Some packing and covering theorems.

By LÁSZLÓ FEJES TÓTH in Budapest.

Let us consider an infinite set of equal circles placed in the plane in such a way that no two circles overlap. According to a remarkable result of A. THUE[1]) the density of such a set is $\leqq \pi/\sqrt{12} = 0{,}9069\ldots$[2]). The dual counterpart of this result is due to R. KERSHNER[3]) and it states the fact that the density of an infinite set of equal circles covering the plane is $\geqq 2\pi/\sqrt{27} = 1{,}209\ldots$[4]).

The density of a set of domains strewn over the plane is defined here by a suitable limit value. It can be interpreted as the sum of the areas of the domains pro the unit of the area of the plane, or the sum of the areas of the domains divided by the area of the whole plane.

In the present paper we are going to extend these results in different directions. Instead of equal circles we shall consider first arbitrary congruent convex domains, then incongruent circles. In addition to these generalisations we shall consider, instead of the whole plane, a finite region of the plane, namely a convex polygon having at most six sides. We shall call such a polygon shortly a hexagon.

Our results are as follows[5]).

Theorem 1. *If n is the number of certain congruent convex domains lying in a hexagon $\mathfrak{h}$ so that no two of them overlap, then*

$$n \leqq \mathfrak{h}/h, \tag{1}$$

where h denotes the hexagon of the smallest area circumscribed about a domain.

---

1) A. THUE, Om nogle geometrisk taltheoretiske Theoremer, *Naturforskermode* 1892, pp. 352—353; Über die dichteste Zusammenstellung von kongruenten Kreisen in einer Ebene, *Christiania Videnskaberne Selskabs Skrifter* 1910, p. 9.

2) The constant $\pi/\sqrt{12}$ equals the density of the closest lattice of equal circles, or the ratio of the area of a circle to the area of the circumscribed regular hexagon.

3) R. KERSHNER, The number of circles covering a set, *American Journal of Math.*, **61** (1939), pp. 665—671.

4) $2\pi/\sqrt{27}$ equals the density of the smallest lattice of equal circles covering the plane, or the ratio of the area of a circle to the area of the inscribed regular hexagon.

5) For a domain and its area the same symbol will be used.

Theorem 2. *If N is the number of certain congruent convex domains covering a hexagon $\mathfrak{h}$ in such a way that the boundaries of the domains intersect one another in at most two points*[6]*, then*

$$(2) \qquad N \geqq \mathfrak{h}/H,$$

where H denotes the hexagon of the largest area inscribed in a domain.

Theorem 3. *Let $c_1, \ldots, c_n$ be n circles lying in a hexagon $\mathfrak{h}$ so that no two of them overlap, then for any $\alpha \leqq 1-(4-\sqrt{27}/\pi)^2/24 = 0,77\ldots$*[7]*) we have*

$$(3) \qquad n\left(\frac{c_1^\alpha + \ldots + c_n^\alpha}{n}\right)^{\frac{1}{\alpha}} \leqq \frac{\pi}{\sqrt{12}}\mathfrak{h}.$$

Theorem 4. *Let $C_1, \ldots, C_N$ be N circles covering the hexagon $\mathfrak{h}$, then for any $\beta \geqq 1+(2+\sqrt{27}/\pi)^2/12 = 2,11\ldots$ we have*

$$(4) \qquad N\left(\frac{C_1^\beta + \ldots + C_N^\beta}{N}\right)^{\frac{1}{\beta}} \geqq \frac{2\pi}{\sqrt{27}}\mathfrak{h}.$$

Let us mention some special cases of these inequalities.

In the case of equal circles (1) and (3) and (2) and (4) are equivalent and they imply the results mentioned above.

A further consequence of Theorem 1 is the fact that *the density of an irregular packing of an infinite set of non-overlapping congruent convex domains having a centre of symmetry cannot exceed the density of the closest lattice of the domains*[8]. For, denoting a domain by d, the density of the closest lattice in question equals d/h.[9]) On the other hand, we have by (1) $nd/\mathfrak{h} \leqq d/h$, whence the assertion.

