On a property of lacunary power-series.

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1. Let us consider the power-series

$$f(z)=\sum_{\nu=1}^{\infty} a_\nu z^{\lambda_\nu} \tag{1.1}$$

whose positive integer exponents λ_ν satisfy the condition

$$\frac{\lambda_\nu}{\nu}\to\infty.$$

This condition can be written also in the form

$$\lim_{x\to+\infty}\frac{1}{x}\sum_{\lambda_\nu\leqq x}{}_{\nu}\,1=0. \tag{1.2}$$

The significance of this condition was first discovered by FABRY by his famous gap-theorem. The effect of this condition on integral functions was systematically studied first in the frame of more general questions by PÓLYA[1]) in a fundamental memoir; his most interesting[2]) theorems refer to the case (1.2). So his Theorem VI asserts, vaguely expressed, that an integral function of the form (1.1) satisfying the Fabry-condition (1.2) has in all angles with vertex at $z=0$ an "equal growth". PÓLYA originally measured the growth by the order and type with respect to an angle and to the whole plane. In the important papers of S. MANDELBROJT and L. SCHWARTZ the theorem appeared in the form

$$\max_{|z|=r}|f(z)|\leqq \max_{\substack{r(1-\varepsilon)\leqq|z|\leqq r(1+\varepsilon)\\ \alpha\leqq \operatorname{arc} z\leqq\beta}}|f(z)|^{1+\varepsilon} \tag{1.3}$$

if $\varepsilon>0$, $0\leqq\alpha<\beta\leqq 2\pi$, $r>r_0(f,\varepsilon,\beta-\alpha)$, and it was also extended to DIRICHLET's series. PÓLYA's Theorem VIII replaces in his theorem VI the angle by a more general domain which may be called an angle with a curve extending to ∞ as bisector. The inequality (1.3) was sharpened

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[1]) G. PÓLYA, Lücken und Singularitäten von Potenzreihen, *Math. Zeitschrift,* **29** (1929), pp. 549–640.

[2]) See in particular p. 556 in [1]).

by F. SUNYER i BALAGUER[3]). He showed, for all integral functions $f(z)$ satisfying (1.2), the existence of an $\eta(r)$ tending to 0 with $\frac{1}{r}$ such that to any prescribed continuous $\Theta(r)$ and all sufficiently large r there is a z_r in the domain

$$(1.4) \qquad \frac{r}{1+\eta(r)} \leqq |z_r| \leqq r(1+\eta(r)), \quad |\arc z_r - \Theta(r)| < \eta(r)$$

with

$$(1.5) \qquad \log|f(z_r)| > (1-\eta(r)) \log M(r,f),$$

where, as usual,

$$(1.6) \qquad M(r,f) = \max_{|z|=r} |f(z)|.$$

The essential content of the theorem may be expressed in the simplest case $\Theta(r) = \text{const.}$ by saying that an integral function with Fabry-condition exhibits an "equal" growth in all "not too tight funnels" around an arbitrary ray issuing from the origin as on the whole plane.

2. In these years I have developed an analytical method which I used for various purposes. In my lecture about this method[4]) at the Meeting of the Czechoslovakian and Polish Mathematical Societies in Prague, Sept. 1949, I have risked the assertion, that some of the results of PÓLYA and various refinements are within reach of my method. In what follows I shall show I was right. Using the abbreviation

$$(2.1) \qquad \max_{\substack{|z|=r \\ \alpha \leqq \arc z \leqq \beta}} |f(z)| = M(r,\alpha,\beta,f)$$

I shall prove the

Theorem I. *Given $\alpha, \beta, \varepsilon$ with $0 \leqq \alpha < \beta \leqq 2\pi$, $0 < \varepsilon \leqq \frac{1}{2}$, and an arbitrary integral function $f(z)$ satisfying the Fabry-condition* (1.2), *we have for all $r > r_1 = r_1(f, \varepsilon, \beta-\alpha)$ the inequality*

$$(2.2) \qquad M^{1+\varepsilon}(r,f) \leqq \frac{48\pi}{\beta-\alpha} M^{2\varepsilon}(2r,f)\; M(r,\alpha,\beta,f).$$

If $f(z)$ increases very quickly, the inequality (2.2) can become a triviality. One will see from the subsequent proofs how they have to be modified in order to reach greater generality. Anyway, in the most interesting

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[3]) F. SUNYER i BALAGUER, Propriedades de las funciones enteras representadas por series de Taylor lagunares (orden finito), *Semin. Math. de Barcelona*, 2 (1949), fasc. 1 p. 1—48. He found similar results also in the case of positive maximal density.

