

On abelian groups whose subgroups are endomorphic images.

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To Professor László Kalmár on his 50th birthday.

§ 1. Introduction.

In a previous paper¹⁾ [4] we have stated the problem of determining all abelian groups G with

Property P. *Every subgroup of G is an endomorphic image of G .*

In paper [4] we have made the first step towards the solution of this problem by determining all abelian groups in which every finitely generated subgroup is an endomorphic image. A further progress was made by E. SAŞIADÀ [6] who has considered the same problem replacing the term "finitely generated" by "countable". Now, the present paper has for its aim to give the complete solution of the general problem. We shall arrive at the solution by making extensive use of the methods elaborated in [1], [2], [3], [8] and of the results contained in these papers. However, the present paper may be read also without being acquainted with the cited papers.

We state also the dual of our present problem: to find all abelian groups G every factor group of which is isomorphic to some subgroup of G . Here we omit this dual problem, but we intend to discuss it on another occasion.

The solution of our present problem is almost trivial for torsion free groups. In fact, it is easy to see that such a group G has property P if and only if it contains a direct summand²⁾ which is the direct sum of infinite cyclic groups in number equal to the rank of G (see [6]); hence a torsion free group possesses "rarely" the property P. On the other hand, the solution of the stated problem for torsion or mixed groups is far from being

¹⁾ Numbers in brackets refer to the Bibliography given at the end of this paper.

²⁾ The group operation will be written as addition.

trivial. This assertion is justified by the fact that we need a number of deep results from the structure theory of these groups. Of fundamental importance are in our investigations the basic subgroups of abelian p -groups, discovered by L. KULIKOV [5], as well as two recent results on basic subgroups according to which a basic subgroup B of an abelian p -group G is always an endomorphic image of G , resp. a direct sum of cyclic groups is a homomorphic image of G if and only if it is a homomorphic image of B too ([3], [8]). Without the concept of basic subgroups it seems to be impossible to characterize all torsion and mixed groups with property P, so that also our investigations show how important a structure invariant is the basic subgroup in abelian p -groups. Some methods and results of [2] have also applications in our discussions.

Considering that a torsion group is of property P if and only if every primary component of it has the same property, the investigation of torsion groups may be reduced immediately to p -groups. The solution leads in this case to an interesting result. Namely, it will turn out that a p -group G is of property P if and only if its final rank equals the final rank of its basic subgroup B , the "final rank" being defined as the minimal cardinal number among the ranks of G, pG, p^2G, \dots . Thus the sought criterion is merely the equality of two cardinal invariants of G (for which the sign \cong is true in every p -group), and therefore the possession of property P has no deep effect on the structure of the group; consequently, a great variety of p -groups has property P. In particular, a countable p -group almost always is of property P, the only exceptions are the direct sums of a bounded group and of one or more groups of type p^∞ . Hence it results that every countable p -group is a homomorphic image of any countable unbounded p -group G of property P. More generally, if G is an arbitrary p -group of property P, then every p -group whose power does not exceed the final rank of G , is a homomorphic image of G . This statement will be a simple consequence of the fact that a p -group G is of property P if and only if $B \sim G$ where B is a basic subgroup G .³⁾

In case G is a mixed group of property P, in investigating the structure of G a great role is played by the torsion subgroup T of G and by the factor group G/T , the latter being a torsion free group. It is worth while noticing that a sufficient condition for G being of property P is that both T and G/T have property P.⁴⁾ However, this condition is not necessary: it is

³⁾ Since we have always $G \sim B$, therefore each p -group G of property P which is no direct sum of cyclic groups yields a pair of groups, namely G and B , such that $G \sim B$ and $B \sim G$, but G and B are not isomorphic.

⁴⁾ Of course, a necessary and sufficient condition for G/T to have property P is that G/T contain a direct summand of the form $\sum_r C(\infty)$, and this requirement is equivalent to the same requirement on G in place of G/T . (r is the torsion free rank of G .)

enough to know only of those primary components of T that they have property P whose final ranks exceed the torsion free rank of G . Although this result seems to be rather natural in view of the results on torsion and torsion free groups, its proof is not easy at all, since it needs certain rather deep results on mixed groups.

We may mention an interesting consequence of our results. We may ask for all abelian groups G with

Property P' . *Every subgroup, which is a direct sum of cyclic groups in G , is an endomorphic image of G .*

Evidently, a group G with property P has also property P' . Now our results will imply that the converse is also true and so, for an abelian group, the properties P and P' are equivalent.

