On abelian groups every multiple of which is a direct summand.

To Professor Reinhold Baer on his 50th birthday.

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§ 1. Introduction.

In a previous paper [3] one of us has determined all groups every subgroup of which is a direct summand.[1]) These groups are exactly the elementary abelian torsion groups.[2]) [3]) The concept of endomorphic image being a notion of intermediate character between those of a subgroup and of a direct summand, the problem mentioned splits up into the following two problems:

Problem I. Determine all groups every endomorphic image of which is a direct summand.

Problem II. Determine all groups every subgroup of which is an endomorphic image of the group.

These problems seem to be very difficult, even in the case of abelian groups. The present paper is devoted to Problem I in case of abelian groups, and this problem will be solved completely for torsion groups as well as for torsion-free groups. There will also be given a class of mixed groups, containing all solutions of Problem I; however, we cannot decide for the moment whether each group of this class has the property involved in Problem I.

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1) The numbers in brackets refer to the Bibliography at the end of this paper.

2) For notation and terminology see § 2.

3) We are indebted to Professor R. BAER who has kindly informed us of this result being closely related to one of his results (Theorem 3, p. 504 in [2]). In fact, the concept of "retract" and that of direct summand being identical for abelian groups, the result of BAER and that of [3] coincide in case of abelian groups. For arbitrary groups, however, it is not a priori evident that exactly the same groups exhaust the solutions of both problems. This follows only from the fact that a group which is an "absolute retract" in the sense of BAER, as well as a group every subgroup of which is a direct summand, is proved in [2] resp. [3] to be necessarily commutative.

As a matter of fact, our main result (Theorem 1) gives the complete solution of describing all *abelian* groups with the

Property (P): Every multiple of the group is a direct summand of the group.

By a multiple of an additive abelian group G we mean the subgroup nG for some natural number n, i. e. the set of all elements ng with $g \in G$. Therefore property (P) means that the endomorphic image of the group G under the special endomorphism $g \to ng$ is a direct summand of G for each natural number n. We shall see that for torsion-free groups the groups of property (P) are identical with the solutions of Problem I, and that for torsion groups also the latter are easy to select from among the former. In case of mixed groups, however, we have not yet succeeded completely in effectuating this selection.

Among the abelian groups of the property (P) described completely by Theorem 1, we can distinguish essentially three main categories: the elementary torsion groups, the algebraically closed groups, and some mixed groups which we shall call — by virtue of a generalization given later of the concept of direct sum — "p-direct sums over their torsion subgroups". The groups of the first two categories are well-known. On the other hand, the third category is that of an interesting class of mixed groups which seems to be new, and deserves, for this as well as for other reasons, further investigation. Let us mention but one of their properties: these groups are the simplest and most lucid example of a mixed group the torsion subgroup of which is not a direct summand of the group (see Corollary of Theorem 3). The examples hitherto known in mathematical literature are namely based on more complicated constructions or are groups given by defining relations.

Our investigations are yielding in the way of an additional result also the determination of all mixed abelian groups G, the torsion subgroup T of which is an elementary group, and the factor group G/T algebraically closed (Theorem 3).

§ 2. Preliminaries.

In what follows, by a *group* we shall mean always an additively written *abelian* group with more than one element. Groups will be denoted by Latin capital letters and their elements by $x, a, b, \dots, g$. The other small Latin letters are reserved for rational integers (in particular p and q for prime numbers). We shall denote the endomorphisms of a group by small Greek letters. A subgroup generated by the elements $a, b, \dots$ of a group is denoted by $\{a, b, \dots\}$. A group every element of which is of finite order, is called a *torsion group*. In the contrary case, when every non-zero element of the group is of infinite order, the group is called *torsion-free*. A group which is neither a torsion group nor torsion-free is said to be a *mixed group.* All elements of finite order of a mixed group form a subgroup which we call *the torsion subgroup of the group.*

Let p be an arbitrary prime number. If the group G contains an element of order p, then p is called an *actual prime* for G. The set of all actual primes for G will be called the *actual prime system* of G. If $pG = G$ for each prime p, then G is called *algebraically closed.* Obviously the algebraically closed groups G are exactly those, in which the equation $nx = a$ has a solution $x \in G$ for any $a \in G$ and $n > 0$. It is easy to see that any homomorphic image of an algebraically closed group is at the same time itself an algebraically closed group. According to an important theorem of R. BAER *an algebraically closed group is always a direct summand of every containing group* [1].

In what follows, we shall need a generalization of the concept of the direct sum which, for a finite number of summands, coincides with the usual direct sum, and which has plaid already an important role in investigations on mixed groups with commutative ring of endomorphisms [5].

