Extremum problems for non-negative sine polynomials.

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In various chapters of the theory of Fourier series and elsewhere non-negative trigonometrical polynomials

$$(0.1)\qquad T(\vartheta)\equiv\frac{1}{2}a_0+(a_1\cos\vartheta+b_1\sin\vartheta)+\ldots+(a_n\cos n\vartheta+b_n\sin n\vartheta)$$

play an important rôle. For instance, the non-negative character of the arithmetic means of the polynomials

$$(0.2)\qquad \frac{1}{2}+\cos\vartheta+\ldots+\cos n\vartheta$$

is the basic fact in FEJÉR's theory of summability of Fourier series. Similarly, certain sine polynomials, non-negative for $0\leqq\vartheta\leqq\pi$ (in the range $\langle 0,\pi\rangle$), are frequently of importance. As an example we quote GRONWALL's polynomials

$$(0.3)\qquad \sin\vartheta+\frac{1}{2}\sin 2\vartheta+\ldots+\frac{1}{n}\sin n\vartheta.$$

In 1915, L. FEJÉR and F. RIESZ [2][1]) gave a parametric representation of fundamental importance for non-negative trigonometrical polynomials. By means of this representation, L. FEJÉR and others determined in

$$(0.4)\qquad T(\vartheta)\leqq\frac{1}{2}a_0(n+1);\;\; a_k^2+b_k^2\leqq a_0^2\cos^2\pi\Big/\left(\left[\frac{n}{k}\right]+2\right)$$

the maxima for such polynomials and for their coefficients, when the constant term $\frac{1}{2}a_0$ and the degree n are prescribed. It should be noted that FEJÉR's problem remains essentially the same if the subclass of non-negative cosine polynomials is considered[2]).

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1) The numbers refer to the list of references at the end of this paper.

2) Cf. L. FEJÉR [3] where a survey of the relevant literature can be found. An extension of these results to 'finite' Fourier integrals (which are in a certain sense the analogues of trigonometrical polynomials) has been given more recently by BOAS and KAC [1].

A completely new situation arises if one considers sine polynomials

(0.5) $$S(\vartheta) \equiv b_1 \sin\vartheta + b_2 \sin 2\vartheta + \dots + b_n \sin n\vartheta \qquad (b_n \neq 0)$$

of given degree n which are non-negative in the range $\langle 0, \pi \rangle$. It is the class of these polynomials we discuss in the present paper. Clearly $b_1 \geqq 0$ and $b_1 = 0$ is only possible when $S(\vartheta)$ vanishes identically. We shall usually normalise by assuming that $b_1 = 1$.

First we determine the maximum of $S(\vartheta)$ for fixed ϑ in $\langle 0, \pi \rangle$, and find:

$(0.6)_o$ $$S(\vartheta) \leqq \operatorname{Max} \begin{cases} \dfrac{1}{4\sin^2\vartheta} \{(n+2)\sin\vartheta - \sin(n+2)\vartheta\} \\ \dfrac{1}{4\sin^3\vartheta} \displaystyle\sum_0^{(n-3)/2} \dfrac{\{(k+3)\sin(k+1)\vartheta - (k+1)\sin(k+3)\vartheta\}^2}{(k+1)(k+3)} \end{cases};$$

$(0.6)_e$ $$S(\vartheta) \leqq \operatorname{Max} \begin{cases} \dfrac{1}{2} \dfrac{\cot\frac{1}{2}\vartheta}{\sin^2\vartheta} \displaystyle\sum_0^{(n-2)/2} \dfrac{\{(k+2)\sin(k+1)\vartheta - (k+1)\sin(k+2)\vartheta\}^2}{(k+1)(k+2)} \\ \dfrac{1}{2} \dfrac{\tan\frac{1}{2}\vartheta}{\sin^2\vartheta} \displaystyle\sum_0^{(n-2)/2} \dfrac{\{(k+2)\sin(k+1)\vartheta + (k+1)\sin(k+2)\vartheta\}^2}{(k+1)(k+2)} \end{cases},$$

when n is odd or even, respectively. In particular, when $\vartheta = 0$,

$$1 + 2b_2 + 3b_3 + \dots + n b_n \leqq \begin{cases} (n+1)(n+2)(n+3)/24 & (n \text{ odd}) \\ n(n+2)(n+4)/24 & (n \text{ even}). \end{cases}$$

The determination of the maxima and minima for the coefficients b_k is rather involved[3]). We have computed them in the cases b_2, b_3 and b_{n-1}, b_n. In other cases, in particular for b_4 and b_5, we discuss relevant methods of determination. Our main results are:

(0.7) $$|b_2| \leqq \begin{cases} 2\cos 2\pi/(n+3) & (n \text{ odd}) \\ 2\cos\vartheta_0 & (n \text{ even}), \end{cases}$$

where ϑ_0 is the least positive root of

(0.8) $$(n+4)\sin(n+2)\vartheta/2 + (n+2)\sin(n+4)\vartheta/2 = 0.$$

Next,

$(0.9)_e$ $$1 - 2\cos\pi/(n'+3) \leqq b_3 \leqq 1 + 2\cos 2\pi/(n'+3),$$

$(0.9)_o$ $$1 - 2\cos\vartheta_1 \leqq b_3 \leqq 1 + 2\cos 2\pi/(n'+3),$$

according to whether $n' = [\frac{1}{2}(n-1)]$ is even or odd, respectively. Here ϑ_1 is the least positive root of

