Translation of figures between lattice points.

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We consider in the n-dimensional euclidean space R_n an n-dimensional point-lattice L and a figure F. This figure may be an arbitrary point set, but will be specialized later on. We apply to F all translations of R_n and consider those of these translated figures which do not contain any point of L. There may be points (apart from the points of L) which can not be covered by any of these translated figures, i. e. which can not be covered by a figure arising from F by translation without covering by the same figure a point of L too. The present paper deals with the question what can be said about the set U of these "uncoverable" points[1]).

1. For any figure F in R_n, let $F+\mathbf{v}$ denote the figure arising from F when translated by the vector $\mathbf{v}$.

We call a point set G a *point group* if for each vector $\mathbf{v}$ the sets $G+\mathbf{v}$ and G are identical or have no point in common. The vectors $\mathbf{v}$, for which $G+\mathbf{v}=G$, form an additive group and will be called *G-vectors.* Two points are *G-homologous* if the vector joining them is a G-vector. A point group is a *point-lattice* if it consists of isolated points.

We shall consider a point-lattice L and a point set F, and denote, according to the introduction, by U the set of points P for which each $F+\mathbf{v}$ containing P contains at least one point of L too. Consequently, U contains all points of L. We denote by $[F]$ the union of all sets $F+\mathbf{l}$ where $\mathbf{l}$ is an L-vector. It is possible that U contains all points of R_n. This is the case if and only if $[F]$ is the whole space R_n. As easily seen, U contains all points which are L-homologous to one of its points.

2. *U contains the set $L+\mathbf{u}$ if and only if $[F]+\mathbf{u}$ contains $[F]$.* $U\supset L+\mathbf{u}$ (i. e U contains $L+\mathbf{u}$) means that, translating an arbitrary point P of F in

[1]) In the two dimensional case our problem could be stated as follows: We clean up the floor with a brush. There are outstandig nails in the floor which form a point-lattice. The brush may be slid or lifted up remaining always parallel to its former positions. What can be said about the poin s where the dirt is gathering, which can not be cleaned up?

a point of $L+\mathbf{u}$, there will be a point Q of F which covers after this translation a point of L. The translated Q belongs to L if and only if the translated $Q+\mathbf{u}$ belongs to $L+\mathbf{u}$, consequently, if and only if $Q+\mathbf{u}$ and P are L-homologous points. The supposition $U\supset L+\mathbf{u}$ is therefore equivalent to the fact that there exists to each point P of F a point Q of F for which $Q+\mathbf{u}$ and P are L-homologous. This is just the content of our proposition.

If U contains $L+\mathbf{u}_1$ and $L+\mathbf{u}_2$, it contains $L+(\mathbf{u}_1+\mathbf{u}_2)$ too. In fact, if $[F]+\mathbf{u}_1$ and $[F]+\mathbf{u}_2$ contain $[F]$, we have $[F]+(\mathbf{u}_1+\mathbf{u}_2)=([F]+\mathbf{u}_1)+$ $+\mathbf{u}_2\supset[F]+\mathbf{u}_2\supset[F]$.

We could say therefore that U is a "point semigroup". U would be a point group if it would contain $L-\mathbf{u}$ together with $L+\mathbf{u}$. But this is not in general the case. We show this by the

Example 1. L consists of the points of R_1 with integer coordinates: F consists of the points $p\alpha$, where α is irrational and $p=1,2,\ldots$ The corresponding set U consists of the points $q-p\alpha$, where q is integer and $p=0,1,2,\ldots$.

This example shows that U need not be closed in general.

3. *If F is open, then U is closed.* In fact, the complementary of U is a sum of open sets $F+\mathbf{v}$.

If F is bounded and closed, then U is closed. Let P be the limit point of a sequence P_n. If P does not belong to U, there exists an $F+\mathbf{v}$ containing P but none of the points of L. Since $F+\mathbf{v}$ is closed, each lattice point has a positive distance from it. Since $F+\mathbf{v}$ is bounded, there is only a finite number of lattice points whose distance from it is less than 1 and these distances have a positive lower bound ε. Consequently, if we translate $F+\mathbf{v}$ at a distance less than ε, the new figure contains no lattice point. But these newly translated figures cover the ε-neighbourhood of P and so almost all points of the sequence P_n too. These do not belong to U.