A group G will be called a *direct sum* of its subgroups U_ν, if there exist endomorphisms ε_ν of G such that

1) $\varepsilon_\nu G = U_\nu$;

2) $\varepsilon_\mu \varepsilon_\nu = \begin{cases} \varepsilon_\mu & \text{if } \mu = \nu; \\ 0 & \text{if } \mu \neq \nu; \end{cases}$

3) $g \in G$ and $\varepsilon_\nu g = 0$ for every ν implies $g = 0$.

Among the direct sums of the groups U_ν there exists a "greatest" one G_c satisfying the additional requirement:

4) For any choice of a system of representative elements $g_\nu \in U_\nu$ there exists an element g of G such that $\varepsilon_\nu g = g_\nu$ holds for every ν.

This group G_c having the properties 1)—4) is obviously uniquely determined (up to an isomorphism) by the groups U_ν, and will be called the *complete direct sum* of the U_ν's. We denote it by

$$G_c = \sum_\nu U_\nu. \tag{1}$$

The group G_c may also be described as the set of all possible "vectors" $\langle \ldots, g_\nu, \ldots \rangle$ which contain exactly one component g_ν from each group U_ν and which are added component-wise. Clearly any direct sum of the groups U_ν is a subgroup of (1).

On the other hand, among all direct sums of the groups U_ν there exists a "smallest" one G_d, which is a subgroup of any direct sum. This is characterized as the direct sum satisfying

4*) For any element $g \in G$, there are only a finite number of ν's with $\varepsilon_\nu g \neq 0$.

This group

$$G_d = \sum_{\nu}^{*} U_\nu \tag{2}$$

determined uniquely by the groups U_ν as the group satisfying 1), 2), 3) and 4*), is called the *discrete direct sum* of the U_ν's. The group (2) may also be described as the set of all vectors $\langle \ldots, g_\nu, \ldots \rangle$ having only a finite number of components $\neq 0$.

In terms of the complete and discrete direct sums the direct sums of the U_ν's may be characterized as the groups G for which $G_d \subseteq G \subseteq G_c$. For a finite number of U_ν's we have $G_d = G_c$; thus in this case there exists only one direct sum of the U_ν's. Therefore the concept of the direct summand in the generalized sense is the same as that in the old sense.

Let us mention an important example. It is well known that a torsion group T may be represented as the discrete direct sum of its uniquely determined primary components T_p, where T_p is a p-group, i. e. a group containing only elements of p-power order:

$$T = \sum^{*} T_p. \tag{3}$$

Therefore the complete direct sum

$$\bar{T} = \sum T_p \tag{4}$$

is uniquely determined be T; it may be called the *complete p-direct sum over T*. In accordance with this, the groups between T and $\bar{T}$ (T and $\bar{T}$ included), in other words, the direct sums of the groups T_p, may be called the *p-direct sums over T*. It is obvious that, if the actual prime system of T contains an infinity of primes, then all of these, except T, are mixed groups and their torsion subgroup is just T.

We denote by R the additive group of all rational numbers, by $C(p^m)$ the cyclic group of order p^m for an arbitrary natural number m, and by $C(p^\infty)$ the additive group of all rational numbers mod 1 whose denominators are powers of p. It is known [4] that any algebraically closed group A can be represented in the form

$$A = \sum^{*} C(p_\nu^\infty) + \sum^{*} R \tag{5}$$

where the p_ν's are arbitrary primes (distinct or not) and one of the two Σ^*'s of the right member may vanish.

The torsion group T is called an *elementary group* if the order of each element in T is a square-free number. Each primary component of such a group is of the form

$$T_p = \sum^{*} C(p), \tag{6}$$

so that an elementary group T can be represented as a discrete direct sum

$$T = \sum^{*} C(p_\nu) \tag{7}$$

where the p_ν's are arbitrary primes (distinct or not).

§ 3. The main result.

