Contributions to the Theory of Minimal Surfaces.[1])

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Introduction.

The present paper contains results which I obtained while studying the work of H. A. Schwarz on minimal surfaces.

Schwarz constructed several examples of portions of minimal surfaces which do *not* have a smallest area if compared with surfaces bounded by the same curve. By very beautiful computations he obtains a convenient formula for the second variation of the area-integral and constructs then his examples by using certain deep existence theorems concerning the partial differential equation $\Delta\psi + \lambda p\psi = 0$, where λ is a parameter and p is a given positive function. The results of Schwarz concerning this equation and the methods he develops are of the highest importance.[2]) It seemed to me however that it would be interesting to finish up the construction of Schwarz in an elementary way. In § 1 of this paper I show in a very elementary way that if we consider, for instance, a piece of the much studied minimal surface of Enneper[3])

$$x = 3u + 3uv^2 - u^3,$$
$$y = -3v - 3u^2v + v^3,$$
$$z = 3u^2 - 3v^2,$$

[1]) Parts of this paper have been presented to the American Mathematical Society at the meetings in Minneapolis, September 1931, and Chicago, April 1932.

[2]) The papers of Schwarz, concerned with these subjects, are the following ones. Beitrag zur Untersuchung der zweiten Variation von Minimalflächenstücken im Allgemeinen und von Teilen der Schraubenfläche im besonderen, and Über ein die Flächen kleinsten Flächeninhaltes betreffendes Problem der Variationsrechnung, pp. 151—167 and pp. 223—269 respectively in the *Gesammelte Mathematische Abhandlungen* of Schwarz (Berlin, 1890).

[3]) See G. Darboux, *Théorie générale des surfaces* (Paris, 1887), vol. 1, pp. 372—376, where a picture of the surface is also given.

corresponding to $0 \leqq u^2 + v^2 \leqq r^2$, where $1 < r < \sqrt{3}$, then this piece is bounded by a curve without multiple points and its area is *not* a minimum if compared with surfaces bounded by the same curve. This example is chosen as an illustration; the elementary method used permits to obtain also one of the general theorems of SCHWARZ, namely that if the spherical image of a minimal surface $\mathfrak{M}$ comprises half of the unit sphere in its interior, then the area of $\mathfrak{M}$ is not a minimum.

On several occasions, SCHWARZ stated that a minimal surface is generally not determined by its boundary curve. He also stated that if the boundary is a skew quadrilateral, then the minimal surface is univocally determined. I was unable to find in the Collected Papers of SCHWARZ or in the more recent literature a proof of this statement. In a previous paper[4]) I proved the uniqueness for the case when the boundary curve has a simple covered convex curve as its orthogonal projection upon some plane. In § 2 of the present paper, I prove the uniqueness for the case when the boundary curve has a simply covered convex curve as its central projection from some point upon some plane. The minimal surfaces considered are supposed to be continuous images of the circle, otherwise they are allowed to have multiple points and any singularities.

Several applications of the preceding results are considered in § 3. The conclusions obtained are very immediate, still I thought that their interest might justify their explicit statement.

§. 1. Elementary discussion of the second variation of the area-integral.

1. Let R be a JORDAN region in the (u, v)-plane bounded by an analytic Jordan curve. Consider a surface

$$S: x = x(u, v), \quad y = y(u, v), \quad z = z(u, v), \quad (u, v) \text{ in } R,$$

where $x(u, v)$, $y(u, v)$, $z(u, v)$ have all necessary differential coefficients in the closed region R. Suppose that the above equations carry the boundary curve of R in a one-to-one way into a JORDAN curve Γ in the (x, y, z)-space. Put, as usual

$$E = x_u^2 + y_u^2 + z_u^2, \quad F = x_u x_v + y_u y_v + z_u z_v, \quad G = x_v^2 + y_v^2 + z_v^2,$$

[4]) Some remarks on the problem of PLATEAU, *Proceedings of the National Academy of Sciences*, **16** (1930), pp. 242—248; see in particular p. 247.

and suppose that

$$W^2 = EG - F^2 > 0$$

in the closed region R. This assumption secures the existence of a normal for every point of the surface S. The direction cosines of the normal will be denoted by X, Y, Z.

2. Denote now by ε a parameter and by $\psi(u, v)$ a function having continuous first partial derivatives in the closed region R and *vanishing* on the boundary of R. Define a surface $\bar{S}$ by the equations

$$\left.\begin{aligned} x &= x(u, v) + \varepsilon\psi(u, v) X(u, v), \\ y &= y(u, v) + \varepsilon\psi(u, v) Y(u, v), \\ z &= z(u, v) + \varepsilon\psi(u, v) Z(u, v) \end{aligned}\right\} \quad (u, v) \text{ in } R.$$

The area of $\bar{S}$ is a function $A(\varepsilon)$ of ε; we are going to compute $A'(0)$ and $A''(0)$. If $\bar{E}, \bar{F}, \bar{G}$ are the first fundamental quantities relative to $\bar{S}$, we find

$$\bar{E}\bar{G} - \bar{F}^2 = a_0 + a_1\varepsilon + a_2\varepsilon^2 + \ldots, \tag{1.1}$$

where

$$\left\{\begin{aligned} & a_0 = W^2 = EG - F^2, \; a_1 = -2\psi(EN - 2FM + GL), \\ & a_2 = \psi^2[Eg - 2Ff + Ge + 4(LN - M^2)] + \\ & \qquad\qquad + E\psi_v^2 - 2F\psi_u\psi_v + G\psi_u^2. \end{aligned}\right. \tag{1.2}$$