[4]) See P. TURÁN, On a new method in the analysis with applications, *Časopis pro pešt. mat. a fys.*, 74 (1949), pp. 125—131. A detailed exposition of this method with numerous new applications will be given in a forthcoming book.

case when $f(z)$ is of finite order and of normal type, the inequality (2. 2) is not at all trivial.

How is theorem I connected with PÓLYA's results? We impose upon $f(z)$ first the restriction

$$\frac{M(2r,f)}{M^{c_1}(r,f)} \leqq c_2 \tag{2. 3}$$

with suitable numerical c_1 and c_2 for all sufficiently large r. Then (2. 2) assumes for $r > r_1(f, \varepsilon, \beta-\alpha)$, the form

$$M^{1-(2c_1-1)\varepsilon}(r, f) \leqq \frac{48\pi c_2}{\beta-\alpha} M(r, \alpha, \beta, f), \tag{2. 4}$$

or putting $2c_2-1 = c_3$ for all $r > \varrho_0(f, \beta-\alpha, \varepsilon)$

$$\frac{\beta-\alpha}{48\pi c_2} M^{1-c_3\varepsilon}(r, f) \leqq M(r, \alpha, \beta, f) \leqq M(r, f). \tag{2. 5}$$

Next we suppose only the existence of a sequence

$$r_1 < r_2 < \ldots \to +\infty \tag{2. 6}$$

and the existence of constants c_4 and c_5 independent of ν such that

$$\frac{M(2r_\nu, f)}{M^{c_4}(r_\nu, f)} \leqq c_5. \tag{2. 7}$$

As we shall show in **8** this condition is fulfilled for all integral functions of finite order, and if $f(z)$ is of order k then moreover the limitation

$$2r_\nu \leqq r_{\nu+1} \leqq 2r_\nu^{k+2} \tag{2. 8}$$

can be given. The above reasoning gives in the case (2. 7) the inequality

$$\frac{\beta-\alpha}{48\pi c_2} M^{1-c_3\varepsilon}(r_\nu, f) \leqq M(r_\nu, \alpha, \beta, f) \leqq M(r_\nu, f). \tag{2. 9}$$

Hence we obtain the

Corollary. *If an integral function of finite order satisfies the Fabry-condition* (1. 2) *then there is a sequence of concentric circles* $|z| = r_\nu$ *with the restriction* (2. 8) *on which the inequality* (2. 9) *holds.*

This is one way of refining PÓLYA's theorem and both, this and theorem I, are obviously not contained in MANDELBROJT's and L. SCHWARTZ's theorems. Another way of refinement is to replace the angle by a narrower domain as in SUNYER i BALAGUER's theorem. Some results in this direction will be given in **9**. By suitable changes in the proof of theorem I, $M(2r, f)$ could have been replaced by $M((1+\delta)r, f)$, but we shall not treat it as well as its extensions to Dirichlet's series.

3. The *systematic* study of integral functions with gaps started with PÓLYA's paper. In an interesting way no attention was given sofar to the corresponding harmonic expansions which usually followed the functiontheoretical developments. In what follows we shall show that an analogous theorem holds also for harmonic expansions. Let

$$h(r,\varphi)=\sum_{\nu=1}^{\infty} r^{\lambda_\nu}(a_\nu \cos\lambda_\nu\varphi + b_\nu \sin\lambda_\nu\varphi) \tag{3.1}$$

be a harmonic function converging on the whole plane with positive integer increasing exponents λ_ν satisfying the Fabry condition (1.2) and

$$\begin{aligned} \max_{\varphi} |h(r,\varphi)| &= H(r,h),\\ \max_{\alpha\leqq\varphi\leqq\beta} |h(r,\varphi)| &= H(r,\alpha,\beta,h). \end{aligned} \tag{3.2}$$