§ 2. Preliminaries.

By a group we shall mean throughout an additive abelian group with more than one element. Groups will be denoted by Latin capitals, their elements by the letters x, a, \dots, g , while i, j, k, m, n will mean as usual rational integers, in particular, p a prime and the sequence p_1, p_2, p_3, \dots the set of all rational primes. Small Gothic types such as m, p, r denote cardinal numbers.

By $O(a)$ we denote the order of a group element a . For a subset S of a group G , $\{S\}$ and $|S|$ will denote the subgroup generated by S and the cardinality of S , respectively. The sign "+" is used to denote (besides group operation) direct sum and, for a cardinal number m ,

$$\sum_m M$$

means the (discrete) direct sum of m isomorphic copies of the given group M .

A group every element of which is of finite order is called a *torsion group*. In the contrary case, namely if every element is of infinite order, the group is said to be *torsion free*. A group which is neither a torsion group nor torsion free is a *mixed group*. In a mixed group G the elements of finite order form a subgroup T called the *torsion subgroup* of G . A torsion group is the direct sum of its uniquely determined *primary components*, these being *p-groups*, i. e. groups in which the orders of the elements are powers of a fixed prime p . If the torsion group T contains an element of a maximal order, then it is a *bounded group*, otherwise *unbounded*. A *p-group* H is called *p^k-bounded* if it contains an element of order p^k but it contains no element of order p^{k+1} .

The cyclic group of order s will be denoted by $C(s)$ for $1 < s \leq \infty$, while $C(p^\infty)$ serves to denote the quasicyclic group (i. e. the group of type

p^∞), the latter being isomorphic to the additive group of all rational numbers with p -power denominators, reduced modulo 1 (p a fixed prime).

We shall denote by A_p the direct sum of p groups, each isomorphic to the direct sum of cyclic groups of order p, p^2, \dots respectively, i. e.

$$(1) \quad A_p = A_p(p) = \sum_p \sum_{k=1}^{\infty} C(p^k).$$

An arbitrary subset $S = (a_i)$ of a group G such that $0 \notin S$ is said to be *independent* if for any finite subset a_1, \dots, a_k of S a relation

$$n_1 a_1 + \dots + n_k a_k = 0 \quad (n_i \text{ rational integers})$$

implies $n_1 a_1 = \dots = n_k a_k = 0$, i. e. $n_i = 0$ in case $O(a_i) = \infty$ and $O(a_i) | n_i$ in case $O(a_i)$ is finite. By the *rank* of G , denoted by

$$(2) \quad \text{rank}(G),$$

we mean the cardinality of a maximal independent system in G containing but elements of infinite and/or prime-power order. (2) is an invariant of G and it is easy to see that it is equal to $|G|$ unless (2) is finite. By the *torsion free rank* of G we shall mean the cardinality of a maximal independent system, containing but elements of infinite order, in G . This is again an invariant of G , being equal to $\text{rank}(G/T)$ where T is the torsion subgroup of G .

If G is a p -group, then the monotone decreasing sequence of cardinals,

$$\text{rank}(G) \geq \text{rank}(pG) \geq \dots \geq \text{rank}(p^n G) \geq \dots$$

arrives (after a finite number of steps) at a minimal value, say, $\text{rank}(p^n G)$ which we shall call the *final rank* of G and denote by

$$\min_n \text{rank}(p^n G).$$

In the remaining part of this section let G denote an arbitrary abelian group containing elements of order p with a fixed prime p . We shall say that H is a p^k -regular subgroup of G if H is a p^k -bounded p -group and for each $a \in H$ we have

$$a \in \frac{p^k}{O(a)} H.$$

In [7] one of us has proved that a p^k -regular subgroup H of an arbitrary abelian group G is a direct summand of G if and only if

$$(3) \quad H \cap p^k G = 0.^5)$$

This result implies the existence of a maximal p^k -regular direct summand of any group G (having such a summand at all), since p^k -regularity and property (3) of a subgroup H are of inductive character. Applying this to

⁵⁾ Hence it follows easily that the p^k -regular groups coincide with the groups $\sum_{\mathfrak{n}} C(p^k)$ where \mathfrak{n} is an arbitrary cardinal number.