In these formulas, L, M, N, e, f, g are the second and third fundamental quantities relative to the original surface S; the explicit expressions are

$$L = \mathfrak{X}\mathfrak{x}_{uu}, \; M = \mathfrak{X}\mathfrak{x}_{uv}, \; N = \mathfrak{X}\mathfrak{x}_{vv},$$
$$e = \mathfrak{X}_u^2, \; f = \mathfrak{X}_u\mathfrak{X}_v, \; g = \mathfrak{X}_v^2,$$

where $\mathfrak{x}$ and $\mathfrak{X}$ stand for the vectors (x, y, z) and (X, Y, Z) respectively. We obtain then

$$(\bar{E}\bar{G} - \bar{F}^2)^{\frac{1}{2}} = b_0 + b_1\varepsilon + b_2\varepsilon^2 + \ldots,$$

where the coefficients b are obtained by squaring and comparing with (1. 1). It follows

$$b_0^2 = a_0, \; 2b_0b_1 = a_1, \; b_1^2 + 2b_0b_2 = a_2. \tag{1.3}$$

As

$$A(\varepsilon) = \iint\limits_R (\bar{E}\bar{G} - \bar{F}^2)^{\frac{1}{2}} du\, dv,$$

we get

$$A'(0)=\iint_R b_1\,du\,dv,\quad A''(0)=\iint_R 2b_2\,du\,dv.$$

3. Suppose now that the original surface S is a minimal surface, that is to say that its mean curvature

$$H=\frac{EN-2FM+GL}{2(EG-F^2)}$$

vanishes identically. From (1. 2) it follows then that $a_1=0$, and (1. 2) and (1. 3) yield the formulas

$$\text{(1. 4)}\quad \begin{aligned} A'(0)=0,\ A''(0)=\iint_R \frac{1}{W}&[\psi^2(Eg-2Ff+Ge)+\\ &+4\psi^2(LN-M^2)+E\psi_v^2-2F\psi_u\psi_v+G\psi_u^2]\,du\,dv.\end{aligned}$$

If we can choose $\psi(u,v)$, subject to the condition of vanishing on the boundary curve of the region R, so as to make $A''(0)$ negative, the function $A(\varepsilon)$ will have a relative maximum for $\varepsilon=0$, that is to say the given minimal surface S will certainly not have a minimum area if compared with surfaces bounded by the same curve.

4. Suppose that the minimal surface S is given by the formulas of WEIERSTRASZ

$$\text{(1.5)}\qquad \begin{cases} x=\Re\int(1-w^2)\,\mu(w)\,dw,\\ y=\Re\int i(1+w^2)\,\mu(w)\,dw,\\ z=\Re\int 2w\,\mu(w)\,dw,\end{cases}$$

where $w=u+iv$ and where $\mu(w)$ is an analytic function of w in the closed region R. For the quantities $E, F, G, L, M, N, e, f, g$ we obtain the expressions

$$\begin{aligned} &E=G=\mu\bar{\mu}(1+w\bar{w})^2,\ F=0,\\ &L=-N=-(\mu+\bar{\mu}),\ M=-i(\mu-\bar{\mu}),\\ &e=g=\frac{4}{(1+w\bar{w})^2},\ f=0,\end{aligned}$$

where the bar denotes the conjugate complex number. Substituting in (1. 4) we obtain *the formula of Schwarz for the second variation:*

$$\text{(1. 6)}\qquad A''(0)=\iint_R\left[\psi_u^2+\psi_v^2-\frac{8\psi^2}{(1+u^2+v^2)^2}\right]du\,dv.$$

The function $\mu(w)$ does not figure in $A''(0)$, which depends only upon the region R and the function ψ.

5. So we have to find

a) a JORDAN region R and a function $\psi(u, v)$ in R, having continuous first partial derivatives in the closed region R and vanishing on the boundary of R, and such that the integral (1.6), extended over R, be *negative*, and

b) an analytic function $\mu(w)$ in R, such that the formulas (1.5) of WEIERSTRASZ carry the boundary curve of R in a one-to-one and continuous way into a JORDAN curve in the (x, y, z)-space.

It is possible to give examples for such a situation without reference to the deep existence theorems used by SCHWARZ,[5]) as we are going to show presently.

6. Consider the function

$$\psi(u, v) = \frac{u^2+v^2-r^2}{u^2+v^2+r^2} \quad \text{for} \quad 0 \leqq u^2+v^2 \leqq r^2. \tag{1.7}$$

Then

$$\psi(u, v) = 0 \quad \text{for} \quad u^2+v^2 = r^2. \tag{1.8}$$

Put

$$\lambda(r) = \iint\limits_{(r)} \left[\psi_u^2 + \psi_v^2 - \frac{8\psi^2}{(1+u^2+v^2)^2}\right] du\, dv, \tag{1.9}$$

where $\iint\limits_{(r)}$ means integration over $0 \leqq u^2+v^2 < r^2$. First we observe that $\lambda(1) = 0$. Indeed, using (1.8), we find by partial integration

$$\lambda(r) = -\iint\limits_{(r)} \psi \left[\Delta\psi + \frac{8\psi}{(1+u^2+v^2)^2}\right] du\, dv,$$

and an easy computation shows that the function

$$\frac{u^2+v^2-1}{u^2+v^2+1},$$

to which ψ reduces for $r = 1$, satisfies the partial differential equation

$$\Delta\psi + \frac{8\psi}{(1+u^2+v^2)^2} = 0.$$

So the relation

$$\lambda(1) = 0 \tag{1.10}$$

is proved. We show next that

$$\lambda'(1) < 0. \tag{1.11}$$

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[5]) See, in particular, the second paper of SCHWARZ quoted under [2]).

This can be seen as follows. Substituting (1.7) into (1.9), we obtain

$$\lambda(r)=\iint\limits_{(r)}\frac{16(u^2+v^2)r^4}{(u^2+v^2+r^2)^4}\,du\,dv-\iint\limits_{(r)}\frac{8\,(u^2+v^2-r^2)^2}{(u^2+v^2+r^2)^2(u^2+v^2+1)^2}\,du\,dv.$$

We introduce new variables α,β by $u=\alpha r$, $v=\beta r$ and we get

$$\lambda(r)=\iint\limits_{(1)}\frac{16\,(\alpha^2+\beta^2)}{(\alpha^2+\beta^2+1)^4}\,d\alpha\,d\beta-\tag{1.12}$$
$$-\iint\limits_{(1)}\frac{8\,(\alpha^2+\beta^2-1)^2\,r^2}{(\alpha^2+\beta^2+1)^2\,(\alpha^2r^2+\beta^2r^2+1)^2}\,d\alpha\,d\beta,$$

where the domain of integration (namely the unit circle) is now fixed. Hence we can compute $\lambda'(1)$ by differentiating under the integral sign. We obtain

$$\lambda'(1)=16\iint\limits_{(1)}\frac{(\alpha^2+\beta^2-1)^3}{(\alpha^2+\beta^2+1)^5}\,d\alpha\,d\beta.$$

As $\alpha^2+\beta^2<1$ in the domain of integration, the integrand is obviously negative, and (1.11) is proved. From $\lambda(1)=0$ and $\lambda'(1)<0$ it follows that we have a $\sigma>1$ such that

$$\lambda(r)<0 \quad \text{for} \quad 1<r<\sigma. \tag{1.13}$$

It should be observed that it follows from (1.12) that

$$\lambda(r)\to\iint\limits_{(1)}\frac{16\,(\alpha^2+\beta^2)}{(\alpha^2+\beta^2+1)^4}\,d\alpha\,d\beta \quad \text{for} \quad r\to+\infty,$$

that is to say that $\lambda(r)$ does not stay permanently negative for $r>1$.

