Abelian groups every finitely generated subgroup of which is an endomorphic image.

By A. KERTÉSZ and T. SZELE in Debrecen.

§ 1. Introduction.

In a previous article one of us has determined all groups every subgroup of which is a direct summand [2][1]). In a subsequent joint paper [3] we have split this problem into the following two, rather difficult problems:

Problem I. Determine all groups every subgroup of which is an endomorphic image of the group.

Problem II. Determine all groups every endomorphic image of which is a direct summand of the group.

These problems seem to be very difficult even in the case of abelian groups. Our paper [3] is devoted to the discussion of Problem II in case of abelian groups, and contains an almost complete solution of this problem.

In the present paper we shall make the first step towards the solution of Problem I in the case of abelian groups, namely we determine all abelian groups G possessing the following property:

(1) *Every finitely generated subgroup of G is an endomorphic image of G.*

All such groups are described in Theorem 1 and 2 (see § 2).

As to notations and terminology we make the following remarks. By a group we always mean an additively written *abelian* group. The letters $x, a, b, c, \ldots, g$ denote elements of groups and the other small Latin letters ordinary integers. The symbol $\{a_1, a_2, \ldots\}$ denotes the group generated by the elements $a_1, a_2, \ldots$ of a group. We denote by $O(a)$ the order of the element a of a group. Then $1 \leqq O(a) \leqq \infty$. An abelian group is called *torsion-free* if it contains no element $\neq 0$ of finite order. In the contrary case, i. e. if any element of the group is of finite order, the group is called a *torsion group*. It may happen that in a torsion group there exists an element of maximal order. Then we say that the group is a *bounded group*. In the contrary case we call the group an *unbounded group*. It is well-known that any (abelian) torsion group splits into the direct sum of its uniquely determined *primary*

[1]) The numbers in brackets refer to the Bibliography at the end of this paper.

components, the latter being p-groups, i. e. groups every element of which is of p-power order where p denotes a fixed prime number. The direct sum of two groups A, B will be denoted by $A+B$. The cyclic group of order r $(1 \leqq r \leqq \infty)$ we denote by $C(r)$. Then, e. g.,

$$\sum_{n} C(\infty) \tag{2}$$

is a direct sum of n cyclic groups of infinite order.

A system of elements (of an arbitrary finite or infinite cardinal number) $a_1, a_2, \ldots$ of a group is said to be independent if any relation (containing an arbitrary finite subset of the system)

$$m_1 a_1 + \cdots + m_k a_k = 0$$

implies

$$m_1 a_1 = \cdots = m_k a_k = 0.$$

The maximal number of elements of an independent system of elements of infinite order in a group G is called the *torsion-free rank* of G.

A group A is called *algebraically closed* (or, in another terminology, complete) if any equation $nx = a$ has a solution $x \in A$ for each element $a \in A$ and for each natural number n. An equivalent of this condition is the requirement $nA = A$ for $n = 1, 2, 3, \ldots$. (Here nA denotes the set of all elements na with $a \in A$.) The union A of all algebraically closed subgroups of an arbitrary group P is obviously itself an algebraically closed group, and we call it the maximal algebraically closed subgroup of P. Since, by a well-known theorem of R. Baer ([1], p. 766), every algebraically closed subgroup of a group is a direct summand of the group, we have the representation

$$P = A + B \tag{3}$$

where the subgroup B of P contains (by the definition of A) no algebraically closed subgroup $\neq 0$. Such a group is called a *reduced group*. While the maximal algebraically closed subgroup A of P is invariantly defined, the reduced subgroup B in (3) is not uniquely determined in general. The structure of B, however, is uniquely determined, since (3) implies the isomorphism

$$B \cong P/A. \tag{4}$$

We make use of the representation (3) only in the case if P is a p-group. Then the subgroup A is defined as the union of all subgroups U of P for which $pU = U$ holds. It is well-known that an algebraically closed p-group U (i. e. a p-group with the property $pU = U$) is always a direct sum of groups of type $C(p^\infty)$, where $C(p^\infty)$ denotes the quasicyclic group of type (p^∞) defined as the additive group modulo 1 of the rational numbers with p-power denominators. Then the representation (3) says that any abelian p-group can be decomposed into a direct sum $A+B$ where the subgroup A is a direct sum of groups of type $C(p^\infty)$ while the subgroup B contains no subgroup of type $C(p^\infty)$.

