Lattice points and Fourier expansions.

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1. Introduction.

We have recently given [4][1]) a new line of reasoning for proving Hardy's identity [8] in the theory of lattice points in a circle, and for the related convergence theorems of Hardy, Landau [8, 9], Walfisz [12, 13], Oppenheim [11], Wilton [15, 16], Dixon and Ferrar [7]. We employed a general summability-theorem, due to Bochner [3, Th. 1], for partial derivatives of multiple Fourier series, and we combined it with a theorem of Ananda-Rau [1] on scales of Riesz summability for general Dirichlet series in which assumptions on the magnitude of the coefficients are made explicitly.

In the present paper we will throw the part due to Ananda-Rau into the differentiability-theorem itself, thus obtaining a much broader theorem on multiple Fourier series in general, from which to deduce the particular lattice-point conclusions by much shorter steps. Actually in § 3 we will first have a relatively simple version of the general differentiability theorem sufficient for the lattice-point conclusions envisaged, and afterwards, in § 5 and § 6, we will enlarge on the differentiability theorem for its own sake. This will bring out its similarity with a criterion of Chandrasekharan and Minakshisundaram [6, Th. 4.1] which was the first attempt towards extending, from one to several variables, a convergence-test for Fourier series due to Hardy and Littlewood [10] in which the order of magnitude of the Fourier coefficients is prescribed; and it will also throw further light on the entire problem of localization of convergence and summability for Fourier series in general [2]; the latter problem is more delicate for multiple series than for simple series, and rather more delicate for formal (partial) derivatives of a series than for the original series proper, and the present paper may also be viewed as a further contribution towards managing this problem in some of its aspects.

1) Numbers in brackets [ ] refer to the bibliography placed at the end of the paper.

2. Notations and Definitions.

Let $f(x)=f(x_1,\ldots,x_k)$ be periodic in each variable with period 2π, and Lebesgue integrable in (x). It has then a Fourier expansion which we indicate by writing

$$f(x_1,\ldots,x_k)\sim\sum\ldots\sum a_{n_1\ldots n_k}e^{i(n_1x_1+\ldots+n_kx_k)}.$$

Let

$$A_n(x)=\sum_{n_1^2+\ldots+n_k^2=n} a_{n_1\ldots n_k}e^{i(n_1x_1+\ldots+n_kx_k)}$$

and

$$S_n(x)=\sum_0^n A_r(x).$$

Define for $l>0$

$$S^l(x_1,\ldots,x_k;R)\equiv S^l(x;R)\equiv S^l(R)=$$

$$=\sum_{n_1^2+\ldots+n_k^2\leqq R^2}\{R^2-(n_1^2+\ldots+n_k^2)\}^l a_{n_1\ldots n_k}e^{i(n_1x_1+\ldots+n_kx_k)}=$$

$$=\sum_{r=0}^n(R^2-r)^lA_r(x)=2l\int_0^R(R^2-u^2)^{l-1}S(u)\,u\,du,$$

where

$$n=[R]\quad\text{and}\quad S(R)=S^0(R)=S_n(x).$$

Let

$$T^l(R)=S^l(R)R^{-2l}.$$

Define

$$f(x,t)=\frac{\Gamma(k/2)}{2(\pi)^{k/2}}\int_\sigma f(x_1+t\xi_1,\ldots,x_k+t\xi_k)\,d\sigma_\xi$$

where σ denotes the unit-sphere $\xi_1^2+\ldots+\xi_k^2=1$ and $d\sigma_\xi$ its $(k-1)$-dimensional volume-element.

3. A Convergence Theorem.

We shall first state a few lemmas, which are needed for the proof of our theorem.

Lemma 1. *Suppose that*

$$\frac{a_n}{l_n-l_{n-1}}=O(l_n^\alpha),$$

where $\{l_n\}$ *is a strictly increasing sequence of positive numbers diverging to* ∞, *and suppose that* $\sum a_nl_n^{-\gamma}$ *is summable by Riesz's means of type* l_n *and of order* r, *briefly: summable* (l,r), γ *being real. Let* $0\leqq s<r$. *Then* $\sum a_nl_n^{-\sigma}$ *is summable* (l,s) *for*

$$\sigma>\frac{(\alpha+1)(r-s)+\gamma(s+1)}{r+1}.$$

This has been proved by ANANDA-RAU [1, Th. 7], and if we choose $l_n = n$, $\gamma = 0$, it reduces to the following

Lemma 1 A. *If $\sum a_n$ is summable (n, r) and $a_n = O(n^\alpha)$ and $0 \leqq s < r$, then $\sum a_n n^{-\sigma}$ is summable (n, s) for*

$$\sigma > \frac{(\alpha+1)(r-s)}{r+1}.$$

Lemma 2. *If $f(x) = f(x_1, \ldots, x_k)$ is a periodic function of class L, or an almost periodic function of Stepanoff class, and*

$$f(x) \sim \sum_n a(n)\, e^{i\Lambda(n,x)} \tag{3.1}$$

where $\Lambda(n, x)$ denotes $n_1 x_1 + \ldots + n_k x_k$, and $a(n) = a(n_1, \ldots, n_k)$ is the Fourier coefficient, and $D^q(n_1, \ldots, n_k)$ is, for any non-negative integer q, a homogeneous polynomial of total degree q in $n_1, \ldots, n_k$, then

