The class of functions which are absolutely convergent Fourier transforms.

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We show that if a locally compact abelian group G has the property that every continuous function vanishing at infinity on it is an absolutely convergent Fourier transform, then G is finite. More precisely, the set of functions on G which are absolutely convergent Fourier transforms is dense (in the uniform topology) in the space of continuous functions on G which vanish at infinity, and is either all of the space, or of first category in it, according as G is finite or not. Thus the difficulties in Fourier analysis which appear to arise from the circumstance that not every continuous function vanishing at infinity on a locally compact abelian group is an absolutely convergent Fourier transform are inherent in any generalization from finite groups. A point of interest from a more classical viewpoint is that our results establish the existence of continuous functions on the reals which tend to zero at infinity, but which are not absolutely convergent Fourier transforms, and also the existence of similar continuous periodic functions on the reals. These results are well known, by virtue of explicit constructions of such functions, but the proof which we give is purely existential, and thereby avoids the computations required to verify that given functions are not representable as absolutely convergent Fourier transforms. (BANACH has shown in a related fashion the existence of nondifferentiable functions.)

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Theorem. *Let G be a locally compact abelian (abbreviated to: "LCA") group. Then the class of all functions on G which are absolutely convergent Fourier transforms is either a dense set of first category in the space of complex-valued continuous functions vanishing at infinity on G, or all of that space, according as G is infinite or finite.*

We recall that a function f on G is said to be an absolutely convergent Fourier transform if there is a function F, integrable on the character group G^* of G, such that $f(x) = \int_{G^*} x^*(x) F(x^*) dx^*$, for $x \in G$ (here and elsewhere dx^* represents the element of Haar measure on G^*, dx that on G, and "integrable" means "integrable relative to Haar measure"). A function φ vanishes at infinity on a topological space Γ if for every positive number ε, the set where φ exceeds ε in absolute value has compact closure. By $C(\Gamma)$ we denote the Banach algebra off all complexvalued continuous functions vanishing at infinity on the space Γ, the norm of a function being defined as its least upper bound. A statement will be said to be true "nearly everywhere" (abbreviated to "n. e.") on a LCA group G if it is a function on the group which is true except on a set whose intersection with any compact set has Haar measure zero, and a numerical function on G will be called "measurable" if the inverse image of any open set under the function is a set in G which meets each compact set in a set which is measurable in the usual sense. For any LCA group G, G^* will denote the character group of G, and $L_p(G)$ will denote the Banach space of pth-power integrable complexvalued functions on G, the norm of a function being defined as the pth root of the integral of the pth power of the absolute value of the function.

Lemma 1. *If K is a regular finite countably-additive set function on a LCA group G, then its Fourier-Stieltjes transform* $\int_G x^*(x)\,dK(x)$ *vanishes identically (on G^*) only if $K=0$.*

If h is an arbitrary continuous function on G which vanishes outside of a compact set, then $\int h(xy^{-1})\,dK(y)$ defines a function on G which is easily shown, by means of the Fubini theorem, to be integrable, and whose Fourier transform is the product of the Fourier-Stieltjes transform of K with the Fourier transform of h, and so vanishes identically. By the uniqueness theorem for Fourier transforms (see e. g. [3]), $\int_G h(xy^{-1})\,dK(y)$ vanishes n. e., so that being a continuous function, it vanishes everywhere on G. By virtue of the arbitrary character of h and the regularity of K, it results that $K=0$.

The following lemma is known, but the simple proof which we give is new.

Lemma 2. *For any LCA group G, the Fourier transforms of the elements of $L_1(G)$ are dense in $C(G^*)$.*

It is known (the generalized Riemann-Lebesgue lemma) that all L_1 transforms are in $C(G^*)$ (see e. g. [4]), and we prove their density by an indirect argument. Assuming that they are not dense, then by the Hahn-Banach theorem there exists a non-vanishing continuous linear functional χ on $C(G^*)$ which vanishes on all transforms. By the known form of continuous linear functionals on $C(\Gamma)$, where Γ is a locally compact Hausdorff space, there exists a finite regular countably-additive set function K on G^* such that for $F \in C(G^*)$, $\chi(F) = \int_{G^*} F(x^*)\,dK(x^*)$. Now if $F(x^*) = \int_G x^*(x) f(x)\,dx$, with

$f \in L_1(G)$. then $\chi(F) = 0 = \int\limits_{G^*}\left[\int\limits_{G} x^*(x) f(x)\, dx\right] dK(x^*)$, which by the Fubini theorem is the same as $\int\limits_{G}\left[\int\limits_{G^*} x^*(x)\, dK(x^*)\right] f(x)\, dx$. Now f is arbitrary in $L_1(G)$, so that it follows that $\int\limits_{G^*} x^*(x)\, dK(x^*) = 0$ n. e. on G, and since the integral represents a continuous function, vanishes identically on G. Lemma 1 now implies $K = 0$, which yields the contradiction $\chi = 0$.