Then we have the

Theorem II. *For the above defined $h(r,\varphi)$ and for any prescribed $\alpha, \beta, \varepsilon$ with $0\leqq\alpha<\beta\leqq 2\pi$, $0<\varepsilon\leqq\frac{1}{2}$ we have, for all $r>r_2(h,\beta-\alpha,\varepsilon)$, the inequality*

$$H(r,h)^{1+\varepsilon} \leqq 32\left(\frac{56\pi}{\beta-\alpha}\right)^4 H^{2\varepsilon}(2r,h)\,H(r,\alpha,\beta,h). \tag{3.3}$$

Applying theorem II to the real and imaginary parts of the function $f(z)$ of theorem I we could deduce immediately (2.2) from (3.3), apart from the factor independent of f. Owing to this fact and since the independent proof of theorem I runs on the same line as that of theorem II we shall give a detailed proof only for theorem II.

4. The method in question consists in a systematic use of two inequalities. What we here actually need, is a consequence of the first of them and asserts that if the m_ν's are integers, then for arbitrary complex coefficients d_ν

$$\max_{0\leqq x\leqq 2\pi}\left|\sum_{\nu=1}^{k} d_\nu e^{i m_\nu x}\right| < \left(\frac{56\pi}{b-a}\right)^k \max_{a\leqq x\leqq b}\left|\sum_{\nu=1}^{k} d_\nu e^{i m_\nu x}\right|. \tag{4.1}$$

For a proof of this inequality see my previous paper[5]). The basis for analogous investigations for Dirichlet's series might be the more general inequality[6])

$$\max_{a_1\leqq x\leqq d_1}\left|\sum_{\nu=1}^{k} d_\nu e^{i\mu_\nu x}\right| \leqq \left(2e\frac{d_1-a_1}{c_1-b_1}\right)^k \max_{b_1\leqq x\leqq c_1}\left|\sum_{\nu=1}^{k} d_\nu e^{i\mu_\nu x}\right|,$$

where $\mu_1<\mu_2<\cdots<\mu_k$ are real and $a_1<b_1<c_1<d_1$. Choosing in (4.1)

$$k=2n+1,\quad a=-b,$$

$$-m_1=m_{2n+1}=k_n,\ -m_2=m_{2n}=k_{n-1},\ \ldots,\ -m_n=m_{n+2}=k_1,\ m_{n+1}=k_0=0;$$

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5) P. Turán, On a theorem of Littlewood, *Journal London Math. Soc.*, 21 (1946), pp. 268—275.

6) This could be easily inferred from the paper 5).

$$d_j = \frac{a_{n+1-j} + i b_{n+1-j}}{2} \text{ for } 1 \leqq j \leqq n, \quad d_{n+1} = a_0,$$

$$d_j = \frac{a_{j-n-1} - i b_{j-n-1}}{2} \text{ for } n+2 \leqq j \leqq 2n+1,$$

we obtain from (4. 1)

$$\max_{0 \leqq x \leqq 2\pi} \left| \sum_{j=0}^{n} (a_j \cos k_j x + b_j \sin k_j x) \right| \leqq$$
$$\leqq \left(\frac{28\pi}{b} \right)^{2n+1} \max_{-b \leqq x \leqq b} \left| \sum_{j=0}^{n} (a_j \cos k_j x + b_j \sin k_j x) \right|.$$

By an obvious change of notation this gives

$$\max_{0 \leqq x \leqq 2\pi} \left| \sum_{j=0}^{n} (a_j \cos k_j x + b_j \sin k_j x) \right| \leqq$$
$$\leqq \left(\frac{56\pi}{b-a} \right)^{2n+1} \max_{a \leqq x \leqq b} \left| \sum_{j=0}^{n} (a_j \cos k_j x + b_j \sin k_j x) \right| \tag{4. 2}$$

if $0 \leqq a < b \leqq 2\pi$; this will be our starting point.