(i) *the operator*

$$D_x^q = D^q\left(\frac{\partial}{i\partial x_1}, \ldots, \frac{\partial}{i\partial x_k}\right)$$

applies to the almost periodic function

$$T_R^\delta(x) = \sum_{|n| \leqq R} \left(1 - \frac{|n|^2}{R^2}\right)^\delta a(n)\, e^{i\Lambda(n,x)}$$

and the resulting function is almost periodic;

(ii) $$D_x^q T_R^\delta(x) = \sum_{|n| \leqq R} \left(1 - \frac{|n|^2}{R^2}\right)^\delta a(n)\, D^q(n) e^{i\Lambda(n,x)};$$

(iii) *for every x at which the condition*

$$\int_0^t |f_x(t)|\, t^{k-1-q}\, dt = o(t^k)$$

is satisfied, we have

$$\lim_{R \to \infty} D_x^q T_R^\delta(x) = 0 \tag{3.2}$$

for $\delta > \frac{k-1}{2} + q$.

This has been proved by BOCHNER [2, Th. I].

Lemma 3. *If $k \geqq 1$, $0 \leqq n < \infty$, and if the numbers $a_{n_1 \ldots n_k}$ are arbitrarily given for $0 \leqq n_1 \leqq n, \ldots, 0 \leqq n_k \leqq n$, then there exists an exponential polynomial*

$$P(x_1, \ldots, x_k) = \sum_{\lambda_1=0}^{n} \cdots \sum_{\lambda_k=0}^{n} \gamma_{\lambda_1 \ldots \lambda_k} e^{i(\lambda_1 x_1 + \ldots + \lambda_k x_k)}$$

such that at the origin

$$\left(\frac{\partial^{n_1+\cdots+n_k} P}{\partial x_1^{n_1}\ldots\partial x_k^{n_k}}\right)_{x=(0)} = a_{n_1\ldots n_k}.$$

Proof. Obviously, if arbitrary numbers $b_{n_1\ldots n_k}$, $0 \leqq n_j \leqq n$, $(j=1,\ldots,k)$ are prescribed, then there exists an (ordinary) polynomial

$$Q(z_1,\ldots,z_k) = \sum_{\mu_1=0}^{n} \cdots \sum_{\mu_k=0}^{n} \delta_{\mu_1\ldots\mu_k} z_1^{\mu_1}\ldots z_k^{\mu_k}$$

such that

$$\left(\frac{\partial^{n_1+\cdots+n_k} Q}{\partial z_1^{n_1}\ldots\partial z_k^{n_k}}\right)_{z=0} = b_{n_1\ldots n_k},$$

namely, $b_{n_1\ldots n_k} = n_1!\ldots n_k!\,\delta_{n_1\ldots n_k}$. Now consider the transformation of variables

$$z_1 = e^{ix_1}-1,\ldots, z_k = e^{ix_k}-1.$$

Obviously it transforms a $P(x)$ into a $Q(z)$ and conversely, under preservation of n, and for prescribed values $a_{n_1\ldots n_k}$ this leads to values $b_{n_1\ldots n_k}$ by ordinary rules of differentiation of a function of functions, and inversely from the b's to the a's, and hence the lemma.

Lemma 4. *If $f(x)$ is a periodic or almost periodic function* (3. 1), *and if in a neighborhood of the point $x=x_0$ the function has continuous derivatives of total order $\leqq q$, then at $x=x_0$ we have for $\delta > \frac{k-1}{2}+q$:*

$$\lim_{R\to\infty} [D_x^q T_R^\delta(x) - D_x^q f(x)] = 0.$$

Proof. The conclusion is obviously trivial for an *exponential* polynomial $P(x)$. In general we put, by lemma 3, $f(x)=P(x)+f^1(x)$ where for $f^1(x)$ all partial derivatives of total order $\leqq q$ are zero at the point $x=x_0$. But $f^1(x)$ has also continuous derivatives of order q in the neighborhood of x_0. From this it follows easily that $f^1(x)$ satisfies assumption (iii) of lemma 2 and hence (3. 2) follows.

Remark. The "modification" referred to in lemma 6 of our previous paper [4, p. 241] is made explicit now; even there, differentiability has to be assumed in a neighborhood of the point in question.

Theorem 1. *If $f(x)$ is defined as in* § 2, *and if*

(3. 3) $$A_n = O(n^\alpha)$$

then at every point x in a neighborhood of which $f(x)$ possesses partial derivatives of all orders, the series $\sum A_n n^h$ is summable (n,δ) for $\delta \geqq 0$, and $\delta > 2\alpha+1+2h$.

