On straight line representation of planar graphs.

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In the present note I shall prove that if a finite graph can be represented on the plane at all, it can be represented with straight segments as edges too, provided that the graph does not contain two edges joining the same nodes[1]).

Let us sketch the main ideas of the proof. Suppose the theorem holds for graphs with n nodes. Let G be a graph with $n+1$ nodes and having no multiple edges. By letting one of its edges shrink into a point we get a graph G^* having n nodes. Suppose first this graph G^* has no multiple edges; then by hypothesis we can draw it with straight segments as edges. Now, stretch the node which corresponds to the shrinken edge of G into a short straight edge. Thus we get G drawn with straight segments as edges. Secondly, if G^* has multiple edges, we can divide G by a circuit of three edges into two subgraphs; drawing these with straight segments as edges, one outside and the other inside of a triangle, we obtain a straight representation of the graph G.

Section **1** is devoted to the most important definitions. After stating our theorem in its exact form we show that it suffices to prove it for graphs, each region of which is bounded by a circuit of three edges (sections **2** and **3**). Section **4** is dealing with such "triangulated" graphs. In section **5** we construct to any given triangulated simple graph another which has one node less, but which is not necessarily simple. Making use of this reduction, we establish the theorem in section **6.**

1. In what follows we call a finite graph *planar* if it is represented

[1]) There is a well known theorem of KURATOWSKI (Sur le problème des courbes gauches en topologie, *Fundamenta Math.*, **15** (1930), pp. 271—283) which gives a necessary and sufficient condition of drawing an abstract graph on the plane. It was Mr. T. SZELE who posed the question what are the necessary and sufficient conditions *that a given abstract graph could be drawn with straight segments as edges.* He conjectured the above theorem.

on the plane with Jordan arcs as edges, meeting only at their endpoints: the *nodes* of the graph. If every edge of a planar graph is a straight segment, we call it a *straight graph.* A graph having no multiple edges (i. e. more than one edge joining the same nodes), will be called a *simple graph.* The plane is divided by the edges of the graph into a number of regions called the *regions of the graph.* A graph each region of which is bounded by a circuit of three edges will be called a *triangulated graph.*

The definition of *adjoining elements* is the following. A node is adjoining an edge if it is an endpoint of the edge. A node or an edge is adjoining to a region if it lies on the boundary of the region.

Definition. The graphs G and G' will be called *equal* if there can be established a *one-to-one correspondence* between the nodes, edges and regions of G and G' such that (1) adjoining elements correspond to adjoining elements, (2) the regions containing the point at infinity correspond to each other.

We shall prove the following

Theorem. *Every simple graph is equal to a straight graph*[2]).

2. Let G' be a given graph, D one of its regions, P and Q two different nodes adjoining to D. Let us join P and Q within D by an edge (Jordan arc) which meets G' only in the points P and Q. The graph thus completed will be denoted by G. If G is equal to a straight graph, then by leaving off the segment (edge) corresponding to the new edge we get a straight graph which is equal to G'. By induction we get easily the

Lemma 1. *If a graph is equal to a straight graph, then any of its subgraphs is also equal to a straight graph.*

As the structure of a graph may be "simplified" by adding new edges, it suffices, by our above remark, to deal with such completed graphs.

As a consequence of Lemma 1, our theorem will be proved if we establish the following two lemmas.

Lemma 2. *Every simple graph is a subgraph of a triangulated simple graph.*

Lemma 3. *Every triangulated simple graph is equal to a straight graph.*

3. In this section we prove Lemma 2.

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[2]) Under topological transformations of the whole plane, graphs are transformed into equal ones. The converse is not generally true. However, we remark that our theorem holds true even if we modify the definition of equality, calling two graphs equal if they may be carried one into the other by a topological transformation of the whole plane.

Let be G a simple graph containing at least four nodes. We construct a triangulated simple graph G' having G as its subgraph.

The construction of G' is as follows. Let us select two nodes on the boundary of a region of G which are connected neither on the boundary of the region nor outside it. Connecting these nodes by an edge (a Jordan arc) which lies, except its endpoints, inside the region, we get a new graph: G_1. By definition, G_1 is a simple graph. Proceeding in a similar way with the graph G_i we construct a graph G_{i+1} $(i = 1, 2, \ldots)$. As the number of nodes in G_i is the same as in G_{i+1}, we arrive at last to a graph G_k to which no G_{k+1} can be constructed, i. e. such that every couple of nodes adjoining to the same region of G_k are connected by an edge.

This graph $G' = G_k$ is connected[3]). For in the opposite case there would exist a region D which is adjoining two different parts of G'. Choosing the nodes P and Q on different parts of G', both adjoining to this region D, they would be connected by an edge. This is a contradiction.

Now we show that this connected simple graph G' is triangulated, i. e. every region D of it is bounded by a circuit of three edges. If there were on the boundary of D only one or two edges[4]), then G' and hence G would contain only three nodes, as G' is connected and simple. This contradicts to our hypothesis. Thus there are at least three edges a, b, c on the boundary of D. If these do not form a circuit, then there are at least four nodes P, Q, R, S on the boundary of the region D. But G' contains edges connecting the nodes adjoining to D, i. e. edges PQ, PR, PS, QR, QS, RS. These edges divide the plane into four regions, on the boundary of each of which there are only three nodes, and so D, which is a part one of the four regions, can not have all four points as boundary points. Having thus arrived to a contradiction, we have proved that all regions of G' are bounded by circuits of three edges.

