The minimum of a binary cubic form[1]).

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1. Let
$$f(x, y) = ax^3 + bx^2y + cxy^2 + dy^3$$
be a binary cubic form with real coefficients and of discriminant
$$D = -27a^2d^2 + 18abcd + b^2c^2 - 4ac^3 - 4db^3,$$
so that $f(x, y)$ has one or three real linear factors according as $D < 0$ or $D > 0$. The problem is to find how small $|f(x, y)|$ can be made for integer values of x, y not both zero, i. e. the lower bound of $|f(x, y)|$ for these x, y.

With such questions, it is not difficult nowadays to prove the existence of results that integers x, y not both zero exist for which
$$|f(x, y)| \leq k|D|^{1/4},$$
where k is a numerical constant, and these have been known for many years. Thus if $D > 0$, ARNDT in 1858 and HERMITE in 1859, showed that the result holds with $k = \left(\frac{4}{27}\right)^{1/4}$.[2]) If $D < 0$, HERMITE showed in 1859 that we can take $k = \frac{1}{2}$. The best possible value of k was neither known nor had any suggestions about its value been made until recently by myself when I proved the following

Theorem.[3]) *If $D > 0$, integers x, y not both zero exist such that*
$$|f(x, y)| \leq \sqrt[4]{\frac{D}{49}}.$$

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[1]) Lecture held in the Bolyai-Institute of the University Szeged, December 16, 1948.

[2]) CH. HERMITE, *Oeuvres*, II (Paris, 1908), pp. 93–99.

[3]) L. J. MORDELL, On numbers represented by binary cubic forms, *Proceedings London Math. Society*, (2) **48** (1943), pp. 198 - 228.

This is a best possible result, and the equality sign is necessary when and only when

$$\sqrt[4]{\frac{49}{D}}\, f(x, y) \sim x^3 + x^2 y - 2xy^2 - y^3,$$

where the right hand side has discriminant 49.

If $D < 0$, *integers* x, y *not both zero exist such that*

$$|f(x, y)| \leqq \sqrt[4]{\frac{|D|}{23}}.$$

This is a best possible result, and the equality sign is necessary when and only when

$$\sqrt[4]{\frac{23}{|D|}}\, f(x, y) \sim x^3 - xy^2 - y^3,$$

where the right hand side has discriminant -23.

The significance of the numbers 49, -23 is clear. Thus 49 is the least positive discriminant of irreducible binary cubic forms with integer coefficients, and so the constant 49 cannot be improved for such forms, i. e. made larger, as then $|f(x, y)| < 1$ and so would be zero. This occurs only when $x = y = 0$. Similarly for -23.

2. Some light may be thrown on the subject if we consider the quadratic case when

$$g(x, y) = ax^2 + bxy + cy^2$$

of discriminant $d = b^2 - 4ac$. It is well known from the work of Lagrange and Gauss that the corresponding best possible results are when

$$d < 0, \qquad g(x, y) \leqq \sqrt{\frac{|d|}{3}},$$

equality arising only when

$$\sqrt{\frac{3}{|d|}}\, g(x, y) \sim x^2 + xy + y^2;$$

and from the work of Markoff[4]), Korkine and Zolotareff[5]) that when

$$d > 0, \qquad g(x, y) \leqq \sqrt{\frac{d}{5}},$$

equality arising only when

$$\sqrt{\frac{5}{d}}\, g(x, y) \sim x^2 + xy - y^2.$$

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4) A. Markoff, Sur les formes quadratiques binaires indéfinies, *Math. Annalen*, **15** (1879), pp. 281-406; **17** (1880), pp. 379-399.

5) A. Korkine—G. Zolotareff, Sur les formes quadratiques, *Math. Annalen*, **6** (1873), pp. 366-389.

If we consider the first of these, a result such as $g(x,y) \leqq \sqrt{\frac{|d|}{l}}$, where l is a numerical constant, has a simple geometric interpretation. It means that a point P whose coordinates are integers x, y; i. e. a lattice point, lies in, i. e. inside or on the boundary of the ellipse $g(x, y) \leqq \sqrt{\frac{|d|}{l}}$. A value of l is given by a fundamental theorem of MINKOWSKI in the geometry of numbers, namely the theorem[6]):

A two dimensional closed, convex region, symmetrical about the origin O and of area $\geqq 4$ contains within it a lattice point other than O.

