Another proof of the Gödel—Rosser incompletability theorem.

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In a paper[1]) which became a source of a series of investigations, GÖDEL has proved a theorem to the effect that for every postulate system satisfying some very general conditions, there is an arithmetical problem unsolvable in that system. One of the conditions for the postulate system requires not only its non-contradictoriness, i. e. the absence of two theorems one of which is the negation of the other, but also its ω-consistency, i. e. the absence of an enumerable series of theorems, one stating that some positive integer has a given property while the others state in succession that 0 does not have that property, 1 does not have that property, etc. While non-contradictoriness is a natural condition for in a contradictory system (containing some parts of logic, e. g. those allowing to form indirect proofs) everything can be proved, hence there are no unsolvable problems, ω-consistency is regarded a rather sophisticated condition. Hence it was a great progress that ROSSER succeeded[2]) in replacing the condition of ω-consistency by non-contradictoriness.

In this paper I shall give a simplified proof for ROSSER's theorem using, with appropriate modifications, the method by which I proved GÖDEL's theorem[3]). At the same time, I shall present the proof with the same degree of generality as I did that of GÖDEL's theorem in two recent publications[4]),

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[1]) K. GÖDEL, Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I, *Monatshefte für Math. und Phys.*, **38** (1931), p. 173—198.

[2]) B. ROSSER, Extensions of some theorems of Gödel and Church, *Journal of symbolic logic*, **1** (1936), p. 87—91, especially theorem II, p. 89. See also D. HILBERT and P. BERNAYS, *Grundlagen der Mathematik*. II (Berlin, 1939), p. 275—276.

[3]) L. KALMÁR, *a)* Egyszerű példa eldönthetetlen aritmetikai problémára, *Mat. és fiz. lapok*, **50** (1943), p. 1—23; *b)* Eine einfache Konstruktion unentscheidbarer Sätze in formalen Systemen, forthcoming in *Methodos*; and see footnote [4]).

[4]) L. KALMÁR, *a)* Une forme du théorème de Gödel avec des hypothèses minimales, *Comptes Rendus Acad. Sci. Paris*, **229** (1949), p. 963—965; *b)* Quelques formes générales du théorème de Gödel, *ibidem*, **229** (1949), p. 1047—1049.

i. e. without making use of the deductive structure of the postulate system in question. This enables me to formulate the proof without supposing any concept of symbolic logic so that it can be understood without preliminary knowledge.

1. Let us call a *theory*[5]) any ordered triad $\Theta = (A, P, \varkappa)$ formed of two arbitrary sets A and P and a function $\varkappa$ of one variable defined over P and taking elements of A as values.

We call the elements of A *assertions*[6]) or *propositions*, those of P *proofs*. If $\mathbf{a} = \varkappa(\mathfrak{p})$ $(\mathbf{a} \in A, \mathfrak{p} \in P)$, we call $\mathbf{a}$ the *conclusion* of $\mathfrak{p}$ and $\mathfrak{p}$ *a proof* of $\mathbf{a}$. A proposition which is the conclusion of a proof is called a *theorem* in Θ.

2. In this paper, we shall deal with special theories which we call *Rosser theories*. A Rosser theory is a theory Θ satisfying the conditions (a) to (e) below.

(a) Θ has to be *adequate to express negation*; i. e. a function ν of one variable has to exist, defined for some propositions and taking propositions as values. Moreover, $\nu(\mathfrak{p}) = \nu(\mathfrak{q})$ has to imply $\mathfrak{p} = \mathfrak{q}$.

If $\nu(\mathbf{a})$ is defined, we call $\mathbf{a}$ a *deniable* proposition and we call $\nu(\mathbf{a})$ the *negation* of $\mathbf{a}$, or the *contrary assertion* to $\mathbf{a}$; also $\mathbf{a}$ is called the contrary assertion to $\nu(\mathbf{a})$. Instead of $\nu(\mathbf{a})$, we shall write $\overline{\mathbf{a}}$. The theory Θ is called *contradictory* if for a deniable proposition $\mathbf{a}$, both $\mathbf{a}$ and $\overline{\mathbf{a}}$ are theorems in Θ; it is called *non-categorical* if for a deniable proposition $\mathbf{a}$, neither $\mathbf{a}$ nor $\overline{\mathbf{a}}$ is a theorem in Θ.

