On the solvability of systems of linear inequalities.

By JÁNOS SURÁNYI in Budapest.

To László Kalmár on his fiftieth birthday.

Introduction.

1. In what follows we shall give some criteria for the solvability of homogeneous and inhomogeneous systems of linear inequalities of the form [1])

(1) $$l_i(x) \equiv a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \geqq 0 \qquad (i = 1, 2, \ldots, m)$$

and

(2) $$L_i(x) \equiv l_i(x) + b_i = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n + b_i \geqq 0 \quad (i = 1, 2, \ldots, m),$$

respectively. By a solution of the systems (2) we mean any point $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ in the n-dimensional Euclidean space satisfying $L_i(\xi) \geqq 0$ for $i = 1, 2, \ldots, m$. In the case of system (1), however, we require of a solution ξ to satisfy not only $l_i(\xi) \geqq 0$ for $i = 1, 2, \ldots, m$ but also for some j $(1 \leqq j \leqq m)$ the strict inequality $l_j(\xi) > 0$.

The problem of homogeneous systems can immediately be reduced to that of inhomogeneous systems. In fact, the requirement that at least one of the linear forms $l_j(x)$ should be positive, is equivalent to the inequality

$$\sum_{i=1}^{m} l_i(x) > 0.$$

Owing to the homogeneity of the forms $l_i(x)$, the last inequality can be replaced as to solvability by the inhomogeneous one

$$l_{m+1}(x) \equiv \sum_{i=1}^{m} l_i(x) \equiv a_{m+1,1}x_1 + a_{m+1,2}x_2 + \cdots + a_{m+1,n}x_n \geqq 1,$$

where

$$a_{m+1,k} = a_{1k} + a_{2k} + \cdots + a_{mk} \qquad (k = 1, 2, \ldots, n).$$

Now, setting

$$L_i^*(x) \equiv l_i(x) \quad \text{for} \quad i = 1, 2, \ldots, m; \quad L_{m+1}^*(x) \equiv l_{m+1}(x) - 1,$$

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[1]) All the numbers of this paper are assumed to be real numbers.

the solvability of the system (1) is equivalent to the solvability of the inhomogeneous system

$$(1^*) \qquad L_i^*(x) \geqq 0 \qquad (i = 1, 2, \ldots, m+1).$$

2. In the present paper I shall give criteria for the solvability of systems of linear inequalities in terms of the coefficient-matrix. Our results can be easily obtained along the lines suggested by the classical results and methods of the theory of linear equations. A criterion of this type was recently found by L. M. BLUMENTHAL.[2]) He obtained his criterion (see theorem VI below) in terms of the symmetrical matrix consisting of the elements

$$\alpha_{ij} = a_{i1}a_{j1} + a_{i2}a_{j2} + \cdots + a_{in}a_{jn} \qquad (i, j = 1, 2, \ldots, m);$$

as to his methods, he emphasized their metrical character.

Starting from BLUMENTHAL's preliminary note I have proved[3]) BLUMENTHAL's theorem by means of the notion of linear independence, i. e. by using affine methods only. The methods used in my paper also furnished some further criteria (theorems I and II) in terms of the original matrix $(a_{ik})_{mn}$, a fact that seems to underline the adequacy of these methods.

Theorem II was found independently by S. N. ČERNIKOV[4]). The basic geometrical idea of his proof is similar to mine, but the elaboration runs along different lines. Therefore, we publish here our former proof in a simplified form, but first we shall give another demonstration by induction.

3. Let us denote the rank of the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}$$

by r. For the solvability (in the above sense) of the system (1), resp. (2), the following criteria hold.

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[2]) L. M. BLUMENTHAL, Two existence-theorems for systems of linear inequalities, *Pacific Journal of Math.*, **2** (1952), 523—530. Preliminary note: Abstract 286*t* in *Bulletin of the Amer. Math. Soc.*, **58** (1952), 380. — TH. MOTZKIN, Beiträge zur Theorie der linearen Ungleichungen, *Dissertation*, Basel, 1936, dealing with various problems concerning linear inequalities, gave also criteria for the solvability, which seem, as the reviews of his paper show, to be analogous to our theorem I. His thesis was not accessible for me.