(0.10) $$(n'+4)\cos(n'+2)\vartheta/2 + (n'+2)\cos(n'+4)\vartheta/2 = 0.$$

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[3]) The estimate

$$|b_k| = \frac{2}{\pi} \left| \int_0^\pi S(\vartheta) \sin\vartheta \frac{\sin k\vartheta}{\sin\vartheta} \, d\vartheta \right| \leqq k b_1 = k$$

is trivial.

Further

(0. 11) $$|b_{n-1}| \leqq 1, \quad -(n-2)/(n+2) \leqq b_{n-1} \leqq 1,$$

according to n being odd or even, respectively. Finally, in the same two cases,

(0. 12) $$-(n-1)/(n+3) \leqq b_n \leqq 1, \quad |b_n| \leqq n/(n+2).$$

The introductory § 1 contains general remarks concerning various methods dealing with problems of our kind. In § 2 we determine the maximum of $S(\vartheta)$, in § 3 the extrema for b_{n-1} and b_n, in § 4 for b_2, and in § 5 for the less simple case b_3. In § 6 some formal properties of orthogonal polynomials are discussed which are useful in dealing with the general b_k. The last § 7 deals, in particular, with b_4 and b_5.

§ 1. General remarks.

1. For given degree n and given $b_1 (=1)$ we put

(1. 1) $$\underline{B}(k, n) = \operatorname{Min} b_k, \quad \bar{B}(k, n) = \operatorname{Max} b_k.$$

Now, if the sine polynomial $S(\vartheta)$ is positive in $\langle 0, \pi \rangle$, then so is $S(\pi - \vartheta) = \Sigma(-1)^{k-1} b_k \sin k\vartheta$. Hence

(1. 2) $$\underline{B}(k, n) = -\bar{B}(k, n) \qquad (k \text{ even}).$$

Also

(1. 3) $$S^*(\vartheta) = \frac{1}{2}\{S(\vartheta) + S(\pi - \vartheta)\} = b_1 \sin\vartheta + b_3 \sin 3\vartheta + \dots$$

is non-negative. If n is even, then S^* is of degree at most $n-1$. It follows that

(1. 4) $$\underline{B}(k, n) = \underline{B}(k, n-1), \quad \bar{B}(k, n) = \bar{B}(k, n-1) \qquad (k \text{ odd}, n \text{ even}).$$

2 Let $0 \leqq \vartheta \leqq \pi$. Any non-negative sine polynomial can be written in the form

(1. 5) $$S(\vartheta) = \sin\vartheta \sum_1^n b_k \sin k\vartheta / \sin\vartheta = \sin\vartheta \; P(\cos\vartheta)$$

where

(1. 6) $$P(\cos\vartheta) = \frac{1}{2} a_0 + a_1 \cos\vartheta + \dots + a_{n-1} \cos(n-1)\vartheta$$

is a non-negative cosine polynomial of degree $n-1$; the converse is also true. Here

(1. 7) $$2b_k = a_{k-1} - a_{k+1}$$

where $a_n = a_{n+1} = 0$. Now, according to L. Fejér and F. Riesz, any non-negative trigonometrical polynomial $P(\cos\vartheta)$ admits of the parametric representation

(1. 8) $$P(\cos\vartheta) = |c_0 + c_1 e^{i\vartheta} + \dots + c_{n-1} e^{i(n-1)\vartheta}|^2,$$

where the c_ν are (arbitrary) real constants. Hence, by (1.7), $b_k = \Phi_k(c_0, c_1, \dots, c_{n-1})$ is a certain quadratic form of the c_ν. In particular,

(1.9) $$b_1 = \Phi_1 = \frac{1}{2}(a_0 - a_2) = c_0^2 + c_1^2 + \dots + c_{n-1}^2 - (c_0c_2 + c_1c_3 + \dots + c_{n-3}c_{n-1})$$

is positive definite.

There are then, theoretically, two possibilities of computing $\underline{B}(k,n)$ and $\overline{B}(k,n)$:

(i) We can either form the characteristic equation

(1.10) $$|\Phi_k - \lambda\Phi_1| = 0$$

and obtain our quantities as the least and greatest roots of this equation.

(ii) Alternatively, we may form the system of linear equations in the c_ν corresponding to (1.10) and solve this system. This method works satisfactorily in the cases b_{n-1} and b_n.

In general, however, the method based on (1.10) is not easily adaptable for obtaining explicit results, in particular when n is large.