Lemma 3. *If T is an open subgroup of a LCA group G, then an element of $L_1(G)$ vanishes n. e. outside of T if and only if its Fourier transform is constant on cosets modulo the annihilator $\tilde{T}$ of T in G^*.*

It is trivial to verify the "only if" part. Now if F is the Fourier transform of an element f of $L_1(G)$, and is constant on cosets modulo $\tilde{T}$, then $F(x^* y^*) = F(x^*)$ for all $y^* \in \tilde{T}$ and $x^* \in G^*$. It results that $\int\limits_{G} x^*(x)(y^*(x) - 1) f(x)\, dx = 0$ for $x^* \in G^*$, which by the uniqueness of the Fourier transform implies that $[y^*(x) - 1] f(x)$ vanishes n. e. If f is continuous, this implies in turn that f vanishes except on the set of x for which $y^*(x) = 1$, and since y^* is arbitrary on $\tilde{T}$, it follows that f then vanishes except on the intersection over $y^* \in \tilde{T}$ of such sets, i. e. f vanishes outside of T. Now in general, there exists a sequence $\{g_n\}$ of bounded functions in $L_1(G)$ which vanish outside of T and are such that $\{f * g_n\}$ converges in $L_1(G)$ to f. The continuous function $f * g_n$ has as its transform the product of the Fourier transforms of f and g_n and so is constant on cosets modulo $\tilde{T}$. Hence $f * g_n$ vanishes outside of T, and since the set of elements of $L_1(G)$ which vanish n. e. outside T is closed, f itself is in this set.

It is convenient at this point to introduce the following notation: a LCA group G has the property Φ, or alternatively G^* has property Φ^*, if every element of $C(G)$ is an L_1 Fourier transform. We note that when G has the property Φ, then the Fourier transform is a homeomorphism of $L_1(G^*)$ onto $C(G)$, by a well-known theorem of BANACH [1].

Lemma 4. *If T^* is an open subgroup of the LCA group G^*, and if G^* has the property Φ^*, then so also does T^*.*

If $f \in C(T^{**})$, then there is a unique function f' in $C(G/\tilde{T})$ which corresponds to f via the natural isomorphism of T^{**} with $G/\tilde{T}^*$, and this function defines via the inverse of the natural mapping on G to $G/\tilde{T}^*$, an element f'' of $C(G)$ which is constant on cosets modulo $\tilde{T}^*$. By the preceding lemma there is an element F'' of $L_1(G^*)$ which vanishes outside T^* and is such that

$f''(x)=\int_{T^*} x^*(x)F''(x^*)\,dx^*$. It follows that $f'(x\widetilde{T}^*)=\int_{T^*} x^*(x)F''(x)\,dx^*$ and hence that $f(u)=\int_{T^*} x^*(u)F''(x^*)\,dx^*$ for $u\in T^{**}$.

Lemma 5. *If a compact metric abelian group G has the property Φ, then G is finite.*

Clearly G^* is either a finite or a countable discrete group, and so weak sequential convergence in $L_1(G^*)$ coincides with strong convergence (see [1]. Hence the same is true in $C(G)$. Since weak sequential convergence in a $C(\Gamma)$, where Γ is compact, is identical with convergence at every point, together with uniform boundedness (this fact follows easily from the known form of continuous linear functionals on a $C(\Gamma)$), it follows that any bounded sequence of elements of $C(G)$ which converges at every point converges uniformly. Since the characteristic function of a closed set in G is the bounded pointwise limit of elements of $C(G)$, it results that every such function is continuous, so that every closed set in G is open, i. e. G is discrete. Being also compact, it is finite.

Lemma 6. *If a compact abelian group G has the property Φ, then G is finite.*

For then G^* is discrete, and is either finite, in which case G is also, or contains a countable subgroup H^*. In the latter event, Lemma 4 would imply that H^* has the property Φ^*, and so by the preceding lemma is finite, a contradiction.

