Some prime-number consequences of the Ikehara theorem.

By NORBERT WIENER and LEONARD GELLERT in Cambridge, Mass.

SELBERG[1] and ERDŐS[2] have recently shown the Prime Number Theorem to be demonstrable by elementary methods[1]). The present paper is devoted, not to elementary proofs of this theorem, but to simple analytical considerations which may throw light on the reasons why such elementary, but not easy, proofs may be expected to function. Our fundamental tool is the Ikehara theorem[3]), to the effect that if for $\Re z > 1$,

$$\int_0^\infty u^{-z} dF(z) = \frac{a}{z-1} + G(z), \tag{1}$$

where $F(z)$ is monotone; and if over every finite range of x,

$$\lim_{x \to 1+0} G(x+iy) = H(y), \tag{2}$$

where $H(y) \in L$ over every finite range; then

$$\lim_{A\to\infty} \frac{1}{A} F(A) = a. \tag{3}$$

We shall consider the two Dirichlet series:

$$\int_0^\infty u^{-z} \log u \; d\bar{\omega}(u) = -\frac{\zeta'(z)}{\zeta(z)}, \tag{4}$$

and

$$-\int \frac{\zeta''(z)}{\zeta(z)} dz = -\frac{\zeta'(z)}{\zeta(z)} + \int \zeta'(z) d\left(\frac{1}{\zeta(z)}\right) = -\frac{\zeta'(z)}{\zeta(z)} - \int \left(\frac{\zeta'(z)}{\zeta(z)}\right)^2 dz =$$

$$= \int_0^\infty u^{-z} \log u \; d\bar{\omega}(u) - \int dz \left(\int_0^\infty u^{-z} \log u \; d\bar{\omega}(u)\right)^2 = \tag{5}$$

$$= \int_0^\infty u^{-z} \log u \; d\bar{\omega}(u) - \int dz \int_0^\infty u^{-z} \log u \; d\bar{\omega}(u) \int_0^\infty v^{-z} \log v \; d\bar{\omega}(v).$$

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1) A. SELBERG, An elementary proof of the prime number theorem, *Annals of Math.*, (2) **50** (1949), pp. 305—313.

2) P. ERDŐS, On a new method in the elementary theory of numbers which leads to an elementary proof of the prime number theorem, *Proceedings National Academy of Sciences U.S.A.*, **35** (1949), pp. 374—384.

3) N. WIENER, Tauberian theorems, *Annals of Math.*, (2) **33** (1933), pp. 1—100.

Now,

$$-\int dz \int_0^\infty u^{-z} \log u \, d\tilde{\omega}(u) \int_0^\infty v^{-z} \log v \, d\tilde{\omega}(v) =$$

$$= -\int dz \int_0^\infty \log u \, d\tilde{\omega}(u) \int_0^\infty w^{-z} \log \frac{w}{u} \, d_v \tilde{\omega}\left(\frac{w}{u}\right) =$$

(6)
$$= -\int dz \int_0^\infty \log u \, d\tilde{\omega}(u) \int_0^\infty w^{-z} d_w \int_0^{\frac{w}{u}} \log v \, d\tilde{\omega}(v) =$$

$$= -\int dz \int_0^\infty w^{-z} d_w \int_{u=0}^{u=\infty} \log u \, d\tilde{\omega}(u) \int_0^{\frac{w}{u}} \log v \, d\tilde{\omega}(v) =$$

$$= \text{const.} + \int_0^\infty w^{-z} \frac{1}{\log w} d_w \int_{u=0}^{u=\infty} \log u \, d\tilde{\omega}(u) \int_0^{\frac{w}{u}} \log v \, d\tilde{\omega}(v).$$

Hence,

(7)
$$-\int \frac{\zeta''(z)}{\zeta(z)} dz = \text{const.} + \int_0^\infty u^{-z} \log u \, d\tilde{\omega}(u) +$$

$$+ \int_0^\infty u^{-z} \frac{1}{\log u} d_u \int_{w=0}^{w=\infty} \log w \, d\tilde{\omega}(w) \int_0^{\frac{u}{v}} \log v \, d\tilde{\omega}(v).$$

Let us now consider $\zeta(z)$ on the line $z = 1 + iy$. It is well known that $\zeta(z)$ is analytic on this line, except at $z = 1$, when it is of the form $\varphi(z) + \frac{1}{z-1}$, where $\varphi(z)$ is analytic. It consequently has on the line only zeros of finite order. Now, if ε is real and positive,

(8)
$$\left| \frac{1}{\zeta(1 + iy + \varepsilon)} \right| \leq \left| \sum \frac{\mu(n)}{n^{1+iy+\varepsilon}} \right| \leq \sum \frac{1}{n^{1+\varepsilon}} \leq \zeta(1+\varepsilon) = O\left(\frac{1}{\varepsilon}\right);$$

so that the zeta function cannot have a pole of higher order than 1 on the 1-line. It follows that $-\frac{\zeta'(z)}{\zeta(z)}$ cannot have a singularity on the 1-line that is not a pole of order 1 with residue -1, except for the singularity at $z = 1$, which is a pole of order 1 with residue 1. Thus either $-\frac{\zeta'(z)}{\zeta(z)}$ has no other singularity on the 1-line than the pole at $z = 1$, in which case

(9)
$$\frac{1}{u} \int_0^u \log u \, d\tilde{\omega}(u) \to 1,$$

or it has a singularity at $1+i\lambda$. In the first case, we may write (9):

$$\frac{\log u\,\varpi(u)}{u}-\frac{1}{u}\int_0^u\frac{\varpi(u)\,du}{u}\to 1; \tag{10}$$

and since it is well known that $\varpi(u)=o(u)$, it follows that

$$\varpi(u)\sim\frac{u}{\log u}, \tag{11}$$

which is equivalent to the prime number theorem.

