Variation Problems of which the Extremals are Minimal Surfaces.

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1. Introduction.

DARBOUX[1]) has shown that *every* differential equation of the second order

$$(1.1) \qquad y'' = g(x, y, y')$$

can be regarded as the EULER differential equation of all those problems of minimizing an integral of the form

$$(1.2) \qquad \int_{x_1}^{x_2} f(x, y, y')\, dx$$

for which f is determined by quadratures from the relation

$$(1.3) \qquad \frac{\partial^2 f}{\partial y'^2} = M(x, y, y'),$$

where M is an expression involving an arbitrary function of two first integrals of the proposed differential equation (1.1).

HIRSCH[2]) has shown that the result of DARBOUX is due to the fact that for every differential equation (1.1) there exists a multiplier, namely the function $M(x, y, y')$ in (1.3), such that the differential equation

$$(1.4) \qquad M[y'' - g(x, y, y')] = 0$$

has a self-adjoint equation of variation.

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[1]) G. DARBOUX, *Théorie des Surfaces* (1887), Tome III, p. 53, §§ 604—605.

[2]) A. HIRSCH, Über eine charakteristische Eigenschaft der Differentialgleichungen der Variationsrechnung, *Mathematische Annalen*, **49** (1897), pp. 49—72, loc. cit., § 7.

On the basis of a result due to KÜRCHÁK[3]) the writer[4]) has found necessary and sufficient conditions for a partial differential equation of the second order

$$(1.5) \qquad F(x_1, \ldots, x_n, z, p_1, \ldots, p_n, p_{11}, \ldots, p_{nn}) = 0,$$

$$p_i = \frac{\partial z}{\partial x_i}, \quad p_{ij} = \frac{\partial^2 z}{\partial x_i \partial x_j},$$

linear in the p_{ij}, to have a self-adjoint equation of variation and hence to be, *as it stands,* the LAGRANGE equation of a problem of minimizing an n-fold integral of the form

$$(1.6) \qquad \int_{(n)} f(x_1, \ldots, x_n, z, p_1, \ldots, p_n)\, dx_1 \ldots dx_n.$$

The simple example of the analytic partial differential equation

$$(1.7) \qquad r + q = 0$$

shows that even in the case $n = 2$ the theorem of HIRSCH is no longer true since the conditions which must be satisfied by the multiplier $M(x, y, z, p, q) \neq 0$ in order that $Mr + Mq = 0$, the most general linear partial differential equation equivalent to (1. 7), shall have a self-adjoint equation of variation, are incompatible.

The present paper treats the inverse problem of DARBOUX for the important partial differential equation of the minimal surfaces for which the conditions just referred to are compatible. It is found that, in contrast to the multiplier of DARBOUX which involved an arbitrary function, the multiplier for this partial differential equation of the second order is uniquely determined up to a constant factor. On the other hand the partial differential equation $r = 0$ is shown to admit a multiplier of even greater generality than that of DARBOUX. It is found that the most general integrand function f of a first order[5]) problem (1. 6) with $n = 2$ for which the extremals are minimal surfaces differs from the familiar integrand function $(1 + p^2 + q^2)^{1/2}$ of the area integral only by an

[3]) J. KÜRCHÁK, Über eine charakteristische Eigenschaft der Differentialgleichungen der Variationsrechnung, *Mathematische Annalen,* **60** (1905), pp. 157—165.

[4]) L. LA PAZ, An Inverse Problem of the Calculus of Variations for Multiple Integrals, *Transactions American Mathematical Society,* **32** (1930), pp. 509—519.

[5]) A problem of minimizing a multiple integral is said to be of order m if the integrand function of the multiple integral involves partial derivatives of the mth order of the function $z(x_1, x_2, \ldots, x_n)$.

additive function of the form $\frac{\partial w_1(x, y, z)}{\partial x} + \frac{\partial w_2(x, y, z)}{\partial y}$. Second order problems for which the extremals are minimal surfaces are considered in the concluding section of the paper.[6])

2. Determination of the most general multiplier.

A surface $z = z(x, y)$ defined and of class C'' in a region R of the xy-plane is a minimal surface if z and its partial derivatives of the first and second orders satisfy in R the partial differential equation

$$(2.1) \qquad F(x, y, z, p, q, r, s, t) \equiv (1 + q^2) r - 2pqs + (1 + p^2) t = 0.$$

