On rational polynomials.

By P. TURÁN in Budapest.

1. E. LAGUERRE[1]) proved the following well-known theorem:

Let $x_1 = 1$ and $x_2 = -1$ be two consecutive roots of the polynomial $f(x)$ of degree n, with real roots only. Then $f'(x) \neq 0$ in the intervals $\left(1-\frac{2}{n}, 1\right)$ and $\left(-1, -1+\frac{2}{n}\right)$.

This theorem is the best possible in the sense that for the polynomials $f_1(x) = (x+1)(x-1)^{n-1}$ and $f_2(x) = (x-1)(x+1)^{n-1}$, the conditions of the theorem are fulfilled and $f_1'(x) = 0$ for $x = -1 + \frac{2}{n}$ and $f_2'(x) = 0$ for $x = 1 - \frac{2}{n}$. Of course, we may suppose the coefficients of $f(x)$ to be real.

The theorem of LAGUERRE was improved by P. MONTEL[2]), who supposed only that the coefficients are real and that $f(x)$ does not vanish in the strip $-1 < \Re z < 1$.[3]) A further generalisation was obtained by J. VON SZ. NAGY[4]), who has shown under the same preliminary hypothesis of real coefficients, that Laguerre's statement holds good even if the roots are supposed to lie outside the circle $|z| < 1$.

The essential of these theorems may be characterized shortly as that under certain hypothesis the places of local maxima cannot lie „too near" to the roots. Of course, the same is true for the values ξ of x, for which

$$\max_{-1 \leqq x \leqq +1} |f(x)| = |f(\xi)|,$$

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[1]) E. CESÀRO, Solution de la Question 1338, *Nouvelles Annales de Math.,* (3) **4** (1885), pp. 328—330.

[2]) P. MONTEL, Sur les zéros des derivées des fonctions analytiques, *Bull. de la Soc. Math. France,* **58** (1930), pp. 105—126.

[3]) $\Re z$ denotes the real part of z.

[4]) J. VON SZ. NAGY, Über die reellen Nullstellen des Derivierten eines Polynoms mit reellen Koeffizienten, *these Acta,* **8** (1936), pp. 42—53.

shortly, for the places ξ of absolute maximum. As Mr. A. Rényi remarked, this fact is true without any hypothesis, even the reality of the coefficients is not necessary and by using Markoff's well known theorem he obtained the first estimation

$$-1+\frac{1}{n^2}<\xi<1-\frac{1}{n^2}.$$

This interval is not the best possible. Our theorems I and II give the narrowest interval for the most important case of real coefficients.

Theorem I. *Let n be even and $f(x)$ a polynomial of degree n with real coefficients such that $f(1)=f(-1)=0$ and $f(x)$ does not vanish for $-1<x<1$. If the absolute maximum within the interval $-1\leqq x\leqq 1$ is attained for $x=\xi$, then the inequality*

$$(1)\qquad -\cos\frac{\pi}{n}\leqq\xi\leqq\cos\frac{\pi}{n}\sim 1-\frac{\pi^2}{2n^2}$$

holds true. The bounds $\pm\cos\frac{\pi}{n}$ are the best possible; for $n=2$ we have obviously always $\xi=0=\pm\cos\frac{\pi}{2}$, while for $n\geqq 4$ $\xi\neq\cos\frac{\pi}{n}$.

Theorem II. *Let n be odd and let $f(x)$ satisfy the conditions of theorem I. Then, with the same notation, the inequality*

$$(2)\qquad -\frac{3\cos\frac{\pi}{n}-1}{1+\cos\frac{\pi}{n}}<\xi<\frac{3\cos\frac{\pi}{n}-1}{1+\cos\frac{\pi}{n}}\sim 1-\frac{\pi^2}{2n^2}$$

holds good. The bounds $\pm\frac{3\cos\frac{\pi}{n}-1}{1+\cos\frac{\pi}{n}}$ are the best possible.[5])