3. We prefer to base our actual discussion on the following theorem of LUKÁCS [**4**, pp. 4—5][4]):

Any polynomial $P(x)$ of degree $\widehat{P} = N$, which is non-negative in $\langle -1, 1\rangle$, can be represented in the form

(1.11)$_e$ $$P(x) = A^2(x) + (1-x^2)B^2(x); \quad \widehat{A} \leqq \frac{N}{2}, \quad \widehat{B} \leqq \frac{N-2}{2},$$

(1.11)$_o$ $$P(x) = (1-x)C^2(x) + (1+x)D^2(x); \quad \widehat{C} \leqq \frac{N-1}{2}, \quad \widehat{D} \leqq \frac{N-1}{2},$$

according to N being even or odd.

Now, by (1.5), if we put $x = \cos\vartheta$,

(1.12) $$S(\vartheta) = \sin\vartheta\, P(x) = (1-x^2)^{1/2} P(x),$$

where $P(x)$ is a non-negative polynomial of degree $N = n-1$ in $\langle -1, 1\rangle$. It is clear that, when N is even, say, we can restrict P to range over polynomials of the type A^2 or $(1-x^2)B^2$. Max $S(\vartheta)$ is then the greater (not smaller) maximum obtained in the two cases. A similar remark applies to $\underline{B}(k,n)$ and $\overline{B}(k,n)$, and to the case when N is odd.

4. We have

(1.13) $$P(x) = \sum_1^n b_k \sin k\vartheta/\sin\vartheta = \sum_1^n b_k U_{k-1}(x).$$

Here

(1.14) $$U_k(x) = U_k(\cos\vartheta) = \sin(k+1)\vartheta/\sin\vartheta = 2^k x^k + A_k x^{k-2} + \dots$$

is the familiar Tchebychev polynomial of the second kind. The polynomials

[4]) This theorem can also be derived from the results of FEJÉR and RIESZ.

$\sqrt{\frac{2}{\pi}}\,U_k$ form an orthonormal system with the weight function $(1-x^2)^{1/2}$ over the range $\langle -1, 1\rangle$. It follows that

$$(1.15)\qquad \int_{-1}^{1} x^k U_k(x)\,(1-x^2)^{1/2}\,dx = \pi 2^{-(k+1)}.$$

Also

$$(1.16)\qquad b_k = \frac{2}{\pi}\int_{-1}^{1} P(x)\,U_{k-1}(x)\,(1-x^2)^{1/2}\,dx.$$

We shall also require orthogonal polynomials over $\langle -1, 1\rangle$ corresponding to the weight functions $w(x)=(1-x^2)^{3/2}$ and $w(x)=(1-x)(1-x^2)^{1/2}$. The former are[5])

$$(1.17)\qquad V_k(x) = (x^2-1)^{-1}\left[\frac{U_{k+2}(x)}{k+3} - \frac{U_k(x)}{k+1}\right] = \frac{2^{k+2}}{k+3}x^k + \ldots,$$

so that, by (1.15),

$$(1.18)\qquad \int_{-1}^{1} V_k^2(x)\,(1-x^2)^{3/2}\,dx = \int_{-1}^{1}\left(\frac{U_k(x)}{k+1} - \frac{U_{k+2}(x)}{k+3}\right)\left(\frac{2^{k+2}}{k+3}x^k + \ldots\right)(1-x^2)^{1/2}dx$$
$$= \frac{2\pi}{(k+1)(k+3)}.$$

Similarly, when $w(x)=(1-x)(1-x^2)^{1/2}$, we have the orthogonal polynomials

$$(1.19)\qquad W_k(x) = (x-1)^{-1}\left(\frac{U_{k+1}(x)}{k+2} - \frac{U_k(x)}{k+1}\right) = \frac{2^{k+1}}{k+2}x^k + \ldots$$

with

$$(1.20)\qquad \int_{-1}^{1} W_k^2(x)\,(1-x)\,(1-x^2)^{1/2}\,dx = \pi/(k+1)(k+2).$$

5. Our problem is of the general type of determining the extrema of a quotient

$$(1.21)\qquad \int_{\alpha}^{\beta} u^2(x)\,h(x)\,w(x)\,dx \;:\; \int_{\alpha}^{\beta} u^2(x)\,w(x)\,dx,$$

where $w(x)$ is a given weight function and $h(x)$ a given polynomial; $u(x)$ is an arbitrary polynomial of given degree whose coefficients vary through all real values not all zero[6]).

In the cases b_2 and b_3 we shall have $h(x)=x$. This is the so called '*problem of the centroid*', first treated by Tchebychev.

[5]) Compare (6.6).
[6]) Actually, we shall have either $\alpha=-1$, $\beta=1$ or $\alpha=0$, $\beta=1$.

Its solution is as follows: [Cf. 4, Theorem 7. 72. 1, p. 183; we follow (apart from slight changes) the notation of **4.**]

Let $w(x)$ be a given weight function over $\langle\alpha, \beta\rangle$ and the $p_k(x)$ be the orthonormal polynomials associated with it. Let $f(x)$ run through all polynomials of given degree N and non-negative in $\langle\alpha, \beta\rangle$. Finally, let $\overline{M}$ and $\underline{M}$ be the maximum and minimum of the quotient

$$\int_\alpha^\beta f(x)\, x\, w(x)\, dx : \int_\alpha^\beta f(x)\, w(x)\, dx. \tag{1.22}$$

If $N=2m$, then $\overline{M}$ is the greatest and $\underline{M}$ is the least zero of $p_{m+1}(x)$. If $N=2m+1$, then $\overline{M}$ is the greatest zero of $p_{m+2}(\alpha)\,p_{m+1}(x)-p_{m+1}(\alpha)\,p_{m+2}(x)$, and $\underline{M}$ is the least zero of $p_{m+2}(\beta)\,p_{m+1}(x)-p_{m+1}(\beta)\,p_{m+2}(x)$.