As a corollary of Theorem 3 let us mention the following fact. If $r_1, \ldots, r_n$ denote the radii of n non-overlapping circles lying e. g. in a square

---

[6]) The restriction about the points of intersection of the curves bordering the domains is probably superfluous and it diminishes not only the beauty of Theorem 2 but disturbs also the duality of Theorem 1 and 2. Therefore it would be very desirable to get rid Theorem 2 from this restriction.

[7]) The constants occurring in Theorems 3 and 4 are not the best possible ones. On the other hand, the set of circles arising from the closest packing of congruent circles by means of placing into each gap a smaller circle yields an example which shows that for values of α such that $1+(2/\sqrt{3}-1)^{2\alpha} > 3^{1-\alpha}$, i. e. $\alpha > 0,94\ldots$, Theorem 3 does not hold any more.

[8]) As I have learnt by the kind information of I. Fáry, this result was also found by K. Mahler who announced this fact during a lecture of Chabauty held in Paris in October 1949. — Let us still note that the tesselation of the plane by oblongs like the parquetry shows that in certain cases the density of the closest lattice can be reached also by not lattice-like packings of the domains.

[9]) Compare Theorem 3 of the paper of Dowker quoted in footnote 11.

S, then $r_1+\ldots+r_n<\sqrt{nS}/\sqrt{12}$. This means that if we will make the total length of the perimeters of a great (but given) number of non-overlapping circles lying in a square (or in an arbitrarily given domain) possibly large then we have to take congruent circles of convenient size and arrange them in "hexagonal close-packing". An analogous statement concerning the areas of the circles does not hold.

The simple idea of the following proofs of (1) and (2) is, apart from some slight modifications, the same used previously by the author in the case of ellipses[10]). The extension to the general case follows by means of certain results of C. H. DOWKER[11]) to which my attention was called recently by P. TURÁN.

The results of DOWKER in question state that if $a(\nu)$ is the area of the ν-gon of the smallest area circumscribed about an arbitrarily given convex domain and $A(\nu)$ the area of the ν-gon of the largest area inscribed in the domain, then the sets $a(\nu)$ and $A(\nu)$ $(\nu=3,4,\ldots)$ are convex. More precisely for any $\nu\geqq 4$ $a(\nu-1)+a(\nu+1)\geqq 2a(\nu)$ and $A(\nu-1)+A(\nu+1)\leqq 2A(\nu)$ hold.

We shall still need the following well known consequence of EULER's formula. If we decompose a hexagon into $n\geqq 1$ convex polygons the number of the sides of which being $\nu_1,\ldots,\nu_n$, then

$$\frac{\nu_1+\ldots+\nu_n}{n}\leqq 6. \tag{5}$$

Let now $d_1,\ldots,d_n$ be the domains satisfying the conditions of Theorem 1. Let us replace d_1 by a convex domain p_1 having no common inner point with $d_2,\ldots,d_n$ so that $d_1\subset p_1\subset\mathfrak{h}$ and that no other convex domain $\supset p_1$ has these properties. p_1 is a convex polygon having, say, ν_1 sides. Let us construct successively to each domain d_i a polygon p_i defined analogously as p_1 the number of the sides of which being ν_i. Although the polygons p_i generally do not fill out the hexagon $\mathfrak{h}$ it is easy to see that the inequality (5) holds unaltered.

Consider now the ν-gon of the smallest area $a(\nu)$ circumscribed about a domain d_i. Since the set $a(\nu)$ is convex we can extend the definition of $a(\nu)$ to any $\nu\geqq 3$ so that the function obtained should be a decreasing function convex from below. Thus, by JENSEN's inequality

---

10) FEJES L., Extremális pontrendszerek a síkban, a gömbfelületen és a térben; *Acta Sci. Math. Naturalium, Kolozsvár* **23** (1944), pp. 15—18. — The case of congruent ellipses seems to involve some interest in crystallography. *Cf.* W. NOWACKI, Über Ellipsenpackungen in der Kristallebene, *Schweizerische Mineralogische und Petrographische Mitteilungen*, **28** (1948), pp. 501—508.