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[5]) It is interesting to note that — contrary to the usual in the theory of polynomals — for $n\geqq 3$ equality can *not* be attained. It is easy to generalize theorems I and II, if we introduce the notion of the Chebishef system, (see S. Bernstein, *Leçons sur les propriétés extrémales* etc., Collection Borel, 1926). One calls a system $\varphi_0(x),\ldots,\varphi_n(x)$ of bounded and continuous functions a Chebishef system, if any $F_n(x=a_0\varphi_0(x)+\ldots+a_n\varphi_n(x)$ with $|a_0|^2+|a_1|^2+\ldots+|a_n|^2\neq 0$ has at most n roots. Here every root is counted as simple, except those in which $F_n(x)$ does not change its sign; these we consider as double ones. The problem to be treated is to consider the minimal distance of those values $x=\xi_\nu$ from two consecutive roots of $F_n(x)$, a and b, in which $F_n(x)$ takes its absolute maximum for $a\leqq x\leqq b$ and $a<\xi_\nu<b$. For n even we obtain that these values ξ_ν cannot approximate a or b better then the least and greatest places of maxima of that $F_n(x)=a_0\varphi_0(x)+\ldots+a_{n-1}\varphi_{n-1}(x)+\varphi_n(x)$, which has the minimal deviation from 0. For n odd the same is furnished by a suitably "dilated" minimizing polynomial. Thus these theorems give a new extremal property of the polynomials with minimal deviation from 0.

2. There is another question of the same kind. Let us fix the point ξ where the polynomial $g(x)$ of degree n has to take its absolute maximum and let us ask for the minimal distance of the next root. This question was investigated by D. LÁZÁR, who however did not publish his results. The corresponding question for trigonometric polynomials was solved long ago by M. RIESZ[6]); he proved that the distance between ξ and any root equals at most $\frac{\pi}{2n}$, equality occuring only for the polynomials $c_1 \cos(n\vartheta + c_2)$. A weaker inequality, with 1 instead of $\frac{\pi}{2}$, follows easily from the well known theorem of S. BERNSTEIN. The knowledge of the best possible factor $\frac{\pi}{2}$, however, has many important applications in the theory of interpolation[7]) and in the theory of uniform distribution[8]).

The theorem of M. RIESZ suggests naturally the following problem. Suppose the polynomial $G(z)$ of degree n takes in $|z| \leqq 1$ the maximum of absolute value at $z = 1$; how near to this point can lie roots (I) on $|z| = 1$, (II) in the circle $|z| \leqq 1$? It follows easily from a theorem of M. RIESZ, that $f(z) \neq 0$ in the domain $|1 - z| < \frac{1}{n}$. The question (I) can be answered by

T h e o r e m III. *The polynomial $G(z)$ definied above does not vanish on that arc of the unit-circle for which $|\text{arc}\, z| < \frac{\pi}{n}$, even if we allow complex coefficients. $G\left(e^{\pm i \frac{\pi}{n}}\right) = 0$ only if $G(z) = c(1 + z^n)$.*

As to question (II), we note that to every point z_0 of the radii $\text{arc}\, z = \pm \frac{\pi}{n}$, $0 \leqq |z| \leqq 1$, there exists an appropriate polynomial of our class vanishing at z_0. In fact, let be $0 \leqq \varrho \leqq 1$ and

$$G_1(z) = \frac{z - \varrho e^{i\frac{\pi}{n}}}{1 - \varrho e^{-i\frac{\pi}{n}} z}\left[1 - \left(\varrho e^{-i\frac{\pi}{n}}\right)^n z^n\right] = \frac{z - \varrho e^{i\frac{\pi}{n}}}{1 - \varrho e^{-i\frac{\pi}{n}} z}(1 + \varrho^n z^n).$$

Then for $|z| = 1$ we have $|G_1(z)| = |1 + \varrho^n z^n| \leqq |G_1(1)|$ and $G_1\left(\varrho e^{i\frac{\pi}{n}}\right) = 0$. The same argument holds for $\varrho e^{-i\frac{\pi}{n}}$.

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6) M. RIESZ, Eine trigonometrische Interpolationsformel etc., *Jahresbericht der Deutschen Math. Vereinigung,* **23** (1915), pp. 354—368.

7) P. ERDŐS and P. TURÁN, On Interpolation, III, *Annals of Math.,* **41** (1940), pp. 510—553.

8) P. ERDŐS and P. TURÁN, On the uniformly dense distribution of certain sequences of points, *Annals of Math.,* 41 (1940), pp. 163—173.

Now we proceed to the proof of our theorems; our method is similar to that used by M. RIESZ.