We note that extremum problems of our type are normally treated by the Gauss-Jacobi method of mechanical quadrature. We use instead, in §§ 6 and 7, certain formal identities for orthogonal polynomials associated with $w(x)$ and $h(x)\,w(x)$.

§ 2. The maximum of $S(\vartheta)$.

1. First, let n be odd, so that the degree $N=n-1$ of $P(x)$, in (1. 13), is even. By $(1.\,11)_e$, we may assume $P=A^2$ or $P=(1-x^2)B^2$. The maximum of $S(\vartheta)$ is then the greater of the two maxima obtained in each case.

(i) Let $P=A^2$ and $A(x)=\sum_0^h \alpha_k U_k(x)$, where $h=\frac{1}{2}(n-1)$. Since the $\sqrt{\frac{2}{\pi}}\,U_k$ are orthonormal with weight $(1-x^2)^{1/2}$, we have, by (1. 16),

$$b_1=\frac{2}{\pi}\int_{-1}^{1} A^2(x)\,(1-x^2)^{1/2}\,dx=\sum_0^h \alpha_k^2=1. \tag{2.1}$$

Hence, by Cauchy's inequality,

$$\begin{aligned} P(x) &\leq \sum \alpha_k^2\cdot\sum U_k^2(x)=\sum U_k^2(x)= \\ &=\sum\left(\frac{\sin(k+1)\vartheta}{\sin\vartheta}\right)^2=\sum_0^h\frac{1-\cos 2(k+1)\vartheta}{2\sin^2\vartheta}= \\ &=\frac{1}{2\sin^2\vartheta}\left[\frac{n+1}{2}+\frac{1}{2}-\frac{\sin(n+2)\vartheta}{2\sin\vartheta}\right], \end{aligned} \tag{2.2}$$

so that by (1. 12)

$$S(\vartheta)\leq\frac{1}{4\sin^2\vartheta}\{(n+2)\sin\vartheta-\sin(n+2)\vartheta\}, \tag{2.3}$$

which is the first inequality $(0.\,6)_o$.

Clearly, equality in (2. 2) and hence in (2. 3) can be attained.

(ii) Let $P=(1-x^2)B^2$ and $B(x)=\sum_0^h \beta_k V_k(x)$ where $h=\frac{1}{2}(n-3)$. Then, by (1.18),

$$(2.4)\qquad b_1=\frac{2}{\pi}\int_{-1}^{1} B^2(x)(1-x^2)^{3/2}dx=4\sum_0^h \frac{\beta_k^2}{(k+1)(k+3)}=1.$$

Hence

$$(2.5)\quad B^2(x)\leqq \sum \frac{\beta_k^2}{(k+1)(k+3)}\cdot\sum(k+1)(k+3)V_k^2(x)=\frac{1}{4}\sum_0^h(k+1)(k+3)V_k^2(x)$$

which is equivalent to the second inequality $(0.6)_o$.

2. If n is even we have the two cases $P(x)=(1\pm x)C^2(x)$. It suffices to consider the case of the factor $1-x$, the two cases changing into each other on replacing x by $-x$, that is ϑ by $\pi-\vartheta$.

Putting $C(x)=\sum_0^h \gamma_k W_k(x)$, where $h=\frac{1}{2}(n-2)$, we have, by (1.20),

$$(2.6)\qquad b_1=\frac{2}{\pi}\int_{-1}^{1} C^2(x)(1-x)(1-x^2)^{1/2}dx=2\sum_0^h\frac{\gamma_k^2}{(k+1)(k+2)}=1.$$

Hence

$$(2.7)\qquad C^2(x)\leqq\frac{1}{2}\sum_0^h(k+1)(k+2)W_k^2(x),$$

which is equivalent to the first inequality $(0.6)_e$. The second is obtained on changing ϑ into $\pi-\vartheta$.

§ 3. The extrema of b_n and b_{n-1}.

1. Let n be odd. Again we have two cases:

(i) Let $P(x)=A^2(x)$ where $A(x)=\sum_0^h \alpha_k U_k(x)$ and $h=\frac{1}{2}(n-1)$. By (1.14), $P(x)=\alpha_h^2 2^{2h}x^{2h}+\dots$ Hence, using (1.15) and (1.16),

$$(3.1)\quad b_n=\frac{2}{\pi}\int_{-1}^{1}P(x)U_{n-1}(x)(1-x^2)^{1/2}dx=\frac{2}{\pi}\alpha_h^2 2^{2h}\int_{-1}^{1}x^{2h}U_{2h}(x)(1-x^2)^{1/2}dx=$$
$$=\frac{2}{\pi}\alpha_h^2 2^{2h}\pi 2^{-(2h+1)}=\alpha_h^2\leqq 1,$$

by (2.1). Also $b_n\geqq 0$.