5. Now we turn to the proof of theorem II. Owing to the gap-condition (1. 2) we can give an $\omega(k)$ tending monotonically to $+\infty$ such that

$$\frac{\lambda_k}{k} > \omega(k). \tag{5. 1}$$

Let

$$s_k(r, \varphi) = \sum_{\nu=1}^{k} r^{\lambda_\nu} (a_\nu \cos \lambda_\nu \varphi + b_\nu \sin \lambda_\nu \varphi). \tag{5. 2}$$

Fixing r and k we may apply the estimation (4. 2) to $s_k(r, \varphi)$. Then we obtain

$$\max_{\varphi} |s_k(r, \varphi)| \leqq \left(\frac{56\pi}{\beta - \alpha} \right)^{2k+1} \max_{\alpha \leqq \varphi \leqq \beta} |s_k(r, \varphi)|. \tag{5. 3}$$

The maximum on the left resp. on the right should be attained at $\varphi = \varphi_1$, resp. $\varphi = \varphi_2$. We have obviously

$$|s_k(r, \varphi_2)| \leqq |h(r, \varphi_2)| + |s_k(r, \varphi_2) - h(r, \varphi_2)| \leqq$$
$$\leqq H(r, \alpha, \beta, h) + |s_k(r, \varphi_2) - h(r, \varphi_2)|.$$

Since we have for all real values φ

$$h(r, \varphi) - s_k(r, \varphi) = \sum_{\nu=k+1}^{\infty} r^{\lambda_\nu} (a_\nu \cos \lambda_\nu \varphi + b_\nu \sin \lambda_\nu \varphi)$$

and

$$a_\nu (2r)^{\lambda_\nu} = \frac{1}{\pi} \int_0^{2\pi} h(2r, \vartheta) \cos \lambda_\nu \vartheta \, d\vartheta,$$

$$b_\nu (2r)^{\lambda_\nu} = \frac{1}{\pi} \int_0^{2\pi} h(2r, \vartheta) \sin \lambda_\nu \vartheta \, d\vartheta,$$

we obtain easily

$$|h(r,\varphi)-s_k(r,\varphi)|=\left|\frac{1}{\pi}\int_0^{2\pi} h(2r,\vartheta)\left(\sum_{\nu=k+1}^{\infty}\frac{1}{2^{\lambda_\nu}}\cos\lambda_\nu(\vartheta-\varphi)\right)d\vartheta\right|\leqq$$

$$\leqq 2H(2r,h)\frac{2}{2^{\lambda_{k+1}}}<\frac{4H(2r,h)}{2^{\lambda_k}},$$

thus

$$\max_{\alpha\leqq\varphi\leqq\beta}|s_k(r,\varphi)|\leqq H(r,\alpha,\beta,h)+\frac{4H(2r,h)}{2^{\lambda_k}}. \tag{5.4}$$

For our fixed r, $h(r,\varphi)$ should attain its maximum at $\varphi=\varphi_3$. Then analogously as before

$$|s_k(r,\varphi_1)|\geqq|s_k(r,\varphi_3)|\geqq|h(r,\varphi_3)|-|h(r,\varphi_3)-s_k(r,\varphi_3)|\geqq H(r,h)-\frac{4H(2r,h)}{2^{\lambda_k}}.$$

Putting this and (5.4) into (5.3) we obtain

$$H(r,h)\leqq\left(\frac{56\pi}{\beta-\alpha}\right)^{2k+1}H(r,\alpha,\beta,h)+ \tag{5.5}$$
$$+\frac{448\pi}{\beta-\alpha}H(2r,h)\left\}\frac{1}{2}\left(\frac{56\pi}{\beta-\alpha}\right)^{\frac{2k}{\lambda_k}}\right\}^{\lambda_k}.$$

6. As said, let $0<\varepsilon\leqq\frac{1}{2}$ and let r be so large that

$$H(r,h)>4 \tag{6.1}$$

and with the above defined ω

$$\varepsilon\,\omega\left(\frac{\varepsilon\log\sqrt{H(r,h)}}{\log\frac{56\pi}{\beta-\alpha}}\right)>8\log\frac{56\pi}{\beta-\alpha}. \tag{6.2}$$