This equation does not have a self-adjoint equation of variation and hence, as it stands, is not the LAGRANGE equation of a problem of minimizing a double integral of the form

$$(2.2) \qquad I = \iint_R f(x, y, z, p, q)\, dx\, dy.$$

For a problem (2.2) with first order integrand function f, the LAGRANGE equation is necessarily a second order partial differential equation[7]) linear in r, s, t and with a self-adjoint equation of variation. Hence, if the integral surfaces $z = z(x, y)$ of $F = 0$ are to be the extremals of such a problem, the partial differential equation $F = 0$ must be equivalent to a partial differential equation of the second order with the properties of linearity and self-adjointness of the equation of variation just described. Since the analytic partial differential equation $F = 0$ is itself linear in r, s, t, it now follows easily that the most general linear equation $a_1 r + 2a_2 s + a_3 t + a_4 = 0$ $(a_i = a_i(x, y, z, p, q))$ equivalent to $F = 0$ is of the form $MF = 0$ where $M \neq 0$ is a function of x, y, z, p, q alone. If, in addition, the equation $MF = 0$ is to have a self-adjoint equation of variation then the multiplier M must satisfy

[6]) Throughout the paper to avoid cumbersome details the expression partial differential equation is understood to mean analytic partial differential equation. Any regional restrictions necessary to justify this interpretation are presupposed.

[7]) Integrand functions f, linear in p and q, are excluded from consideration.

the following relations identically in the variables x, y, z, p, q[8])

$$
(2.3)\quad \left\{
\begin{aligned}
&\frac{\partial(M+Mq^2)}{\partial q}+\frac{\partial(Mpq)}{\partial p}=0,\\
&\frac{\partial(M+Mp^2)}{\partial p}+\frac{\partial(Mpq)}{\partial q}=0,\\
&\frac{\partial(M+Mq^2)}{\partial x}-\frac{\partial(Mpq)}{\partial y}+p\frac{\partial(M+Mq^2)}{\partial z}-q\frac{\partial(Mpq)}{\partial z}=0,\\
&\frac{\partial(Mpq)}{\partial x}-\frac{\partial(M+Mp^2)}{\partial y}+p\frac{\partial(Mpq)}{\partial z}-q\frac{\partial(M+Mp^2)}{\partial z}=0.
\end{aligned}
\right.
$$

Hence the multiplier M must be a solution of the following system of non-homogeneous linear partial differential equations:

$$
(2.4)\quad \left\{
\begin{array}{llllll}
pq\dfrac{\partial M}{\partial p}+(1+q^2)\dfrac{\partial M}{\partial q} & * & * & * & +3qM=0,\\
(1+p^2)\dfrac{\partial M}{\partial p}+\quad pq\dfrac{\partial M}{\partial q} & * & * & * & +3pM=0,\\
* \quad * & -(1+q^2)\dfrac{\partial M}{\partial x}+ & pq\dfrac{\partial M}{\partial y} & -p\dfrac{\partial M}{\partial z} & *=0,\\
* \quad * & pq\dfrac{\partial M}{\partial x} & -(1+p^2)\dfrac{\partial M}{\partial y} & -q\dfrac{\partial M}{\partial z} & *=0.
\end{array}
\right.
$$

To integrate (2.4) we transform it into a system linear and homogeneous in the first partial derivatives of a function $m(x, y, z, p, q, M)$ with $\dfrac{\partial m}{\partial M}\neq 0$ which defines M by means of the equation $m=$ constant. The resulting system is:

$$
(2.5)\quad \left\{
\begin{array}{lllllll}
U_1m= & pq\dfrac{\partial m}{\partial p}+(1+q^2)\dfrac{\partial m}{\partial q} & * & * & * & -3qM\dfrac{\partial m}{\partial M} & =0,\\
U_2m= & (1+p^2)\dfrac{\partial m}{\partial p}+\quad pq\dfrac{\partial m}{\partial q} & * & * & * & -3pM\dfrac{\partial m}{\partial M} & =0,\\
U_3m= & * \quad * & -(1+q^2)\dfrac{\partial m}{\partial x}+ & pq\dfrac{\partial m}{\partial y} & -p\dfrac{\partial m}{\partial z} & * & =0,\\
U_4m= & * \quad * & pq\dfrac{\partial m}{\partial x} & -(1+p^2)\dfrac{\partial m}{\partial y} & -q\dfrac{\partial m}{\partial z} & * & =0.
\end{array}
\right.
$$

The four equations in (2.5) are independent since the fourth order determinant obtained from the matrix of coefficients of this system by deleting the last two columns has the value $(1+p^2+q^2)^2$ and is therefore different from zero, but the system (2.5) is not complete since the commutator

[8]) L. La Paz, loc. cit., p. 512.