(ii) Let $P(x)=(1-x^2)B^2(x)=(1-x^2)\left\{\sum_0^h\beta_k V_k(x)\right\}^2$ where $h=\frac{1}{2}(n-3)$. Then, by (1.17),

$$(3.2)\qquad P(x)=-\beta_h^2\frac{2^{2(h+2)}}{(h+3)^2}x^{2h+2}+\dots$$

Hence, as in (3. 1), since $n-1=2h+2$,

$$(3.3)\quad b_n=-\frac{2}{\pi}\beta_h^2\frac{2^{2(h+2)}}{(h+3)^2}\pi 2^{-(2h+3)}=-\frac{4\beta_h^2}{(h+3)^2}\geqq-\frac{(h+1)(h+3)}{(h+3)^2}=-\frac{n-1}{n+3},$$

by (2. 4). Also $b_n\leqq 0$. This establishes (0. 12), when n is odd.

2. If n is even, we may take

$$P(x)=(1-x)C^2(x)=(1-x)\left\{\sum_0^h\gamma_k W_k(x)\right\}^2 \quad \text{where } h=\frac{1}{2}(n-2).$$

By (1. 19),

$$(3.4)\qquad P(x)=-\gamma_h^2\frac{2^{2(h+1)}}{(h+2)^2}x^{2h+1}+\ldots,$$

and we find, as above, using (2. 6),

$$(3.5)\quad b_n=-\frac{2}{\pi}\gamma_h^2\frac{2^{2(h+1)}}{(h+2)^2}\pi 2^{-2(h+1)}=-\frac{2\gamma_h^2}{(h+2)^2}\geqq-\frac{(h+1)(h+2)}{(h+2)^2}=-\frac{n}{n+2},$$

which establishes (0. 12), when n is even.

3. For b_{n-1} we may assume that n is odd, since for even n the case reduces, by (1. 4), to that of the last coefficient. Again, we have our two cases.

(i) We take

$$(3.6)\quad \begin{aligned}P(x)&=\left(\sum_0^h\alpha_k U_k(x)\right)^2=(\alpha_h 2^h x^h+\alpha_{h-1}2^{h-1}x^{h-1}+\ldots)^2=\\&=2^{2h}\alpha_h^2x^{2h}+2^{2h}\alpha_h\alpha_{h-1}x^{2h-1}+\ldots;\ h=\frac{1}{2}(n-1).\end{aligned}$$

Hence, by (1. 16) and (2. 1),

$$(3.7)\quad \begin{aligned}b_{n-1}&=\frac{2}{\pi}\int_{-1}^{1}P(x)U_{2h-1}(x)(1-x^2)^{1/2}dx=\\&=\frac{2}{\pi}2^{2h}\alpha_h\alpha_{h-1}\pi 2^{-2h}=2\alpha_h\alpha_{h-1}\leqq\alpha_h^2+\alpha_{h-1}^2\leqq 1.\end{aligned}$$

(ii) We take

$$(3.8)\quad \begin{aligned}P(x)&=(1-x^2)\left(\sum_0^h\beta_k V_k(x)\right)^2=(1-x^2)\left[\frac{2^{h+2}}{h+3}\beta_h x^h+\frac{2^{h+1}}{h+2}\beta_{h-1}x^{h-1}+\ldots\right]^2=\\&=-\frac{2^{2(h+2)}}{(h+3)^2}\beta_h^2x^{2h+2}-\frac{2^{2(h+2)}\beta_h\beta_{h-1}}{(h+3)(h+2)}x^{2h+1}+\ldots;\ h=\frac{n-3}{2}.\end{aligned}$$

Hence, by (2. 4),

$$(3.9)\quad \begin{aligned}b_{n-1}&=-\frac{2}{\pi}\frac{2^{2(h+2)}\beta_h\beta_{h-1}}{(h+3)(h+2)}\cdot\pi 2^{-2h}=-\frac{8\beta_h\beta_{h-1}}{(h+3)(h+2)}\geqq\\&\geqq-4\frac{\beta_{h-1}^2+\beta_h^2}{(h+3)(h+2)}>-4\left[\frac{\beta_{h-1}^2}{h(h+2)}+\frac{\beta_h^2}{(h+1)(h+3)}\right]\geqq-1.\end{aligned}$$

This completes the proof of (0. 11) when n is odd. When n is even, (0. 11) follows from the first formula (0. 12) on replacing n by $n-1$.

§ 4. The extrema for b_2.