$k = 1, 2, \dots$, we obtain successively $G = B_1 + D_1 = B_1 + B_2 + D_2 = \dots$,

$$(4) \quad G = B_1 + B_2 + \dots + B_m + D_m$$

where B_k is a maximal p^k -regular direct summand of D_{k-1} (or $B_k = 0$) (we have put $G = D_0$). Thus we get a subgroup of G ,

$$(5) \quad B = B_1 + B_2 + \dots + B_m + \dots$$

which is a direct sum of cyclic p -groups, since $B_k = \sum C(p^k)$, and is by definition a maximal subgroup of G such that it is a "partial-wise" direct summand of G in the sense that the "partial sums" $B_1 + B_2 + \dots + B_m$ are maximal p^m -bounded direct summands of G . We emphasize that D_m in (4) has the property that $d \in D_m$, $O(d) = p$ imply $d \in p^m D_m$.

Any subgroup B of G with the above properties is called a *basic subgroup* of the p -primary component T_p of the torsion subgroup T of G , or briefly, a *p -basic subgroup* of G . It is not hard to show that any two p -basic subgroups of G are isomorphic (see e. g. [8]). In case G is a p -group, then its basic subgroup B can alternatively be defined by the following three properties: B is a direct sum of cyclic groups, it is a serving⁶⁾ subgroup of G , and the factor group G/B is a direct sum of groups $C(p^\infty)$.

As we have mentioned in § 1, in our following investigations an important role is played by the basic subgroups. In particular, we shall often make use of the following two lemmas:

Lemma 1. *A basic subgroup B of an abelian p -group G is a homomorphic image of G .*

Lemma 2. *If a direct sum of cyclic groups is a homomorphic image of an abelian p -group G , then the same direct sum is a homomorphic image of any basic subgroup B of G .*

For the proofs of these lemmas we refer to [8] (see also [3]).

In the description of the structure of mixed groups with property P we shall need the following results.

Lemma 3. *If the torsion subgroup T of an abelian mixed group G is the direct sum of cyclic groups, then G has a decomposition $G = G_1 + G_2$ such that G_1 is a torsion group and $|G_2| \leq \max(r, \aleph_0)$ where r is the torsion free rank of G .*

Lemma 4. *Each mixed abelian group G has a decomposition: $G = G_1 + G_2$ where G_1 is a torsion group whose p_i -components are bounded and G_2 is a mixed group such that, for each prime p_i , the p_i -component of the torsion subgroup of*

⁶⁾ H is a serving subgroup of G if for each $a \in H$, the solvability of an equation $nx = a$ in G implies its solvability in H .

G_2 has a rank not greater than any prescribed cardinal number m , which is $\geq \max(p_i, r, \aleph_0)$ where p_i is the final rank of the p_i -component of the torsion subgroup of G and r is the torsion free rank of G .

Lemma 3 is a special case of Corollary in [2], p. 304, while Lemma 4 is an equivalent form of Lemma 3 of paper [2].

§ 3. The torsion groups with property P.

In this section we investigate the groups with property P and $r=0$, i. e. the torsion groups of property P. Obviously, such a group possesses property P if and only if every primary component of the group possesses this property, so that there is no restriction in considering only p -groups G . B will denote a basic subgroup of G .

Our main result on p -groups is contained in

Theorem 1. *For an abelian p -group G the following statements are equivalent:*

- α) G is a group with property P;
- β) the final rank of G is equal to that of a basic subgroup B of G :
- (6) $\min \text{rank}(p^n G) = \min \text{rank}(p^n B)$;
- γ) G is a homomorphic image of B .

Before proving this result, let us consider some corollaries of this theorem.

Corollary 1. *An abelian p -group G possesses property P if and only if every subgroup of G is a homomorphic image of B .*

Indeed, property P of G implies by γ) that $B \sim G \sim H$ for any subgroup H of G . The converse follows at once from the homomorphism $G \sim B$ (see Lemma 2).

Corollary 2. *An abelian p -group G of infinite final rank has property P if and only if any abelian p -group K satisfying*

$$(7) \quad |K| \leq \min \text{rank}(p^n G)$$

is a homomorphic image of G .

For, if G has property P, then by β) we have $B \sim K$, B being the direct sum of cyclic groups, and therefore $G \sim B$ implies $G \sim K$. Conversely, if for each K satisfying (7) we have $G \sim K$, then, in particular, each subgroup of G is a homomorphic image of G .