Theorem 1. *An abelian group G has the property* (P) — *i. e. nG is a direct summand of G for any natural number n — if and only if G can be represented as a direct sum*

$$G = A + B \tag{8}$$

where:

a) *A is an algebraically closed group or $A = 0$;*
b) *B is a p-direct sum over its torsion subgroup $T \neq 0$ or $B = 0$;*
c) *T is an elementary group;*
d) *the factor group B/T is algebraically closed.*

Remarks. The direct summand A of G as well as the torsion subgroup T of B is completely described by (5) resp. (7). We show that there exist groups B with the properties b)—d) and we give an oversight on all of them. Indeed, the complete p-direct sum over the group (3), i. e. the group (4) — T being now an arbitrary elementary group — has the property that the factor group $\bar{T}/T$ is algebraically closed. To prove this we must show that if the "vector" $c = \langle \ldots, c_k, \ldots \rangle (c_k \in T_{p_k})$ is an arbitrary element of $\bar{T}$ and q is a prime, then there exists an $x \in \bar{T}$ such that $c - qx \in T$. This is obvious, since one may plainly construct a "vector" x with $c - qx = 0$ or $c - qx \in T_{p_j}$, according as $q \neq p_k, (k = 1, 2, 3, \ldots)$ or $q = p_j$. Now, for a given elementary group T, the determination of all groups B with the properties c) and d) is naturally equivalent to giving all algebraically closed subgroups B/T of the algebraically closed group $\bar{T}/T$. Since the group T/T is torsion-free, this process becomes easier by taking into account that if S is an arbitrary subgroup of $\bar{T}/T$ and if we adjoin to S all those elements e of $\bar{T}/T$ for which $re \in S$ with some natural number r, then we obtain an algebraically closed subgroup S_0 of $\bar{T}/T$.

The groups G given by (8) are in general mixed groups. According to Theorem 1 *the only torsion-free groups with the property* (P) *are the groups of type*

$$\sum\nolimits^{*} R, \tag{9}$$

i. e. the algebraically closed torsion-free groups, and *the only torsion groups with the property* (P) *are the groups of type*

$$\sum\nolimits^{*} C(p_\mu^{\alpha}) + \sum\nolimits^{*} C(q_\nu), \tag{10}$$

i. e. the direct sums of an algebraically closed torsion group and of an elementary group. (This is the case $B = T$, with a torsion group A.)

Proof of the necessity of the conditions a)—d) *of Theorem* 1. We shall show that if pG is a direct summand of the group G for any prime p, then G is a group given by (8) with the properties a)—d).

Consider the union A of all algebraically closed subgroups of G. Clearly A itself *is an algebraically closed subgroup of* G and so, by the theorem of Baer mentioned in § 2, the direct decomposition (8) holds with *a suitable group B which contains no algebraically closed subgroup* $\neq 0$. Now we state that B contains the subgroup pB as a direct summand for each prime p. Indeed, by our hypothesis and by (8),

$$G = pG + D = A + pB + D, \tag{11}$$

where, because of $D \cong G/pG$,

$$pD = 0. \tag{12}$$

Hence

$$pG = p^2 G,$$

and consequently, by $B \cong G/A$,

$$pB = p^2 B. \tag{13}$$

Now we can show that for the group $H = pB + D$, pH is a direct summand of H. This follows immediately from

$$pB + D = p(pB + D) + D$$

which is a consequence of (12) and (13). On the other hand, by (8) and (11)

$$B \cong G/A \cong pB + D = H$$

and so our assertion on B is proved.

In what follows, let $p_1, p_2, p_3, \dots$ denote the sequence of *all* primes. Then we have, by the property of B just proved,

$$B = p_k B \dotplus U_k \qquad (k = 1, 2, 3, \dots) \tag{14}$$

where, because of $U_k \cong B/p_k B$,

$$U_k = \sum\nolimits^{*} C(p_k) \quad \text{or} \quad U_k = 0 \tag{15}$$

according as p_k is an actual prime for B or not. From (14) and (15) we obtain

$$p_k B = p_k (p_k B) \tag{16}$$

which of course is the same as (13). Now we are going to show that in (14) the direct summand U_k too is defined invariantly as the subset of B containing all elements of B of order p_k, and the zero. In order to prove this, it is sufficient to show, that $p_k B$ does not contain any element of order p_k. As a matter of fact, in case of $p_k B$ containing an element c_1 of order p_k, by (16) there would exist a chain $c_1, c_2, c_3, \dots$ of elements of B, for which

$$c_1 \neq 0, \quad p_k c_1 = 0, \quad p_k c_2 = c_1, \quad p_k c_3 = c_2, \dots .$$

In that case however $\{c_1, c_2, c_3, \dots\} \cong C(p_k^\infty)$ would be an algebraically closed subgroup of B, which is impossible.

From the above statement thus proved, there follows also that the primary component T_{p_k} of the torsion group T of the group B coincides with the group U_k in (15). This being true for an arbitrary prime p_k, we obtain that T is an elementary group, i. e. the statement c) of Theorem 1 is true for the group B.