3. Proof of theorem I. Suppose at first $n \geqq 4$. We consider the particular polynomial

$$H(x) = \frac{1}{2}(1 - T_n(x)),$$

where $T_n(x)$ denotes, as usual, the n^{th} Chebishef polynomial, i. e. $T_n(\cos\vartheta) = \cos n\vartheta$. For $-1 \leqq x \leqq +1$ we have $0 \leqq H(x) \leqq 1$ and

$$(3) \qquad H\left(\cos\frac{2\nu-1}{n}\pi\right) = 1,\ H\left(\cos\frac{2\nu}{n}\pi\right) = 0, \qquad \nu = 1, 2, \ldots, \frac{n}{2};$$

n being even we have $H(1) = H(-1) = 0$. As to the polynomial $f(x)$, we have $f(1) = f(-1) = 0$ and we may suppose that for $-1 < x < +1$

$$(4) \qquad 0 < f(x) \leqq 1, \quad \max_{-1 \leqq x \leqq +1} f(x) = 1.$$

This, together with (3), gives that the polynomial $f(x) - H(x)$ of degree n has at least one root in every interval

$$\cos\frac{\nu\pi}{n} \leqq x \leqq \cos\frac{(\nu-1)\pi}{n}, \qquad \nu = 2, 3, \ldots, n.$$

It remains only to consider the interval $\cos\frac{\pi}{n} \leqq x \leqq 1$. If $f(x)$ would attain its absolute maximum for the interval $[-1, +1]$ at a point ξ for which $\cos\frac{\pi}{n} < \xi < 1$, then the relations

$$f\left(\cos\frac{\pi}{n}\right) \leqq 1 = H\left(\cos\frac{\pi}{n}\right),\ f(\xi) = 1 > H(\xi),\ f(1) = H(1)\,(=0)$$

would hold and the interval $\cos\frac{\pi}{n} < x \leqq 1$ would contain at least two roots of $f(x) - H(x)$. But then this polynomial of order n would have at least $n+1$ roots and so $f(x)$ would be identical to $H(x)$; this is impossible, for $H(x)$ does not belong to our class of polynomials (the roots $x = \pm 1$ being not *consecutive* ones). This contradiction evidently holds even if $\xi = \cos\frac{\pi}{n}$. So the inequality $\xi < \cos\frac{\pi}{n}$ is proved for $n \geqq 4$; for this case the proof of the inequality $\xi > -\cos\frac{\pi}{n}$ runs on similar lines. For $n \geqq 4$ it remains only to show that the constant $\cos\frac{\pi}{n}$ is the best possible. From (3) and

$$T_n(x) = 2^{n-1}x^n + \ldots, \qquad H(x) = -2^{n-2}x^n + \ldots$$

follows, that

$$H(x)=2^{n-2}(1-x^2)\prod_{\nu=1}^{\frac{n}{2}-1}\left(x-\cos\frac{2\nu\pi}{n}\right)^2.$$

We consider at first the polynomial

$$H_1(x,\varepsilon)=2^{n-2}(1-x^2)\left(x-\cos\frac{2\pi}{n}+i\varepsilon\right)\cdot$$

$$\cdot\left(x-\cos\frac{2\pi}{n}-i\varepsilon\right)\prod_{\nu=2}^{\frac{n}{2}-1}\left(x-\cos\frac{2\nu\pi}{n}\right)^2=$$

$$=2^{n-2}(1-x^2)\left|x-\cos\frac{2\pi}{n}+i\varepsilon\right|^2\prod_{\nu=2}^{\frac{n}{2}-1}\left(x-\cos\frac{2\nu\pi}{n}\right)^2$$

(the last factor being 1 for $n=4$). Since for $\varepsilon\to 0$ $H_1(x,\varepsilon)$ converges to $H(x)$ uniformly in $[-1,+1]$, the places of *relative* maximum of $H_1(x,\varepsilon)$ converge to those of $H(x)$, i. e. to the places $x=\cos\frac{2\nu-1}{2n}\pi$. A simple geometrical consideration shows that, choosing ε sufficiently small, the place of *absolute* maximum is in the neighbourhood of $\cos\frac{\pi}{n}$. Let ε be fixed and consider the polynomial

$$H_2(x,\varepsilon,\eta)=2^{n-2}(1-x^2)\left(x-\cos\frac{2\pi}{n}+i\varepsilon\right)\cdot$$

$$\cdot\left(x-\cos\frac{2\pi}{n}-i\varepsilon\right)\prod_{\nu=2}^{\frac{n}{2}-1}\left(x-\cos\frac{2\nu\pi}{n}+i\eta\right)\left(x-\cos\frac{2\nu\pi}{n}-i\eta\right)=$$

$$=2^{n-2}(1-x^2)\left|\left(x-\cos\frac{2\pi}{n}+i\varepsilon\right)\prod_{\nu=2}^{\frac{n}{2}-1}\left(x-\cos\frac{2\nu\pi}{n}+i\eta\right)\right|^2.$$