Corollary 3. *A bounded p -group has property P.*

This is obvious in view of Theorem 1.

Corollary 4. A countable abelian p -group fails to have property P if and only if it is the direct sum of a nonvoid set of groups $C(p^\infty)$ and of a bounded p -group.

In fact, it is evident that a p -group

$$(8) \quad G = N + \sum_m C(p^\infty) \quad (m > 0)$$

with a bounded subgroup N can not have property P. On the other hand, if a countable p -group G is not of the form (8), then either G is bounded (see Corollary 3) or the basic subgroup of G is unbounded. In both cases G has property P.

Proof of Theorem 1.

α) implies β). Denoting by m the final rank of G , α) ensures the existence of a homomorphism

$$G \sim C_n = \sum_m C(p^n) \quad (n \text{ a fixed integer}),$$

so that by Lemma 2 we conclude $B \sim C_n$ whence $\min \text{rank}(p^n B) \cong m$. The sign \cong being true here for each group G , we arrive at β).

β) implies γ). Supposing β), let us consider the representation (4) of G for a natural integer m satisfying

$$\text{rank}(p^m D_m) = m = \min \text{rank}(p^n G).$$

Since each element of order p of D_m belongs to $p^m D_m$, and since the rank of a p -group can alternatively be defined as the cardinality of a maximal independent set of elements of order p in the group, it follows

$$(9) \quad \text{rank } D_m = \text{rank}(p^m D_m) = m.$$

On the other hand, representation (5) of B shows that for the group $F_m = B_{m+1} + B_{m+2} + \dots$ we have

$$(10) \quad m \cong \text{rank}(p^m B) = \text{rank}(p^m F_m) = \text{rank } F_m.$$

F_m being a direct sum of cyclic groups, (9) and (10) imply $F_m \sim D_m$ and hence we obtain $B = B_1 + \dots + B_m + F_m \sim B_1 + \dots + B_m + D_m = G$, as stated.

γ) implies α). Let G be a p -group with $B \sim G$ and H a subgroup of G . Then we have $B' \sim H$ for a suitable subgroup B' of B . But B being the direct sum of cyclic groups, a homomorphism $B \sim B'$ exists, so that $G \sim B$ (Lemma 1) implies $G \sim B \sim B' \sim H$, i. e. G is a group with property P.

§ 4. The mixed groups with property P.

In this section we suppose that the torsion free rank r of the group G under consideration is greater than 0. Our main purpose is to prove

Theorem 2. *An abelian group G of torsion free rank $r > 0$ possesses property P if and only if*

(i) *in case $r < \aleph_0$ the group G is of the form*

$$(11) \quad G = T + \sum_r C(\infty)$$

where T is a torsion group with property P (covered by Theorem 1);

(ii) *in case $r \geq \aleph_0$ the group G contains a direct summand*

$$(12) \quad \sum_r C(\infty)$$

and in the torsion subgroup T of G each primary component T_i of final rank $> r$ is a p_i -group with property P.

First, let us mention the following two immediate corollaries.

Corollary 5. *For a torsion free group G we have the trivial result that G has property P if and only if G is either a direct sum of a finite number of infinite cyclic groups or has a direct summand of type*

$$\sum_{|G|} C(\infty).$$

Corollary 6. *A necessary and sufficient condition for a countable mixed group G to have property P is that it can be represented either in the form (11) with a finite r and with a torsion group T of property P (covered by Corollary 4), or in the form*

$$G = U + \sum_{\aleph_0} C(\infty)$$

with an arbitrary countable abelian group U .

Proof of Theorem 2. Let us first consider the case $\aleph_0 > r = r$. Now a group G in (11) surely possesses property P, for any subgroup of G is the direct sum of a subgroup of T and of a direct sum of groups $C(\infty)$ in number $\leq r$.