More generally, this theorem is still true if we define a lattice point to be one whose coordinates x, y are of the form

$$x = \alpha X + \beta Y, \; y = \gamma X + \delta Y,$$

where X, Y are integers and α, β, γ, δ are any real constants with determinant

$$\Delta = \alpha\delta - \beta\gamma > 0,$$

if in the theorem we replace 4 by 4Δ. We then call the aggregate of such points (x, y) a lattice of determinant Δ, but here we need only consider lattices of determinant unity.

An application of this result to the ellipse shows that a lattice point not O lies in it if

$$\frac{2\pi}{\sqrt{|d|}} \sqrt{\frac{|d|}{l}} \geqq 4, \quad \text{or} \quad l \leqq \left(\frac{\pi}{2}\right)^2 < 3.$$

This is worse than the best possible value $l = 3$. MINKOWSKI[7]) has shown, however, that the best possible value can be deduced by finding the minimum value of the area of a parallelogram with one vertex at O and the other three on the boundary of the ellipse. There is of course no number theory involved in solving the minimum problem. These problems are simple in theory but generally very difficult to solve.

When $d > 0$, the region $|g(x, y)| \leqq \sqrt{\frac{d}{l}}$ is an infinite region bounded by four hyperbolic arcs having for asymptotes the lines given by $g(x, y) = 0$. There is no corresponding theorem for infinite regions, but an estimate $l = 4$ may be found by inscribing in the region a parallelogram whose centre is at the origin with vertices on the asymptotes

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[6]) H. MINKOWSKI, *Diophantische Approximationen* (Leipzig, 1907), p. 29.

[7]) H. MINKOWSKI, *Ibidem*, pp. 51—55.

and choosing l so that its area is 4. Then the parallelogram will contain a lattice point not O by MINKOWSKI's theorem, and so also will the infinite region. There is also now no method of finding the best possible result by inscribing minimum parallelograms as in the case of convex regions. In fact $|g(x, y)| \leq \sqrt{d/l}$ was the only simple infinite region for which a best possible result was known for l.

3. The problem of the minimum of a binary cubic can be reduced to a question in the geometry of numbers. It is easily shown that any binary cubic $f(x, y)$ of discriminant D can be transformed by a linear substitution with real coefficients and determinant unity into any other binary cubic $g(x, y)$ of discriminant D. On dividing by an appropriate factor, we may assume that $D = -23$ when $D < 0$, or $D = 49$ when $D > 0$. We write

$$g(x, y) = x^3 - xy^2 - y^3 \qquad \text{of discriminant } -23,$$

and

$$h(x, y) = x^3 + x^2y - 2xy^2 - y^3 \qquad \text{of discriminant } 49.$$

Hence for appropriate real $\alpha, \beta, \gamma, \delta$ with $\alpha\delta - \beta\gamma = 1$, we can write

$$f(X, Y) = g(\alpha X + \beta Y, \gamma X + \delta Y) \qquad \text{if } D < 0,$$
$$f(X, Y) = h(\alpha X + \beta Y, \gamma X + \delta Y) \qquad \text{if } D > 0.$$

Now the points

$$x = \alpha X + \beta Y, \quad y = \gamma X + \delta Y$$

describe a lattice Λ, say, of determinant unity when X, Y run through all integer values. Our result takes the from: *Every lattice Λ of determinant unity has at least one of its points other than the origin O in each of the regions*

$$|g(x, y)| \leq 1, \quad |h(x, y)| \leq 1.$$

The constant on the right hand side is the best possible as is obvious from the lattice $x = X$, $y = Y$.

Let us consider the region $|g(x, y)| \leq 1$, say R. This is an infinite region bounded by the two curves $g(x, y) = \pm 1$ which have a common asymptote $x - \vartheta y = 0$ where ϑ is the real root of $t^3 - t - 1 = 0$. The asymptote is a line of symmetry of the region. It is soon seen that the parallelogram, really the square, $|x| \leq 1$, $|y| \leq 1$ is of special importance. The square has all its vertices and all the middle points of its sides on the boundary B of R. Its sides $x = \pm 1$ are tangents to the boundary at $x = \pm 1$, and further the square lies entirely in R except for a small region R_1 abutting the line $y = 1$ with $0 < x < 1$, and of course also for the image of R_1 in the origin O. This square, having its centre at O and of area 4, contains a point P other than O of every lattice Λ of determinant unity. If P is not an inner point of R, and this we may assume since otherwise the theorem is proved, it must be one of the vertices or middle points of the sides of the square, or lie in R_1. In the first two cases, it is easily shown that Λ has a point not O as an inner point of R except when Λ is the critical lattice $x=\xi$, $y=\eta$ which obviously has points on the boundary of R. In the third case, a point P of Λ is contained in R_1 and we include its boundary in R_1 since we wish to find points of Λ which are inner points of R.