[3]) The new proof of BLUMENTHAL's theorem was first presented in the seminary of P. TURÁN, August 1952. A form completed with new criteria was read at the „Bolyai János Mathematical Society", January 31, 1953. See *Matematikai Lapok,* 4 (1953), 196, and published in Hungarian: Egyenlőtlenségrendszerek megoldhatóságáról, *Az Eötvös Loránd Tudományegyetem Természettudományi Karának Évkönyve (Annals of the University Eötvös Loránd in Budapest, Faculty of Nat. Sci.),* 1952/53, pp. 19—25.

[4]) C. H. Черников, Систем линейных неравенств, Успехи Матем. Наук, **8** (1953), fasc. 2, pp. 8—73. See esp. pp. 17—29.

Theorem I. *System* (1) *is solvable if and only if* $2r-1$ *subscripts* $i_1, i_2, \ldots, i_{r-1}, k_1, k_2, \ldots, k_r$ *can be found so that the determinants*

$$(3) \qquad \Delta_j = \begin{vmatrix} a_{i_1k_1} & a_{i_1k_2} & \ldots & a_{i_1k_r} \\ \cdot & \cdot & \cdot & \cdot \\ a_{i_{r-1}k_1} & a_{i_{r-1}k_2} & \ldots & a_{i_{r-1}k_r} \\ a_{jk_1} & a_{jk_2} & \ldots & a_{jk_r} \end{vmatrix} \qquad (j=1,2,\ldots,m)$$

are either all non-negative, or all non-positive, but not all 0.

Theorem II. *System* (2) *is solvable if and only if* $2r$ *subscripts* $i_1, i_2, \ldots, i_r, k_1, k_2, \ldots, k_r$ *can be found so that*

$$(4) \qquad \Delta = \begin{vmatrix} a_{i_1k_1} & a_{i_1k_2} & \ldots & a_{i_1k_r} \\ a_{i_2k_1} & a_{i_2k_2} & \ldots & a_{i_2k_r} \\ \cdot & \cdot & \cdot & \cdot \\ a_{i_rk_1} & a_{i_rk_2} & \ldots & a_{i_rk_r} \end{vmatrix} \neq 0$$

and, for $j=1,2,\ldots,m$,

$$(5) \qquad \frac{D_j}{\Delta} = \frac{1}{\Delta} \begin{vmatrix} a_{i_1k_1} & a_{i_1k_2} & \ldots & a_{i_1k_r} & b_{i_1} \\ a_{i_2k_1} & a_{i_2k_2} & \ldots & a_{i_2k_r} & b_{i_2} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{i_rk_1} & a_{i_rk_2} & \ldots & a_{i_rk_r} & b_{i_r} \\ a_{jk_1} & a_{jk_2} & \ldots & a_{jk_r} & b_j \end{vmatrix} \geqq 0.$$

By making use of our remark in **1** we can deduce Theorem I immediately from Theorem II. Indeed, let us assume that Theorem II is already proved. We shall see that, if system (1) — and so also system (1*) — is solvable, then the subscript i_r occurring in Theorem II may be chosen equal to $m+1$. Indeed, one of the subscripts $i_1, i_2, \ldots, i_r$ is equal to $m+1$, for in the opposite case we should have

$$\frac{D_{m+1}}{\Delta} = \frac{1}{\Delta} \begin{vmatrix} a_{i_1k_1} & a_{i_1k_2} & \ldots & a_{i_1k_r} & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{i_rk_1} & a_{i_rk_2} & \ldots & a_{i_rk_r} & 0 \\ a_{m+1,k_1} & a_{m+1,k_2} & \ldots & a_{m+1,k_r} & -1 \end{vmatrix} = -1$$

contrary to the requirements of Theorem II. Hence we may suppose $m+1=i_r$, because a permutation of the subscripts has the same effect on the sign of Δ and that of the D_j's.

Now, if $i_r = m+1$, then we have by (5)

$$\frac{D_j}{\Delta} = \frac{1}{\Delta} \begin{vmatrix} a_{i_1k_1} & a_{i_1k_2} & \ldots & a_{i_1k_r} & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{i_{r-1}k_1} & a_{i_{r-1}k_2} & \ldots & a_{i_{r-1}k_r} & 0 \\ a_{m+1,k_1} & a_{m+1,k_2} & \ldots & a_{m+1,k_r} & -1 \\ a_{jk_1} & a_{jk_2} & \ldots & a_{jk_r} & 0 \end{vmatrix} = \frac{\Delta_j}{\Delta} \geqq 0$$

for $j=1,2,\ldots,m$. Thus the conditions of Theorem I are also proved to be necessary.