Proof. By lemma 4, if we choose D as the Laplace operator

$$\left(\frac{\partial^2 f}{\partial x_1^2}+\ldots+\frac{\partial^2 f}{\partial x_k^2}\right)$$

and apply it q times to the function f, where q is a non-negative integer, we obtain that $\sum A_n n^q$ is summable (n, δ) for $\delta > \frac{k-1}{2}+2q$. Since $A_n n^q = O(n^{\alpha+q})$, it follows by lemma 1 A, that $\sum A_n n^h$ is summable (n, η) for $\eta \geqq 0$, and

$$q-h > \frac{(\alpha+q+1)(\delta-\eta)}{\delta+1}$$

or

$$\eta > \delta - \frac{(q-h)(\delta+1)}{\alpha+q+1}.$$

Since δ may be any number greater than $\frac{k-1}{2}+2q$, this implies that any

$$\eta > \frac{\left(\frac{k-1}{2}\right)(\alpha+1+h)+h+2q\left(\alpha+\frac{1}{2}+h\right)}{\alpha+q+1}$$

is admissible. Given k, α, h since q may be chosen as large as we please, the theorem will be true for $\eta > 2\alpha+1+2h$.

Remarks. It should be noticed that there is no restriction on α. However, if

$$a_{n_1 \ldots n_k} = O\left(\frac{1}{(n_1^2+\ldots+n_k^2)^{k/2}}\right) \tag{3. 4}$$

then at every point of mean-continuity (etc.) we have convergence of $\sum A_n$. See [6, p. 741]. The significance of the theorem is that even though only something less than (3. 4) is satisfied, a stronger hypothesis on the function than continuity will still lead to summability, and, in special cases, to convergence. We will show in the next section how the above theorem is entirely adequate to obtain the most complete results on the summation of certain series of Bessel functions occurring in the theory of lattice points.

4. Application to summations over lattice points.

Let $r_k(n) = \sum_{n_1^2+\ldots+n_k^2=n} 1$ for integral values of n_k, representation of n which differ only in sign or order being counted as distinct. Let

$$R_k(x) = \sum_{n \leqq x}' r_k(n)$$

the last term $r_k(x)$ in the sum being replaced by $\frac{1}{2} r_k(x)$ if x is an integer.

Then it is known that $R_k(x)$ can be "represented" as a series of Bessel functions; in particular, if $k=2$, we have HARDY's identity [8]: if x is non-integral,

$$R_2(x)=\pi x - x^{1/2} \sum \frac{r_2(n) J_1(2\pi \sqrt{nx})}{n^{1/2}}. \tag{4.1}$$

Here J_1 stands for the Bessel function of order 1. When $k>2$, the expansion corresponding to the right of (4.1) is no longer convergent, but can be summed by RIESZ's means. WALFISZ has proved that the corresponding series in k-dimensions, namely

$$\sum \frac{r_k(n) J_{k/2}(2\pi \sqrt{nx})}{n^{k/4}} \tag{4.2}$$

is summable (n,δ) for $\delta > \frac{k-3}{2}$, and *not* summable for $\delta = \frac{k-3}{2}$. More complete results of this type were obtained by DIXON and FERRAR [7] and in a recent paper we obtained the following result [4, p. 248]: if

$$r_k(n,h) = \sum_{n_1^2+\dots+n_k^2=n} e^{2\pi i (n_1 h_1 + \dots + n_k h_k)},$$

then

$$\sum r_k(n,h) J_\mu(2\pi\xi\sqrt{n})\, n^l \tag{4.3}$$

is summable (n,η) for $\eta \geqq 0$ and $l < \frac{3}{4} - \frac{k}{2} + \frac{\eta}{2}$, $\mu > -1$ whenever ξ^2 is non-integral. (4.3) not only yields WALFISZ's result when $h_1 = \dots = h_k = 0$, but is actually sharper since μ does not depend on k. We will now show that a result which includes WALFISZ's and HARDY's, can be deduced as a direct consequence of theorem 1.

Corollary to theorem 1. *If ξ^2 is non-integral,*

$$\sum r_k(n) J_{k/2+\beta}(2\pi\xi\sqrt{n})\, n^l \tag{4.4}$$

is summable (n,η) for $\eta \geqq 0$ and $l < \frac{3}{4} - \frac{k}{2} + \frac{\eta}{2}$, $\beta > -1$.