By (1.2), it suffices to determine $\overline{B}(2, n)$. Since $U_1(x) = 2x$ we have

$$(4.1) \qquad b_2 : b_1 = 2\int_{-1}^{1} xP(x)(1-x^2)^{1/2}dx : \int_{-1}^{1} P(x)(1-x^2)^{1/2}dx,$$

which is a special case of the problem of the centroid (1.2). If n is odd, $\widehat{P} = n-1$ is even, and $\overline{B}(2, n)$ is twice the greatest zero of

$$U_{\frac{n+1}{2}}(x) = \frac{\sin\frac{n+3}{2}\vartheta}{\sin\vartheta}, \text{ i. e. } \overline{B}(2, n) = 2\cos\frac{2\pi}{n+3}.$$

If n is even, then $\overline{B}(2, n) = 2\cos\vartheta_0$, where $x_0 = \cos\vartheta_0$ is the greatest root of

$$(4.2) \qquad U_{\frac{n+2}{2}}(-1)\,U_{\frac{n}{2}}(x) - U_{\frac{n}{2}}(-1)\,U_{\frac{n+2}{2}}(x) = 0,$$

which is equivalent to (0.8).

§ 5. The extrema for b_3.

By (1.3) and (1.4) we may assume that n is odd and that $P(x) = Q(x^2)$ where $\widehat{Q} = n' = \frac{1}{2}(n-1)$. Since $U_2(x) = 4x^2 - 1$ we have.

$$(5.1) \quad b_3 : b_1 = \int_0^1 (4t-1)\,Q(t)(1-t)^{1/2}t^{-1/2}dt : \int_0^1 Q(t)(1-t)^{1/2}t^{-1/2}dt = 4T-1,$$

say. Thus our problem is again a special case of the problem of the centroid.

Now the $p_k(t) = U_{2k}(\sqrt{t})$ are, plainly, orthogonal polynomials over $\langle 0, 1\rangle$ associated with the weight function $(1-t)^{1/2}t^{1/2}$.

If $n' = 2m$, then $p_{m+1}(t) = U_{n'+2}(\sqrt{t})$ and hence

$$(5.2) \quad \text{Max } T = \cos^2\frac{\pi}{n'+3}, \quad \text{Min } T = \cos^2\left(\frac{\pi}{2}\,\frac{n'+2}{n'+3}\right) = \sin^2\frac{\pi}{2(n'+3)},$$

$$(5.3) \qquad \overline{B}(3, n) = 1 + 2\cos\frac{2\pi}{n'+3}, \quad \underline{B}(3, n) = 1 - 2\cos\frac{\pi}{n'+3}.$$

If $n' = 2m+1$, we need the greatest zero of

$$(5.4) \qquad U_{2m+4}(0)\,U_{2m+2}(\sqrt{t}) - U_{2m+2}(0)\,U_{2m+4}(\sqrt{t}) =$$

$$= (-1)^{m+2}\frac{\sin(2m+3)\vartheta + \sin(2m+5)\vartheta}{\sin\vartheta} = (-1)^{m+2}\frac{2\sin(3m+4)\vartheta\cos\vartheta}{\sin\vartheta},$$

where $t = \cos^2\vartheta$, which leads to the right half of $(0.9)_0$; the least zero of

$$(5.5) \qquad U_{2m+4}(1)\,U_{2m+2}(\sqrt{t}) - U_{2m+2}(1)\,U_{2m+4}(\sqrt{t}) =$$

$$= \frac{(2m+5)\sin(2m+3)\vartheta - (2m+3)\sin(2m+5)\vartheta}{\sin\vartheta}$$

similarly gives the left half of $(0.9)_0$.

§ 6. Identities involving orthogonal polynomials

1. Let $w(x)$ be a weight function over $\langle\alpha,\beta\rangle$, and let the $p_m(x)=k_m x^m+\ldots$, where $k_m>0$, be the associated orthonormal polynomials. We introduce the moments

$$c_m=\int_\alpha^\beta x^m w(x)\,dx \tag{6.1}$$

The determinants $D_m=[c_{p+q}]_0^m$ are then positive, and we have, for $m\geq 1$,[7])

$$p_m(x)=(D_{m-1}D_m)^{-1/2}[c_{p+q}x-c_{p+q+1}]_0^{m-1}, \tag{6.2}$$

$$k_0=D_0^{-1/2},\quad k_m=(D_{m-1}/D_m)^{1/2},\quad D_m=(k_0k_1\ldots k_m)^{-2}. \tag{6.3}$$

We wish to generalise these formulae.

2. Let $\alpha_1,\alpha_2,\ldots,\alpha_l$ be real or complex constants chosen so that the polynomial

$$u(x)=(\alpha_1-x)(\alpha_2-x)\ldots(\alpha_l-x) \tag{6.4}$$

is real and non-negative in $\langle\alpha,\beta\rangle$. This will be, for instance, the case when the α_i are sufficiently large positive. We assume, moreover, that the determinants

$$\Delta_m=\begin{vmatrix} p_m(\alpha_1) & p_{m+1}(\alpha_1) & \ldots & p_{m+l-1}(\alpha_1)\\ p_m(\alpha_2) & p_{m+1}(\alpha_2) & \ldots & p_{m+l-1}(\alpha_2)\\ \cdot & \cdot\ \cdot\ \cdot & & \cdot\\ p_m(\alpha_l) & p_{m+1}(\alpha_l) & \ldots & p_{m+l-1}(\alpha_l)\end{vmatrix},\quad m=0,1,2,\ldots,N, \tag{6.5}$$