Suppose now that the conditions of Theorem I are fulfilled, i. e. all the determinants Δ_i $(1\leqq i\leqq m)$ different from 0 have the same sign, and some of them does not vanish. Since these determinants differ from each other but in their last row, we have

$$0\neq\sum_{i=1}^{m}\Delta_i=\begin{vmatrix} a_{i_1k_1} & a_{i_1k_2} & \ldots & a_{i_1k_r} \\ \cdot & \cdot & \cdot & \cdot \\ a_{i_{r-1}k_1} & a_{i_{r-1}k_2} & \ldots & a_{i_{r-1}k_r} \\ a_{m+1,k_1} & a_{m+1,k_2} & \ldots & a_{m+1,k_r}\end{vmatrix}=\Delta.$$

Furthermore

$$\Delta_j=\begin{vmatrix} a_{i_1k_1} & a_{i_1k_2} & \ldots & a_{i_1k_r} \\ \cdot & \cdot & \cdot & \cdot \\ a_{i_{r-1}k_1} & a_{i_{r-1}k_2} & \ldots & a_{i_{r-1}k_r} \\ a_{jk_1} & a_{jk_2} & \ldots & a_{jk_r}\end{vmatrix}=\begin{vmatrix} a_{i_1k_1} & a_{i_1k_2} & \ldots & a_{i_1k_r} & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{i_{r-1}k_1} & a_{i_{r-1}k_2} & \ldots & a_{i_{r-1}k_r} & 0 \\ a_{m+1,k_1} & a_{m+1,k_2} & \ldots & a_{m+1,k_r} & -1 \\ a_{jk_1} & a_{jk_2} & \ldots & a_{jk_r} & 0\end{vmatrix}=D_j$$

for $j=1,2,\ldots,m$, whence

$$\frac{D_j}{\Delta}=\frac{\Delta_j}{\Delta}\geqq 0$$

by the hypotheses of Theorem I. Thus the conditions of Theorem II hold true for the system (1*) with the row-subscripts $i_1, i_2, \ldots, i_{r-1}, m+1$. This means that system (1*), and so also system (1), is solvable.

In accordance with what has been said, we may restrict ourselves to the proof of Theorem II.

First proof, by induction on r.

4. If $r=0$, i. e. all the coefficients vanish, then $\Delta=1$ and the conditions of Theorem II reduce to $b_j\geqq 0$ $(j=1,2,\ldots,m)$, which are in fact necessary and sufficient in this case.

a) Suppose now that $r>0$ and the theorem is already proved for systems with coefficient-matrices of rank $r-1$.

Let us assume first that the system (2) is solvable and ξ is one of its solutions. Then we can find also a solution for which some of the linear forms $L_j(x)$ vanish, but not identically. Indeed, if this is not the case for the original ξ, then supposing a_{ik_1} is a non-vanishing coefficient, we diminish or augment the value of x_{k_1} according as a_{ik_1} is positive, or negative, until one of the linear forms vanishes. This happens for $x_{k_1}=\xi'_{k_1}$ say. Put $\xi'=$ $=(\xi_1, \xi_2, \ldots, \xi_{k_1-1}, \xi'_{k_1}, \xi_{k_1+1}, \ldots, \xi_n)$. If we have e. g. $L_{i_1}(\xi')=0$, then, expressing ξ_{k_1}, and substituting it into the other linear forms, we find that the system

$$(2') \quad L_i'(y) = \left(a_{i1} - \frac{a_{i_1 1}}{a_{i_1 k_1}} a_{ik_1}\right) y_1 + \cdots + \left(a_{i,k_1-1} - \frac{a_{i_1,k_1-1}}{a_{i_1 k_1}} a_{ik_1}\right) y_{k_1-1} + \\ + \left(a_{i,k_1+1} - \frac{a_{i_1,k_1+1}}{a_{i_1 k_1}} a_{ik_1}\right) y_{k_1+1} + \cdots + \left(a_{in} - \frac{a_{i_1 n}}{a_{i_1 k_1}} a_{ik_1}\right) y_n + b_i - \frac{b_{i_1}}{a_{i_1 k_1}} a_{ik_1} \geqq 0$$