7. The inequality (1.13) expresses that if $1<r<\sigma$, then there exists a function $\psi(u,v)$, having continuous first derivatives for $0\leqq u^2+v^2\leqq r^2$, vanishing for $u^2+v^2=r^2$, and such that

$$\iint\limits_{(r)}\left[\psi_u^2+\psi_v^2-\frac{8\psi^2}{(1+u^2+v^2)^2}\right]du\,dv<0.$$

It is easy to complete this result and to show that such a ψ exists even for $r\geqq\sigma$. Consider an $r\geqq\sigma$. Choose r_0,δ so that

$$1<r_0<\sigma,\ \ \delta>0,\ \ r_0+\delta<\sigma,$$

and define, for $0\leqq u^2+v^2\leqq r^2$, a function ψ as follows.

$$(1.14)\quad \begin{cases} \psi = \dfrac{u^2+v^2-r_0^2}{u^2+v^2+r_0^2} \quad \text{for} \quad u^2+v^2 \leqq r_0^2, \\ \psi = \dfrac{1}{r_0\delta^2}\left[(u^2+v^2)^{\frac{1}{2}} - r_0\right]\left[(u^2+v^2)^{\frac{1}{2}} - r_0 - \delta\right]^2 \\ \qquad\qquad\qquad\qquad \text{for} \quad r_0^2 \leqq u^2+v^2 \leqq (r_0+\delta)^2, \\ \psi = 0 \quad \text{for} \quad (r_0+\delta)^2 \leqq u^2+v^2 \leqq r^2. \end{cases}$$

This ψ vanishes for $u^2+v^2=r^2$, and an easy computation shows that it has continuous first derivatives for $u^2+v^2 \leqq r^2$. Consider now

$$(1.15)\quad \iint\limits_{(r)} \left[\psi_u^2+\psi_v^2 - \frac{8\psi^2}{(1+u^2+v^2)^2}\right] du\,dv = \iint\limits_{(r_0)} + \iint\limits_{(r_0,\, r_0+\delta)} + \iint\limits_{(r_0+\delta,\, r)}.$$

The first integral on the right-hand side of (1.15) is equal to $\lambda(r_0)$, and so this term is negative, on account of $r_0 < \sigma$. The last term on the right-hand side of (1.15) is zero, and for the second term a rough evaluation gives the bound

$$\frac{9}{r_0^2}(2r_0+\delta)\,\delta,$$

and so this term will be as small as we please, if $\delta > 0$ is chosen properly. So we see that by a proper choice of $\delta > 0$ the function ψ defined by (1.14) will make the integral (1.6) negative.

8. It follows then, on account of § 1, No. **3** and **4**, that the formulas (1.5) of WEIERSTRASZ, if considered for $|w| \leqq r$, $1 < r$, give a minimal surface which does *not* have a minimum area, no matter how the analytic function $\mu(w)$ is chosen. In order to have a clear-cut example, this $\mu(w)$ must be chosen so that the minimal surface obtained be bounded by a curve without multiple points. A case which permits a simple discussion is $\mu(w) \equiv 3$. The corresponding surface is called the *minimal surface of Enneper*. From (1.5) we obtain, putting $\mu(w) \equiv 3$, the equations of the surface in the form

$$(1.16)\quad \begin{cases} x = 3u + 3uv^2 - u^3, \\ y = -3v - 3u^2v + v^3, \\ z = 3u^2 - 3v^2. \end{cases}$$

We are going to show that if $r < \sqrt{3}$, then the image of the circle $u^2+v^2=r^2$ by the equations (1.16) is a curve without multiple points. Write $u = r\cos\varphi$, $v = r\sin\varphi$, where r is fixed according to $r > \sqrt{3}$. Then x, y, z become functions $x(\varphi)$, $y(\varphi)$, $z(\varphi)$ of φ, the explicit expressions being

$$(1.17)\qquad \begin{cases} x(\varphi)=[3-r^2(\cos^2\varphi-3\sin^2\varphi)]\,r\cos\varphi, \\ y(\varphi)=-[3-r^2(\sin^2\varphi-3\cos^2\varphi)]\,r\sin\varphi, \\ z(\varphi)=3r^2\cos 2\varphi. \end{cases}$$

We show that for

$$(1.18)\qquad 0\leqq\varphi_1<\varphi_2<2\pi$$

the equations

$$x(\varphi_1)=x(\varphi_2),\ y(\varphi_1)=y(\varphi_2),\ z(\varphi_1)=z(\varphi_2)$$

cannot be satisfied simultaneously if $r<\sqrt{3}$. Indeed, from $z(\varphi_1)=z(\varphi_2)$ it follows that $\cos 2\varphi_1-\cos 2\varphi_2=2\sin(\varphi_2+\varphi_1)\sin(\varphi_2-\varphi_1)=0$, that is to say that $\sin(\varphi_2+\varphi_1)=0$ or $\sin(\varphi_2-\varphi_1)=0$. The only possibilities, consistent with (1.18), are

I. $\varphi_2=\varphi_1+\pi$.

II. $\varphi_2=-\varphi_1+\pi,\ 0\leqq\varphi_1<\frac{\pi}{2}$.

III. $\varphi_2=-\varphi_1+2\pi,\ 0<\varphi_1<\pi$.

IV. $\varphi_2=-\varphi_1+3\pi,\ \pi<\varphi_1<\frac{3\pi}{2}$.

The equations $x(\varphi_1)=x(\varphi_2)$, $y(\varphi_1)=y(\varphi_2)$ give then, corresponding to these four cases, the following relations:

I*. $x(\varphi_1)=-x(\varphi_1),\ y(\varphi_1)=-y(\varphi_1)$, that is to say $x(\varphi_1)=y(\varphi_1)=0$.