We remark that U need not be closed if F is only measurable. This is shown by example 1.

If U is closed, then U is a point group. Let $\mathbf{u}$ be a vector for which $L+\mathbf{u}$ belongs to U. It contains consequently, for any positive integer m, the set $L+m\mathbf{u}$ too. According to Dirichlet's theorem we can find to any positive ε an integer m for which the points of $L+m\mathbf{u}$ are contained in the ε-neighbourhoods of the lattice points. The points of $L+(m-1)\mathbf{u}$ are, therefore, in the ε-neighbourhoods of the points of $L-\mathbf{u}$. The latters are then limit points of U and belong to U. The vectors $\mathbf{u}$, for which $L+\mathbf{u}\subset U$, form an additive group, i. e. U is a point group.

Resuming our statement, we have the following result:

If F is open, or bounded and closed, then U is a closed point group.

Since, in the considered case, $[F]+\mathbf{u}\supset[F]$ implies in succession $U\supset L+\mathbf{u}$, $U\supset L-\mathbf{u}$, $[F]-\mathbf{u}\supset[F]$, and this is equivalent to $[F]+\mathbf{u}\subset[F]$, we proved:

If F is open, or bounded and closed, then U consists of the lattices $L+\mathbf{u}$ for which $[F]+\mathbf{u}=[F]$.

4. We call a set of R_n *k-dimensional* if it is contained in a k-dimensional linear subspace of R_n but is not contained in any linear subspace of lower dimension.

An n dimensional closed point group G of R_n is either a point-lattice, or coincides with R_n itself, or else it consists of isolated parallel linear k-dimensional subspaces ($k=1, 2, \ldots, n-1$).

We consider a point P of G and the set of points of G which are in an ε-neighbourhood of P. The dimension of this point set can not increase with decreasing ε. If ε is sufficiently little, this dimension becomes minimal. We denote this local dimension of G by k and the k-dimensional linear subspace containing this k-dimensional neighbourhood by S. If $k=0$, P is isolated and G is a point-lattice. Since k is minimal, there are k points of G in every prescribed neighbourhood of P in S which form with P a k-dimensional set. The lattices generated by these sets belong to G and assure that G is everywhere dense in S. As G is closed, it contains all points of S. If $k=n$, G is identical with R_n. If $0<k<n$, the k-dimensional subspace S is isolated in G, since otherwise the local dimension at P could not be k. Because G is a point group, it consists of isolated subspaces parallel to S.

We have obtained the following result:

If F is open or if F is bounded and closed, then U is either a point-lattice, or coincides with R_n, or else it consists of isolated parallel linear subspaces.

It is easy to find special figures which realize all these different possibilities. The case $U=R_n$ has no special interest. The case in which U consists of parallel subspaces may be easily reduced to the case where U is a point-lattice. This can be done by projecting the lattice L and the figure F on an $(n-k)$-dimensional linear subspace T which is totally orthogonal to S. The projected figure F_T and the projected lattice L_T define the set U_T. As easily proved, U_T is a point-lattice and U consists of subspaces parallel to S through the points of U_T. We are therefore interested only in the case where U is a point-lattice.

5. If U is a point-lattice, the question arises if it can have other points than the points of L. This may happen even if F is a simply connected region. We show this by the

Example 2. Let L be two dimensional and we define F by deleting from a fundamental parallelogram of L the ε-neighbourhoods of the vertices and of the midpoints of two opposite sides. The corresponding U is the lattice generated by L and these midpoints, whether F is open or closed.

We specialize the figure F and examine the case of (open or closed) convex figures.

If the set U corresponding to the open or closed convex figure F is a point-lattice, then U is identical with L if the dimension n is equal to 1 *or* 2, *but it may be different from L if $n \geqq 3$.*

The one dimensional case is obvious.