$(i = 1, 2, \ldots, m)$ admits the solution

$$(6) \qquad (\xi_1, \xi_2, \ldots, \xi_{k_1-1}, \xi_{k_1+1}, \ldots, \xi_n).$$

Since the coefficient-matrix of (2′) has obviously the rank $r-1$, hence by the induction hypothesis we can find subscripts $i_2, \ldots, i_r, k_2, \ldots, k_r$ for which

$$(4') \qquad \Delta' = \begin{vmatrix} a_{i_2 k_2} - \frac{a_{i_1 k_2}}{a_{i_1 k_1}} a_{i_2 k_1} & \ldots & a_{i_2 k_r} - \frac{a_{i_1 k_r}}{a_{i_1 k_1}} a_{i_2 k_1} \\ \cdot & \cdot & \cdot \\ a_{i_r k_2} - \frac{a_{i_1 k_2}}{a_{i_1 k_1}} a_{i_r k_1} & \ldots & a_{i_r k_r} - \frac{a_{i_1 k_r}}{a_{i_1 k_1}} a_{i_r k_1} \end{vmatrix} \neq 0,$$

and

$$(5') \qquad \frac{D_j'}{\Delta'} = \frac{1}{\Delta'} \begin{vmatrix} a_{i_2 k_2} - \frac{a_{i_1 k_2}}{a_{i_1 k_1}} a_{i_2 k_1} & \ldots & a_{i_2 k_r} - \frac{a_{i_1 k_r}}{a_{i_1 k_1}} a_{i_2 k_1} & b_{i_2} - \frac{b_{i_1}}{a_{i_1 k_1}} a_{i_2 k_1} \\ \cdot & \cdot & \cdot & \cdot \\ a_{i_r k_2} - \frac{a_{i_1 k_2}}{a_{i_1 k_1}} a_{i_r k_1} & \ldots & a_{i_r k_r} - \frac{a_{i_1 k_r}}{a_{i_1 k_1}} a_{i_r k_1} & b_{i_r} - \frac{b_{i_1}}{a_{i_1 k_1}} a_{i_r k_1} \\ a_{j k_2} - \frac{a_{i_1 k_2}}{a_{i_1 k_1}} a_{j k_1} & \ldots & a_{j k_r} - \frac{a_{i_1 k_r}}{a_{i_1 k_1}} a_{j k_1} & b_j - \frac{b_{i_1}}{a_{i_1 k_1}} a_{j k_1} \end{vmatrix} \geqq 0$$

for $j = 1, 2, \ldots, m$.

But the determinants Δ' and D_j' in (4′) and (5′) are equal to $a_{i_1 k_1}^{-1}$ times the determinants Δ and D_j in (4) and (5), respectively, and so the necessity of our conditions has been proved.

b) Assuming again the theorem proved for $r-1$, suppose the conditions (4) and (5) are fulfilled for a system (2) of rank r $(r > 0)$. If e. g. $a_{i_1 k_1} \neq 0$, then the conditions (4′), (5′) also hold, and so the system (2′) of rank $r-1$ admits a solution of the form (6). Let us choose

$$\xi_{k_1} = -\frac{a_{i_1 1}}{a_{i_1 k_1}} \xi_1 - \cdots - \frac{a_{i_1 k_1 - 1}}{a_{i_1 k_1}} \xi_{k_1-1} - \frac{a_{i_1 k_1+1}}{a_{i_1 k_1}} \xi_{k_1+1} - \cdots - \frac{a_{i_1 n}}{a_{i_1 k_1}} \xi_n - \frac{b_{i_1}}{a_{i_1 k_1}},$$

then for $\xi = (\xi_1, \ldots, \xi_{k_1-1}, \xi_{k_1}, \xi_{k_1+1}, \ldots, \xi_n)$ we have $L_{i_1}(\xi) = 0$, and the other linear forms will assume the same value in the point ξ, as the corresponding linear form in the point (6). This completes the proof of the sufficiency of the stated condition.

5. The practical use of the criteria of theorems I and II can be facilitated by the following two remarks:

Remark I. *If the conditions of Theorem I, resp. Theorem II, are satisfied for some $r' < r$, then the system* (1) *resp. the system* (2) *is solvable.*

Thus, e. g., if the non-vanishing coefficients of an unknown occurring in system (1) are all of the same sign, respectively if none of the constants b_i in (2) is negative, then the corresponding system is solvable.