(2.6) $$(U_1 U_4) m = -p(1+2q^2)\frac{\partial m}{\partial x} + 2p^2 q \frac{\partial m}{\partial y} + (1+q^2)\frac{\partial m}{\partial z} \equiv U_5 m$$

is not a linear combination of U_i $(i=1,\ldots,4)$. However, it is found that on adjoining $U_5 m$ to (2.5) there results a complete system of equations

(2.7) $$U_i m = 0 \qquad (i = 1, \ldots, 5).$$

A particular solution of (2.7) is:

(2.8) $$m_1 = M \cdot (1+p^2+q^2)^{\frac{3}{2}}.$$

Hence, since the general integral of a complete system of n equations in $(n+1)$ independent variables is an arbitrary function of a single particular integral of the system,[9] the most general solution of (2.6) is given by

(2.9) $$m = A(m_1),$$

where A is an arbitrary function of its argument. The most general non-singular solution M of the original system of equations (2.4) is now obtained by solving for M the relation $A(m_1) =$ constant, and is found to have the value

(2.10) $$M = c \cdot (1+p^2+q^2)^{-\frac{3}{2}}$$

where c is a constant different form zero since M by hypothesis is not identically zero. Since multipliers which differ by a non-zero constant factor are not regarded as distinct we thus reach:

Theorem I. *The partial differential equation $F=0$ of the minimal surfaces admits the unique multiplier*

(2.11) $$M = (1+p^2+q^2)^{-\frac{3}{2}}$$

for which the equivalent partial differential equation $MF=0$ has a self-adjoint equation of variation.

The most general integrand function of a problem (2.2) associated with the partial differential equation $MF=0$ where M has the value (2.11) is now secured by use of the case $n=2$ of a general theorem recently proved by the writer.[10] We thus obtain:

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[9] G. Goursat, *Leçons sur l'Intégration des Équations aux Dérivées Partielles du Premier Ordre* (1921), p. 71.

[10] L. La Paz, loc. cit., p. 513.

Theorem II. *The most general integral* (2. 2) *for which the extremal surfaces are minimal surfaces has an integrand function f of the form*

$$f = (1+p^2+q^2)^{\frac{1}{2}} + \frac{\partial w_1}{\partial x} + \frac{\partial w_2}{\partial y} \tag{2.12}$$

where the functions w_1 and w_2 are arbitrary functions of x, y, z alone.

Hence, in view of a familiar result[11]) the problem of minimizing the classical area integral is essentially the only first order problem (2. 2) for which the extremals are minimal surfaces.

3. The most general multiplier for $r=0$.

The generality of the multiplier of Darboux does not depend on the form of the function $g(x, y, y')$ defining the ordinary differential equation (1. 1). On the contrary for partial differential equations of the second order it is no longer true that the generality of the multiplier is independent of the form of the equation. Thus, on the one hand, the example (1.7) of the introduction shows that a given partial differential equation may possess no multiplier whatever; while, on the other hand, the example discussed in the present section proves the existence of partial differential equations which admit multipliers of even greater generality than that of Darboux.[12])

Consider the partial differential equation

$$r = 0. \tag{3.1}$$

If there is to exist a multiplier $M(x, y, z, p, q) \neq 0$ such that

$$M \cdot r = 0, \tag{3.2}$$

the most general partial differential equation of the second order linear in r, s, t which is equivalent to $r=0$, has a self-adjoint equation of variation, then M must satisfy the following conditions[13])

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[11]) Compare O. Bolza, *Vorlesungen über Variationsrechnung* (1900), p. 659.

[12]) On the basis of the degree of generality of the multiplier M a complete classification of analytic partial differential equations of the second order linear in the second partial derivatives has been obtained by the writer and will be incorporated in a later paper.

[13]) See reference in footnote [8]).

$$(3.31) \qquad \frac{\partial M}{\partial q} = 0,$$

$$(3.32) \qquad \frac{\partial M}{\partial x} + p \frac{\partial M}{\partial z} = 0.$$

From the first of the above relations M is seen to be a function of x, y, z and p alone. Hence the most general solution M of the homogeneous linear partial differential equation (3.32) is an arbitrary function of any three particular independent solutions of this equation. Three such solutions are

$$(3.4) \qquad M_1 = y, \quad M_2 = p, \quad M_3 = z - px.$$

Consequently we have

Theorem III. *The partial differential equation $r = 0$ admits as its most general multiplier the function*

$$M(x, y, z, p) = A(y, p, z - px)$$

where A is different from zero but is otherwise an arbitrary function of its three arguments.

Application of the case $n = 2$ of a theorem previously referred to[14]) leads to

Theorem IV. *The most general integral* (2.2) *for which the* LAGRANGE *partial differential equation is*

$$A(y, p, z - px) \cdot r = 0$$

has an integrand function of the form

$$f = \int_{p_0}^{p} \int_{p_0}^{p} A(y, p, z - px)\, dp\, dp + \frac{\partial w_1}{\partial x} + \frac{\partial w_2}{\partial y}$$

where w_1 and w_2 are arbitrary functions of x, y and z.