In two dimensions we distinguish two cases. Since we shall need the first part of this argument later on also in three dimensions, we shall speak in this part simultaneously from two and three dimensions. Since F is convex and U is a lattice, F is necessarily bounded.

We consider first the case in which the boundary of $[F]$ does not consist of isolated points. We denote by $\mathbf{u}$ a U-vector which is not an L-vector and by $\mathbf{l}$ the L-vectors. We consider a part B of the boundary of $[F]$, of the same topological dimension as this boundary. This part B is covered by parts of the boundaries of the figures $F+\mathbf{l}$, and simultaneously, since $[F]=[F+\mathbf{u}]$, by parts of the boundaries of the figures $F+\mathbf{u}+\mathbf{l}$. The number of these parts is finite, because F is bounded. We can choose, therefore, an interior point P in the common part of two of them in which $[F]$ has a tangent line (plane). P has L-homologous points P_1 resp. $P_2+\mathbf{u}$ on the boundaries of F resp. $F+\mathbf{u}$, the tangent lines (planes) in which exist and are parallel to the tangent line (plane) in P. The resulting points P_1 and P_2 are on the boundary of F and of $[F]$, are U-homologous to each other, and F has parallel tangent lines (planes) in them. It follows that all the points of the chord P_1P_2 are on the boundary of F.

We could have chosen, instead of B, any other part of the boundary of $[F]$. Since U is a lattice, $[F]$ must have boundaries in all directions, more precisely: there exist to each prescribed direction parts of the boundary of $[F]$, all the supporting half-planes (-spaces) in the points of which contain the prescribed direction in their interior, i. e. contain an interior half line of the prescribed direction. In each of these parts we can find points to take the part of P. In the two dimensional case our reasoning leads to at least three pairs of U-homologous points on the boundary of F which are all different from each other. They define an inscribed hexagon $P_1P_2Q_1Q_2R_1R_2$ in F.

If the boundary of $[F]$ consists of isolated points, each of them is on the boundary of at least three figures $F+\mathbf{l}$ and defines three L-homologous points on the boundary of F (necessarily open in this case) An other isolated point of the boundary of $[F]$, U-homologous but not L-homologous to the former one, defines three L-homologous points on the boundary of F which are U-homologous to the former ones and different from them. Thus we have in this case an inscribed hexagon in F with U-homologous vertices.

It is sufficient therefore to prove the impossibility of the hexagon of the first case. We do this with the help of following elementary geometrical

Lemma. *A convex hexagon has at least one vertex with the property that its symmetric with respect to the midpoint of the diagonal joining the two adjacent vertices is in the interior of the hexagon.*

If the sum of the angles at A and B of the convex quadrangle $ABCD$ exceeds π, it is immediately seen that the symmetric of A or of B with respect to the midpoint of BD resp. AC is in the interior of the quadrangle, or in the interior of the side CD. Since the total sum of the angles of our hexagon is 4π, there must be two adjacent angles whose sum exceeds π. One of them has consequently the asserted property.

If e. g. the symmetrical of P_2 with respect to the midpoint of P_1Q_1 is in the interior of the hexagon, the latter is an interior point of F and U-homologous to Q_1. We have two U-homologous points, one of them is an interior point of $[F]$, the other on its boundary. This contradicts the fact that we have $[F]+\mathbf{u}=[F]$ for each U vector $\mathbf{u}$.

In three dimensions we give two examples where U is not identical with L. Both examples can be easily checked.

Example 3. F is the open tetrahedron with vertices $(2, \pm 4, 0)$, $(-2, 0, \pm 4)$. L is formed by the points with odd integer coordinates whose sum is $\equiv 1 \pmod 4$. The corresponding U consists of all points with odd integer coordinates. U remains the same if we take instead of F a (not too much) diminished, open or *closed* homothetic of F.

Example 4. F is the open octahedron with vertices $(\pm 3, 0, 0)$, $(0, \pm 3, 0)$, $(0, 0, \pm 3)$. L is the same as in example 3 and the corresponding U is the same too. The set U corresponding to any homothetically diminished figure is in this case identical with L.