11) C. H. DOWKER, On minimum circumscribed polygons, *Bulletin American Math. Society*, **50** (1944), pp. 120—122.

$$\mathfrak{h} \geqq p_1 + \ldots + p_n \geqq a(\nu_1) + \ldots + a(\nu_n) \geqq n\, a\left(\frac{\nu_1 + \ldots + \nu_n}{n}\right) \geqq n\, a(6), \text{ q. e. d.}$$

The proof of (2) is analogous to the previous one. Let $D_1, \ldots, D_N$ be the domains considered in Theorem 2. Let us replace D_1 by the least convex domain D_1' such that the domains $D_1', D_2, \ldots, D_N$ should cover $\mathfrak{h}$ and that the boundary of D_1' intersect the boundaries of the domains $D_2, \ldots, D_N$ in at most two points. By constructing successively to each domain D_i the domain D_i' defined analogously as D_1', it may occur that some of the domains D_i' overlap. In this case let us continue the above process until each domain D_i will contract to a polygon P_i having no common inner point with another one. Denoting the number of the sides of P_i by μ_i, we have $\mu_1 + \ldots + \mu_N \leqq 6N$. Hence, $A(\mu)$, $\mu \geqq 3$, being an increasing function concave from below such that for $\mu = 3, 4, \ldots$ the function $A(\mu)$ equals the maximum of the areas of the μ-gons inscribed in a domain D_i, we have

$$\mathfrak{h} = P_1 + \ldots + P_N \leqq A(\mu_1) + \ldots + A(\mu_N) \leqq N A\left(\frac{\mu_1 + \ldots + \mu_N}{N}\right) \leqq N A(6), \text{ q. e. d.}$$

To the proofs of (3) and (4) we make two remarks.

Remark 1. The function $\Phi(x, y)$ of the two variables $x \geqq 0$, $y \geqq 3$ defined by

$$\Phi(x, y) = x^{\frac{1}{\alpha}} \varphi(y) \quad , \quad \varphi(y) = \frac{y}{\pi} \tan \frac{\pi}{y},$$

is for any $\alpha \leqq 0{,}77 \ldots$ $(\alpha \neq 0)$ convex from below.

For, the condition of the convexity is

$$\Phi_{xx} \Phi_{yy} - \Phi_{xy}^2 = \alpha^{-2} x^{2\alpha - 2} [(1-\alpha)\varphi \varphi'' - \varphi'^2] \geqq 0,$$

i. e.

$$\pi^2 y^2 \cos^2 \frac{\pi}{y} [(1-\alpha)\varphi\varphi'' - \varphi'^2] = 2\pi^2 (1-\alpha) \sin^2 \frac{\pi}{y} - \left(\pi - y \sin \frac{\pi}{y} \cos \frac{\pi}{y}\right)^2 \geqq 0$$

or

$$2\pi^2 (1-\alpha) \geqq [f(y)]^2 \, , \quad f(y) = \pi \operatorname{cosec} \frac{\pi}{y} - y \cos \frac{\pi}{y}.$$

But $f(y)$ being for $y \geqq 3$ a decreasing function[12]) of y, the condition of the convexity is satisfied if $2\pi^2(1-\alpha) \geqq [f(3)]^2$.

Remark 2. The function $\Psi(x, y)$ of the two variables $x \geqq 0$, $y \geqq 3$ defined by

$$\Psi(x, y) = x^{\frac{1}{\beta}} \psi(y) \quad , \quad \psi(y) = \frac{y}{2\pi} \sin \frac{2\pi}{y},$$

is for any $\beta \geqq 2{,}11 \ldots$ concave from below.

