a,0)\in L$ and $(b,\varrho)\in V$ (ϱ belongs to the residue class ring of the integers mod p^k) imply $(b,\varrho)(a,0)= =(ba+\varrho a,0)\in L$, i. e. L is a left ideal also of V. A simple application of Hopkins' result to V completes the proof.

Obviously Hopkins' theorem is a special case of Theorem 6, since by Theorem 4 an Artinian ring with onesided unity element can not contain any subgroup of type p^∞.

On account of the fact that in an Artinian ring any element contained in a group of type p^∞ is necessarily a twosided annihilator of the ring, we find

Corollary 4. *If an Artinian ring contains no annihilator, then for its left ideals the maximum condition holds.*

Considering a ring R as an additive group with the left operator domain R, it is known that a composition series exists if and only if the left ideals satisfy both the minimum and the maximum condition. From Theorem 6 we conclude:

Theorem 7. *The left ideals of a ring have a composition series if and only if it is an Artinian ring containing no subgroup of type p^∞.*

As the left ideals of a ring form a modular lattice and therefore the Jordan—Hölder theorem holds, it follows that for each Artinian ring A without subgroups of type p^∞ — and only for these rings — there exist a unique natural integer l, the length of A, and l simple A-modules such that all maximal chains of left ideals have the same length l;

$$A=L_0\supset L_1\supset\cdots\supset L_l=0,$$

and the factor groups L_{i-1}/L_i ($i=1,\dots,l$) are, up to order, isomorphic to the simple A-modules in question. It is easy to see that the additive structure of any simple A-module G is either $\sum \mathfrak{R}$ or $\sum \mathfrak{C}(p)$ (p a fixed prime), since pG being an A-submodule of G, either it coincides with G or reduces to 0; in the torsion case the first alternative can not occur, for a simple A-module[17]) can not contain any subgroup of type p^∞. From this simple remark we may at once obtain a lower bound for l in terms of the additive structure of A. In (3) let C_i be a p_i-ring for which the least upper bound of the orders of its elements is $p_i^{k_i}$. Then a composition series for the left ideals of C_i is of length $\geqq k_i$, considering that $C_i \supset p_i C_i \supset \cdots \supset p_i^{k_i} C_i = 0$ is a properly descending chain of left ideals. Consequently, the length l of A satisfies the inequality

$$l \geqq k_1 + \cdots + k_r \text{ or } l \geqq 1 + k_1 + \cdots + k_r$$

according as A is a torsion ring or not. Of course, the same inequality must hold for the length l' of a composition series of right ideals, if it exists.

§ 6. The radical of an Artinian ring.

Let A be an Artinian ring and N the radical of A. The factor ring A/N is always Artinian (moreover, semisimple), but the radical N — considered as a ring — need not be Artinian. We seek for a necessary and sufficient condition for N to be again an Artinian ring.

If the radical N of an Artinian ring A is itself an Artinian ring, then N is a nilpotent Artinian ring and therefore it has a structure described in [6]. Consequently, N is a torsion ring with minimum condition for subgroups, i. e. the direct sum of a finite number of groups $\mathfrak{C}(p^k)$ with $1 \leqq k \leqq \infty$.

Conversely, if the radical N of an Artinian ring A possesses this additive structure, then N satisfies the minimum condition for subgroups and therefore is itself an Artinian ring. We have thus proved

T h e o r e m 8. *The radical N of an Artinian ring A is itself an Artinian ring if and only if it is the direct sum of a finite number of groups $\mathfrak{C}(p^k)$ with $1 \leqq k \leqq \infty$. — If A contains no subgroup of type p^∞, this condition reduces to the finiteness of N.*

With the aid of this result it is easy to construct an Artinian ring (for example, using Theorem 1) in which the radical is not an Artinian ring.

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17) Any simple A-module not annihilated by A is known to be isomorphic to some minimal left ideal of the semisimple ring A/N (N the radical of A); see e. g. [8], p. 170.

Finally, let us mention that the Artinian character of the radical implies that in the third summand of (2) there is but a finite number of subgroups $\mathfrak{C}(p^k)$ with $k > 1$.

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