To prove our remark, observe that if the conditions are satisfied with the subscripts $i_1, i_2, \dots, i_{r'-1}, k_1, k_2, \dots, k_{r'}$, resp. $i_1, i_2, \dots, i_{r'}, k_1, k_2, \dots, k_{r'}$, then the subsystems

$$a_{ik_1}x_{k_1} + \cdots + a_{ik_{r'}}x_{k_{r'}} \geqq 0 \qquad (i = 1, 2, \dots, m)$$

and

$$a_{ik_1}x_{k_1} + \cdots + a_{ik_{r'}}x_{k_{r'}} + b_i \geqq 0 \qquad (i = 1, 2, \dots, m)$$

are solvable owing to Theorems I and II, respectively, and their solutions, completed by values 0 for the further unknowns, furnish solutions for the corresponding original system.

Even more useful is[5])

Remark II. *If the column vectors with subscripts $k_1, k_2, \dots, k_r$ in the coefficient-matrix of rank r are linearly independent, then in looking for determinants satisfying the conditions of our theorems, we may restrict ourselves to these columns with subscripts $k_1, k_2, \dots, k_r$.*

Let us denote by $\mathbf{a}_1, \mathbf{a}_2, \dots \mathbf{a}_n, \mathbf{b}$ the column vectors; then the systems (1) and (2) may be written in the form

$$\mathbf{a}_1 x_1 + \mathbf{a}_2 x_2 + \cdots + \mathbf{a}_n x_n > 0,$$

resp.

$$\mathbf{a}_1 x_1 + \mathbf{a}_2 x_2 + \cdots + \mathbf{a}_n x_n + \mathbf{b} \geqq 0,$$

where $\mathbf{u} \geqq 0$ means that $\mathbf{u}$ is a vector without negative components in the given coordinate system, and $\mathbf{v} > 0$ means that $\mathbf{v} \geqq 0$ and $\mathbf{v}$ has at least one positive component.

By hypothesis, we have some vector equations

$$\mathbf{a}_k = c_{k1}\mathbf{a}_{k_1} + c_{k2}\mathbf{a}_{k_2} + \cdots + c_{kr}\mathbf{a}_{k_r} \qquad (k = 1, 2, \dots, n).$$

Thus if ξ is a solution of system (1) or (2), then $(\eta_1, \eta_2, \dots, \eta_r)$ with

$$\eta_\varrho = c_{1\varrho}\xi_1 + c_{2\varrho}\xi_2 + \cdots + c_{n\varrho}\xi_n \qquad (\varrho = 1, 2, \dots, r)$$

is a solution of the system

$$\mathbf{a}_{k_1} y_1 + \mathbf{a}_{k_2} y_2 + \cdots + \mathbf{a}_{k_r} y_r > 0,$$

or

$$\mathbf{a}_{k_1} y_1 + \mathbf{a}_{k_2} y_2 + \cdots + \mathbf{a}_{k_r} y_r + \mathbf{b} \geqq 0,$$

respectively. Consequently, the conditions of the theorems must be satisfied also for the columns in question, q. e. d.

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[5]) Cf. theorem IV of my paper quoted under [3]), pp. 22—23, and the second note to theorem 3 of the paper quoted under [4]), pp. 27—28.

Second proof.

6. Our Theorem II can be derived from the obvious inhomogeneous analogue of H. MINKOWSKI's[6]) classical result concerning homogeneous systems. Although we restrict ourselves to Theorem II, we prove MINKOWSKI's theorem for homogeneous systems too, because BLUMENTHAL's result will be deduced from this theorem. For the sake of completeness, we shall reproduce his theorem together with the original proof. We shall employ the notations as used above.

Theorem III (MINKOWSKI). *System* (1) *is solvable if and only if it admits a solution ξ for which an independent system of $r-1$ linear forms*

$$l_{i_1}(x), l_{i_2}(x), \ldots, l_{i_{r-1}}(x)$$

exists, which vanishes at the point ξ.

(This implies $l_i(\xi)>0$ for every linear form l_i independent from the froms $l_{i_1}, l_{i_2}, \ldots, l_{i_{r-1}}$, for ξ has to be a solution of (1) in the sense given in **1**.)

Theorem IV. *System* (2) *is solvable if and only if it admits a solution ξ, for which an independent system of linear forms*

$$l_{i_1}(x), l_{i_2}(x), \ldots, l_{i_r}(x)$$

exists so that the corresponding inhomogeneous forms vanish:

$$L_{i_1}(\xi)=L_{i_2}(\xi)=\cdots=L_{i_r}(\xi)=0.$$

Such a solution ξ will be called an extremal solution ("äusserste Lösung" in the terminology of MINKOWSKI).