On the other hand, let

$$\zeta(1+i\lambda)=0. \tag{12}$$

Then

$$-\frac{\zeta'(z)}{\zeta(z)}-\frac{1}{2}\frac{\zeta'(z+i\lambda)}{\zeta(z+i\lambda)}-\frac{1}{2}\frac{\zeta'(z-i\lambda)}{\zeta(z-i\lambda)}= \tag{13}$$
$$=\int_0^\infty u^{-z}(1+\cos\lambda\log u)\log u\,d\varpi(u)$$

will be a Dirichlet series with non-negative coefficients with no singularity at $z=1$. It is easy to show that it can then have no singularity on $\Re z=1$. Thus by a very weak form of the Ikehara theorem,

$$\frac{1}{u}\int_0^u(1+\cos\lambda\log u)\log u\,d\varpi(u)\to 0. \tag{14}$$

This is to say that

$$\varpi(u)=\varpi_1(u)+\varpi_2(u), \tag{15}$$

where

$$\lim_{u\to\infty}\frac{1}{u}\int_0^u\log u\,d\varpi_1(u)=0, \tag{16}$$

and where $\varpi_2(u)$ only increases over the intervals

$$1+\cos\lambda\log u<\varepsilon, \tag{17}$$

or

$$|\lambda\log u-n\pi|<\varepsilon. \tag{18}$$

On the other hand, $-\int\frac{\zeta''(z)}{\zeta(z)}dz$ is a function which may be shown to behave like

$$\frac{\dfrac{2}{(z-1)^2}}{\dfrac{1}{z-1}}=\frac{2}{z-1} \tag{19}$$

at $z=1$, and to be analytic elsewhere in $x\geqq 1$, except for possible logarithmic singularities where $\zeta(z)=0$. These do not interfere with Lebesgue integrability.

Hence the Ikehara theorem applies, and

$$(20)\qquad \frac{1}{u}\int_0^u \log u\, d\tilde{\omega}(u) + \frac{1}{u}\int_0^u \frac{1}{\log u}\, d_u \int_{w=0}^{w=\infty} \log w\, d\tilde{\omega}(w) \int_0^{\frac{u}{w}} \log v\, d\tilde{\omega}(v) \to 2.$$

Hence

$$(21)\qquad \int_{|\lambda \log u - n\pi| < \varepsilon_1} \left(\log u\, d\tilde{\omega}(u) + \frac{1}{\log u}\, d_u \int_{w=0}^{w=\infty} \log w\, d\tilde{\omega}(w) \int_0^{\frac{u}{w}} \log v\, d\tilde{\omega}(v)\right) \sim = 2e^{\frac{\pi n}{\lambda}}\left(e^{\frac{\varepsilon_1}{\lambda}} - e^{-\frac{\varepsilon_1}{\lambda}}\right),$$

so that

$$(22)\qquad \int_{|\lambda \log u - n\pi| < \varepsilon_1} \log u\, d\tilde{\omega}(u) \leqq 2e^{\frac{n\pi}{\lambda}}\left(e^{\frac{\varepsilon_1}{\lambda}} - e^{-\frac{\varepsilon_1}{\lambda}}\right).$$

Combining this with (16), (17), and (18), we see that for large u's

$$(23)\qquad \frac{1}{u}\int_0^u \log u\, d\tilde{\omega}(u) \leqq \frac{1}{u}\left(e^{\frac{\varepsilon_1}{\lambda}} - e^{-\frac{\varepsilon_1}{\lambda}}\right)\frac{2ue^{\varepsilon_1 \pi}}{1 - e^{-\pi/\lambda}},$$

which is asymptotically not greater than

$$(24)\qquad \frac{2\left(e^{\frac{\varepsilon_1}{\pi}} - e^{-\frac{\varepsilon_1}{\pi}}\right)}{1 - e^{-\pi/\lambda}}.$$

Since ε_1 is arbitrary, we see that

$$(25)\qquad \lim_{u\to\infty} \frac{1}{u}\int_0^u \log u\, d\tilde{\omega}(u) = 0.$$

This is however inconsistent with the known elementary Chebychev theorem, to the effect that

$$(26)\qquad \underline{\lim}_{u\to\infty} \frac{1}{u}\int_0^u \log u\, d\tilde{\omega}(u) > 0,$$

so that we have succeeded in eliminating the hypothesis that there is a λ for which $\zeta(1 + i\lambda) = 0$, and have proved the prime number theorem.

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