Proof. It is known that the series

$$A + B \sum \frac{r_k(n) J_{k/2+\beta}(2\pi\xi\sqrt{n})}{n^{k/4+\beta/2}} \tag{4.5}$$

for suitable constants A and B, is the (spherical) multiple Fourier series of the function

$$f(x_1,\dots,x_k) = \sum [\xi^2 - \{(n_1^2+x_1^2)+\dots+(n_k^2+x_k^2)\}]^\beta \tag{4.6}$$

at the origin $x=(0,\dots,0)$, for $\beta > -1$. [4, p. 243 (3.4)]. If ξ^2 is non-integral, the function given in (4.6) is infinitely differentiable in a neighborhood of the origin, and the terms of its Fourier series (4.5) satisfy the condition (in our notation of § 2)

$$(4.7) \qquad A_n = O\left(n^{\frac{k-2}{2}+\varepsilon-\frac{k}{4}-\frac{\beta}{2}-\frac{1}{4}}\right) = O\left(n^{\frac{k-5}{4}-\frac{\beta}{2}+\varepsilon}\right)$$

since $J_\mu(x) = O(x^{-1/2})$ as $x \to \infty$, $\mu > -1$ and $r_k(n) = O\left(n^{\frac{k-2}{2}+\varepsilon}\right)$. Hence we apply theorem 1, and deduce that

$$\sum \frac{r_k(n)\, J_{k/2+\beta}(2\pi\xi\sqrt{n})}{n^{k/4+\beta/2}}\, n^p$$

is summable (n, η) for $\eta > \frac{k-3}{2} - \beta + 2p$. Setting $l = p - \frac{k}{4} - \frac{\beta}{2}$ we obtain that

$$\sum r_k(n)\, J_{k/2+\beta}(2\pi\xi\sqrt{n})\, n^l$$

is summable (n, η) for $\eta > 2l + k - \frac{3}{2}$ which is the required result.

Remark. The corollary will still hold when the order of the Bessel function in (4.4) is not necessarily $\frac{k}{2} + \beta$ but any $\mu > -1$; in order to see that, we have only to refer to the reasoning given in our previous paper [4, p. 246], which closely follows that of DIXON and FERRAR [7].

5. An improvement on theorem 1.

Theorem 1 was concerned with the case when the function $f(x)$ was infinitely differentiable in a neighborhood of a given point; we shall now prove a similar result in the case where the function has partial derivatives upto an assigned order which is finite; if this order exceeds the number $\frac{1}{4}(k-1)$, (where k is the dimension-number) then we already can reach the conclusion of theorem 1, without having to assume infinite differentiability. We shall however have a restriction on $\{a_{n_1 \ldots n_k}\}$ instead of on A_n. For the proof of the theorem we need the following

Lemma 5. *For given $\varepsilon > 0$ and $\varepsilon \leqq x \leqq 2\varepsilon$, let $\psi(x)$ be a function defined in the following way:*

(i) $\psi(\varepsilon) = 1$, $\psi(2\varepsilon) = 0$;

(ii) *$\psi(x)$ possesses derivatives of all orders in $\varepsilon \leqq x \leqq 2\varepsilon$;*

(iii) $\left(\frac{d^r\psi}{dx^r}\right)_{x=\varepsilon} = 0$, $\left(\frac{d^r\psi}{dx^r}\right)_{x=2\varepsilon} = 0$, *for* $r = 1, 2, 3, \ldots$.

Let $g(y)$ be defined in the following way:

(iv) $g(y) = 1$ *for* $|y| \leqq \varepsilon$;

(v) $g(y) = \psi(y)$ *for* $\varepsilon \leqq |y| \leqq 2\varepsilon$;

(vi) $g(y) = 0$ *for* $2\varepsilon \leqq |x| \leqq \pi$;

(vii) $g(y + 2\pi) = g(y)$.

Let $g(x_1, \ldots, x_k) = \prod_{r=1}^{k} g(x_r)$ *and let the Fourier expansion of* $g(x_1, \ldots, x_k)$ *be*

$$g(x_1, \ldots, x_k) \sim \sum_{-\infty}^{\infty} \cdots \sum b_{n_1 \ldots n_k} e^{i(n_1 x_1 + \ldots + n_k x_k)}.$$

Then for every $\beta > 0$ *we have:*

$$b_{n_1 \ldots n_k} = O\left(\frac{1}{(n_1^2 + \ldots + n_k^2)^\beta}\right). \tag{5.1}$$

Now let $f(x_1, \ldots, x_k)$ *be any periodic function having the period* 2π *in each variable and Lebesgue integrable, and let*

$$a_{n_1 \ldots n_k} = O\left(\frac{1}{(n_1^2 + \ldots + n_k^2)^\alpha}\right) \tag{5.2}$$

where $\{a_{n_1 \ldots n_k}\}$ *are the Fourier coefficients of* f. *If* $\{c_{n_1 \ldots n_k}\}$ *are the Fourier coefficients of the product* $f.g$, *then*

$$c_{n_1 \ldots n_k} = O\left\{\frac{1}{(n_1^2 + \ldots + n_k^2)^\alpha}\right\}. \tag{5.3}$$

Proof. An explicit example of a function ψ satisfying our requirements is found in Wiener [14, p. 562], where the interval $(0, 1)$ is considered instead of $(\varepsilon, 2\varepsilon)$.