In the case of a homogeneous system, the geometrical meaning of the theorem is as follows. Each inequality determines in n-dimensional Euclidean space a closed half-space bounded by a hyperplane [$n-1$-dimensional linear manifold] through the origin. The solutions are the points common to all the half-spaces, i. e. lie in a pyramid with its vertex in the origin (for $r=n$), or in a "trough" with an "$n-r$-dimensional edge" through the origin. The extremal solutions are the points of the $n-r+1$-dimensional boundary faces of the pyramid or trough. In the case of an inhomogeneous system, the situation is quite analogous.

In order to find an extremal solution, we approach, starting from an arbitrary solution, to the boundary hyperplane of a half-space, until we reach some boundary hyperplane. Then, remaining in this hyperplane, we approach towards another, and so on, and thus we reach boundary faces of lower and lower dimensions. This argument shows the existence of an extremal solution. The following algebraic proof rests on the same idea.

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[6]) H. MINKOWSKI, *Geometrie der Zahlen*, 2. Aufl. (Leipzig, 1910), pp. 39—45.

7. The sufficiency of the condition in Theorems III and IV is obvious.

Let us assume, conversely, that the system (1) admits a solution ξ. We choose a maximal linearly independent system

$$(7) \qquad l_{i_1}(x),\ l_{i_2}(x), \ldots,\ l_{i_s}(x)$$

in the set of all linear forms vanishing in the point ξ. (If all the forms are positive in the point ξ, then the set (7) is empty, $s=0$.)

Since for $s=r$ all the linear forms would vanish in the point ξ, hence s cannot exceed the value $r-1$. For $s=r-1$ the solution ξ is already an extremal solution, and the set (7) constitutes a desired set.

In case $s<r-1$, we choose a form $l_{i_{s+1}}(x)$ with $l_{i_{s+1}}(\xi)>0$; $l_{i_{s+1}}(x)$ is clearly independent of the forms (7). Considering that $s+1<r$, there exists a further form $l_{i_{s+2}}(x)$ independent of $l_{i_1}(x), l_{i_2}(x), \ldots, l_{i_{s+1}}(x)$ and, consequently, different from 0 in the point ξ. Proceeding in this way, we arrive at a set of forms

$$(7') \qquad l_{i_{s+1}}(x), l_{i_{s+2}}(x), \ldots, l_{i_r}(x)$$

which are all different from 0 in the point ξ. (7) and (7') form a maximal independent system among the forms $l_i(x)$ $(i=1,2,\ldots,m)$. This implies that every $l_i(x)$ may be expressed in the form

$$l_i(x)=c_{i1}l_{i_1}(x)+c_{i2}l_{i_2}(x)+\cdots+c_{ir}l_{i_r}(x) \qquad (i=1,2,\ldots,m)$$

with appropriate constants c_{ik}.

8. Let us consider the linear forms in t:

$$A_i(t)=c_{i,s+1}t+c_{i,s+2}l_{i_{s+2}}(\xi)+\cdots+c_{i,r}l_{i_r}(\xi) \quad (i=1,2,\ldots,m).$$

For $t=t_0=l_{i_{s+1}}(\xi)$ all these forms are positive, except those vanishing identically. Since we have $A_{i_{s+1}}(t)=t$, at least one of these forms will diminish if we reduce the value of t, starting from the value t_0. So proceeding until one of them vanishes, we arrive at a value $t=t_1$, for which one of the linear forms which were non-vanishing for t_0, will vanish; let one of these be the form $A_{i_0}(t)$.

In view of the linear independence of the set (7), (7'), the system

$$l_{i_1}(x)=l_{i_2}(x)=\cdots=l_{i_s}(x)=0,\ l_{i_{s+1}}(x)=t_1,\ l_{i_{s+2}}(x)=l_{i_{s+2}}(\xi),\ldots,l_{i_r}(x)=l_{i_r}(\xi)$$

is solvable. A solution η of it is at the same time a solution of the system of inequalities (1), since the inequalities $l_i(\eta)=A_i(t_1)\geqq 0$ $(i=1,2,\ldots,m)$ and $l_{i_r}(\eta)=l_{i_r}(\xi)>0$ hold. Since $l_{i_0}(\xi)\neq 0$,

$$l_{i_0}(x),\ l_{i_1}(x),\ l_{i_2}(x),\ \ldots,\ l_{i_s}(x)$$

are linearly independent and all vanish in the point η, thus the maximum number of linearly independent forms among those vanishing in point η is greater than the number of those vanishing in the point ξ. Repeating this process a sufficient number of times, we obtain an extremal solution.