§ 2. Results.

Our results are contained in the following two theorems.

Theorem 1. *Let G be an arbitrary abelian group containing only elements of finite order. Then every finitely generated subgroup of G is an endomorphic image of G if and only if for each primary component P of G the following requirement is fulfilled: if the maximal algebraically closed subgroup of A is not 0, then P/A is an unbounded group.*

Remark. Consequently a reduced torsion group G possesses always the property (1).

Theorem 2. *Let G be an arbitrary abelian group containing elements of infinite order. Then every finitely generated subgroup of G is an endomorphic image of G if and only if either G is a direct sum of a finite number of infinite cyclic groups and of a torsion group described in Theorem 1, or G contains a direct summand (2) for each n $(=1, 2, 3, \ldots)$.*

Remark. In particular, a torsion-free group G possesses the property (1) if and only if either G itself is finitely generated, or it contains a finitely generated direct summand of rank n for each n $(=1, 2, 3, \ldots)$.

§ 3. Proof of Theorem 1.

We start with the following

Lemma.[2]) *If a is an element of order p^r of an abelian p-group N such that*

(5) $$\{a\} \cap p^r N = 0,$$

then the cyclic subgroup $\{a\}$ is a direct summand of N.

Proof. We show that (5) implies

(6) $$N = \{a\} + M$$

with a suitable subgroup M of N. For this purpose we define M as a *maximal* subgroup of N possessing the following properties:

(7) $$p^r N \subseteq M, \quad \{a\} \cap M = 0.$$

(The existence of such a group M is assured by ZORN's lemma.) If (6) is not true, then there exists an element $x \in N$ such that

(8) $$x \notin \{a\} + M$$

but

$$px \in \{a\} + M$$

i. e.

(9) $$px = ta + d \qquad (d \in M).$$

Then we have by (9)

$$p^r x = p^{r-1} ta + p^{r-1} d.$$

---

[2]) This is a special case of a Theorem in [4].

On the other hand, we infer from (7) that $p^r x \in M$, i. e., by (7) and the last equation $p^{r-1} ta = 0$. Thus $t = pt'$, so that we get by (9)

$$p(x - t'a) = d,$$

i. e., for the element

(10) $$x' = x - t'a$$

$$x' \notin M, \quad px' = d \in M$$

hold. This implies, by the maximality of M,

$$\{a\} \cap \{M, x'\} \neq 0.$$

Consequently $x' \in \{a\} + M$, which involves by (10)

$$x \in \{a\} + M.$$

This contradicts (8), completing so the proof of the Lemma.

Proof of the *necessity* of the conditions in Theorem 1. Let G be an arbitrary abelian torsion group with property (1), and P a primary component of G. Since the p-group P is an endomorphic image of G, P is also a group with property (1). Starting from this fact we show that P satisfies the requirement formulated in Theorem 1. Moreover, we shall prove this even under the weaker assumption that *every cyclic subgroup of P is an endomorphic image of P*. Suppose that, contrary to our assertation, for the representation (3) of P

(11) $$A \neq 0, \quad p^{m-1}B \neq 0, \quad p^m B = 0$$

hold with some natural number m. Then it follows from $A \neq 0$ that P contains a subgroup $C(p^{m+1})$. We show that this subgroup cannot be an endomorphic image of P. Indeed, the existence of a subgroup $H \subset P$ with

$$P/H \sim C(p^{m+1})$$

would imply $p^{m+1} P \subset H$. On the other hand, we have by (3) and (11)

$$p^{m+1} P = p^{m+1} A + p^{m+1} B = p^{m+1} A = A,$$

so that $A \subset H$. Then, however,

$$B \sim P/A \sim P/H \sim C(p^{m+1})$$

which is, by (11), impossible, proving so the necessity of the conditions in Theorem 1.