Since $g(x_r)$ is infinitely differentiable, it follows by a well-known result in Fourier series that its Fourier coefficient

$$a_n^{(r)} = O\left(\frac{1}{n^{\beta_1}}\right)$$

for every $\beta_1 > 0$; from this it follows that (5.1) is satisfied for every $\beta > 0$. Further, we have

$$c_{n_1 \ldots n_k} = \sum_{(m)=-\infty}^{\infty} \cdots \sum b_{m_1 \ldots m_k} a_{n_1 - m_1, \ldots, n_k - m_k}.$$

Hence

$$|c_{n_1 \ldots n_k}| = O\left(\sum \cdots \sum [\{1 + (n_1 - m_1)^2 + \ldots + (n_k - m_k)^2\}^{-\alpha} (1 + m_1^2 + \ldots + m_k^2)^{-\beta}]\right)$$

$$= O\left(\int_{-\infty}^{\infty} \cdots \int \{1 + (n_1 - \xi_1)^2 + \ldots + (n_k - \xi_k)^2\}^{-\alpha} \{1 + \xi_1^2 + \ldots + \xi_k^2\}^{-\beta} d\xi_1 \ldots d\xi_k\right).$$

If we subject the above integrand to an orthogonal transformation

$$\eta_r = \sum_s d_{rs} \xi_s$$

with determinant $+1$, and

$$d_{11} : d_{12} : \ldots : d_{1k} = n_1 : n_2 : \ldots : n_k,$$

then we have

$$\Sigma \xi_r^2 = \Sigma \eta_r^2, \ \Sigma n_r \xi_r = \sqrt{\Sigma n_r^2}\, \eta_1.$$

Hence

$$|c_{n_1 \dots n_k}| = O\left(\int \dots \int [\{1 + (\sqrt{\Sigma n_r^2} - \eta_1)^2 + \dots + \eta_k^2\}^{-\alpha} \{1 + \eta_1^2 + \dots + \eta_k^2\}^{-\beta}]\, d\eta_1 \dots d\eta_k\right)$$
$$= O\left(\int \dots \int \{1 + (x - \eta_1)^2 + \dots + \eta_k^2\}^{-\alpha} \{1 + \eta_1^2 + \dots + \eta_k^2\}^{-\beta}\, d\eta_1 \dots d\eta_k\right)$$

where $x = \sqrt{\Sigma n_r^2}$. Setting $|\eta| = \sqrt{\Sigma \eta_r^2}$, we have

$$|c_{n_1 \dots n_k}| = O\Big(\int_{|\eta| \leqq x/2} \dots \int + \int_{|\eta| > x/2} \dots \int\Big)$$

where

$$\int_{|\eta| \leqq x/2} \dots \int = O\left(\left\{1 + \left(\frac{x}{2}\right)^2\right\}^{-\alpha}\right) \int \dots \int \frac{d\eta_1 \dots d\eta_k}{(1 + \Sigma \eta_r^2)^\beta} = O(x^{-2\alpha}) = O[(\Sigma n_r^2)^{-\alpha}],$$

since β may be assumed large. Again,

$$\int_{|\eta| > x/2} \dots \int = O\Big(\int_{|\eta| > x/2} \dots \int \{1 + \Sigma \eta_r^2\}^{-\beta} d\eta_1 \dots d\eta_k\Big) =$$
$$= O\Big(\int_{t > x/2} t^{k-1} (1 + t^2)^{-\beta} dt\Big) = O(x^{k-2\beta}) = O(x^{-2\alpha}),$$

if we choose $\beta = \alpha + \frac{k}{2}$, and hence the lemma.

Theorem 2. *If $f(x)$ which is defined as in § 2 has continuous derivatives of total order $\leqq 2q$, where q is a non-negative integer, at a point x, and if*

$$a_{n_1 \dots n_k} = O((n_1^2 + \dots + n_k^2)^\beta),$$

then at that point, the series ΣA_n is summable (n, δ), $\delta \geqq 0$ and $\delta = \max(\eta, \gamma)$ where

$$\eta > \frac{\left(\frac{k-1}{2}\right)\left(\beta + \frac{k}{2}\right) - q}{\beta + \frac{k}{2} + q}$$

and $\gamma > 2\beta + k - 1$; in particular, if $q \geqq \frac{k-1}{4}$, then it is summable (n, δ) for any $\delta > 2\beta + k - 1$, $\delta \geqq 0$.