II*. $x(\varphi_1)=-x(\varphi_1)$, that is to say $x(\varphi_1)=0$.

III*. $y(\varphi_1)=-y(\varphi_1)$, that is to say $y(\varphi_1)=0$.

IV*. $x(\varphi_1)=-x(\varphi_1)$, that is to say $x(\varphi_1)=0$.

The brackets in (1.17) are both $\geqq 3-r^2$, and consequently $\neq 0$ on account of $r<\sqrt{3}$. Hence, from $x(\varphi_1)=0$ it follows that $\cos\varphi_1=0$, and from $y(\varphi_1)=0$ it follows that $\sin\varphi_1=0$. It is then obvious that I* is impossible, and that II and II*, III and III*, IV and IV* respectively are incompatible.

§ 2. A uniqueness theorem.

1. If the functions $x(u,v)$, $y(u,v)$, $z(u,v)$ are continuous in a closed Jordan region R, we shall say that the equations

$$x=x(u,v),\ y=y(u,v),\ z=z(u,v),\quad (u,v) \text{ in } R$$

define a *continuous surface S of the type of the circle.* This definition does not require that the correspondence between the points (u,v) and (x,y,z) be one-to-one.

Suppose we map, by equations $u=u(\alpha,\beta)$, $v=v(\alpha,\beta)$, the JORDAN region R in a one-to-one and continuous way upon another JORDAN region R^*. The functions $x(u,v)$, $y(u,v)$, $z(u,v)$ are transformed into the new functions $x^*(\alpha,\beta)=x[u(\alpha,\beta), v(\alpha,\beta)]$ and so on, and we say that the two sets of equations

$$x=x(u,v),\ y=y(u,v),\ z=z(u,v),\quad (u,v) \text{ in } R,$$

and

$$x=x^*(\alpha,\beta),\ y=y^*(\alpha,\beta),\ z=z^*(\alpha,\beta),\quad (\alpha,\beta) \text{ in } R^*$$

define the same surface, or that these two sets of equations are parametric representations of the same surface.

In the sequel, by a *surface* we always mean a continuous surface of the type of the circle.

2. Consider a surface

$$S:\ x=x(u,v),\ y=y(u,v),\ z=z(u,v),\quad (u,v) \text{ in } R.$$

If these equations carry the boundary of R in a one-to-one and continuous way into a JORDAN curve Γ in the (x, y, z)-space, we shall say that S is bounded by Γ.

3. Given a surface

$$S:\ x=x(u,v),\ y=y(u,v),\ z=z(u,v),\quad (u,v) \text{ in } R,$$

consider an interior point (u_0, v_0) of R. Map some vicinity of (u_0, v_0) in a one-to-one and continuous way upon a domain of an (α,β)-plane by equations $u=u(\alpha,\beta)$, $v=v(\alpha,\beta)$. The functions $x(u,v)$, $y(u,v)$, $z(u,v)$ become then functions of α,β and we say that we introduced *new parameters in the vicinity of* (u_0, v_0).

Suppose then that it is possible to introduce new parameters α,β in the vicinity of *every* interior point (u_0, v_0) of R in such a way that

I. x, y, z become harmonic functions of α,β, and

II. x, y, z as functions of α,β satisfy the relations $E=G$, $F=0$, where

$$E=x_\alpha^2+y_\alpha^2+z_\alpha^2,\quad F=x_\alpha x_\beta+y_\alpha y_\beta+z_\alpha z_\beta,\quad G=x_\beta^2+y_\beta^2+z_\beta^2.$$

Under these circumstances, we shall say that S is a *minimal surface.* Parameters α,β, satisfying the conditions I, II for the vicinity of an interior point (u_0, v_0) of R, will be called *local typical parameters for* (u_0, v_0).

4. If we should add the condition $EG-F^2>0$, the minimal surfaces in the sense of the preceding definition would become identical to the minimal surfaces considered in differential geometry. In differential geometry, the condition $EG-F^2>0$ is standard. On the one hand, this condition secures the existence of the tangent plane, and on the other hand, $EG-F^2$ appears in the denominators of most of the important quantities studied in differential geometry. Our purpose in omitting the condition $EG-F^2>0$ and in requiring the existence of typical parameters *in the small* only, is to secure the generality necessary for the applications.

5. Uniqueness theorem. *Let there be given, in the (x, y, z)-space, a Jordan curve Γ. Suppose that Γ has a simply covered convex curve as its central or parallel projection upon some plane. Then Γ cannot bound more than one minimal surface.*

6. We are going to state first a lemma which will be used in the proof. Let $h(u, v)$ be a continuous function in a JORDAN region R. Let (u_0, v_0) be an interior point of R, and map some vicinity of (u_0, v_0) in a one-to-one and continuous way upon a domain in an (α, β)-plane by equations $u=u(\alpha, \beta)$, $v=v(\alpha, \beta)$. The function $h(u, v)$ is then transformed, in the vicinity of (u_0, v_0), into the function $h^*(\alpha, \beta)=h[u(\alpha, \beta), v(\alpha, \beta)]$, and we say that we introduced *new variables α, β in the vicinity of* (u_0, v_0).

Suppose it is possible to introduce new variables (α, β) in the vicinity of *every* interior point (u_0, v_0) of R in such a way that the transformed function $h^*(\alpha, \beta)$ is harmonic. Then $h(u, v)$ will be called a *generalized harmonic function;* the variables α, β as described above will be called *local typical variables.*

Lemma. *Let $h(u, v)$ be a generalized harmonic function in a Jordan region R. Suppose that after introduction of local typical variables α, β for the vicinity of some interior point (u_0, v_0) of R, the transformed function and its first partial derivatives with respect to α and β vanish at the image (α_0, β_0) of (u_0, v_0). Then the function $h(u, v)$ vanishes in at least four distinct points of the boundary of R. Moreover, $h(u, v)$ takes on both positive and negative values on the boundary of R, except in the trivial case when $h(u, v)$ vanishes identically.*

In the particular case when R is a circle and $h(u, v)$ is harmonic in the usual sense, this lemma has been proved in a previous paper of the author.[6]) The proof extends easily to the

[6]) The problem of the least area and the problem of PLATEAU, *Math. Zeitschrift*, **32** (1930), pp. 763—796; see in particular § 2, No 3.

above general statement, and the reader is asked to think through himself the necessary modifications.