Finally, we prove our statement for higher than three dimensions. We consider a three dimensional subspace R_3 of R_n. We construct in R_3, according to the examples 3 or 4, a lattice L_3 and an open resp. closed figure F_3. They define a lattice $U_3 \neq L_3$. We choose an n-dimensional lattice L in R_n which contains the points of L_3, and which has no further points in R_3. The points of R_n whose orthogonal projection on R_3 belongs to F_3, and which have a distance $<\varepsilon$ resp. $\leqq \varepsilon$ from R_3, form an open resp closed figure F. It is easily seen that the set U corresponding to L and F contains all the points of U_3, and that U is a lattice if ε is sufficiently small. This completes our proof for each dimension.

6. We consider next the further specialization of central symmetrical convex figures. As to open central symmetrical convex figures the result of 5 can not be improved. This is shown by Example 4.

If the set U corresponding to the closed, central symmetrical, convex figure F is a point-lattice, then U is identical with L for the dimensions $n = 1, 2, 3$, but may be different from L if $n \geqq 4$.

The cases of one and two dimensions are contained in the result of 5.

In three dimensions we apply the argument of 5. This argument provides two points P_1 and P_2 on the boundary of F which are on the boundary of $[F]$, are U-homologous to each other, F has parallel tangent planes in them, and the chord P_1P_2 is on the boundary of F. Since F is symmetrical with respect to O, the symmetrical points P_1' and P_2' of P_1 and P_2 with respect to O have the same properties. We apply the argument of 5 again to get four points Q_1, Q_2, Q_1', Q_2' which have the same properties, and the tangent planes of F in which are not parallel with P_1P_2. Consequently the lines Q_1Q_2 and $Q_1'Q_2'$ are not parallel with P_1P_2 and $P_1'P_2'$. Since F has parallel tangent planes at the ends of these lines, the other sides of the parallelograms $P_1P_2P_1'P_2'$ and $Q_1Q_2Q_1'Q_2'$ run in the interior of F. The vertices of these parallelograms are necessarily all different.

We consider the polyhedron which is the least convex cover of both parallelograms. We draw a straight line through O, parallel to P_1P_2. It cuts the surface of the polyhedron in two points A and A' which are in the interior or on the boundary of two, with respect to O symmetrically situated, polygons belonging to the surface of the polyhedron. We consider first the case that A and A' are not on the periphery of the parallelogram $P_1P_2P_1'P_2'$. The distance of the planes of the two polygons, measured in the direction of P_1P_2, is in this case greater than P_1P_2. It is easily seen that, if two central symmetrically situated polygons in the plane have common points, then each of them has at least one vertex in the interior or on the boundary of the other. Applying this to the parallel projections of our two polygons, formed by rays parallel to P_1P_2, we obtain one of the vertices of our parallelograms, the vector P_1P_2 traced out from which leads in an interior point of the polyhedron. Consequently, this vertex has a U-homologous in the interior of F. This is against the fact that the vertices of our parallelograms are on the boundary of $[F]$.

We have therefore to discuss only the case that A and A' are on the periphery of the parallelogram $P_1P_2P_1'P_2'$. In this case the whole periphery of this parallelogram is on the surface of the polyhedron. By the same reason we are entitled to suppose this also from the parallelogram $Q_1Q_2Q_1'Q_2'$. Both these parallelograms have a couple of opposite sides running on the boundary of F and the other two sides running in the interior of F. Obviously, the sides on the boundary can not cut the sides running in the interior. Even two sides on the boundary can not cut each other; since the tangent plane in Q_1 and Q_2 (containing the line Q_1Q_2) is not parallel with P_1P_2. The only possibility is (employing a convenient notation) that the sides P_2P_1' and Q_2Q_1', resp. P_1P_2' and Q_1Q_2' cut each other. The two plane quadrangles $P_2Q_2P_1'Q_1'$ and $P_1Q_1P_2'Q_2'$ are then on the surface of the polyhedron. These quadrangles are not parallelograms, because they are central symmetrically situated and the lines P_1P_2 and Q_1Q_2 are not parallel. Therefore, according to the proof of our lemma in 5, there is a vertex of $P_2Q_2P_1'Q_1'$, e. g. P_1', whose symmetric X with respect to the midpoint of Q_2Q_1' is on the closed quadrangle itself and is not one of its vertices. Consequently, X is in the interior of F and U-homologous to P_1 since the vectors P_1X and Q_1Q_2 are equal. Namely, the triangles $P_1Q_1Q_2'$ and XQ_2Q_1' are congruent and parallel, since both are central symmetrically situated to the triangle $P_1'Q_1'Q_2$. Our result that P_1 has a U-homologous in the interior of F is against the fact that P_1 is on the boundary of $[F]$. This impossibility proves our statement for three dimensions.