Proof. Without loss of generality we can assume the point in question to be the origin. We write the function $f(x)$ as follows:

$$f(x_1, \dots, x_k) = f(x_1, \dots, x_k)\, g(x_1, \dots, x_k) + [1 - g(x_1, \dots, x_k)]\, f(x_1, \dots, x_k)$$
$$= \varphi_1(x_1, \dots, x_k) + \varphi_2(x_1, \dots, x_k), \text{ say},$$

where $g(x_1, \dots, x_k)$ is defined as in lemma 5. It follows from that lemma that $\varphi_2(x)$ is infinitely differentiable in a neighborhood of the origin (since it vanishes there), while $\varphi_1(x)$ is continuously differentiable $2q$ times everywhere. If we now write

$$\varphi_1 \sim \sum c_{n_1 \ldots n_k} e^{i(n_1 x_1 + \ldots + n_k x_k)}$$

and

$$\varphi_2 \sim \sum d_{n_1 \ldots n_k} e^{i(n_1 x_1 + \ldots + n_k x_k)}$$

by the same lemma 5, we have

$$\left.\begin{matrix} c_{n_1 \ldots n_k} \\ d_{n_1 \ldots n_k} \end{matrix}\right\} = O\left((n_1^2 + \ldots + n_k^2)^\beta\right).$$

Theorem 1 is now applicable to φ_2, and so it follows that *its* Fourier expansion (summed spherically) is summable (n, γ) for

$$\gamma > 2\left(\beta + \frac{k-2}{2}\right) + 1. \tag{5.4}$$

For φ_1 we proceed as follows. If we write

$$C_n = \sum_{n_1^2 + \ldots + n_k^2 = n} c_{n_1 \ldots n_k} e^{i(n_1 x_1 + \ldots + n_k x_k)}$$

and apply the Laplace-operator q times, then owing to the continuity of the derivatives, it follows that $\Sigma C_n n^q$ is summable (n, δ) for $\delta > \frac{k-1}{2}$, at the origin. [2, Th. VI.] Hence it follows, as in the proof of theorem 1, that ΣC_n is summable (n, η) for

$$\eta > \delta - \frac{q(\delta+1)}{\left(\beta + \frac{k-2}{2}\right) + q + 1}$$

or,

$$\eta > \frac{\left(\frac{k-1}{2}\right)\left(\beta + \frac{k}{2}\right) - q}{\beta + \frac{k}{2} + q}. \tag{5.5}$$

The first part of our theorem results from (5.4) and (5.5). If $2\beta + k - 1 < 0$ and $\dfrac{\left(\frac{k-1}{2}\right)\left(\beta + \frac{k}{2}\right) - q}{\beta + \frac{k}{2} + q} < 0$, then $\eta = \gamma = 0$ so that $\delta = 0$.

In order to prove the second part we note that summability (n, δ) of ΣA_n for *some* $\delta > 2\beta + k - 1$ could only fail if

$$\frac{\left(\frac{k-1}{2}\right)\left(\beta + \frac{k}{2}\right) - q}{\beta + \frac{k}{2} + q} > 2\beta + k - 1. \tag{5.6}$$

If we set

$$\beta + \frac{k-2}{2} = \alpha, \quad \frac{k-1}{2} = r \tag{5.7}$$

we see that (5. 6) is equivalent to

(5. 8) $$r(\alpha+1)>(\alpha+1)(2\alpha+1+2q).$$

Let us now discuss the following cases separately: (i) $\alpha+1=0$, (ii) $\alpha+1<0$, (iii) $0<\alpha+1<\frac{1}{2}$, (iv) $\alpha+1\geqq\frac{1}{2}$:

If $\alpha+1=0$ then the strict inequality in (5. 8) is impossible, and hence our theorem is proved in this case. If $\alpha+1<0$ then $-\beta>\frac{k}{2}$ so that ΣA_n converges absolutely, and our theorem is true trivially in this case. If $0<\alpha+1<\frac{1}{2}$ then we have $2\beta+k-1=2\alpha+1<0$; and since $\alpha+1+q>0$, and $r(\alpha+1)<q$ if $q\geqq\frac{k-1}{4}$, we also have

$$\frac{\left(\frac{k-1}{2}\right)\left(\beta+\frac{k}{2}\right)-q}{\beta+\frac{k}{2}+q}=\frac{r(\alpha+1)-q}{\alpha+1+q}<0.$$

Hence in this case $\eta=\gamma=0$, provided that $q\geqq\frac{k-1}{4}$, and we have convergence of ΣA_n, so the theorem is true. Finally if $\alpha+1\geqq\frac{1}{2}$ then $2\alpha+1\geqq0$: and if $q\geqq\frac{k-1}{4}$ then $r\leqq2q$; so that we have $(2\alpha+1+2q)\geqq\frac{k-1}{2}$ which contradicts (5. 8); hence in this case also the theorem is proved.

6. Another convergence theorem.

We shall now establish a theorem more general than that of CHANDRASEKHARAN and MINAKSHISUNDARAM [6, Th. 4. 1]. We need the following lemmas.