Now we can show that B is a p-direct sum over T. By the uniqueness proved above of both terms on the right hand of (14) we conclude that each element b of B may be written in exactly one way as the sum of an element $\varepsilon_k b$ in U_k and of an element in $p_k B$. It is clear that the mapping $b \to \varepsilon_k b$ is an endomorphism of B. The endomorphisms thus defined possess obviously the following properties:

1) $\varepsilon_k B = U_k$;

2) $\varepsilon_i \varepsilon_k = \begin{cases} \varepsilon_i \text{ if } i = k; \\ 0 \text{ if } i \neq k; \end{cases}$

3) if $b \in B$ and $\varepsilon_k b = 0$ for every k, then $b = 0$.

Only the third statement requires a proof. $\varepsilon_k b = 0$ being equivalent to $b \in p_k B$, it is sufficient to show that the cross cut of the groups $p_1 B, p_2 B, p_3 B, \ldots$ contains the only element 0. Assume this is not true and let c be an element of infinite order or of prime number order common to all groups $p_k B$ $(k = 1, 2, 3, \ldots)$. Then there exists, by (16), a chain of elements $c_1^{(k)}, c_2^{(k)}, \ldots$ of B for each natural number k such that

$$p_k c_1^{(k)} = c, \; p_k c_2^{(k)} = c_1^{(k)}, \; p_k c_3^{(k)} = c_2^{(k)}, \ldots$$

Clearly $\{c_1^{(1)}, c_2^{(1)}, \ldots, \ldots, c_1^{(k)}, c_2^{(k)}, \ldots, \ldots\}$ is a subgroup of type R or of type $C(p^\infty)$ of B according as c is an element of infinite order or of order p. This is however a contradiction, since B contains no algebraically closed subgroup $\neq 0$. This completes the proof of the statement b) of Theorem 1.

Finally consider the factor group B/T. We have, by $U_k \subseteq T$ and by (14),

$$p_k B \cong B/U_k \sim B/T.$$

Hence it follows from (16)

$$p_k(B/T) = B/T$$

for every prime number p_k, i. e. also d) of Theorem 1 holds for the group B.

Proof of the sufficiency of the conditions a)—d) *of Theorem* 1. We are going to show that if G is a group given by (8) and having the properties a)—d) of Theorem 1, then nG is a direct summand of G for each natural number n. We have by $nA = A$

$$nG = A + nB$$

and so it is enough to show that nB is a direct summand of B. Let $n = p_1^{l_1} \ldots p_r^{l_r}$ be the prime power decomposition of n. Then we prove the validity of

$$B = (U_1 + \cdots + U_r) + nB. \tag{17}$$

Clearly

$$nB \cap (U_1 + \cdots + U_r) = 0$$

as the group $U_1 + \cdots + U_r$ contains only elements of order which is a divisor of $p_1 \ldots p_r$ while nB contains no such element $\neq 0$. On the other hand, we have to show that any element $b \in B$ has a representation

$$b = d + nb' \tag{18}$$

with $d \in U_1 + \cdots + U_r$ and $b' \in B$. From condition d) we infer the existence of an element $b_0 \in B$ such that

$$b - nb_0 = d_0 \in T. \tag{19}$$

Now, according to

$$T = (U_1 + \cdots + U_r) + T',$$

the representation

$$d_0 = d + d' \qquad (d \in U_1 + \cdots + U_r,\ d' \in T') \tag{20}$$

holds. Here $d' = nd''$ with $d'' \in T'$ (the order of d' being prime to n), consequently, by (19) and (20), the validity of (18) is proved with $b' = b_0 + d''$. This completes the proof of (17) and at the same time that of Theorem 1.

§ 4. On abelian groups every endomorphic image of which is a direct summand.

Concerning Problem I mentioned in § 1, we prove the following

Theorem 2. *In order that an abelian group G may contain each of its endomorphic images as a direct summand, it is necessary that G be a group described in Theorem 1, with the additional property:*

e) *There exist no elements $a \in A$, $b \in B$ with the same finite order > 1 (i. e. the actual prime systems of A and B contain no prime in common).*

Remarks. Condition e) says that for the torsion subgroup (10) of G $p_\mu \neq q_\nu$ holds. — We conjecture that the conditions a)—e) for the group G are always sufficient for every endomorphic image of G to be a direct summand of G. For the moment, however, we can prove this only in case of G being either a torsion-free group (i. e. a group of type (9)), or a torsion group (i. e. a group of type (10) with $p_\mu \neq q_\nu$). In these cases the mentioned property of G is an immediate consequence of the fact that an arbitrary endomorphic image of an algebraically closed group is itself algebraically closed (i. e., in virtue of the theorem of Baer quoted in § 2, a direct summand), and that every subgroup of an elementary group is a direct summand. It is an open question whether the groups B, characterized by the conditions b)—d) of Theorem 1, always have each of their endomorphic images as a direct summand. At present we know only that the answer to this question is affirmative whenever B is a complete p-direct sum.