(b) Θ has to be *adequate to express precedence relation as well as its negation*; i. e. a set F has to exist and a function of three variables $\mathbf{a} = \pi(\mathbf{f}, k, l)$ defined for $\mathbf{f} \in F$ and for non-negative integers k, l, and taking deniable propositions as values. Moreover, $\pi(\mathbf{f}, k, l) = \overline{\pi(\mathbf{g}, i, j)}$ has to imply $\mathbf{f} = \mathbf{g}$, $k = i$, $l = j$; and $\pi(\mathbf{f}, k, l)$ has always to differ from $\overline{\pi(\mathbf{g}, i, j)}$.

We call the elements of F *functionals*. Instead of $\pi(\mathbf{f}, k, l)$ we shall write $k \prec_{\mathbf{f}} l$ (read: k precedes l in the course of values of $\mathbf{f}$); we call such

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5) I avail myself of this very convenient term, introduced for a slightly different purpose by A. CHAUVIN, Structures logiques, *Comptes Rendus Acad. Sci. Paris*, **228** (1949), p. 1085—1087. I do not say "postulate system" for I do not suppose that a theory is based on postulates; I do not say "formal system" for I do not suppose that a theory is formalized; I do not say "logic" for I do not suppose that a theory admits the usual logical inferences.

6) Of course, the theorems to be proved are true also if A is (e. g.) the complex plane, P the unit circle and $\varkappa$ a function analytical in the unit circle, provided the below conditions are fulfilled (as to the condition (d), see footnote [12])). Nevertheless, I use the terms "assertion," "proof" etc. for the only interesting cases of the theorems to be proved I know are those in which the elements of A are assertions indeed, those of P proofs etc. in the everyday sense; at least, I cannot regard the fact that neither of two values which are in a certain relation are taken by a function as surprising and so as interesting as the fact that neither of two contrary assertions can be proved in a theory.

a proposition a *precedence assertion* or a *positive precedence assertion*, its negation $\overline{k \prec_{\mathbf{f}} l}$ a *negative precedence assertion*.

(c) Θ has to be an *interpreted* theory; i. e. to each[7]) functional an arithmetical function[8]) has to be attached, called its *interpretation*.

If an arithmetical function φ is the interpretation of a functional $\mathbf{f}$, we say, φ is *representable* in Θ and we call $\mathbf{f}$ a[9]) *representation* of φ. A non-negative integer j is called a *counter-example* for a precedence assertion $k \prec_{\mathbf{f}} l$ and also for the negation $\overline{l \prec_{\mathbf{f}} k}$ of the converse precedence assertion $l \prec_{\mathbf{f}} k$ if we have[10])

$$\varphi(0) \neq k,\ \varphi(1) \neq k, \ldots, \varphi(j-1) \neq k,\ \varphi(j) = l$$

for the interpretation φ of the functional $\mathbf{f}$. A positive or a negative precedence assertion is called *false* if there is a counter-example for it[11]). The theory Θ is called *incorrect* if a false, positive or negative, precedence assertion is a theorem in it.

(d) Θ has to be an *enumerable* theory, i. e. its functionals as well as its proofs[12]) have to form a finite or an enumerable infinite set.

Consider a one-to-one correspondence ψ between the functionals of Θ and a subset of the *positive* integers (0 excluded!) as well as a one-to-one correspondence χ between the proofs of Θ and a subset of the positive integers. We call the positive integer attached to a functional by ψ or to a proof by χ is *Gödel number*. A positive or negative precedence assertion of the form $2i-1 \prec_{\mathbf{f}} 2i$ or $\overline{2i-1 \prec_{\mathbf{f}} 2i}$ for which i is the Gödel number of the

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[7]) For applications to particular theories, it would be convenient to loosen this condition by requiring an interpretation for *some* functionals only; however, this would have the same effect as to replace F by a subset of F.

[8]) We call a function *arithmetical* if it is defined for non-negative integers and takes non-negative integers as values.

[9]) Of course, an arithmetical function can have several representations. In general, it is a hard problem to decide if two different functionals represent the same arithmetical function.