Choosing η sufficiently small, $H_2(x,\varepsilon,\eta)$ takes its absolute maximum in the neighbourhood of $\cos\frac{\pi}{n}$ and does not vanish in $-1<x<+1$. Finally the polynomial

$$H_3(x,\varepsilon,\eta)=\frac{H_2(x,\varepsilon,\eta)}{\max\limits_{-1\leq x\leq+1}|H_2(x,\varepsilon,\eta)|}$$

belongs to our class and takes its absolute maximum in $[-1,+1]$ arbitrary near to $\cos\frac{\pi}{n}$. Thus the case $n\geqq 4$ is settled; for $n=2$ our class reduces obviously to the polynomials $c(1-x^2)$ (c real), which attain their absolute maximum at $x=0=\cos\frac{\pi}{2}$, qu. e. d.

4. Proof of theorem II. This runs on the same lines as that of theorem I, but we need another auxiliary polynomial instead of $H(x)$. Let be $n \geqq 5$, $T_n(x)$ the same as before and

$$H_4(x) = \frac{1}{2}\left[1 - T_n\left(\frac{1-\cos\frac{\pi}{n}}{2} + \frac{1+\cos\frac{\pi}{n}}{2}x\right)\right]. \tag{5}$$

This polynomial has the following properties. For $-1 \leqq x \leqq +1$

$$0 \leqq H_4(x) \leqq 1, \tag{6}$$

$H_4(x)$ vanishes if and only if

$$x = \frac{2\cos\frac{2\nu\pi}{n} - 1 + \cos\frac{\pi}{n}}{1+\cos\frac{\pi}{n}} = \eta_\nu, \qquad \nu = 0, 1, \ldots, \frac{n-1}{2}. \tag{7}$$

The equation $H_4(x) = 1$ holds if

$$x = \frac{2\cos\frac{(2\nu-1)\pi}{n} - 1 + \cos\frac{\pi}{n}}{1+\cos\frac{\pi}{n}} = \varrho_\nu, \qquad \nu = 1, 2, \ldots, \frac{n-1}{2}. \tag{8}$$

We have obviously

$$1 = \eta_0 > \varrho_1 > \eta_1 > \ldots > \varrho_{\frac{n-1}{2}} > \eta_{\frac{n-1}{2}} = -1,$$

further

$$H_4(1) = H_4(-1) = 0, \qquad H_4'(-1) = 0.$$

As to the polynomial $f(x)$ to be considered, we may suppose, as before, relations (4) and $f(1) = f(-1) = 0$. According to (6), (8) and (7), the polynomial $f(x) - H_4(x)$ of degree n has in each interval

$$\eta_\nu \leqq x \leqq \varrho_\nu, \qquad \varrho_{\mu+1} \leqq x \leqq \eta_\mu,$$
$$\nu = 1, 2, \ldots, \frac{n-3}{2}; \quad \mu = 1, 2, \ldots, \frac{n-3}{2}$$

at least one, and so together at least $n-3$ roots. Again, let us consider the interval $-1 = \eta_{\frac{n-1}{2}} \leqq x \leqq \varrho_{\frac{n-1}{2}}$. The polynomial $f(x) - H_4(x)$ has here at last two new roots, since $H_4(\varrho_{\frac{n-1}{2}}) = 1$, $H_4(-1) = H_4'(-1) = 0$. Let $f(x)$ attain its absolute maximum, within the interval $[-1, +1]$, at $x = \xi$ and suppose, contrary to our statement, that $\xi > \frac{3\cos\frac{\pi}{n} - 1}{1+\cos\frac{\pi}{n}} = \varrho_1$.