7. Consider now a minimal surface

$$\mathfrak{M}: \ x = x(u, v), \ y = y(u, v), \ z = z(u, v), \quad (u, v) \text{ in } R,$$

bounded by a JORDAN curve Γ. Suppose that the central projection of Γ from some point upon some plane is a simply covered convex curve. We can obviously suppose that the center of projection is the point (0, 0, 1), that the plane is the (x, y)-plane, and that Γ is below the plane $z = 1$, that is to say that $z < 1$ on the whole curve Γ. From this it follows that

$$z(u, v) < 1 \text{ in } R. \tag{2.1}$$

Indeed, by the definition of a minimal surface, $z(u, v)$ is a generalized harmonic function, and it is immediate that such a function takes on its maximum on the boundary.

Denote by R^* the JORDAN region bounded by the central projection Γ^* of Γ. The central projection of a point $x(u, v)$, $y(u, v)$, $z(u, v)$ of the minimal surface $\mathfrak{M}$ from the point (0, 0, 1) upon the (x, y)-plane is a point with coordinates $x = a$, $y = b$, where

$$a = \frac{x(u, v)}{1 - z(u, v)}, \quad b = \frac{y(u, v)}{1 - z(u, v)}.$$

We are going to show that these equations carry the JORDAN region R in a one-to-one and continuous way into the JORDAN region R^*.

8. Consider an interior point (u_0, v_0) of R; we shall first show that the transformation is one-to-one in the vicinity of (u_0, v_0). We introduce, for the minimal surface $\mathfrak{M}$, local typical parameters in the vicinity of (u_0, v_0). The vicinity of (u_0, v_0) is then transformed in a one-to-one and continuous way into a vicinity of the image (α_0, β_0) of (u_0, v_0), and the functions x, y, z become functions $x(\alpha, \beta)$, $y(\alpha, \beta)$, $z(\alpha, \beta)$ of α, β. Hence it is sufficient to prove that the equations

$$a = \frac{x(\alpha, \beta)}{1 - z(\alpha, \beta)}, \quad b = \frac{y(\alpha, \beta)}{1 - z(\alpha, \beta)} \tag{2.2}$$

carry the vicinity of (α_0, β_0) in a one-to-one and continuous way into a vicinity of

$$a_0 = \frac{x_0}{1 - z_0}, \quad b_0 = \frac{y_0}{1 - z_0},$$

where $x_0 = x(\alpha_0, \beta_0)$ and so on.

As α, β are local typical parameters for the minimal surface $\mathfrak{M}$, the functions $x(\alpha, \beta)$, $y(\alpha, \beta)$, $z(\alpha, \beta)$ are harmonic and consequently analytic functions of α, β. As $z(\alpha, \beta) < 1$, a and b are also analytic functions of α, β and the assertion that the transformation is one-to-one in the small will be proved if we show that

$$a_\alpha^{(0)} b_\beta^{(0)} - a_\beta^{(0)} b_\alpha^{(0)} \neq 0,$$

where $a_\alpha^{(0)} = a_\alpha(\alpha_0, \beta_0)$ and so on. Suppose $a_\alpha^{(0)} b_\beta^{(0)} - a_\beta^{(0)} b_\alpha^{(0)} = 0$. Then we have two numbers λ, μ which satisfy the relations

$$\lambda a_\alpha^{(0)} + \mu b_\alpha^{(0)} = 0, \ \lambda a_\beta^{(0)} + \mu b_\beta^{(0)} = 0, \ \lambda^2 + \mu^2 > 0.$$

Substituting the values of $a_\alpha^{(0)}, \dots, b_\beta^{(0)}$ obtained from (2. 2), we get

$$(2.3) \qquad \begin{aligned} (\lambda x_\alpha^{(0)} + \mu y_\alpha^{(0)})(1 - z_0) + (\lambda x_0 + \mu y_0) z_\alpha^{(0)} = 0, \\ (\lambda x_\beta^{(0)} + \mu y_\beta^{(0)})(1 - z_0) + (\lambda x_0 + \mu y_0) z_\beta^{(0)} = 0. \end{aligned}$$

Consider now, in the JORDAN region R, the function

$$(2.4) \quad h(u, v) = (\lambda x_0 + \mu y_0)(z(u, v) - 1) - (\lambda x(u, v) + \mu y(u, v))(z_0 - 1).$$

If we introduce, in the vicinity of (u_0, v_0), the new variables α, β, then h and its first partial derivatives with respect to α, β vanish at (α_0, β_0), as it follows from (2. 4) and (2. 3). On the other hand, $h(u, v)$ is a generalized harmonic function. Indeed, typical local parameters for the minimal surface $\mathfrak{M}$ transform $x(u, v)$, $y(u, v)$, $z(u, v)$ into harmonic functions simultaneously, and consequently h, as a linear combination of x, y, z with constant coefficients, is also transformed into a harmonic function.

Applying then the lemma of § 2, No. **6** to $h(u, v)$, it follows that $h(u, v)$ vanishes in at least four distinct points of the boundary of R, and that $h(u, v)$ takes on both positive and negative values on the boundary of R, except if $h(u, v)$ vanishes identically. This means that the plane with equation

$$(2.5) \qquad (\lambda x_0 + \mu y_0)(z - 1) - (\lambda x + \mu y)(z_0 - 1) = 0$$

intersects the boundary curve Γ of $\mathfrak{M}$ in at least four distinct points, and that either there are points of $\mathfrak{M}$ on both sides of this plane, or else Γ is entirely situated in this plane. As however the plane (2. 5) obviously passes through the center of projection (0, 0, 1), these conclusions are in contradiction with the assumption that the projection of Γ from (0, 0, 1) upon the (x, y)-plane is a simply covered convex curve.