4. Second order problems of which the extremals are minimal surfaces.

In the preceding sections we have been concerned alone with the determination of the most general first order problems associated with a given partial differential equation of the second order linear in r, s, t. In the present section we seek to determine

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[14]) See footnote [10]).

the most general problem of minimizing a double integral of the form

$$\iint h(x, y, z, p, q, r, s, t)\, dx\, dy \tag{4.1}$$

for which the LAGRANGE equation is a second order equation equivalent to the partial differential equation of the minimal surfaces and hence for which the extremals are minimal surfaces.

If the LAGRANGE equation of (4.1) is to reduce from the fourth order to the second order, JELLETT[15]) has shown that the integrand function h must have the special form

$$h = \varphi(rt - s^2) + ar + 2bs + ct + d \tag{4.2}$$

where φ, a, b, c, d are functions of x, y, z, p, q alone. When h has the form (4.2) it is found that the LAGRANGE partial differential equation is itself of MONGE—AMPERE type

$$\Phi(rt - s^2) + Ar + 2Bs + Ct + D = 0, \tag{4.3}$$

where Φ, A, B, C, D are expressions involving the first and second partial derivatives of the functions φ, a, b, c, d with respect to the variables x, y, z, p, q. If an equation of the form (4.3) is to be equivalent to the analytic linear equation $F = 0$, Φ must vanish and the resulting linear equation $Ar + 2Bs + Ct + D = 0$ must have the form $MF = 0$, where $M = M(x, y, z, p, q) \neq 0$. If furthermore the equation $MF = 0$ is to have a self-adjoint equation of variation then we conclude as in section **2** that M must have the value (2.11). Hence, our problem is to determine the most general integrand function of the form (4.2) for which the coefficients in the associated LAGRANGE partial differential equation (4.3) have the values

$$\Phi = D = 0,\ A = (1 + q^2)\,\delta,\ B = -pq\,\delta,\ C = (1 + p^2)\,\delta, \tag{4.4}$$

where $\delta = (1 + p^2 + q^2)^{-3/2}$.

On calculating the explicit values of these coefficients Φ, A, B, C, D in terms of the coefficients φ, a, b, c, d of the integrand function h we reach:

Theorem V. *The most general problem* (4.1) *for which the* LAGRANGE *equation is a second order equation equivalent to the partial differential equation $F = 0$ of the minimal surfaces has an integrand function of the form* (4.2) *where φ, a, b, c, d are solutions*

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[15]) J. H. JELLETT, *Calculus of Variations* (1852), p. 344.

of the following system of second order partial differential equations:

$$
(4.5)\quad \left\{
\begin{aligned}
&\varphi_{px}+\varphi_{pz}p+\varphi_{qy}+\varphi_{qz}q+3\varphi_z+2b_{pq}-a_{qq}-c_{pp}=0,\\
&\varphi_{yy}+2\varphi_{yz}q+\varphi_{zz}q^2+2a_z+a_{px}+a_{pz}p-a_{qy}-a_{qz}q+\\
&\qquad\qquad\qquad +2b_{py}+2b_{pz}q-d_{pp}=(1+q^2)\,\delta,\\
&\varphi_{xy}+\varphi_{xz}q+\varphi_{yz}p+\varphi_{zz}pq-a_{qx}-a_{qz}p-\\
&\qquad\qquad\qquad -2b_z-c_{py}-c_{pz}q+d_{pq}=pq\,\delta,\\
&\varphi_{xx}+2\varphi_{xz}p+\varphi_{zz}p^2+2b_{qx}+2b_{qz}p+\\
&\qquad\qquad +2c_z+c_{qy}+c_{qz}q-c_{px}-c_{pz}p-d_{qq}=(1+p^2)\,\delta,\\
&a_{xx}+2a_{xz}p+a_{zz}p^2+2b_{xy}+2b_{yz}p+2b_{xz}q+2b_{zz}pq+\\
&\qquad +c_{yy}+2c_{yz}q+c_{zz}q^2+d_z-d_{px}-d_{pz}p-d_{qy}-d_{qz}q=0.
\end{aligned}
\right.
$$

The solution

$$
(4.6)\qquad \varphi=\frac{x^2+y^2+(xp+yq)^2}{2\,(1+p^2+q^2)^{3/2}},\qquad a=b=c=d=0,
$$

shows that the system (4.5) is compatible, but available theory appears inadequate to achieve its integration.

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