By the argument used in proving theorem I, the polynomial $f(x) - H_4(x)$ of degree n would have at least two roots in $[\varrho_1, 1]$ and so this polynomial would admit $(n+1)$ roots at least, whence $f(x) = H_4(x)$; this is impossible, for $H_4(x)$ does not belong to our class. So we have got

$$\xi < \varrho_1 = \frac{3\cos\frac{\pi}{n} - 1}{1+\cos\frac{\pi}{n}}.$$

By the same argument $\xi > -\dfrac{3\cos\frac{\pi}{n}-1}{1+\cos\frac{\pi}{n}}$ for $n \geqq 5$, and we can see that the interval $\left[-\dfrac{3\cos\frac{\pi}{n}-1}{1+\cos\frac{\pi}{n}}, \dfrac{3\cos\frac{\pi}{n}-1}{1+\cos\frac{\pi}{n}}\right]$ is the narrowest one. For the case $n=3$ the proof is quite obvious.

5. Proof of theorem III. Let us consider the expression $|G(z)|^2$ with $z=e^{i\vartheta}$; this is a non-negative trigonometric polynomial of order n, which, according to our hypothesis, attains its absolute maximum at $\vartheta=0$, i. e. $z=1$. Without loss of generality, we may suppose $|G(1)|=1$ and $n\geqq 2$. We consider the auxiliary polynomial $K(z)=\dfrac{1+z^n}{2}$. Again

$$(9) \qquad |K(z)|^2_{z=e^{i\vartheta}} = \frac{1}{4}|1+z^n|^2 = \cos^2\frac{n\vartheta}{2} = \frac{1+\cos n\vartheta}{2} \equiv H_5(\vartheta),$$

and

$$H_5\left(\frac{2\nu\pi}{n}\right)=1, \quad H_5\left(\frac{2\nu-1}{n}\pi\right)=0 \qquad (\nu=0,1,\ldots,n-1).$$

Since the curve $y=|G(e^{i\vartheta})|^2$ runs in the strip $0\leqq y\leqq 1$, the trigonometric polynomial $|G(e^{i\vartheta})|^2 - H_5(\vartheta)$ of order n has at least one root in every interval $\dfrac{l\pi}{n}\leqq\vartheta\leqq\dfrac{(l+1)\pi}{n}$ $(l=1,2,\ldots,2n-2)$. Since $|G(e^{i\vartheta})|^2$ and $H_5(\vartheta)$ attain their absolute maxima at $\vartheta=0$, this point is at least a double root of $|G(e^{i\vartheta})|^2-H_5(\vartheta)$. If we suppose $G(e^{i\vartheta_0})=0$ with $0<\vartheta_0<\dfrac{\pi}{n}$, than according to the inequalities

$$|G(e^{i\vartheta_0})|^2 - H_5(\vartheta_0) \leqq 0, \quad \left|G\left(e^{i\frac{\pi}{n}}\right)\right|^2 - H_5\left(e^{i\frac{\pi}{n}}\right) > 0,$$

$|G(e^{i\vartheta})|^2-H_5(\vartheta)$ would have one more root between ϑ_0 and $\dfrac{\pi}{n}$. Collecting these results we obtain $|G(e^{i\vartheta})|^2\equiv H_5(\vartheta)$. It is well known[9]), that given a non-negative trigonometric polynomial $H_6(\vartheta)$, there exists at

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[9]) Theorem of Fejér—Riesz; see L. Fejér, Über trigonometrische Polynome, *Journal für die reine und angewandte Math.*, **146** (1915), pp. 53–82.

least one, but generally more than one rational polynomial $G_1(z)$ of degree n such that $|G_1(e^{i\vartheta})|^2 \equiv H_6(\vartheta)$. But in the case $H_6(\vartheta) = \frac{1+\cos n\vartheta}{2}$ (or generally, when the roots of $H_6(\vartheta)$ are real), there is only one $G_1(z)$ of this kind. Indeed, taking (9) into account, it follows, that

$$(10) \qquad \left| \frac{G_1(z)}{1+z^n} \right|^2_{z=e^{i\vartheta}} \equiv \frac{1}{4}$$

i. e. $G_1\left(e^{i\frac{\pi}{n}}\right) = G_1\left(e^{\frac{3i\pi}{n}}\right) = \ldots = G_1\left(e^{(2n-1)\frac{i\pi}{n}}\right) = 0$. But this means that $G_1(z) \equiv c(1+z^n)$ and compared with (10), this gives $G_1(z) = \frac{1+z^n}{2}$, qu. e. d.

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