[10]) For $j=0$, read $\varphi(j)=l$.

[11]) It would be natural to define a precedence assertion $k \prec_{\mathbf{f}} l$ to be false also in the case that neither k nor l is a value of the interpretation φ of $\mathbf{f}$; and to define a (positive or negative) precedence assertion to be *true* if and only if its contrary is false. Then, $k \prec_{\mathbf{f}} l$ would be true if (a) k and l are values of φ and the least integer j for which $\varphi(j)=k$ is less than the least integer i for which $\varphi(i)=l$; (b) k is a value of φ, l not; and it would be false in the other cases, i. e. if (c) k and l are values of φ and the least integer j for which $\varphi(j)=k$ is greater than or (in the case $k=l$) equal to the least integer i for which $\varphi(i)=l$; (d) l is a value of φ, k not; (e) neither k nor l is a value of φ. However, I do not need more of the concept of truth and falsehood of precedence assertions than defined above.

[12]) One could loosen this condition by requiring only that the functionals, and for an appropriate correspondence between the functionals and the positive integers, the diagonal proofs and disproofs (as defined below) form a finite or enumerable infinite set.

functional **f** is called a *diagonal proposition* (a positive or a negative one, respectively); the integer i is called its *index*. A proof the conclusion of which is a diagonal proposition is called a *diagonal proof* or a *diagonal disproof* according as its conclusion is positive or negative; the index of its conclusion is called the *index* of the diagonal proof or disproof too. The arithmetical function[13])

$$\varrho(m) = \begin{cases} 2i & \text{if the proof of Gödel number } m \text{ exists and is a diagonal proof of index } i, \\ 2i-1 & \text{if the proof of Gödel number } m \text{ exists and is a diagonal disproof of index } i, \\ 1 & \text{if the proof of Gödel number } m \text{ does not exists or is neither a diagonal proof nor a diagonal disproof} \end{cases}$$

is called the *index function* of Θ (belonging to the correspondences ψ and χ).

(e) Θ has to be *adequate to express its own index function*, i. e., for an appropriate choice of the correspondences ψ and χ, its index function has to be representable in Θ.[14])

3. Now we have the following

First form of the theorem of Rosser. *A Rosser theory is either contradictory or incorrect or non-categorical.*

Indeed, let **g** be the representation of the index function ϱ of a Rosser theory Θ and denote r its Gödel number. Consider the diagonal propositions $2r-1 \prec_g 2r$ and $\overline{2r-1 \prec_g 2r}$ of index r. If

$2r-1 \prec_g 2r$	$\overline{2r-1 \prec_g 2r}$

is a theorem in Θ, denote $\mathfrak{p}$ one of its proofs and s the Gödel number of $\mathfrak{p}$. Then $\mathfrak{p}$ is a diagonal

proof	disproof

of index r. By the definition of ϱ, we have

$\varrho(s) = 2r$.	$\varrho(s) = 2r-1$.

If for a positive integer $t < s$ we have

$\varrho(t) = 2r-1$,	$\varrho(t) = 2r$,

then, by the definition of ϱ, the proof $\mathfrak{q}$ of Gödel number t must exist and be a diagonal

disproof	proof

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[13]) By definition, we have $\varrho(0) = 0$ for 0 is not the Gödel number of any proof.

[14]) For the theories considered by Rosser (l. c.) and for the theories which are in general use in mathematics (e. g. the Peano postulate system for arithmetic, the Zermelo—Fraenkel postulate system for set theory, etc.) one shows easily be means of an enumeration method due to Gödel that, for an appropriate choice of the correspondences ψ and χ, their index functions are recursive or even elementary (see l. c. [3]) *a)*, or [3]) *b)*, footnote [9])), and, as easily seen, every elementary (or even recursive) function is representable in them; hence, condition (e) is fulfilled for these theories.

of index r, for

$2r-1 \neq 0,$ | $2r \neq 0,$

the integer 0 being

even. | not the Gödel number of any functional.

Thus the conclusion of $\mathfrak{q}$, the

negative | positive

diagonal proposition of index r, i. e.