9. The formulas

$$a=\frac{x(u,v)}{1-z(u,v)},\quad b=\frac{y(u,v)}{1-z(u,v)}$$

define a transformation of the region R. On account of $z(u,v)<1$ this transformation is continuous. In the preceding No. **8** we proved that the transformation is one-to-one in the small. From the assumption that the boundary curve of R is carried in a one-to-one way into Γ and that Γ is projected in a one-to-one way into Γ^*, it follows that the transformation carries the boundary curve of R in a one-to-one way into the projection Γ^* of Γ. Hence we have a transformation

$$T:\ a=a(u,v),\ b=b(u,v)$$

with the following properties. T is continuous in the JORDAN region R. T carries the boundary curve of R in a one-to-one way into a JORDAN curve Γ^*, and T is one-to-one in the vicinity of every interior point of R. On account of the *monodromy theorem* in topology,[7] it follows then that T carries the JORDAN region R in a one-to-one and continuous way into the JORDAN region R^* bounded by Γ^*. Consequently, we can express u, v as single-valued and continuous functions of a, b and we obtain then the equations of the minimal surface $\mathfrak{M}$ in the form

$$\mathfrak{M}:\ x=x(a,b),\ y=y(a,b),\ z=z(a,b),\quad (a,b) \text{ in } R^*,$$

where $x(a,b)$, $y(a,b)$, $z(a,b)$ are single-valued continuous functions in R^* and satisfy the relations

$$z(a,b)<1,\ a=\frac{x(a,b)}{1-z(a,b)},\quad b=\frac{y(a,b)}{1-z(a,b)}.$$

It is important to observe that $x(a,b)$, $y(a,b)$, $z(a,b)$ are actually *analytic* functions of a, b in the interior of R^*. Indeed, if we introduce local typical parameters α, β in the vicinity of an interior point (u_0, v_0) of R for the minimal surface $\mathfrak{M}$, then the equations

$$x=x(a,b),\ y=y(a,b),\ z=z(a,b)$$

are obtained by eliminating α, β from the equations

$$x=x(\alpha,\beta),\ y=y(\alpha,\beta),\ z=z(\alpha,\beta)$$

and

$$a=\frac{x(\alpha,\beta)}{1-z(\alpha,\beta)},\quad b=\frac{y(\alpha,\beta)}{1-z(\alpha,\beta)}.$$

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[7]) See, for instance, B. VON KERÉKJÁRTÓ, *Vorlesungen über Topologie*, vol. 1 (Berlin, 1923), p. 175.

As $x(\alpha, \beta)$, $y(\alpha, \beta)$, $z(\alpha, \beta)$ are harmonic, and as we proved that $a_\alpha b_\beta - a_\beta b_\alpha$ is different from zero, $x(a, b)$, $y(a, b)$, $z(a, b)$ come out to be analytic functions of a, b, on account of well-known theorems on implicit functions.

10. As $1-z(a, b)>0$ in R^*, we can write

$$1-z(a, b)=e^{\zeta(a, b)},$$

where $\zeta(a, b)$ is single-valued continuous in R^* and analytic in the interior of R^*. The equations of $\mathfrak{M}$ appear then in the form

$$\text{(2.6)}\quad \mathfrak{M}:\ x=a e^{\zeta(a, b)},\ y=b e^{\zeta(a, b)},\ z=1-e^{\zeta(a, b)},\quad (a, b) \text{ in } R^*.$$

The first fundamental quantities

$$E=x_a^2+y_a^2+z_a^2,\ F=x_a x_b+y_a y_b+z_a z_b,\ G=x_b^2+y_b^2+z_b^2$$

have then the expressions

$$E=e^{2\zeta}[(1+a^2+b^2)\zeta_a^2+2a\zeta_a+1],$$
$$F=e^{2\zeta}[(1+a^2+b^2)\zeta_a\zeta_b+a\zeta_b+b\zeta_a],$$
$$G=e^{2\zeta}[(1+a^2+b^2)\zeta_b^2+2b\zeta_b+1].$$

It follows that

$$W^2=EG-F^2=e^{4\zeta}[\zeta_a^2+\zeta_b^2+(1+a\zeta_a+b\zeta_b)^2],$$

and consequently

$$W^2=EG-F^2>0.$$

Thus $\mathfrak{M}$, which is a minimal surface in the general sense of the definition of § 2, No. **3,** is also a minimal surface in the usual sense of differential geometry. Hence the mean curvature

$$H=\frac{EN-2FM+GL}{2(EG-F^2)}$$

vanishes. For the second fundamental quantities L, M, N we obtain from (2.6) the expressions

$$L=\frac{1}{W}e^{\zeta}(\zeta_a^2-\zeta_{aa}),$$

$$M=\frac{1}{W}e^{\zeta}(\zeta_a\zeta_b-\zeta_{ab}),$$

$$N=\frac{1}{W}e^{\zeta}(\zeta_b^2-\zeta_{bb}),$$

and $H=0$ gives then for ζ the partial differential equation

$$\text{(2.7)}\quad \begin{aligned}&[(1+a^2+b^2)\zeta_b^2+2b\zeta_b+1]\zeta_{aa}-2[(1+a^2+b^2)\zeta_a\zeta_b+a\zeta_b+b\zeta_a]\zeta_{ab}+\\&+[(1+a^2+b^2)\zeta_a^2+2a\zeta_a+1]\zeta_{bb}-(\zeta_a^2+\zeta_b^2)=0.\end{aligned}$$

11. Thus we find that our minimal surface $\mathfrak{M}$ admits of a representation

$$\mathfrak{M}: \; x = a e^{\zeta}, \; y = b e^{\zeta}, \; z = 1 - e^{\zeta}, \quad (a, b) \text{ in } R^*,$$

where $\zeta = \zeta(a, b)$ is single-valued continuous in R^* and analytic in the interior of R^*, and satisfies in the interior of R^* the equation (2.7). The boundary values of ζ on the boundary curve Γ^* of R^* are determined by the boundary curve Γ. Consequently, the uniqueness of the minimal surface bounded by the given Jordan curve Γ will be proved if the solutions of (2.7) are univocally determined by their boundary values, and this follows directly from general uniqueness theorems.

Indeed, (2.7) has the form

$$P(a, b, \zeta, p, q, r, s, t) = 0 \tag{2.8}$$

where P is a polynomial of its arguments and $p, \ldots, t$ stand for $\zeta_a, \ldots, \zeta_{bb}$ (it would be sufficient if P would be a sufficiently regular function of its arguments). For a partial differential equation of the form (2.8) we have then the theorem that the solutions are univocally determined by their boundary values provided

$$P_r P_t - P_s^2 > 0, \quad P_\zeta \leq 0.^{8)}$$

In our case $P_\zeta = 0$, and

$$P_r P_t - P_s^2 = p^2 + q^2 + (1 + ap + bq)^2.$$

This expression is obviously positive.