Assume, conversely, that G (with torsion free rank $r < \aleph_0$) has property P. Choose in G an independent system of elements g_1, \dots, g_r of infinite order. By property P, there exists a homomorphism η mapping G onto its subgroup $\{g_1\} + \dots + \{g_r\}$. If g'_i is an arbitrary inverse image of g_i under η ($i = 1, \dots, r$) and T is the kernel of η , then obviously

$$G = T + \{g'_1\} + \dots + \{g'_r\}.$$

r being the torsion free rank of G , T does not contain elements of infinite

order, i. e., T coincides with the torsion subgroup of G . In order to show that T has property P, take an arbitrary subgroup T' of T . By property P of G , there exists a homomorphism η' of G onto $T' + \{g'_1\} + \dots + \{g'_r\}$. If g'_i is an inverse image of g'_i under η' ($i=1, \dots, r$) and K is the set of all elements of G sent into T' by η' , then we get

$$G = K + \{g'_1\} + \dots + \{g'_r\}.$$

As before, we conclude $K = T$ showing that η' induces a homomorphism of T onto T' , i. e. T is a group of property P, indeed.

Turning our attention to the case $r \geq \mathbf{N}_0$, suppose G has property P. As before we can see that G contains a direct summand (12). Assume T_i is a p_i -primary component of the torsion subgroup T of G such that

$$(13) \quad p_i = \min_n \text{rank}(p_i^n T_i) > r.$$

We show that T_i has property P.

For this purpose let us consider a subgroup A of T_i such that $A \cong A_{p_i}(p_i)$ (see (1)), and let T'_i be the image of T_i under some fixed homomorphism $\eta: G \sim A$. If we shall have proved the inequality

$$(14) \quad \text{rank}(p_i^n T'_i) \geq p_i \quad (n=1, 2, \dots),$$

then we shall be ready, for this ensures that A is a homomorphic image of T'_i (since T'_i is — as a subgroup of A — itself the direct sum of cyclic groups), and $T_i \sim T'_i \sim A$ implies, owing to Lemma 2, $B_i \sim A$ where B_i is a basic subgroup of T_i ; finally, hence we obtain that the final ranks of B_i and T_i are equal, i. e. T_i has property P (cf. Theorem 1).

Now, in order to establish (14), take into account that η induces a homomorphism $\eta_n: p_i^n G \sim p_i^n A$. Under η_n the whole torsion subgroup $p_i^n T$ of $p_i^n G$ is mapped upon $p_i^n T'_i$ ($n=1, 2, \dots$), considering that η maps T_i upon T'_i and all other primary components of T upon 0. Hence η_n maps a coset of $p_i^n G$ modulo $p_i^n T$ upon a coset of $p_i^n A$ modulo $p_i^n T'_i$. Since the cardinality of the cosets of $p_i^n G$ modulo $p_i^n T$ clearly equals r , therefore the image of $p_i^n G$ under η_n must be of power $\leq r \cdot |p_i^n T'_i|$. On the other hand, this image $p_i^n A$ has, by the definition of A , the power p_i , consequently,

$$p_i \leq r |p_i^n T'_i|$$

whence by (13) we obtain (14), in fact.

To complete the proof of Theorem 2, it remains to prove the sufficiency of the condition in case (ii). Assume the group G has a rank $r \geq \mathbf{N}_0$, contains a direct summand G_3 of the form (12) and each primary component T_i of its torsion subgroup T which has a final rank $p_i > r$ is a p_i -group of property P. Put $G = G' + G_3$, where $T \subseteq G'$ and apply Lemma 4 to G' with $m_i = \max(p_i, r)$ to obtain

$$(15) \quad G = G_1 + G_2 + G_3$$

where G_1 is a torsion group with bounded p_i -components, G_2 has the property that the p_i -components of its torsion subgroup are of rank $\leq m_i$ and $G_3 = \sum_r C(\infty)$.

If H is an arbitrarily given subgroup of G , then its torsion free rank s is evidently $\leq r$ and, for every prime p_i , the final rank of the p_i -component U_i of its torsion subgroup U is clearly $\leq p_i$. Now apply again Lemma 4 with the same $m_i = \max(p_i, r)$ to get a decomposition

$$(16) \quad H = H_1 + H_2$$

where H_1 is a torsion group with bounded p_i -components and the p_i -components U'_i of the torsion subgroup U' of H_2 are of rank $\leq m_i$. This property of H_2 does not alter if we separate from H_1 those of its p_i -components whose rank does not exceed m_i as well as those cyclic direct summands of the other p_i -components (H_1 is a direct sum of cyclic groups!) whose order p_i^k satisfies $\text{rank}(p_i^k U'_i) = p_i$, and incorporate all these subgroups of H_1 into H_2 . Then, denoting again by H_1 and H_2 the arising groups, H_1 becomes isomorphic to some subgroup of G_1 and hence $G_1 \sim H_1$. Therefore, it suffices to establish the existence of a homomorphism $G_2 + G_3 \sim H_2$. But H_2 is a homomorphic image of the group