Proof of Theorem 2. Let G be a group every endomorphic image of which is a direct summand. Then by Theorem 1 G is a group with the properties a)—d). Now we suppose that the condition e) is not true for G. Then there exists a prime p_k which is actual for both groups A and B. This means, by (8), (5), (14) and (15), that G can be represented in the form

$$G = C(p_k) + C(p_k^\infty) + H.$$

Hence there exists an endomorphism of G which maps G onto the subgroup of order p_k of $C(p_k^\infty)$. In this case, however, the endomorphic image is not a direct summand of G. This contradiction proves the validity of Theorem 2.

§ 5. On a special class of mixed groups.

In conclusion we give a full oversight on all mixed groups G the torsion subgroup T of which is an elementary group so that G/T is algebraically closed, and we show that these groups form a part of the class of groups G described by Theorem 1. More exactly, there holds the following

Theorem 3. *If the torsion subgroup T of an abelian mixed group G is elementary and the factor group G/T is algebraically closed, then*

$$G = A + B, \tag{21}$$

where A is an algebraically closed torsion-free group (i. e. a group of type (9)) or $A = 0$, and B is a p-direct sum over T such that B/T is algebraically closed.

Remarks. It is well known that each primary component (6) of an elementary group T can be uniquely characterized by the cardinal number of its direct summands $C(p)$. Accordingly the group T itself is completely described by a system of cardinal numbers as its invariants. Similarly, the algebraically closed torsion-free group G/T is in abstracto uniquely determined by the cardinal number of the direct components R in the representation (9). The question remains however open, whether or not the group B described in Theorem 3 is (up to an isomorphism) uniquely determined by the full system of invariants of T and B/T.

Since the group B contains obviously no algebraically closed subgroup $\neq 0$ (the equation $px = b \in B$ being not solvable for each element $b \neq 0$ in B with a suitable prime p), the main assertion of Theorem 3 is the following: *if the torsion subgroup T of an abelian mixed group B containing no algebraically closed subgroup $\neq 0$ is an elementary group and B/T is algebraically closed, then B is a p-direct sum over T.* — On the other hand, the p-direct sums B over an elementary torsion group T with algebraically closed B/T (especially the complete p-direct sum $\overline{T}$ defined by (4)), are very simple examples of mixed groups, the torsion subgroup of which is not a direct summand. As a matter of fact, if for such a group B the representation $B = T + V$ would hold, then by $V \cong B/T$, V would be an algebraically closed subgroup of B which is impossible.

P r o o f o f T h e o r e m 3. Let G be a mixed group the torsion subgroup T of which is elementary and suppose that G/T is algebraically closed. The union A of all algebraically closed subgroups of G being itself an algebraically closed subgroup, the decomposition (21) holds with a suitable group B which contains no algebraically closed subgroup $\neq 0$. The torsion subgroup T of G being elementary, A must be a torsion-free group. Therefore T is a subgroup of B. Now we state that B/T is algebraically closed. This follows immediately from

$$G/T = (A + B)/T \cong A + (B/T)$$

and from the fact that G/T and A are algebraically closed.

In what follows, we have only to prove that B is a p-direct sum over T. Let p_k be an arbitrary prime and U_k the p_k-primary component of T or $U_k = 0$ in case of p_k being no actual prime for B. Clearly

$$p_k B \cap U_k = 0. \tag{22}$$

On the other hand, B/T being algebraically closed, for any element $b \in B$ there exists an element $b' \in B$ such that

$$b - pb' = d \in T. \tag{23}$$

Now, according to $T = U_k + T'$, the representation $d = d_k + d'$ $(d_k \in U_k,\ d' \in T')$ holds. Here $d' = pd''(d'' \in T')$; consequently, by (23),

$$\begin{aligned} b = d + pb' = d_k + d' + pb' = \\ = d_k + p(b' + d'') \in \{U_k, p_k B\}. \end{aligned}$$

Thus, owing to (22), we have shown that

$$B = p_k B + U_k. \tag{24}$$

where on the right hand both direct summands are uniquely determined by p_k. The equation (24) being the same as the equation (14), from this point onwards the proof is identical with the next to last paragraph of the proof of the necessity of the conditions of Theorem 1 in § 3.

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