For system (2) we may proceed just in the same way; here it is allowed that all the forms $L_i(x)$ may vanish, so that a solution required by Theorem IV can also be found.

9. Theorem II may be derived immediately from Theorem IV. Let ξ be an extremal solution of the system (2),

$$L_{i_1}(x),\ L_{i_2}(x),\ \ldots,\ L_{i_r}(x)$$

the corresponding linear forms, and Δ a maximal non-vanishing determinant of their coefficient-matrix.

Let us put $L_i(\xi)=T_i$, so that

$$T_{i_1}=T_{i_2}=\cdots=T_{i_r}=0 \quad \text{and} \quad T_j\geqq 0 \quad \text{for } j=1,2,\ldots,m.$$

Since the system of equations

$$a_{i1}x_1+a_{i2}x_2+\cdots+a_{in}x_n=T_i-b_i$$

is solvable, thus, bordering the determinant Δ (and using the notation adopted in (5)) we obtain

$$\begin{vmatrix} a_{i_1k_1} & a_{i_1k_2} & \cdots & a_{i_1k_r} & -b_{i_1} \\ a_{i_2k_1} & a_{i_2k_2} & \cdots & a_{i_2k_r} & -b_{i_2} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{i_rk_1} & a_{i_rk_2} & \cdots & a_{i_rk_r} & -b_{i_r} \\ a_{jk_1} & a_{jk_2} & \cdots & a_{jk_r} & T_j-b_j \end{vmatrix} = T_j\Delta-D_j=0,$$

or

$$\frac{D_j}{\Delta}=T_j\geqq 0.$$

Thus the necessity of the conditions has been proved.

If, conversely, the conditions (4) and (5) hold, then the system of equations

$$a_{ik_1}x_{k_1}+a_{ik_2}x_{k_2}+\cdots+a_{ik_r}x_{k_r}=\frac{D_i}{\Delta}-b_i \qquad (i=1,2,\ldots,m)$$

is solvable. Denoting by $(\xi_{k_1},\xi_{k_2},\ldots,\xi_n)$ a solution,

$$\xi=(0,\ldots,0,\xi_{k_1},0,\ldots,0,\xi_{k_2},0,\ldots,0,\xi_{k_r},0,\ldots,0)$$

is a solution of system (2) too. In fact, we have

$$L_i(\xi)=l_i(\xi)+b_i=\frac{D_i}{\Delta}\geqq 0,$$

establishing the sufficiency of our conditions.

Blumenthal's criterion.

10. In order to prove BLUMENTHAL's theorem, we need a simple consequence of MINKOWSKI's Theorem III which has some interest in itself as well.

Theorem V. *System* (1) *is solvable, if and only if a maximal independent system of linear forms,*

$$l_{i_1}(x),\ l_{i_2}(x),\ \ldots,\ l_{i_r}(x)$$

exists such that in the relations

$$l_i(x) = c_{i1} l_{i_1}(x) + c_{i2} l_{i_2}(x) + \cdots + c_{ir} l_{i_r}(x) \qquad (i = 1, 2, \ldots, m) \tag{8}$$

the coefficients c_{ir} *of* $l_{i_r}(x)$ *are positive or* 0.

This condition is necessary. To see this we have only to choose an extremal solution ξ, a corresponding independent system $l_{i_1}(x), l_{i_2}(x), \ldots, l_{i_{r-1}}(x)$, and a further form $l_{i_r}(x)$ independent of them. Thus, owing to

$$l_{i_1}(\xi) = l_{i_2}(\xi) = \cdots = l_{i_{r-1}}(\xi) = 0, \quad l_{i_r}(\xi) > 0,$$

we have

$$0 \leqq l_i(\xi) = c_{i_r} l_{i_r}(\xi),$$

and therefore,

$$c_{ir} = \frac{l_i(\xi)}{l_{i_r}(\xi)} \geqq 0.$$

If, on the other hand, $c_{ir} \geqq 0$ for $i = 1, 2, \ldots, m$ then a solution ξ of the system of equations,

$$l_{i_1}(x) = l_{i_2}(x) = \cdots = l_{i_{r-1}}(x) = 0, \quad l_{i_r}(x) = 1$$

will also be a solution of the system of inequalities (1), in view of the inequalities

$$l_{i_r}(\xi) = 1\,(>0), \quad l_i(\xi) = c_{ir} l_{i_r}(\xi) = c_{ir} \geqq 0 \qquad (i = 1, 2, \ldots, m).$$