$$(17) \quad \sum_r C(\infty) + \sum_{r < p_i}^* A_{p_i}(p_i)$$

(the asterisk indicates that the summation is extended only over those primes p_i for which $r < p_i$); in fact, each subgroup of G the p_i -components of whose torsion subgroup are of rank $\leq \max(r, p_i)$ has a generator system containing at most r elements of infinite order and at most $\max(r, p_i)$ elements whose order is a power of p_i , for each i . Thus it will be enough to show that (17) is a homomorphic image of $G_2 + G_3$, or more simply, that

$$M = \sum_{r < p_i}^* A_{p_i}(p_i)$$

is a homomorphic image of G_2 .

By hypothesis, any T_i with final rank $p_i > r$ has property P, i. e. a basic subgroup of T_i has the same final rank p_i . Since on account of (15) the final ranks of the p_i -components of the torsion subgroups of G and G_2 as well as the final ranks of the respective basic subgroups are the same, we infer that the p_i -components T'_i of the torsion subgroup T' of G_2 must have property P for each i with $p_i > r$, i. e. for these i we have

$$T'_i/Q_i \cong A_{p_i}$$

for some subgroup Q_i of T'_i . Let

$$Q = \sum_{p_i > r}^* Q_i + \sum_{p_j \leq r} T'_j;$$

then

$$T'/Q \cong \sum_{r < p_i} A_{p_i}$$

is the torsion subgroup of G_2/Q . Now T'/Q being the direct sum of cyclic groups, we may apply Lemma 3 to the group G_2/Q and then obtain

$$G_2/Q = X/Q + Y/Q$$

where X/Q is a subgroup of T'/Q and hence again the direct sum of cyclic groups, while Y/Q is of power $\leq r$. Consequently, the p_i -components of X/Q must have the final rank p_i for the primes p_i satisfying $p_i > r$ and thus there exists a homomorphism $X/Q \sim M$. Now X/Q is a direct summand of G_2/Q and therefore $G_2 \sim G_2/Q \sim X/Q \sim M$ imply $G_2 \sim M$ which completes the proof of Theorem 2.

§ 5. The groups with property P'.

Having solved the problem of characterizing all groups of property P, it is easy to get a complete solution of the problem of finding the structure of all abelian groups with property P'. This problem is settled by the following

Theorem 3. *An arbitrary abelian group G has property P' if and only if it has property P. Consequently, all groups with property P' are covered by Theorems 1 and 2.*

It suffices to verify that property P' implies property P.

An essential observation is that an arbitrary abelian group G contains a subgroup H such that (i) H is the direct sum of cyclic groups, (ii) the torsion free rank of H is equal to the torsion free rank r of G ; (iii) the p_i -components T_i, U_i of the torsion subgroups T, U of G and H respectively, satisfy:

$$(18) \quad \text{rank}(p_i^k U_i) = \text{rank}(p_i^k T_i) \quad \text{for all } p_i, k,$$

unless the cardinal number on the right hand side is the finite final rank > 0 of T_i when the left hand side may vanish. In this exceptional case T_i is namely the direct sum of a bounded group and a finite number of groups $C(p_i^\infty)$, and it is clear that in all other cases the p_i -group T_i contains a direct sum U_i of cyclic p_i -groups satisfying (18). Consequently, there exists an H with properties (i)–(iii).

Now assume that G possesses property P'. Then there exists an endomorphism $G \sim H$ where H is a group of the preceding paragraph. If the torsion free rank r is finite, then the final rank of T_i can not be a nonzero integer, for then $C(p_i^s) + \sum_{t=1}^s C(\infty)$, if s is sufficiently large, can not be a homomorphic image of G . Hence, in case of finite r , any subgroup F of G

satisfies (ii) and (iii) with \cong in (18) (and with no exceptional case), and besides G has the form $G = T + \sum_r C(\infty)$; consequently, F is a homomorphic image of H . This shows that G has property P if its torsion free rank is finite.

In case $r > \aleph_0$, each subgroup F of G satisfies (ii) and (iii) with \cong in (18) and therefore there exists again a homomorphism $H \sim F$, q. e. d.

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