We can now apply the same argument to other parallelograms of area 4, e. g. one whose sides are $x=\pm 1$ and the tangents at $(0, \pm 1)$, and find that Λ has a point say P_1 in a small curvilinear triangle near the point $(-1, 1)$. The question now suggests itself whether it is possible to find points which are linear combinations of P_1, P_2, such as $P_1 \pm P_2$ etc., which are inner points of R. For this, however a new idea is required suggested at once by the symmetry of the region R about the asymptote. The binary cubic is transformed into itself, and so also the region R, by a linear substitution with real coefficients and of determinant unity. Hence the parallelogram $|x| \leqq 1$, $|y| \leqq 1$ is changed into another one with the same characteristic properties used in the preceding argument. On considering the vertices, and middle points of its sides, we are led to the further critical lattice

$$(3\vartheta^2-1)\,x=-\xi-(\vartheta+3)\,\eta,\quad (3\vartheta^2-1)\,y=-3\vartheta\xi+\eta,$$

and it is easily verified that

$$|f(x, y)|=f(\xi, \eta),$$

and so $|f(x, y)| \geqq 1$ for integers ξ, η not both zero.

The new two small regions corresponding to the original two now lead to points P_2, P_3 of Λ not in R but near to R. These points may not be both different from the previous one, and in fact one of them say P_3 can be proved to be identical with the point P_1. We have now far more possibilities in considering linear combinations of these points, and in doing so, we require a more detailed numerical knowledge of the region e. g. the minimum ordinate of the points of the boundary lying in the square $|x| \leqq 1$; $|y| \leqq 1$, but this presents no difficulty. After many efforts, I succeeded in finding smaller and smaller regions external and near to R and containing points of Λ, and finally was able to show that a linear combination of these points led to a point not O of Λ, any lattice not one of the two critical lattices, which was an inner point of R.

I considered next the corresponding problem for the region S,

$$|h(x, y)| \equiv |x^3+x^2y-2xy^2-y^3| \leqq 1.$$

This, however, introduced fresh difficulties. For first, the boundary had three asymptotes complicating the shape of the region. But a much more important difficulty is the situation of the unit square $|x| \leqq 1$, $|y| \leqq 1$ with respect to S. The square is contained in S except for two small regions one abutting $x=1$ with $y<0$, and the other $y=-1$ with $0<x<1$, and of course their images in O. The square contains a point P not O of every lattice Λ of determinant unity and so if P is not an inner point of S, it may lie in either of two small regions. I was able to show, however, that we could exclude the region abutting $x=1$. Taking into account now that S was unchanged by three essentially distinct linear substitutions, I was able to proceed as before and finally succeeded in proving the theorem.

Subsequently much simpler geometrical proofs were given by DAVENPORT[8]) who clothed his proof in arithmetical form, and by myself[9]). I have also given a proof when $D<0$ by considering the more symmetrical region $|x^3+y^3| \leqq 1$, and have thus reduced the numerical details to a minimum[10]).

4. After these results were found, DAVENPORT discovered arithmetical proofs of surprising simplicity based on ideas related to those used by HERMITE nearly ninety years ago. There is no loss of generality on dividing out by a factor in writing

$$f(x, y) = ax^3 + bx^2y + cxy^2 + dy^3,$$

and supposing that if $D>0$, $D=49$, and if $D<0$, $D=-23$.

Take first $D>0$. Write the Hessian or quadratic covariant of $f(x, y)$ as

$$Ax^2 + Bxy + Cy^2 = (bx+cy)^2 - (3ax+by)(cx+3dy).$$

This is a positive definite form of negative discriminant

$$B^2 - 4AC = -3D;$$

and so by the usual method of reduction, we can transform the Hessian by a unimodular substitution with integer coefficients into another with $C \geqq A \geqq B \geqq 0$. On applying the same substitution to the cubic, we may suppose that its Hessian is so reduced. Then he proved the

Theorem[11]). *Either* $|f(1, 0)| \leqq 1$, *or* $|f(0, 1)| \leqq 1$, *or* $|f(1, 1)| \leqq 1$, *or* $|f(1, -1)| \leqq 1$; *an inequality sign holds except when*

$$\pm f(x, y) = x^3 + x^2y - 2xy^2 - y^3 \quad or \quad x^3 + 2x^2y - xy^2 - y^3.$$

[8]) H. DAVENPORT, The minimum of a binary cubic form, *Journal London Math. Society*, **18** (1943), pp. 168–176.