Proof of the *sufficiency* of the conditions in Theorem 1. In proving the sufficiency of the conditions, we can clearly restrict ourselves to the case in which $G = P$ is a p-group. Let us suppose that P is an abelian p-group, and P/A is an unbounded group, inasmuch as the maximal algebraically closed subgroup A of P is $\neq 0$. We have to show that every finite subgroup of P is an endomorphic image of P. If in the representation (3) B is a bounded group, then $A = 0$ and for the group $P = B$ our assertion follows immediately from the previous Lemma.[3]) Therefore in the sequel we have to consider only the case in which B *is an unbounded group*. Since, by (3), B is a homomorphic image of P, our above assertion follows from the fact that *an arbitrary finite abelian p-group is a homomorphic image of* B. This can be verified as follows. Let D be the cross cut of all groups

$$B,\ pB,\ p^2B, \ldots, p^nB, \ldots.$$

Obviously D is identical with the set of all elements *of infinite height* in B.[4]) We state that $\bar{B}=B/D$ *is an unbounded abelian p-group without elements of infinite height*. Indeed, $p^m\bar{B}=0$ would imply $p^mB\subseteq D$, i. e., $p^mB=p^{m+1}B=p(p^mB)$. But this is impossible since B is a reduced group, and thus the algebraically closed subgroup p^mB of B must be $=0$, in contradiction to our assumption that B is an unbounded group. On the other hand, $\bar{B}$ contains no element $\neq 0$ of infinite height. For let us suppose that the coset $\bar{b}=b+D$ $(b\in B)$ is an element of infinite height in the group $\bar{B}$. Then the congruence

$$p^n x\equiv b \pmod D$$

has a solution $x\in B$ for every natural number n. Since this congruence is equivalent to $p^nx-b\in D$ and every element of D is of infinite height in B, we obtain that b is an element of infinite height in B, i. e. $b\in D$ and therefore $\bar{b}=0$.

Thus we have reached our aim by proving the following statement: *Every finite abelian p-group is a homomorphic image of the unbounded abelian p-group* $\bar{B}$ *containing no element of infinite height*. This follows immediately by repeated application of the following fact: If s is an arbitrary natural number, then

$$\bar{B}=C(p^r)+M \qquad (r\geqq s) \tag{12}$$

holds with a suitable natural number $r\geqq s$ and with suitable subgroups $C(p^r)$, M of $\bar{B}$. We show the validity of this statement by the aid of the previous Lemma. Let b be an element of $\bar{B}$ with $O(b)=p^t\geqq p^s$. From $p^{t-1}b\neq 0$ we infer that there exists a maximal natural number h for which the equation $p^hx=p^{t-1}b$ can be solved in $\bar{B}$. Let $x=a\in\bar{B}$ be a solution of this equation, i. e.

$$p^ha=p^{t-1}b \qquad (h\geqq t-1).$$

Clearly

$$p^r=O(a)=p^{h+1}\geqq p^t\geqq p^s. \tag{13}$$

---

[3]) The Lemma implies, namely, that $\{a\}$ is a direct summand of P for each element $a\in P$ of maximal order. The repeated application of this procedure leads to the fact that every finite subgroup of P is an endomorphic image of P.

[4]) The element b of a p-group B is called an element of infinite height in B if the equation $p^nx=b$ has a solution $x\in B$ for each natural number n.

On the other hand,

(14) $$\{a\} \cap p^r \bar{B} = 0$$

holds. As a matter of fact, if (14) is not true we get

$$p^h a = p^{r-1} a = p^r b_1 \neq 0 \qquad (b_1 \in B)$$

which says that the equation $p^n x = p^h a = p^{r-1} b$ would have a solution $x = b_1 \in \bar{B}$ for $n = r = h + 1$, contrary to the maximal property of h. Hence (14) and (13) prove the validity of (12), completing so the proof of Theorem 1.