Lemma 7. *If a LCA group G which is generated by a compact neighborhood of the identity has property Φ, then G is finite.*

Evidently any continuous linear functional on $L_1(G^*)$ induces via the isomorphism between $L_1(G^*)$ and $C(G)$ a continuous linear functional on $C(G)$. By the known form for such functionals, it results that for any bounded measurable function k on G^*, there exists a finite bounded regular countably-additive set function K on G^* such that $\int k(x^*)f(x^*)\,dx^*=\int F(x)\,dK(x)$ for all $f\in L_1(G^*)$, where F is the Fourier transform of f (we note that G is σ-finite, so that the Riesz representation theorem for continuous linear functionals on L_1 is valid). It is easy to conclude as in the proof of Lemma 2 that $k(x^*)=\int x^*(x)\,dK(x)$ n. e. on G^*, and hence everywhere on G^*, if k is continuous. From this we conclude that every continuous almost periodic (c. a. p.) function on G^* has an absolutely convergent Fourier series. It suffices to show that if $p(x^*)=\int x^*(x)\,dJ(x)$ is a c. a. p. function on G^*, where J has the same properties as K, and also vanishes on points, then $p=0$. Now according to a result of Lyubarskiĭ [2] (stated for connected groups, but whose proof is valid for groups generated by a compact neighborhood of the identity), there exists a sequence $\{C_n^*\}$ of compact subsets of G^* such that the von Neumann mean $M(q)$, for any c. a. p. function q on G^*, is given by the equation $M(q)=\lim_n m_n^{-1}\int_{C_n^*} q(x^*)\,dx^*$, where $m_n=\int_{C_n^*} dx^*$. It results that for any $y\in G$, $M[x^*(y)p(x^*)]=\int_G m_n^{-1}\left[\int_{C_n^*} x^*(xy)\,dx^*\right]dJ(x)$. Since $m_n^{-1}\int_{C_n^*} x^*(xy)\,dx^*$ converges boundedly to $M[x^*(xy)]$, which is 0 or 1 according as $x\neq y^{-1}$ or $x=y$, and since J vanishes on points, it follows that $M[x^*(y)p(x^*)]=0$. As this is true for all y, $p=0$.

Thus every c. a. p. function on G^* has an absolutely convergent Fourier series. Now it is known that the set of c. a. p. functions on G^* is isomorphic to the set of all continuous functions on a compact group G' which contains a subgroup algebraically isomorphic with G^*, and in such a way as to preserve Fourier series (see e. g. [3]). Then G' is a compact group with property Φ, and hence is finite, from which it follows that G is finite.

Completion of the proof of the theorem. By the theorem of BANACH quoted earlier, either G has property Φ, or the set of Fourier transforms is of first category in $C(G)$; and in any case dense in $C(G)$, by Lemma 2. Now if G has property Φ, let H^* be the open subgroup of G^* generated by a compact neighborhood of the identity in G^*. Then H^* has property Φ^* and so by the preceding lemma in finite. It follows that G^* is discrete, so that G is compact, and hence finite by Lemma 6.

We mention that the foregoing theorem has an analog for non-commutative locally compact groups G: Let $L_1(G)$ be complete relative to the norm $\|f\|'=\|T_f\|$, where $f\in L_1(G)$ and T_f is the operator on $L_2(G)$ given by the equation $T_f g=f*g$, $g\in L_2(G)$; then G is finite. The validity of this analog was recently established by the author in collaboration with I. KAPLANSKY. A different type of analog, whose validity is an open question, is as follows: If G is a LCA group and $1<p<2$, then the Fourier transform maps $L_p(G)$ into a dense subset of $L_q(G^*)$, where $p^{-1}+q^{-1}=1$, which is of second category only if G is finite. The theorem proved here corresponds to the limiting case $p=1$. We note finally that the separable case of our theorem can be established much more briefly than the general case.

References

1. S. BANACH, *Théorie des opérations linéaires* (Warsaw, 1932).
2. G. YA. LYUBARSKIĬ, On the integration in the mean of almost periodic functions on topological groups, *Uspehi Mat. Nauk*, N. S. **3** (1948), pp. 195—201.
3. I. E. SEGAL, The group algebra of a locally compact group, *Transactions American Math. Society*, **61** (1947), pp. 69 - 105.
4. A. WEIL, *L'intégration dans les groupes topologiques et ses applications* (Paris, 1940).

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