12. The case of the central projection being thus settled, let us suppose that the boundary curve Γ of the minimal surface $\mathfrak{M}$ has a simply covered convex curve as its projection when projected parallel to some direction upon some plane. The projection of Γ in the same direction and upon a plane perpendicular to that direction is then again a simply covered convex curve and so it is sufficient to consider the case of the *orthogonal* projection.[9]

Suppose that we project upon the (x, y)-plane. Denote by Γ^* the projection of Γ and by R^* the JORDAN region bounded

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[8] A beautiful treatment of this theorem and of related subjects is given in the paper of E. HOPF, Elementare Bemerkungen über die Lösungen partieller Differentialgleichungen zweiter Ordnung vom elliptischen Typus, *Sitzungsberichte der preußischen Akademie der Wissenschaften, mathematisch-physikalische Klasse*, 1927, pp. 147—152.

[9] This case has already been considered, in a somewhat restricted form, in the paper of the author referred to under [4].

by Γ^*. A reasoning similar to that used in the case of the central projection leads in the present case to the following first result which we state, with regard to later application, as a

Lemma. *Suppose that the boundary curve Γ of a minimal surface $\mathfrak{M}$ has a simply covered convex curve Γ^* as its orthogonal projection upon the (x, y)-plane. Denote by R^* the Jordan region bounded by Γ^*. Then $\mathfrak{M}$ admits of a representation*

$$\mathfrak{M}: \quad z = z(x, y), \quad (x, y) \text{ in } R^*,$$

where $z(x, y)$ is single-valued continuous in R^*, analytic in the interior of R^*, and satisfies in the interior of R^* the partial differential equation

$$(1+q^2)r - 2pqs + (1+p^2)t = 0, \tag{2.9}$$

where $p, \ldots, t$ stand for $z_x, \ldots, z_{yy}$.

The boundary values of $z(x, y)$ are univocally determined by the given JORDAN curve Γ. Consequently the uniqueness of the minimal surface $\mathfrak{M}$, bounded by the given JORDAN curve Γ, follows from the fact that the solutions of (2.9) are univocally determined by their boundary values. This fact has been proved in many ways[10]) and follows, in particular, from the general uniqueness theorem used in § 2, No. **11.**

§ 3. Applications.

1. We shall combine the preceding results with the following

Existence theorem.[11]) *Let Γ be a Jordan curve in the (x, y, z)-space. Consider all the continuous surfaces S of the type of the circle, bounded by Γ, and denote by $a(\Gamma)$ the greatest lower bound of their areas $A(S)$, where $A(S)$ is the area as defined by Lebesgue.*

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[10]) A very simple proof might be obtained by using certain convexity properties of the area-integral $\iint (1 + p^2 + q^2)^{1/2}\, dx\, dy$; see A. HAAR, Über reguläre Variationsprobleme, *these Acta*, **3** (1927), pp. 224—234. It would be interesting to investigate the possibility of a similar method for the partial differential equation (2.7).

[11]) T. RADÓ, loc. cit. [6]); J. DOUGLAS, Solution of the problem of PLATEAU, *Transactions of the American Mathematical Society*, **33** (1931), pp. 263—321. — For an exposition of the theory of the area, in the sense of LEBESGUE, see the authors paper, Über das Flächenmaß rektifizierbarer Flächen, *Math. Annalen*, **100** (1928), pp. 445—479.

If $a(\Gamma) < +\infty$, that is to say if Γ bounds some continuous surface of the type of the circle with a finite area, then there exists a minimal surface, bounded by Γ, the area of which is equal to $a(\Gamma)$.

The condition $a(\Gamma) < +\infty$ is satisfied, in particular, if Γ is such that it has a simply covered convex curve as its central or parallel projection upon some plane.[12]) In this case, on account of the uniqueness theorem of § 2, No. **5**, the minimal surface is also *unique*.

2. Consider now the piece of surface

$$\left.\begin{aligned} x &= 3u+3uv^2-u^3, \\ y &= -3v-3u^2v+v^3, \\ z &= 3u^2-3v^2, \end{aligned}\right\} \quad 0 \leqq u^2+v^2 \leqq r^2, \ 1<r<\sqrt{3}. \tag{3.1}$$

This is (see § 1, No. **8**) a minimal surface, bounded by a JORDAN curve, the area of which is *not* a minimum. On the other hand, as $x_u, \ldots, z_v$ are bounded, the area of this surface is finite, and so the existence theorem of § 3, No. **1** guarantees the existence of a minimal surface, bounded by the same JORDAN curve, the area of which *is* a minimum. That is to say, *the equations*

$$\left.\begin{aligned} x &= 3r\cos\varphi-r^3\cos3\varphi \\ y &= -3r\sin\varphi-r^3\sin3\varphi, \\ z &= 3r^2\cos2\varphi, \end{aligned}\right\} \quad 0 \leqq \varphi < 2\pi, \tag{3.2}$$

determine, provided $1<r<\sqrt{3}$, a Jordan curve which bounds at least two distinct minimal surfaces.

While the catenoids give explicit elementary examples of distinct *multiply connected* minimal surfaces with the same boundary curves,[13]) it seems that no elementary example has yet been given for *simply connected* minimal surfaces. In our own example, one of the two minimal surfaces, bounded by the JORDAN curve (3. 2), is explicitly given by the equations (3. 1), while a second minimal surface is only known to *exist*.[14]) It would be interesting to find an *elementary explicit example* of two minimal surfaces, *of the type of the circle,* bounded by the same JORDAN curve.

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[12]) Loc. cit. [6]), in particular p. 265.

[13]) See, for instance, the beautiful chapter IV in the book of G. A. BLISS, *Calculus of Variations*, No. 1 of the Carus Mathematical Monographs (Chicago, 1925).

[14]) The situation is similar in the examples which I was able to find in the literature.