are positive. Then the orthonormal polynomials $q_m(x)$ associated with the weight function $u(x)w(x)$ over $\langle\alpha,\beta\rangle$ are, for $n=0,1,2,\ldots,N-1$, given by the formula

$$u(x)q_m(x)=\left(\frac{k_m}{k_{m+l}\Delta_m\Delta_{m+1}}\right)^{1/2}\begin{vmatrix} p_m(x) & p_{m+1}(x) & \ldots & p_{m+l}(x)\\ p_m(\alpha_1) & p_{m+1}(\alpha_1) & \ldots & p_{m+l}(\alpha_1)\\ \cdot & \cdot\ \cdot\ \cdot & & \cdot\\ p_m(\alpha_l) & p_{m+1}(\alpha_l) & \ldots & p_{m+l}(\alpha_l)\end{vmatrix}. \tag{6.6}$$

For the proof cf. **4**, Theorem 2.5, pp. 28—29, where the orthogonality of these polynomials is shown. As for the normalisation we note that the highest term of $q_m(x)$ is $(k_mk_{m+l})^{1/2}(\Delta_m/\Delta_{m+1})^{1/2}x^m+\ldots$, and that

$$u(x)\,q_m(x)=\left(\frac{k_m}{k_{m+l}}\right)^{1/2}\left(\frac{\Delta_{m+1}}{\Delta_m}\right)^{1/2}p_m(x)+A_1p_{m+1}(x)+\ldots+A_lp_{m+l}(x). \tag{6.7}$$

Hence

$$\int_\alpha^\beta q_m^2(x)\,u(x)\,w(x)\,dx=\int_\alpha^\beta\left(\frac{k_m}{k_{m+l}}\right)^{1/2}\left(\frac{\Delta_{m+1}}{\Delta_m}\right)^{1/2}p_m(x)\,(k_mk_{m+l})^{1/2}\left(\frac{\Delta_m}{\Delta_{m+1}}\right)^{1/2}x^m w(x)\,dx=1. \tag{6.8}$$

[7]) Cf. **4**, (2.2 9), p. 26.

3. Let $h(x)$ be a given polynomial with real coefficients. We want to determine the extrema of the quotient

(6.9) $$\int_{\alpha}^{\beta} h(x) f^2(x) w(x)\,dx : \int_{\alpha}^{\beta} f^2(x) w(x)\,dx,$$

where the coefficients of the polynomial $f(x)$, of degree m, take arbitrary real values $u_0, u_1, \ldots, u_m$ not all zero. These maxima and minima are then characterised as the greatest and least zeros of the discriminant $H_m(\varrho)$ of the quadratic form (in the u_i)

(6.10) $$\int_{\alpha}^{\beta} (h(x) - \varrho)\,[u_0 + u_1 x + \ldots + u_m x^m]^2\, w(x)\,dx.$$

In order to compute $H_m(\varrho)$ we choose first the real numbers ε and ϱ so that $u(x) = \varepsilon(h(x) - \varrho)$ satisfies the above conditions. The highest coefficient of $u(x)$ in (6.4) has to be $(-1)^l$, so that ε depends only on the highest coefficient of h. By (6.3),

(6.11) $$\varepsilon^{-(m+1)} H_m(\varrho) = (k_0' k_1' \ldots k_m')^{-2},$$

where k_i' is the highest coefficient of the orthonormal polynomial $q_i(t)$ associated with the weight function $u(x)\,w(x)$. By (6.6)

(6.12) $$k_i' = (k_i k_{i+l})^{1/2} (\varDelta_i / \varDelta_{i+1})^{1/2}$$

so that

(6.13) $$\varepsilon^{-(m+1)} H_m(\varrho) = (k_0 k_1 \ldots k_m)^{-1} (k_l k_{l+1} \ldots k_{l+m})^{-1}\, \varDelta_{m+1} / \varDelta_0.$$

The quotient $\varDelta_{m+1}/\varDelta_0$ is a symmetric polynomial in $\alpha_1, \alpha_2, \ldots, \alpha_l$ which are the roots of $h(x) - \varrho$. Hence it is a polynomial of degree $m+1$ in ϱ, and *the equation* (6.13) *is an identity in* ϱ. The greatest and least zeros of the polynomial $\varDelta_{m+1}/\varDelta_0$ in ϱ yield the extrema in question.

4. The two simplest cases are $l=1$ and $l=2$ (compare (6.5)). If $l=1$, we have

(6.14) $$\varDelta_{m+1}/\varDelta_0 = k_0^{-1} p_{m+1}(\alpha_1).$$

If $l=2$, then

(6.15) $$\frac{\varDelta_{m+1}}{\varDelta_0} = \frac{p_{m+1}(\alpha_1)\, p_{m+2}(\alpha_2) - p_{m+1}(\alpha_2)\, p_{m+2}(\alpha_1)}{k_0 k_1 (\alpha_2 - \alpha_1)} = (k_0 k_1)^{-1} K_{m+1}(\alpha_1, \alpha_2),$$

where K_m is the 'kernel function' [cf. **4,** (3.2.3), p. 42].

