On the discreteness of the spectrum of a differential equation.

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It was proved by WEYL[1]) that the spectrum associated with the differential equation

$$\frac{d^2\varphi}{dx^2}+\{\lambda-q(x)\}\varphi=0 \qquad (0\leqq x<\infty) \tag{1}$$

is discrete if $q(x)$ is bounded in any finite interval and tends to infinity as $x\to\infty$. His proof is reproduced in my book *Eigenfunction Expansions Associated with Second Order Differential Equations* (Oxford, 1946), § 5. 12. Other proofs have since been given[2]).

The following is another simple proof. Let $\varphi(x,\lambda)$ be the solution of (1) which satisfies a given boundary condition

$$\varphi(0,\lambda)\cos\alpha+\varphi_x(0,\lambda)\sin\alpha=0 \tag{2}$$

at $x=0$. Then it can be proved as in § 5. 12 of my book that, for every real λ, either $\varphi(x,\lambda)$ is L^2 (in which case $\varphi(x,\lambda)$ and $\varphi_x(x,\lambda)$ both tend to zero as $x\to\infty$), or $\varphi(x,\lambda)\to\infty$, or $\varphi(x,\lambda)\to-\infty$. Consider any finite interval $a\leqq\lambda\leqq b$, and denote the sub-sets of this interval where φ has the above properties by E_0, E_1, and E_2 respectively.

If λ' belongs to E_1, $\varphi(x,\lambda')\to\infty$, $\varphi_{xx}(x,\lambda')\to\infty$ (by (1)), and so $\varphi_x(x,\lambda')\to\infty$. Hence for some ξ, with $q(\xi)>\lambda'$, $\varphi(\xi,\lambda')>0$ and $\varphi_x(\xi,\lambda')>0$. Hence also $\varphi(\xi,\lambda)>0$ and $\varphi_x(\xi,\lambda)>0$ if $\lambda-\lambda'$ is sufficiently small. This, however, implies that $\varphi(x,\lambda)\to\infty$. Hence any point of E_1 is an interior point of an interval of points of E_1, and so E_1 is an open set. Similarly E_2 is an open set. Hence E_0 is a closed set.

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1) H. WEYL, Über gewöhnliche Differentialgleichungen mit Singularitäten und die zugehörigen Entwicklungen willkürlicher Funktionen, *Math. Annalen*, **68** (1910), pp. 220–269.

2) K. FRIEDRICHS, Spektraltheorie halbbeschränkter Operatoren und Anwendung auf die Spektralzerlegung von Differentialoperatoren. II., *Math. Annalen*, **109** (1934), pp. 685–713. Criteria for the discrete character of the spectra of ordinary differential operators, *Studies and Essays presented to R. Courant* (New York, 1948), pp. 145–160; a proof by E. C. TITCHMARSH will be published in the *Annali di Matematica*.

The above argument also shows that, if λ' is a point of E_1, then, for some ξ, $\varphi(\xi, \lambda) \geqq m > 0$, while $\varphi(x, \lambda)$ is steadily increasing for $x \geqq \xi$, if λ is in some interval $|\lambda - \lambda'| \leqq \eta$. Hence $\varphi(x, \lambda) \geqq m$ for $x \geqq \xi$, $|\lambda - \lambda'| \leqq \eta$. It follows as on p. 116 of my book that the function $k(\lambda)$ is constant throughout the interval $|\lambda - \lambda'| < \eta$, and so in fact is constant throughout each interval of E_1.

To prove the theorem, we have now to show that E_0 consists at most of a finite number of points.

Suppose on the contrary that there is a sequence of values of λ tending to (but different from) a limit μ, such that these λ's and μ all belong to E_0. Let x_1 be such that $q(x) \geqq \mu + \delta$ $(\delta > 0)$ for $x \geqq x_1$ (such an x_1 exists if $q(x) \to \infty$). Let $x_2 > x_1$ be such that $\varphi(x_2, \mu) \neq 0$, and suppose e.g. that $\varphi(x_2, \mu) > 0$. As in § 5.12 of my book, this implies (since $\varphi(x, \mu)$ is L^2) that $\varphi(x, \mu)$ decreases steadily to zero for $x \geqq x_2$, and in particular that $\varphi(x, \mu) > 0$ for $x \geqq x_2$.

Now $\varphi(x, \lambda)$ is a continuous function of both variables in any finite region (cf. § 1.5 of my book), and so $\varphi(x, \lambda) \to \varphi(x, \mu)$ as $\lambda \to \mu$, uniformly over $0 \leqq x \leqq x_2$. Hence

$$\lim_{\lambda \to \mu} \int_0^{x_2} \varphi(x, \lambda)\, \varphi(x, \mu)\, dx = \int_0^{x_2} \{\varphi(x, \mu)\}^2\, dx > 0,$$

and so

$$\int_0^{x_2} \varphi(x, \lambda)\, \varphi(x, \mu)\, dx > 0$$

if λ is sufficiently near to μ.

Also

$$\varphi(x_2, \lambda) \to \varphi(x_2, \mu) > 0$$

and so $\varphi(x_2, \lambda) > 0$ if λ is sufficiently near to μ. Since $q(x) - \lambda > 0$ if $x \geqq x_1$ and λ is sufficiently near to μ, this implies, as before, that $\varphi(x, \lambda) > 0$ for $x \geqq x_2$ and λ sufficiently near to μ. Hence

$$\int_{x_2}^{\infty} \varphi(x, \lambda)\, \varphi(x, \mu)\, dx > 0$$

if λ is sufficiently near to μ. Altogether

$$\int_0^{\infty} \varphi(x, \lambda)\, \varphi(x, \mu)\, dx > 0 \tag{3}$$

if λ is sufficiently near to μ.

This, however, is impossible; for on multiplying (1), and the corresponding equation with μ, by $\varphi(x, \mu)$, $\varphi(x, \lambda)$, respectively and subtracting, we obtain

$$(\lambda-\mu)\,\varphi(x,\lambda)\,\varphi(x,\mu)=\frac{\partial}{\partial x}\{\varphi(x,\lambda)\,\varphi_x(x,\mu)-\varphi(x,\mu)\,\varphi_x(x,\lambda)\}.$$

Using (2) it follows that

$$(\lambda-\mu)\int_0^X\varphi(x,\lambda)\,\varphi(x,\mu)\,dx=\varphi(X,\lambda)\,\varphi_x(X,\mu)-\varphi(X,\mu)\,\varphi_x(X,\lambda),$$

which tends to 0 as $X\to\infty$. Since $\lambda\neq\mu$ it follows that

(4) $$\int_0^\infty\varphi(x,\lambda)\,\varphi(x,\mu)\,dx=0.$$

This gives a contradiction, and the theorem follows.

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