3. SCHWARZ, after having shown that a minimal surface generally does not have a minimum area, asked for conditions under which a given piece of a minimal surface does have a minimum area. While his methods are restricted to the case of a *relative* minimum,[15] we have the following theorem concerning an *absolute* minimum, as an obvious consequence of the statements in § 3, No. **1** and § 2, No. **5.**

Theorem. *Let there be given a minimal surface $\mathfrak{M}$ bounded by a Jordan curve Γ, such that Γ has a simply covered convex curve as its central or parallel projection upon some plane. Then the area of $\mathfrak{M}$ is a minimum if compared with the areas of all continuous surfaces, of the type of the circle, bounded by Γ.*

Consider then, in particular, a minimal surface $\mathfrak{M}$ in the usual sense of differential geometry. The vicinity of every point of $\mathfrak{M}$ can be then represented, if for instance the (x, y)-plane is parallel to the tangent plane in that point, by an equation $z = z(x, y)$, where $z(x, y)$ is single-valued in the vicinity of the (x, y)-projection of the point under consideration. Consequently we have on $\mathfrak{M}$ a JORDAN curve Γ, surrounding the given point, the projection of which is convex upon the (x, y)-plane. Hence, every point on a minimal surface, in the usual sense of differential geometry, is comprised in a portion of the surface the area of which is a minimum if compared with the areas of all continuous surfaces of the type of the circle and with the same boundary. Shortly: *the area of a minimal surface is an absolute minimum in the small.*

4. As a last application, we are going to discuss a general statement of S. BERNSTEIN concerning the partial differential equation

$$(3.3) \qquad (1+q^2)\,r-2pqs+(1+p^2)\,t=0.$$

Given, in the (x, y)-plane, a JORDAN curve C, and a continuous set of values on C, the boundary value problem for (3.3) requires the determination of a function $z(x, y)$, continuous in and on C, analytic inside, and which takes on the given set of values on C. S. BERNSTEIN stated, without complete proof, that the problem is always solvable if C is convex, and that the problem is generally not solvable if C is not convex.[16] We can easily verify this statement.

[15]) See loc. cit. [2]).

[16]) S. BERNSTEIN, Sur les équations du Calcul des Variations, *Annales de l'École Normale*, (3) **29** (1912), pp. 431—485; see in particular pp. 484—485.

Concerning the second, negative half of the statement, an example which allows a simple and complete discussion may be obtained, for instance, in the following way. Denote by A, B, C, D the vertices of a regular tetrahedron and by Γ the quadrilateral with the sides AB, BC, CD, DA. The orthogonal projection of Γ upon properly chosen planes is then obviously simply covered and convex, and consequently (see § 3, No. 1) there exists a *unique* minimal surface $\mathfrak{M}$ bounded by Γ. Let us first show that $\mathfrak{M}$ *posses through the center of the tetrahedron* A, B, C, D.[17]) Denote by p the plane through A, C and parallel to the edge BD. The orthogonal projection of Γ upon p is then the simply covered perimeter of a square, that is to say the projection is convex. Hence, it follows from the lemma in § 2, No. **12**, that every point interior to this square is the projection of exactly one point of $\mathfrak{M}$. Applying this remark to the center of the square, it follows that the straight line g, connecting the centers of the edges AC and BD, intersects $\mathfrak{M}$ in exactly one point, which we denote by P. Rotate now the figure around the axis g through 90 degrees, and then reflect upon the plane passing through the center of the tetrahedron and perpendicular to g. Then Γ is carried into itself, and consequently $\mathfrak{M}$ is carried into itself. For $\mathfrak{M}$ is carried into a minimal surface, bounded again by Γ, and $\mathfrak{M}$ is the *only* minimal surface bounded by Γ. The point P is therefore carried into a point P' also situated on $\mathfrak{M}$. As g is obviously carried into itself, P' is also situated on g. But g intersects $\mathfrak{M}$ in exactly one point, and thus P and P' must coincide, which is the case if and only if P coincides with the center O of the tetrahedron A, B, C, D. So we see that $\mathfrak{M}$ passes through O.

Choose now the plane through A, B, C as the (x, y)-plane, and let the z-axis be perpendicular to this plane. The projection of the vertex D of the tetrahedron upon the (x, y)-plane is then the center O^* of the equilateral triangle ABC, and O^* is also the projection of the center O of the tetrahedron A, B, C, D.

17) This surface has been explicitly determined by SCHWARZ in his paper, Bestimmung einer speciellen Minimalfläche, pp. 6—91 of the *Gesammelte mathematische Abhandlungen*. He also stated, without proof, that the surface is unique. The fact that the surface passes through the center of the tetrahedron can also be seen by using the explicit equations of the surface in terms of elliptic functions. It seemed interesting to me that this fact also follows by elementary geometry from the uniqueness of the surface.

The orthogonal projection of Γ upon this (x, y)-plane is the simply covered quadrilateral AB, BC, CO^*, O^*A. The z-coordinate of a variable point of Γ is then a continuous function on this quadrilateral, and we assert that *using this function as the given boundary function, the corresponding boundary value problem is not solvable for the partial differential equation* (3. 3). Indeed, if the solution $z(x, y)$ would exist, the equation $z = z(x, y)$ would define a minimal surface, bounded by Γ and such that no straight line parallel to the z-axis intersects the surface in more than one point. This minimal surface would coincide with $\mathfrak{M}$, as $\mathfrak{M}$ is the unique minimal surface bounded by Γ,[18]) and $\mathfrak{M}$ is intersected by the parallel to the z-axis through the center of the tetrahedron A, B, C, D in two points, one of which is the vertex D, and the other one the center O.

The proof of the first, positive half of the statement of S. BERNSTEIN is immediate. If the curve C, bearing the given continuous boundary values, is convex, then the boundary value problem requires to determine a minimal surface, given by an equation $z = z(x, y)$ and bounded by a JORDAN curve Γ of which C is the simply covered convex orthogonal projection. The existence theorem in § 3, No. 1 and the lemma in § 2, No. 12 secure therefore directly the existence of the solution of the boundary value problem.[19])

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[18]) Several examples for the non-existence of the solution of the boundary value problem, discussed in the literature without using arguments similar to our uniqueness theorem in § 2, No 5 and our lemma in § 2, No 12, seem to be incomplete.

[19]) While we only require the mere continuity of the given boundary values, previous results have been obtained under more restrictive conditions. See, also for references, A. HAAR, Über das Plateausche Problem, *Math. Annalen*, 97 (1927), pp. 124—258.