11. In order to obtain a criterion in terms of the coefficients, we have to do nothing else but to calculate the coefficients c_{ir} for each given i $(1 \leqq i \leqq m)$ from the equation system arising from (8):

$$a_{i_1 k} c_{i1} + a_{i_2 k} c_{i2} + \cdots + a_{i_r k} c_{ir} = a_{ik} \qquad (k = 1, 2, \ldots, n). \tag{9}$$

By a direct solution of the system (9), we arrive at theorem I again. When, however, we transform the system by using a procedure applied to integral inequalities by A. HAAR[7]), then we get BLUMENTHAL's criterion concerning the matrix

$$(\alpha_{ij})_{mm} \quad \text{with} \quad \alpha_{ij} = a_{i1} a_{j1} + a_{i2} a_{j2} + \cdots + a_{in} a_{jn}.$$

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[7]) A. HAAR, Über lineare Ungleichungen, *these Acta*, **2** (1923—24), 1—14.

Theorem VI. *System* (1) *is solvable if and only if for appropriate subscripts* $i_1, i_2, \ldots, i_r$

$$\delta = \begin{vmatrix} a_{i_1 i_1} & a_{i_1 i_2} & \ldots & a_{i_1 i_r} \\ a_{i_2 i_1} & a_{i_2 i_2} & \ldots & a_{i_2 i_r} \\ \cdot & \cdot & \cdot & \cdot \\ a_{i_r i_1} & a_{i_r i_2} & \ldots & a_{i_r i_r} \end{vmatrix} > 0,$$

and substituting the last row by the corresponding elements of any row, the resulting determinants are non-negative:

$$\delta_i = \begin{vmatrix} a_{i_1 i_1} & a_{i_1 i_2} & \ldots & a_{i_1 i_r} \\ a_{i_2 i_1} & a_{i_2 i_2} & \ldots & a_{i_2 i_r} \\ \cdot & \cdot & \cdot & \cdot \\ a_{i_{r-1} i_1} & a_{i_{r-1} i_2} & \ldots & a_{i_{r-1} i_r} \\ a_{i i_1} & a_{i i_2} & \ldots & a_{i i_r} \end{vmatrix} \geqq 0 \qquad \text{for} \qquad i = 1, 2, \ldots, m.$$

Indeed let us multiply the k^{th} equation of (9) by $a_{i_\varrho k}$ for a given $\varrho = 1, 2, \ldots, r$, and add the equations thus obtained; then we find that the factors $c_{i1}, c_{i2}, \ldots, c_{ir}$ satisfy the equations

$$a_{i_1 i_\varrho} y_1 + a_{i_2 i_\varrho} y_2 + \cdots + a_{i_r i_\varrho} y_r = a_{i i_\varrho} \qquad (\varrho = 1, 2, \ldots, r). \tag{10}$$

Their matrix is identical with that of the quadratic form

$$\sum_{\varrho=1}^{r} \sum_{\sigma=1}^{r} a_{i_\varrho i_\sigma} u_\varrho u_\sigma = \sum_{\varrho=1}^{r} \sum_{\sigma=1}^{r} \sum_{k=1}^{n} a_{i_\varrho k} a_{i_\sigma k} u_\varrho u_\sigma = \sum_{k=1}^{n} \left(\sum_{\varrho=1}^{r} a_{i_\varrho k} u_\varrho \right)^2,$$

which is seen to be either positive definite or semidefinite. However, this quadratic form must be definite, for in the opposite case the linear forms $l_{i_1}(x), l_{i_2}(x), \ldots, l_{i_r}(x)$ would be dependent. Thus we get $\delta > 0$, and therefore the system (10) admits a unique solution which must coincide with the factors in question, in particular,

$$c_{ir} = \frac{\delta_i}{\delta}.$$

In view of $\delta > 0$, and on account of Theorem V this completes the proof of Theorem VI.

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