Both requirements are evidently fulfilled for $r>\varrho_1(h,\beta-\alpha,\varepsilon)$. For such values r let[7])

$$k=1+\left[\frac{\varepsilon}{\log\frac{56\pi}{\beta-\alpha}}\log\left\{\frac{896\pi}{\beta-\alpha}\frac{H(2r,h)}{H(r,h)}+\sqrt{H(r,h)}\right\}\right]. \tag{6.3}$$

We have first to estimate $\frac{k}{\lambda_k}$ from above. Owing to the monotony of $\omega(x)$ and (6.2) we have

$$\frac{k}{\lambda_k}\leqq\frac{1}{\omega(k)}\leqq\frac{1}{\omega\left(\frac{\varepsilon}{\log\frac{56\pi}{\beta-\alpha}}\log\sqrt{H(r,h)}\right)}\leqq\frac{\varepsilon}{8\log\frac{56\pi}{\beta-\alpha}}$$

[7]) The square-bracket denotes here and only here in this paper the greatest integer.

i. e.

$$\frac{1}{2}\left(\frac{56\pi}{\beta-\alpha}\right)^{\frac{2k}{\lambda_k}} \leqq \frac{1}{2} e^{\frac{\varepsilon}{4}} < \frac{3}{4}.$$

Hence the last member on the right of (5. 5) is less than

$$\frac{448\pi}{\beta-\alpha} H(2r,h)\left(\frac{3}{4}\right)^{\lambda_k}. \tag{6. 4}$$

Since from the definition of k, using the monotony of $\omega(x)$,

$$\lambda_k \geqq k\omega(k) \geqq \omega(k) \frac{\varepsilon}{\log \frac{56\pi}{\beta-\alpha}} \log\left(\frac{896\pi}{\beta-\alpha} \cdot \frac{H(2r,h)}{H(r,h)}\right) >$$

$$> \frac{\varepsilon}{\log \frac{56\pi}{\beta-\alpha}} \log\left(\frac{896\pi}{\beta-\alpha} \frac{H(2r,h)}{H(r,h)}\right) \omega\left(\frac{\varepsilon \log \sqrt{H(r,h)}}{\log \frac{56\pi}{\beta-\alpha}}\right)$$

and, using again (6. 2),

$$\lambda_k > 8 \log\left(\frac{896\pi}{\beta-\alpha} \frac{H(2r,h)}{H(r,h)}\right),$$

we obtain from (6. 4) for the member in question the upper bound

$$\frac{448\pi}{\beta-\alpha} H(2r,h) \exp\left\{-8 \log \frac{4}{3} \log\left(\frac{896\pi}{\beta-\alpha} \frac{H(2r,h)}{H(r,h)}\right)\right\} <$$

$$< \frac{448\pi}{\beta-\alpha} H(2r,h) \exp\left\{-\log\left(\frac{896\pi}{\beta-\alpha} \frac{H(2r,h)}{H(r,h)}\right)\right\} = \frac{1}{2} H(r,h).$$

Hence from (5. 5) we obtained

$$H(r,h) < 2\left(\frac{56\pi}{\beta-\alpha}\right)^{2k+1} H(r,\alpha,\beta,h). \tag{6. 5}$$

7. To complete the proof of theorem II we write (6. 5) in the form

$$H(r,h) < \frac{112\pi}{\beta-\alpha}\left(\frac{56\pi}{\beta-\alpha}\right)^{2k} H(r,\alpha,\beta,h) \tag{7. 1}$$

and replace k by its value from (6. 3). Using the fact that for $a \geqq 2, b \geqq 2$ we have $\log(a+b) \leqq \log a + \log b$, from (7.1) and (6. 1) we obtain

$$\left(\frac{56\pi}{\beta-\alpha}\right)^{2k} \leqq \left(\frac{56\pi}{\beta-\alpha}\right)^{2}\left(\frac{56\pi}{\beta-\alpha}\right)^{\frac{2\varepsilon}{\log \frac{56\pi}{\beta-\alpha}}\left\{\log\left(\frac{896\pi}{\beta-\alpha} \frac{H(2r,h)}{H(r,h)}\right) + \frac{1}{2}\log H(r,h)\right\}} =$$

$$= \left(\frac{56\pi}{\beta-\alpha}\right)^{2}\left(\frac{896\pi}{\beta-\alpha} \frac{H(2r,h)}{H(r,h)}\right)^{2\varepsilon} H^{\varepsilon}(r,h)$$

i. e., from (7. 1) and from $\varepsilon \leqq \frac{1}{2}$,

$$H^{1+\varepsilon}(r,h) \leqq 32\left(\frac{56\pi}{\beta-\alpha}\right)^{4} H^{\varepsilon}(2r,h)\, H(r,\alpha,\beta,h).$$

Thus theorem II is proved.