---

[12]) We have $y^2 \tan \frac{\pi}{y} \sin \frac{\pi}{y} f'(y) = \pi^2 - \sin^2 \frac{\pi}{y} \left(y^2 + \pi y \tan \frac{\pi}{y}\right) < \pi^2 - \sin^2 \frac{\pi}{y} \left(y^2 + \pi^2\right) =$
$= \pi^2 \cos^2 \frac{\pi}{y} - y^2 \sin^2 \frac{\pi}{y} < 0.$

The condition of the concavity is namely

$$\Psi_{xx}\Psi_{yy}-\Psi_{xy}^2=-\beta^{-2}x^{2\beta-2}[(\beta-1)\psi\psi''+\psi'^2]\geqq 0.$$

This is equivalent with

$$-4\pi^2y^2[(\beta-1)\psi\psi''+\psi'^2]=4\pi^2(\beta-1)\sin^2\frac{2\pi}{y}-\left(y\sin\frac{2\pi}{y}-2\pi\cos\frac{2\pi}{y}\right)^2\geqq 0$$

or with

$$4\pi^2(\beta-1)\geqq[g(y)]^2,\quad g(y)=y-2\pi\cot\frac{2\pi}{y}.$$

But $g(y)$ being for $y\geqq 3$ a decreasing function[13]) of y the above condition is satisfied if $4\pi^2(\beta-1)\geqq[g(3)]^2$.

Now, preserving the notations of the above proofs of (1) and (2), we have $p_i\geqq c_i\varphi(\nu_i)$. This follows from the fact that among the ν-gons containing a circle the circumscribed regular ν-gon has the minimal area. Hence, applying JENSEN's inequality to the function $\Phi(x,y)$ of Remark 1

$$\mathfrak{h}\geqq p_1+\ldots+p_n\geqq(c_1^\alpha)^{\frac{1}{\alpha}}\varphi(\nu_1)+\ldots+(c_n^\alpha)^{\frac{1}{\alpha}}\varphi(\nu_n)\geqq$$

$$\geqq n\left(\frac{c_1^\alpha+\ldots+c_n^\alpha}{n}\right)^{\frac{1}{\alpha}}\varphi\left(\frac{\nu_1+\ldots+\nu_n}{n}\right)\geqq n\left(\frac{c_1^\alpha+\ldots+c_n^\alpha}{n}\right)^{\frac{1}{\alpha}}\varphi(6).$$

But this is just the inequality (3) to be proved.

We can proceed analogously at the proof of (4). Since among the μ-gons containing a circle the inscribed regular μ-gon has the maximal area, we have $P_i\leqq C_i\psi(\mu_i)$. Hence Remark 2 and JENSEN's inequality imply that

$$\mathfrak{h}=P_1+\ldots+P_N\leqq(C_1^\beta)^{\frac{1}{\beta}}\psi(\mu_1)+\ldots+(C_N^\beta)^{\frac{1}{\beta}}\psi(\mu_N)\leqq$$

$$\leqq N\left(\frac{C_1^\beta+\ldots+C_N^\beta}{N}\right)^{\frac{1}{\beta}}\psi\left(\frac{\mu_1+\ldots+\mu_N}{N}\right)\leqq N\left(\frac{C_1^\beta+\ldots+C_N^\beta}{N}\right)^{\frac{1}{\beta}}\psi(6).$$

This concludes the proof.

The above proofs show that Theorems 1 and 2 hold also under the more general condition that the domains, instead of being congruent, arise from a convex domain by affine transformations preserving the area. Analogously, we can take in Theorems 3 and 4, instead of circles, arbitrary elliptical discs, provided that no two ellipses intersect one another in more than two points.

At last let us note that by means of the above considerations inequalities analogous to (3) and (4) can also be derived between the surface area and the radii of the incircles or circumcircles of the faces of a polyhedron. For

---

[13]) We have $\sin^2\frac{2\pi}{y}\cdot g'(y)=\sin^2\frac{2\pi}{y}-\frac{4\pi^2}{y^2}<0.$

example, if $r_1, \ldots, r_f$ denote the radii of the incircles of the faces of a polyhedron of area F having e edges and f faces, then

$$(r_1 + \ldots + r_f)^2 \leqq \frac{f^2 F}{2e} \cot \frac{\pi f}{2e}.$$

Equality holds only if all faces are congruent regular polygons.

For Eulerian polyhedra we have

$$(r_1 + \ldots + r_f)^2 \leqq \frac{f^2 F}{6f-12} \cot \frac{\pi f}{6f-12} \quad \left(< \frac{fF}{\sqrt{12}} \right),$$

where equality holds only for the regular tetrahedron, hexahedron and dodecahedron.

(Received August 25, 1949, revised January 17, 1950.)