4. Before proving the Lemma 3, we shall deal in this section with triangulated graphs and prove the following

L e m m a 4. *Let G be a simple triangulated graph which has at least four nodes. If $PP_1, PP_2, \ldots, PP_k$ are all the edges starting from P, in their cyclic order, then the edges $P_1P_2, P_2P_3, \ldots, P_kP_1$ are contained in G and form a circuit C_P which separates P from every other node of G.*

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3) A graph is connected if it is connected as a point set. If in a graph every region is bounded by a circuit of three edges then it is connected. Conversely, if the graph is connected, then its regions are simply connected.

4) On the boundary of a region, there is always an edge. If there are only two nodes on the boundary of the region, then the graph, being connected, consists of this single edge.

Proof. Let us suppose first that P does not lie on the boundary of the region containing the point at infinity. In this case, P lies inside some circuits of the graph. Hence there exists an innermost circuit C'_P, with the nodes $P'_1, P'_2, \ldots, P'_m$, which contains P in its interior. The region adjoining to $P'_i P'_{i+1}$ inside C'_P has a third node on its boundary, say P'. We assert that $P' = P$. For, if $P' \neq P$, then $[P'_1P'_2, P'_2P'_3, \ldots, P'_iP', P'P'_{i+1}, \ldots, P'_mP'_1]$ is a circuit having P in its interior and lying inside C'_P which is a contradiction. Thus the triangular regions adjoining to the edges of C'_P have the common third node P, and as G has no multiple edges, these regions form a simply connected region, i. e. they fill out the interior of C'_P. As G has no node inside C'_P except P, C'_P coincides necessary with C_P and the lemma is proved for this case.

Analogously, if P is adjoining to the unbounded region, then $C_P = [P_1P_2, P_2P_3, \ldots, P_kP_1]$ is the outermost circuit which contains P in its exterior. The proof runs on the same lines as above, and hence our lemma is completely established.

5. Let G be a triangulated simple graph having more than three nodes. To prepare the proof by induction we construct from G a not necessarily simple graph G^* having one node less.

Construction of G^*. Let P be a node of G which does not adjoin to the unbounded region. If we leave off the edges starting from P, the graph thus obtained will coincide with G outside C_P and the interior of C_P will be empty. Let us join the point P_1 with the points $P_3, P_4, \ldots, P_{k-1}$ by lines not crossing each other and lying inside C_P. Denote by G^* the graph thus obtained.

The connection between G and G^* is stated in the following

Lemma 5. *If G^* is not a simple graph, then G has a circuit of three edges which separates two nodes.*

Proof. Suppose G^* is not simple. This can be the case only if two nodes are connected by a new edge and an edge running outside C_P, hence there is a node P_i $(2 \leq i \leq k-1)$ which is joined with P_1 by an edge outside C_P. As the nodes P_1, P_2, P_i, P_k follow in this order on C_P and are all different, the circuit $[P_1P, PP_i; P_iP_1]$ (P_iP_1 *outside* C_P) separates P_2 from P_k. Hence our lemma is proved.

6. Now we are able to prove Lemma 3. As our lemma is trivial for graphs having three nodes, suppose it holds for graphs having ν nodes, where $3 \leq \nu \leq n$. Let G be a graph with $n+1$ nodes. Let P be a node of G, not adjoining to the unbounded region and construct the graph G^* as above. As P does not lie on the boundary of the region which contains the point at infinity, this region is bounded by the same circuit in G^* as in G. We distinguish two cases according to as G^* is simple or not.

If G^* is simple, then by hypothesis there exists a straight graph $\bar{G}^*$ equal to $\bar{G}^*$. Denote by $\bar{X}$ the node of G^* corresponding to the node X of G^*. Consider the circuit $\bar{C}_P=[\bar{P}_1 P_2, \ldots, \bar{P}_k \bar{P}_1]$ of the straight graph $\bar{G}^*$. The straight lines through $\bar{P}_i \bar{P}_{i+1}$ $(2 \leqq i \leqq k-1)$ do not pass through $\bar{P}_1$ as otherwise the edges $\bar{P}_1 \bar{P}_i$ and $\bar{P}_1 \bar{P}_{i+1}$ would have a common segment. One of the two half-planes defined by the straight line $\bar{P}_i \bar{P}_{i+1}$ $(2 \leqq i \leqq k-1)$ is thus characterised by the point $\bar{P}_1$. The intersection of these half-planes (for $i=2, 3, \ldots, k-1$) is a convex region K' with inner points. Let K be the common part of K' and of the interior of $\bar{C}_P$. It can be easily seen that any segment connecting an arbitrary inner point of K with a boundary point of $\bar{C}_P$ lies inside $\bar{C}_P$ but its endpoint. Leaving off the segments $\bar{P}_1 \bar{P}_3, \bar{P}_1 \bar{P}_4, \ldots, \bar{P}_1 \bar{P}_{k-1}$ from $\bar{G}^*$, the interior of $\bar{C}_P$ will be empty. Choosing a point $\bar{P}$ inside K and drawing the segments $\bar{P}\bar{P}_1, \bar{P}\bar{P}_2, \ldots, \bar{P}\bar{P}_k$, we get a graph which is equal to G.

If G^* is not a simple graph, then, by Lemma 5, there exists a circuit of three edges Δ in G which separates two nodes. In this case Δ and the edges outside it form a triangulated graph G_1. Similarly Δ and the edges inside it form another triangulated graph G_2. Both graphs have at most n nodes. Let $\bar{G}_i$ be a straight graph equal to G $(i=1, 2)$. The triangle Δ_1 corresponding to Δ in $\bar{G}_1$ is empty; $\bar{G}_2$ lies inside Δ_2 corresponding to Δ in $\bar{G}_2$ (this holds by the property (2) of equality given in section 1). By an aptly chosen affine transformation we can transform Δ_2 into Δ_1 so that adjoining regions in G shall be transformed into adjoining regions. Thus Lemma 3 has been proved and this concludes the demonstration of our theorem.

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