8. To prove that (2. 7) is fulfilled for all integral functions of finite order we suppose this order should be k. Then for $r > \varrho_2(f)$ we have

$$|f(z)| \leqq 2^{|z|^{k+\frac{1}{2}}}, \qquad M(r, f) \geqq 2. \tag{8. 1}$$

Choosing

$$r_1 = \max(5, \varrho_2(f)), \tag{8. 2}$$

$$c_4 = 25^{k+1}, \qquad c_5 = \max\left(1, \frac{M(2r_1, f)}{M^{25^{k+1}}(r_1, f)}\right), \tag{8. 3}$$

(2. 7) is fulfilled for $\nu = 1$. Suppose $r_1, \ldots, r_\nu$ already satisfy (2. 8) so that (2. 7) is true with the choice (8. 3). Since $r_\nu \geqq r_1 \geqq 5$, the annulus

$$2r_\nu \leqq |z| \leqq 2r_\nu^{k+2} \tag{8. 4}$$

exists. If our assertion were untrue then we would have

$$\begin{aligned} &M(2^2 r_\nu) > M^{25^{k+1}}(2r_\nu) \\ &M(2^3 r_\nu) > M^{25^{k+1}}(2^2 r_\nu) > M^{25^{2(k+1)}}(2r_\nu), \\ &\quad\vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\ &M(2^l r_\nu) > M^{25^{(l-1)(k+1)}}(2r_\nu) \geqq M^{25^{(l-1)(k+1)}}(r_\nu) \geqq 2^{25^{(l-1)(k+1)}}. \end{aligned}$$

Now we determine the integer l so that

$$2^{l-1} \leqq r_\nu^{k+1} < 2^l \tag{8. 5}$$

i. e.

$$2^l r_\nu \leqq 2r_\nu^{k+2}.$$

Then we have

$$2^l > r_\nu^{k+1} \geqq r_1^{k+1} > r_1 \geqq 5,$$

i. e.

$$l > 2 \quad \text{and} \quad l-1 > \frac{l}{2}.$$

Thus

$$M(2^l r_\nu) > 2^{5^{l(k+1)}}$$

or from (8. 5) owing to the monotony of $M(x)$

$$M(2^{l+\frac{l}{k+1}}) > 2^{5^{l(k+1)}}.$$

From (8. 1) we should have

$$2^{5^{l(k+1)}} < 2^{\left(2^{l+\frac{l}{k+1}}\right)^{k+\frac{1}{2}}}$$

i. e. a fortiori

$$5^l < 2^{l+\frac{l}{k+1}} < 2^{2l} = 4^l$$

which is a contradiction. It is likely that the inequality

$$\frac{M(2r)}{M^{c_4}(r)} \leqq c_5$$

holds, with sufficiently large, from r independent c_4 and c_5, for a much larger set of r-values if f is of finite order.

9. In theorems I and II the angle $\beta-\alpha$ was fixed however arbitrary small. One expects that this angle can tend to 0 not too quickly with $\frac{1}{r}$. Indeed in the proof of theorem II we had the only restriction (6. 2) in this direction. (6. 2) can also be written in the form

(9. 1) $$\frac{8}{\varepsilon}\log\frac{56\pi}{\beta-\alpha}\cdot\omega^{-1}\left(\frac{8}{\varepsilon}\log\frac{56\pi}{\beta-\alpha}\right)<4\log H(r, h)$$

if $\omega^{-1}(u)$ denotes the inverse function of $\omega(u)$. Then $u\omega^{-1}(u)$ is an increasing function of u; let $u=\Omega(y)$ be the solution of

$$u\omega^{-1}(u)=y.$$

Thus from (9. 1) we have putting $\beta-\alpha=\delta(r)$

$$\frac{8}{\varepsilon}\log\frac{56\pi}{\delta(r)}<\Omega(4\log H(r, h))$$

i. e.