Lemma 6. *Let $W(x)$ be a positive non-decreasing function of x, and $V(x)$ any positive function of x, both defined for $x>0$. Let $A(t)$ be a function of bounded variation in every finite interval, and*

$$A^k(t)=k\int_0^t(t-u)^{k-1}A(u)\,du,\ \ k>0.$$

Then

$$A(x+t)-A(x)=O[t^\gamma V(x)];\quad \gamma>0,\ t>0$$

and

$$A^k(x)=o(W)$$

together imply

$$A(x)=o\left(V^{\frac{k}{k+\gamma}}W^{\frac{\gamma}{k+\gamma}}\right).$$

If further, $V^{\frac{k}{k+\gamma}} W^{\frac{\gamma}{k+\gamma}}$ *is nondecreasing, then*

$$A^r(x) = o\left(V^{\frac{k-r}{k+\gamma}} W^{\frac{\gamma+r}{k+\gamma}}\right)$$

for $0 \leqq r \leqq k$.

This is a consequence of a convexity theorem of M. RIESZ and the proof follows on well known lines. See [6, lemma 4. 2].

Lemma 7. (i) *If* $\delta > \frac{k-1}{2} + q$, *we have*

$$D_x^q T_R^\delta(x) = cR \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x+t)\, D_t^q [V_{k/2+\delta}(|t|R)]\, dt_1 \ldots dt_k \tag{6.1}$$

where $|t| = (t_1^2 + \ldots + t_k^2)^{1/2}$, $V_\delta(x)$ *stands for* $J_\delta(x)/x^\delta$, J_δ *stands for the Bessel function of order* δ, *and* $D_x^q, f(x)$ *have the same meaning as in lemma* 2;

(ii) *if* $\delta > \frac{k-1}{2} + q$, *then*

$$R \int_{-\infty}^{\infty} \cdots \int |f(x+t)\, D_t^q V_{k/2+\delta}(|t|R)|\, dt_1 \ldots dt_k \leqq \tag{6.2}$$

$$\leqq c\left[R^{k+q} \int_0^{1/R} |f(x,t)|\, t^{k-1} dt + R^{q+\frac{k}{2}-\delta-\frac{1}{2}} \int_{1/R}^{\infty} \frac{|f_x(t)|\, dt}{t^{\delta-\frac{k}{2}+\frac{3}{2}}}\right];$$

(iii) *if*

$$F(x,t) = \int_0^t |f(x,s)|\, s^{k-1-q} ds = o(t^{k+\theta}), \quad \theta > 0, \tag{6.3}$$

as $t \to 0$, *then*

$$D_x^q T_R^\delta(x) = o\left(\frac{1}{R^\theta}\right)$$

as $R \to \infty$, *provided that* $\delta > \frac{k-1}{2} + q + \theta$.

Proof. Parts (i) and (ii) are contained in BOCHNER's paper [2, lemma 6, p. 349]; the argument for part (iii) runs parallel to CHANDRASEKHARAN's [5, Th. V]. We have only to consider the right side of (6. 2). Assumption (6. 3) yields

$$R^{k+q} \int_0^{1/R} |f(x,s)|\, s^{k-1} ds = o(R^{-\theta}), \tag{6.4}$$

and as for the second integral, we split it in two; setting $\varrho = \delta - q - \frac{k}{2} + \frac{1}{2}$, we have

$$R^{-\varrho}\left[\int_{1/R}^{\eta} + \int_{\eta}^{\infty}\right] = \varphi_1 + \varphi_2, \text{ say.}$$

To estimate φ_2 we have only to use the fact that

$$G(x,u)=\int_0^u t^{k-1}|f(x,t)|\,dt=O(u^k)$$

as $u\to\infty$. See [5, p. 213 (2. 11)]. For,

$$\varphi_2=\frac{1}{R^\varrho}\int_\eta^\infty \frac{t^{k-1}|f(x,t)|\,dt}{t^{k+q+\varrho}}=\frac{1}{R^\varrho}\left[\left\{\frac{G(x,t)}{t^{k+q+\varrho}}\right\}_\eta^\infty+c\int_\eta^\infty\frac{G(x,t)\,dt}{t^{k+q+\varrho+1}}\right]=O\left(\frac{1}{R^\varrho}\right)=o\left(\frac{1}{R^\theta}\right)$$

provided that

(6. 5) $$\varrho>\theta,\quad\text{or}\quad \delta>\frac{k-1}{2}+q+\theta.$$

As for φ_1,

$$\varphi_1=R^{-\varrho}\int_{1/R}^{\eta} t^{-k-\varrho}\,dF(t),$$

and we now integrate by parts, and use (6. 3) in the same manner as in [5, p. 219 (3. 26)], thus obtaining

$$\varphi_1=o(R^{-\theta})\quad\text{if}\quad \varrho>\theta.$$

This concludes the proof of the lemma.