It will be shown by the example 5 that in higher than three dimensions the case $U \neq L$ is possible for lattices U corresponding to convex central symmetrical figures.

7. We specialize at last the figure F to be a parallelotope. The case in which F is a cube, would not be a real further specialization, since the whole problem of this paper is an affine geometrical one.

If the set U corresponding to the open or closed parallelotope F is a lattice, then U is identical with L if the dimension n is 1, 2, *or* 3, *but it may be different from L if $n \geqq 4$.*

The cases $n = 1, 2$ and that of a closed parallelepiped in R_3, are contained in our previous results.

Let F be an open parallelepiped in R_3 with the corresponding lattice $U \neq L$. If the boundary of $[F]$ does not consist of isolated points, the impossibility may be shown by the argument of 6.

We restrict ourselves therefore to the case in which the boundary of $[F]$ consists of isolated points. We choose a face of F. Four parallel edges of F are cut by the plane of this face. We draw a parallel to these edges through a point P of the boundary of $[F]$. Since in a neighbourhood of P all the points of this parallel belong to $[F]$, there must be an interior point P_1 of the chosen face which is L-homologous to P. If $\mathbf{u}$ denotes a U-vector which is not an L-vector, we find by the same way an L-homologous P_2 of $P+\mathbf{u}$ in the interior of the same face. P_1 and P_2 are necessarily different. We apply our argument again to a face of F which is not parallel to P_1P_2. We obtain two points Q_1, Q_2 on this other face with the same properties, the distance of which is not parallel to P_1P_2. We are able now to apply the argument of 6 also in this case, since the points defined here have all the properties of the points used there.

For dimensions higher than three we can apply the last remark of 5.

The proof of our statement will therefore be completed by an example in four dimensions.

Example 5. F is an open or closed cube in R_4 whose edges are parallel to the coordinate axes and are of length 2. The lattice L is formed by the points with integer coordinates x_1, x_2, x_3, x_4 for which

$$a = \frac{x_1 + x_3}{2} + x_4 \quad \text{and} \quad b = \frac{x_2 + x_4}{2} + x_3$$

are even integers. The corresponding U is the lattice formed by the points of integer coordinates for which a and b are both even or odd In fact, if the vertices of F have integer coordinates, we may dissect all the cubes $F + 1$ which form $[F]$ in 16 unit cubes whose vertices have integer coordinates. We may therefore, in order to obtain the vectors $\mathbf{u}$ for which $[F] + \mathbf{u} = [F]$, replace F by the set S of 16 points whose coordinates are all 0 or 1. There are only two couples $(0, 0, 0, 0)$, $(1, 1, 1, 1)$ and $(1, 0, 1, 0)$, $(0, 1, 0, 1)$ among these 16 points which are L-homologous to each other. As easily seen, there are 16 classes of L-homologous points with integer coordinates. 14 from these are represented in S. The points $(2, 0, 0, 0)$ and $(0, 0, 2, 0)$ determine the remaining two classes. These two classes form a lattice arising by translation from the lattice U defined above.

At the end of all our specializations we may point out the surprising fact that in higher dimensions, even in case of most regular figures, it may happen that some points, apart from the lattice points, can not be covered by a translated figure without covering a lattice point too, however the whole neighbourhood of the lattice points can be covered by it.

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