(9. 2) $$\delta(r)\geqq 56\pi\exp\left\{-\frac{\varepsilon}{8}\Omega(4\log H(r, h))\right\}$$

which must be true for all $r>\varrho_3(f, \varepsilon)$; in this case the inequality (3. 3) holds of course for all $r>\varrho_3(f, \varepsilon)$. To avoid the possibility that the right side of (9. 2) is too small we require also

$$\delta(r)\geqq H^{-\varepsilon}(2r, h)$$

i. e. we choose

(9. 3) $$\delta_0(r)=\max\left[H^{-\varepsilon}(2r, h);\ 56\pi\exp\left\{-\frac{\varepsilon}{8}\Omega(4\log H(r, h))\right\}\right].$$

Then theorem II will assert that for $r>\varrho_3(f, \varepsilon)$, $0<\varepsilon\leqq\frac{1}{2}$, we will have with $\beta-\alpha=\delta_0(r)$ the inequality

(9. 4) $$H^{1+\varepsilon}(r, h)\leqq 32(56\pi)^4 H^{6\varepsilon}(2r, h)\ H(r, \alpha, \beta, h).$$

Similarly we will have for $|z|\geqq\varrho_4(f, \varepsilon)$, $0<\varepsilon\leqq\frac{1}{2}$ with the above $\delta_0(r)$ the inequality

(9. 5) $$M^{1+\varepsilon}(r, f)\leqq 48\pi M^{3\varepsilon}(2r, f)\, M(r, \alpha, \beta, f).$$

The interest of this remark lies obviously in the fact that $\delta_0(r)$ depends only upon $\min\limits_{k \leqq r} \frac{\lambda_k}{k}$ and not upon the finer distribution of the exponents. Let e. g. $\omega(x) = x$. Then

$$\omega^{-1}(x) = x, \quad \Omega(y) = \sqrt{y}$$

and the reasoning of 2 gives the

Corollary II. *If the exponents of the integral function*

$$f(z) = \sum_{\nu=1}^{\infty} a_\nu z^{\lambda_\nu}$$

of finite order satisfy the inequality

$$\lambda_\nu \geqq \nu^2 \qquad (\nu = 1, 2, \ldots)$$

and if, with fixed $0 < \varepsilon \leqq \frac{1}{2}$ *and* $0 \leqq \gamma \leqq 2\pi$, *U denotes the domain*

$$(9.6) \qquad |z| = r, \quad |\text{arc}\, z - \gamma| \leqq 56\pi \exp\left\{-\frac{\varepsilon}{4}\sqrt{\log M(r, f)}\right\},$$

$$\varrho_4(f, \varepsilon) \leqq |z| \leqq \infty,$$

then there is a sequence r_ν *tending to infinity such that the inequality*

$$c_6 M^{1-c_7\varepsilon}(r_\nu, f) \leqq M(r_\nu, U, f) \qquad (\nu = 1, 2, \ldots)$$

holds with c_6, c_7 *depending only upon* f; *i. e.* $f(x)$ *increases in U as fast as over the whole plane.*

If $f(z)$ is of finite *positive* order k then for all sufficiently large values of r we have

$$\log M(r, f) \leqq r^{k+\delta} \qquad (\delta \text{ arbitrary}, > 0)$$

i. e. the arc of the circle $|z| = r$ assumes the form

$$|\text{arc}\, z - \gamma| \leqq 56\pi \exp\left(-\frac{\varepsilon}{4} r^{\frac{k-\delta}{2}}\right).$$

The length of the arc is then less than, or equal to,

$$112\pi r \exp\left(-\frac{\varepsilon}{4} r^{\frac{k-\delta}{2}}\right)$$

which tends to zero rapidly with $\frac{1}{r}$. Hence the domain (9.6) is like a rather tight funnel which has e. g. a finite area. This shows anyway that replacing the angles by the domain U means in general very considerable reduction of the domain.

(Received April 15, 1952.)