Theorem 3. *If $f(x)$ is defined as in § 2, and if at a given point x,*

(6. 6) $$\frac{1}{t^k}\int_0^t |f(x,s)|\,s^{k-1-2q}\,ds=o(t^\theta),\ \theta\geqq 0$$

as $t\to 0$, where q is a non-negetive integer, and if

(6. 7) $$A_n=O(n^\alpha)$$

then $\Sigma A_n n^q$ is summable (n,r) for $\delta\geqq r\geqq 0$ provided that θ, α, q and r satisfy the relation

(6. 8) $$2(\delta-r)(\alpha+q+1)-\theta(1+r)=0$$

for some $\delta>\frac{k-1}{2}+2q+\theta$.

Proof. By lemma 7 (iii), assumption (6. 6) implies that

$$\Delta_x^q T_R^\delta(x)=o(R^{-\theta})$$

or

(6. 9) $$\Delta_x^q S_R^\delta(x)=o(R^{2\delta-\theta})$$

where $\delta>\frac{k-1}{2}+2q+\theta$ and $\Delta_x^q=\left(\frac{\partial^2}{\partial x_1^2}+\dots+\frac{\partial^2}{\partial x_k^2}\right)^q.$

If we now set

$$B_n = A_n n^q, \qquad B(R) = \sum_{n < R^2} B_n, \qquad B^\delta(R) = 2\delta \int_0^R (R^2 - u^2)^{\delta-1} B(u)\, u\, du, \ \delta > 0,$$

as in § 2, then (6. 9) implies

(6. 10) $$B^\delta(\sqrt{R}) = o(R^{\delta-\theta/2})$$

for $\delta > \frac{k-1}{2} + 2q + \theta$. On the other hand, $B_n = O(n^{\alpha+q})$ and hence

(6. 11) $$|B(\sqrt{\omega+t}) - B(\sqrt{\omega})| \leq \sum_{\omega \leqq n \leqq \omega+t} B_n = O\Big(\sum_{\omega \leqq n \leqq \omega+t} n^{\alpha+q}\Big) = O(t\omega^{\alpha+q}).$$

From (6. 10) and (6. 11) it follows that we can apply lemma 6 if we choose $B(\sqrt{x}) = A(x)$, $x^{\alpha+q} = V(x)$ and $x^{\delta-\theta/2} = W(x)$, and then we obtain

(6. 12) $$B^r(\sqrt{R}) = o(R^\beta)$$

where $0 \leqq r \leqq \delta$ and

(6. 13) $$\beta = \frac{(\alpha+q)(\delta-r)}{\delta+1} + \frac{(1+r)(\delta-\theta/2)}{\delta+1}.$$

Let us write (6. 12) in the form

(6. 14) $$\frac{B^r(R)}{R^{2r}} = o(R^{2\beta-2r}) = o(R^\eta), \text{ say.}$$

If $\eta = 0$, then it follows that $\Sigma A_n n^q$ is summable (n, r); this will be the case if

(6. 15) $$2(\delta - r)(\alpha + q + 1) - \theta(1 + r) = 0$$

where $\delta > \frac{k-1}{2} + 2q + \theta$.

Remarks. (1) Let us write relation (6. 15) in the form

(6. 16) $$r = \frac{2\delta(\alpha+q+1) - \theta}{\theta + 2(\alpha+q+1)}.$$

Now if $\theta = 0$ then $r = \delta$, where $\delta > \frac{k-1}{2} + 2q$. Thus we obtain BOCHNER's result [2, Th. I] as a special case.

(2) Let $q = 0$ and $k = 2$. Then it follows from (6. 16) that $r = 0$, if $2\delta(\alpha+1) = \theta$ where $\delta > \theta + \frac{1}{2}$; and this will be the case if $\alpha < \frac{\theta}{2\theta+1} - 1$.

Suppose now that

(6. 17) $$a_{n_1 n_2} = O\left(\frac{1}{(n_1^2 + n_2^2)^p}\right);$$

then $A_n = O(n^{\varepsilon-p})$ for every $\varepsilon > 0$, since $r_k(n) = O\left(n^{\frac{k-2}{2}+\varepsilon}\right)$. Hence under the assumption (6. 6) with $q = 0$, $k = 2$ and the assumption (6. 17) we conclude that ΣA_n converges if $\varepsilon - p < \frac{\theta}{2\theta+1} - 1$ for every $\varepsilon > 0$ or if

(6. 18) $$p > 1 - \frac{\theta}{2\theta+1},$$

which is exactly the theorem of CHANDRASEKHARAN and MINAKSHISUNDARAM [6, Th. 4. 1].

(3) Though in the assumption (6. 6) we have q as an integer, we can, if necessary, determine the order of summability of $\Sigma A_n n^h$ for *arbitrary h* by applying ANANDA-RAU's theorem (lemma 1). We choose not to repeat this kind of computation.

(4) Our hypothesis (6. 6) differs from the hypothesis in theorems 1 and 2 in as much as it governs the behaviour of the function $f(x)$ at a given point x, and not in a whole neighborhood of it.

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