[9]) L. J. MORDELL, The minimum of a binary cubic form, *Ibidem*, **18** (1943), pp. 201–210, 210–217.

[10]) L. J. MORDELL, Lattice points in the region $|x^3+y^3| \leqq 1$, *Ibidem*, **19** (1944), pp 92–99.

[11]) H. DAVENPORT, The reduction of a binary cubic form. I., *Ibidem*, **20** (1945), pp. 14–22.

A similar result holds when $D<0$, and so we can take $D=-23$. The cubic $f(x, y)$ has now one real linear factor and can be written as,

$$f(x, y)=(x+\vartheta y)(Px^2+Qxy+Ry^2),$$

where ϑ, P, Q, R are real. We may suppose that the quadratic form $Px^2+Qxy+Ry^2$ is positive definite on considering $-f(x, y)$ if need be instead of $f(x, y)$, and then that it is reduced, i. e.

$$|Q| \leqq P \leqq R;$$

and finally that $Q>0$ by writing $-y$ for y if need be. By a unimodular integral substitution on the cubic, we may suppose that $f(x, y)$ is such that these conditions are satisfied for the quadratic. Then DAVENPORT proved the

Theorem[12]). *Either* $|f(1,0)| \leqq 1$, *or* $|f(0, 1)| \leqq 1$, *or* $|f(1,-1)| \leqq 1$, *or* $|f(1, 2)| \leqq 1$. *An inequality sign holds except when*

$$f(x, y)=x^3+x^2y+2xy^2+y^3,$$

which on putting $x=X$, $y=-X-Y$ *becomes* $X^3-XY^2-Y^3$.

5. A flood of results followed from my method, for the application of the geometry of numbers to the minimum of a binary cubic meant that corresponding questions for nonconvex regions were no longer intractable. An obvious region to investigate was

$$|x|^p+|y|^p \leqq 1.$$

which for $p \geqq 1$ is convex and had been studied by MINKOWSKI[13]). When $p<1$, it is not convex and had not been previously considered by mathematicians. I found that my methods applied not only to this region but to the more general one

$$f(|x|, |y|) \leqq 1,$$

where for $x \geqq 0$, $y \geqq 0$, $f(x, y)$ is defined, is symmetrical in x, y and homogeneous of dimension 1 say. We suppose that the region $f(x, y) \geqq f(1, 1)$, $x \geqq 0$, $y \geqq 0$ is convex and terminates in the axes or has them as asymptotes. Then just as for the binary cubic, parallelograms can be constructed whose vertices and middle points of sides all lie on the boundary of the region. Their existence follows since it can be proved that unique numbers a, b, c with $a>b>c$ are defined by the equations

$$f(a+b, a-b)=f(a, b)=cf(1, 1),$$
$$a^2+b^2=2.$$

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[12]) H. DAVENPORT, The reduction of a binary cubic form. II, *Journal London Math. Society*, **20** (1945), pp. 139—157.

[13]) H. MINKOWSKI, l. c. [6]), pp. 21—58.

By considering various regions in which lattice points must lie and utilising the ideas developed for the binary cubic, I was then able[14]) to reduce the question to a minimum problem of the type considered by MINKOWSKI. Further there existed many regions for which the minimum problem could be solved. Thus for lattices of determinant 1, best possible results were found of the form $|x|^p + |y|^p \leqq 2c^p$, $0{\cdot}33 \ldots \leqq p < 1$; $|x^4 - y^4| \leqq \frac{4}{\sqrt{17}}$, also for a star shaped octagon. etc.

Similar methods apply to the region

$$|x^n + y^n| \leqq c^n, \qquad n \geqq 4.$$

I conclude by saying that the success of these methods led MAHLER to his general and important theory of lattice points in star shaped regions, a fruitful theory which has recently added so much to our knowledge of the geometry of numbers and has also been the starting point of many new results.

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Note. In 1945, B. DELAUNAY published a paper entitled "Local methods in the geometry of numbers", *Bulletin Acad. Sci. URSS*, Série Math., **9** (1945), pp. 241—256 (in Russian). He finds a new and simple solution for the minimum of a binary cubic of positive discriminant by an extension of MINKOWSKI's method of continually diminishing the determinant of a lattice which has no point other than the origin in a region.

(Added June 20, 1949.)

[14]) L. J. MORDELL, On the geometry of numbers in some nonconvex regions, *Proceedings London Math. Society*, (2) **48** (1945), pp. 339—390.