$\overline{2r-1 \prec_{g} 2r},$ | $2r-1 \prec_{g} 2r,$

is also a theorem in Θ and Θ is *contradictory.* If, on the contrary, we have

$\varrho(t) \neq 2r-1$ | $\varrho(t) \neq 2r$

for $t=1, 2, \ldots, s-1$ (and, on account of $\varrho(0)=0$, for $t=0$ too), then we have the counter-example s for the theorem

$2r-1 \prec_{g} 2r$ | $\overline{2r-1 \prec_{g} 2r}$

and Θ is *incorrect.* If neither $2r-1 \prec_{g} 2r$ nor $\overline{2r-1 \prec_{g} 2r}$ is a theorem in Θ, then Θ is *non-categorical.*

4. The case that the theory Θ is incorrect can be eliminated (i. e. replaced by contradictoriness) if we make some more conditions enabling us to prove, by means of a counter-example for a positive or negative precedence assertion, the contrary assertion. Thus we call a theory Θ a Rosser theory in the strong sense if, besides being a Rosser theory, it is satisfying the conditions (f) to (h) below.

(f) Θ has to be *adequate to express equality as well as inequality;* i. e. a set N has to exist and two functions $\mathbf{a}=\varepsilon(\mathbf{n}, k)$ and $\mathbf{n}=v(\mathbf{f}, l)$ of two variables, the former defined for $\mathbf{n} \in N$ and for non-negative integers k, and taking deniable propositions as values, whereas the latter defined for functionals $\mathbf{f}$ and for non-negative integers l, and taking elements of N as values.

We call the elements of N *numerals.* Instead of $\varepsilon(\mathbf{n}, k)$, we shall write $\mathbf{n}=k$; we call such a proposition an equation, its negation $\overline{\mathbf{n}=k}$, which we shall write $\mathbf{n} \neq k$, an *inequality.* Instead of $v(\mathbf{f}, l)$, we shall write $\mathbf{f}(l)$; we call such a numeral a *function value.*

(g) Θ has to be *deductively interpreted;* i. e. for any functional $\mathbf{f}$, for its representation φ and for any non-negative integer l, the equation[15]) $\mathbf{f}(l)=\varphi(l)$, and for any non-negative integer $k \neq \varphi(l)$, the inequality $\mathbf{f}(l) \neq k$ have to be theorems in Θ.

(h) Θ has to *admit inference from a counter-example for a positive or negative precedence assertion to the contrary assertion;* i. e., for any functional $\mathbf{f}$ and for any non-negative integers j, k, l for which $\mathbf{f}(0) \neq k$, $\mathbf{f}(1) \neq k, \ldots,$ $\mathbf{f}(j-1) \neq k$, and $\mathbf{f}(j)=l$ are theorems in Θ, the same has to hold for the propositions $l \prec_{\mathbf{f}} k$ and $\overline{k \prec_{\mathbf{f}} l}$.

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[15]) $\mathbf{f}(l)=\varphi(l)$ is an equation indeed, for $\mathbf{f}(l)$ is a numeral and $\varphi(l)$ is a non-negative integer.

5. Now we have the following

Second form of the theorem of Rosser. *A Rosser theory in the strong sense is either contradictory or non-categorical.*

Indeed, let Θ be a Rosser theory in the strong sense. By the first form of the theorem of ROSSER, Θ is either contradictory or incorrect or non-categorical. In the case Θ is incorrect, there is a theorem in Θ which is a false positive or negative precedence assertion; i. e. it has either the form $k \prec_{\mathbf{f}} l$ or the form $\overline{l \prec_{\mathbf{f}} k}$ with non-negative integers k, l and a functional $\mathbf{f}$ for which we have for some non-negative integer j

$$\varphi(0) \neq k,\ \varphi(1) \neq k, \ldots, \varphi(j-1) \neq k,\ \varphi(j) = l,$$

φ denoting the interpretation of f. By (g) we see that

$$\mathbf{f}(0) \neq k,\ \mathbf{f}(1) \neq k, \ldots, \mathbf{f}(j-1) \neq k,\ \mathbf{f}(j) = l$$

are theorems in Θ; hence, by (h), $l \prec_{\mathbf{f}} k$ and $\overline{k \prec_{\mathbf{f}} l}$ are also theorems in Θ and thus, Θ is contradictory.

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