§ 4. Proof of Theorem 2.

In this section G denotes an abelian group containing elements of infinite order.

Proof of the *necessity* of the conditions in Theorem 2. Let G be first an arbitrary abelian group with property (1) and with the torsion-free rank $r < \infty$, and let $g_1, \ldots, g_r$ be an independent system of elements of infinite order in G. Then by our assumption

(15) $$G \sim \{g_1\} + \cdots + \{g_r\}.$$

Let g_i' be an inverse image of g_i under the homomorphism (15) $(i = 1, 2, \ldots, r)$. Then we have obviously

$$G = \{g_1'\} + \cdots + \{g_r'\} + T$$

where T is the kernel of the homomorphism (15). Clearly T is identical with *the torsion subgroup* of G (i. e. T consists of all elements of finite order of G) for T contains no element of infinite order, r being the torsion-free rank of G.

By Theorem 1 there remains only to be proved that T is a group with property (1). Let therefore F be an arbitrary finite subgroup of T. Then

(16) $$S = \{g_1'\} + \cdots + \{g_r'\} + F$$

is a finitely generated subgroup of G, and consequently

(17) $$G \sim S.$$

We have to show that this homomorphism maps the subgroup T onto F. If $g_i'' \in G$ is an inverse image of g_i' under this homomorphism $(i = 1, 2, \ldots, r)$, then we have — as before —

(18) $$G = \{g_1''\} + \cdots + \{g_r''\} + K$$

where K consists of all elements of G which are mapped into F under the homomorphism (17). (Namely K is the kernel of the product of two homomorphisms α and β where α denotes the homomorphism (17) and β denotes the "projection" of S onto $\{g_1'\} + \cdots + \{g_r'\}$ corresponding to the direct representation (16).) Since $g_1'', \ldots, g_r''$ are elements of infinite order, we have in (18) $K = T$ which shows that the homomorphism (17) maps only elements of T into F. This means that the image of T under the homomorphism (17) is exactly F. Thus we have completed the proof for the case, in which the torsion-free rank of G is finite.

Now, if the torsion-free rank of G is infinite, then let $g_1, g_2, \ldots$ be an infinite independent system of elements of infinite order in G. Since, by the property (1) of G, for an arbitrary natural number n the subgroup

$$\{g_1\} + \cdots + (g_n\}$$

is an endomorphic image of G, we get — as before —

$$G = \{g_1'\} + \cdots + \{g_n'\} + H,$$

H being the kernel of the endomorphism in question. This completes the proof of the necessity of the conditions in Theorem 2.

P r o o f of the *sufficiency* of the conditions in Theorem 2. Let be first

$$G = \{g_1\} + \cdots + \{g_r\} + T \tag{19}$$

where $\{g_i\}$ is an infinite cyclic group $(i = 1, 2, \ldots, r)$ and T is an abelian group with property (1). Since each finitely generated subgroup S of G is a direct sum of a finite subgroup F of T and of a free abelian group of rank $\leqq r$, the representation (19) shows immediately that S is a homomorphic image of G.

On the other hand, if C contains a direct summand (2) for each natural number n, then it is obvious that an arbitrary finitely generated abelian group is a homomorphic image of G. This completes the proof of Theorem 2.

Bibliography.

[1] R. Baer, The subgroup of the elements of finite order of an abelian group, *Annals of Math.*, (2) **37** (1936), 766—781.

[2] A. Kertész, On groups every subgroup of which is a direct summand, *Publicationes Math. Debrecen*, **2** (1951), 74—75.

[3] A. Kertész and T. Szele, On abelian groups every multiple of which is a direct summand, *these Acta*, **14** (1952), 157—166.

[4] T. Szele, On direct decomposition of abelian groups, *Journal London Math. Soc.*, **28** (1953), 247—250.

(Received May 22, 1953.)