§ 7. The coefficients b_4 and b_5.

1. In the case b_4 we have

$$(7.1)\qquad b_4 : b_1 = \int_{-1}^{1} P(x)\, U_3(x)\,(1-x^2)^{1/2}\,dx : \int_{-1}^{1} P(x)\,(1-x^2)^{1/2}\,dx.$$

By (1.2), it suffices to consider $\overline{B}(4, n)$ in each of the cases

$$(7.2)\qquad \begin{array}{ll} P(x) = A^2,\ P(x) = (1-x^2)\,B^2 & (n \text{ odd}\ ; \\ P(x) = (1 \pm x)\,C^2 & (n \text{ even}). \end{array}$$

Now $U_3(x) = 8x^3 - 4x$, so that, on using the method of § 6, we are in the case $l = 3$, and we have to solve the equation $\Delta_{m+1}/\Delta_0 = 0$, that is an equation of the form

$$(7.3)\qquad \frac{1}{(\alpha_1-\alpha_2)(\alpha_2-\alpha_3)(\alpha_3-\alpha_1)} \begin{vmatrix} p_{m+1}(\alpha_1) & p_{m+2}(\alpha_1) & p_{m+3}(\alpha_1) \\ p_{m+1}(\alpha_2) & p_{m+2}(\alpha_2) & p_{m+3}(\alpha_2) \\ p_{m+1}(\alpha_3) & p_{m+2}(\alpha_3) & p_{m+3}(\alpha_3) \end{vmatrix} = 0,$$

where the $\alpha_1, \alpha_2, \alpha_3$ are the roots of $U_3(x) - \varrho = 0$; $m = \frac{1}{2}(n-1), \frac{1}{2}(n-3), \frac{1}{2}(n-2)$, respectively; and the polynomials $p_k(x)$ are associated with the weights $(1-x^2)^{1/2}$, $(1-x^2)^{3/2}$, $(1 \pm x)(1-x^2)^{1/2}$, respectively.

If we denote the maximal ϱ by $U_3(\zeta)$ where $-1 \leqq \zeta \leqq 1$, then

$$(7.4)\qquad U_3(x) - \varrho = (x-\zeta)(8x^2 + 8x\zeta + 8\zeta^2 - 4),$$

so that $\alpha_1, \alpha_2, \alpha_3$ have the values

$$(7.5)\qquad \zeta,\ \frac{1}{2}\left(-\zeta \pm \sqrt{2-3\zeta^2}\right).$$

Inserting these values in (7.3) we obtain an equation in ζ.

2. The case b_5 is in some respect even simpler. Here we may assume n odd and $P(x) = Q(x^2)$ where $\widehat{Q} = n' = \frac{1}{2}(n-1)$ (compare § 5).

Since $U_4(x) = 16x^4 - 12x^2 + 1$ we have

$$(7.6)\qquad b_4 : b_1 = \int_0^1 Q(t)\,(16t^2 - 12t + 1)\,(1-t)^{1/2} t^{-1/2}\,dt : \int_0^1 Q(t)\,(1-t)^{1/2} t^{-1/2}\,dt.$$

Now putting $s = 2t - 1$, $Q(t)$ becomes a polynomial $Q^*(s)$ non-negative in $\langle -1, 1\rangle$. Applying the theorem of LUKÁCS, we find that we may restrict $Q(t)$ to the subclasses

$$(7.7)\qquad \begin{array}{lll} A^2(t), & t(1-t)\,B^2(t) & (n' \text{ even}); \\ tC^2(t), & (1-t)\,D^2(t) & (n' \text{ odd}). \end{array}$$

We are in the case $l = 2$, and the equations to be solved are, by (6.15) of the form

$$(7.8)\qquad \frac{1}{\alpha_1-\alpha_2} \begin{vmatrix} p_{m+1}(\alpha_1) & p_{m+2}(\alpha_1) \\ p_{m+1}(\alpha_2) & p_{m+2}(\alpha_2) \end{vmatrix} = 0,$$

where the $p_k(t)$ are associated with the weights $(1-t)^{1/2}t^{-1/2}$, $(1-t)^{3/2}t^{1/2}$, $(1-t)^{1/2}t^{1/2}$, $(1-t)^{3/2}t^{-1/2}$, respectively; and m being $\frac{1}{2}n'$, $\frac{1}{2}(n'-2)$, $\frac{1}{2}(n'-1)$, respectively. Also α_1 and α_2 are the roots of $16t^2-12t+1-\varrho$. Hence, putting $\varrho=16\tau^2-12\tau+1$, these roots are τ and $\frac{3}{4}-\tau$, so that (7. 8) becomes

$$\frac{1}{2\tau-\frac{3}{4}}\begin{vmatrix} p_{m+1}(\tau) & p_{m+2}(\tau) \\ p_{m+1}\left(\frac{3}{4}-\tau\right) & p_{m+2}\left(\frac{3}{4}-\tau\right) \end{vmatrix}=0, \tag{7. 9}$$

or $K_{m+1}\left(\tau, \frac{3}{4}-\tau\right)=0$ [compare (6. 15)].

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