Laplacians and continuous linear functionals.[1])

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A classical theorem of F. RIESZ[2]) gives as the general representation for the continuous linear functional of a function $f(x)$ of the single variable x the STIELTJES integral

$$\int_a^b f(x)\,d\,\alpha(x),$$

where $\alpha(x)$ is a function of limited total variation. EVANS and others have given an analogous representation of the linear functional of a function $f(x_1, \dots, x_n)$ of n variables, in the form of a multiple STIELTJES integral

$$\iint \dots \int f(x_1, x_2, \dots, x_n)\, d_{x_1, x_2, \dots, x_n}\, \alpha(x_1, x_2, \dots, x_n),$$

where α is again in a certain sense a function of limited total variation.[3]) This representation suffers under the disadvantage of involving the particular choice of axes $x_1, x_2, \dots, x_n$ that is, of

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[1]) Following a conversation I had with Prof. NORBERT WIENER about the subharmonic functions and their roll in the theory of the potential, he had the kindness to write at my request the present note for my own use, in which he gives an outline of the methods invented by him in researches covering a period of several years. In the hope, that Prof. WIENER will give a systematical exposé of these researches, I asked him to consent to the publication of the present note. *F. R.*

[2]) Sur les opérations fonctionnelles linéaires, *Comptes rendus de l'Académie des Sciences de Paris,* 29 nov. 1909; Sur certains systèmes singuliers d'équations intégrales, *Annales de l'École normale supérieure,* t. 28, 1911, pp. 33—62; Démonstration nouvelle d'un théorème concernant les opérations fonctionnelles linéaires, *ibid.* t. 31, 1914, pp. 9—14.

[3]) See for instance CH. DE LA VALLÉE POUSSIN, Les fonctions à variation bornée et les questions qui s'y rattachent, *Bulletin des sciences mathématiques,* (2), 44, pp. 267—296, 1920.

not being vectorial. The problem of this paper is to give a vectorial, invariantive representation of the continuous linear functional of a function of n variables. We shall take for conveniance $n=3$.

The simplest vectorial differential operator in three variables is the Laplacian Δ. The simplest vectorial integral operator on a scalar function of a vector x is the anti-Laplacian, which yields

$$\varphi(x) = -\frac{1}{4\pi}\iiint\limits_{-\infty}^{\infty} \frac{f(y)}{|y-x|}\,dy.$$

The operator which bears to the Laplaeian the same relation which the difference operator $\frac{1}{h}(f(x+h)-f(x))$ does to the derivative is the operator

$$\frac{6}{h^2}\left\{\frac{1}{4\pi h^2}\iint\limits_{|\mathfrak{x}|=h} f(x+\mathfrak{x})\,dS - f(x)\right\}.$$

In order to see this last fact, let us suppose that $f(x)$ is representable by the triple Fourier integral

$$f(x) \sim \iiint\limits_{\infty}^{\infty} F(u)e^{iu.x}\,du.$$

Then (at least formally)

$$\Delta f(x) \sim \iiint\limits_{-\infty}^{\infty} -|u|^2 F(u)\,e^{iu.x}\,du,$$

and

$$\frac{6}{h^2}\left\{\frac{1}{4\pi h^2}\iint\limits_{|\mathfrak{x}|=h} f(x+\mathfrak{x})\,dS - f(x)\right\} \sim$$

$$\sim \frac{6}{h^2}\iiint\limits_{-\infty}^{\infty}\left\{\frac{\sin h|u|}{h|u|}-1\right\} F(u)\,e^{iu.x}\,du.$$

If both $F(u)$ and $|u^2|\,F(u)$ are summable and of summable square, it is then easy to prove, by appealing to the three-dimensional form of the Plancherel theory of Fourier transforms,[4]) that

$$\underset{h\to 0}{\text{l. m.}}\ \frac{6}{h^2}\left\{\frac{1}{4\pi h^2}\iint\limits_{|\mathfrak{x}|=h} f(x+\mathfrak{x})\,dS - f(x)\right\} = \Delta f(x).$$

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[4]) Contribution à l'étude de la représentation d'une fonction arbitraire par des intégrales définies, *Rendiconti di Palermo*, 30 (1910), pp. 289—335.

The total variation of a function $f(x)$ which always has $f(x)$ between $f(x+0)$ and $f(x-0)$ may be written in the form

$$\overline{\lim_{h\to 0}} \int_{-\infty}^{\infty} \frac{1}{h} \left| f(x+h) - f(x) \right| dx.$$

It is hence reasonable to consider

$$V(f) = \overline{\lim_{h\to 0}} \iiint_{-\infty}^{\infty} \frac{6}{h^2} \left| \frac{1}{4\pi h^2} \iint_{|\mathfrak{x}|=h} f(x+\mathfrak{x})\, dS - f(x) \right| dx$$

as the three-dimensional total variation of a function $f(x)$. Similarly to the STIELTJES integral

$$\int_{-\infty}^{\infty} f(x)\, d\alpha(x) = \lim_{h\to 0} \int_{-\infty}^{\infty} \frac{f(x)}{h} (\alpha(x+h) - \alpha(x))\, dx,$$

there corresponds what we shall write

$$\Gamma f(x) \Delta \alpha(x) = \lim_{h\to 0} \iiint_{-\infty}^{\infty} \frac{6 f(x)}{h^2} \left(\frac{1}{4\pi h^2} \iint_{|\mathfrak{x}|=h} \alpha(x+\mathfrak{x})\, dS - \alpha(x) \right) dx.$$

We wish to prove the following analogue of RIESZ' theorem: *Let $\alpha(x)$ be a function of limited total variation. Then*

$$\Gamma f(x) \Delta \alpha(x)$$

is defined and finite for every function $f(x)$ which is bounded and continuous and vanishes as $|x| \to \infty$ by any route. We have

$$\Gamma f(x) \Delta \alpha(x) \leqq \max |f(x)|\, V(\alpha).$$

Conversely, let $F\{f\}$ be a linear functional defined for all functions $f(x)$ which are bounded and continuous and vanish as $|x| \to \infty$ by any route, and let there be a number K such that

$$F\{f\} \leqq K \max |f(x)|.$$

Then there is a function $\alpha(x)$ of limited total variation such that

$$F\{f\} = \Gamma f(x) \Delta \alpha(x).$$

This function $\alpha(x)$ is unique except for an additive arbitrary harmonic function.

Proof. To begin with, let us find a function $g(x)$ such that $g(x) = 0$ for all sufficiently large values of x, that Δg exists everywhere and is bounded and continuous, and that

$$|f(x) - g(x)| < \varepsilon$$

for all x. This is possible, by general theorems on approximation. We have

$$\begin{aligned} V g(x) \Delta \alpha(x) &= \lim_{h\to 0} \iiint_{-\infty}^{\infty} \frac{6g(x)}{h^2}\left(\frac{1}{4\pi h^2}\iint_{|\mathfrak{x}|=h} \alpha(x+\mathfrak{x})\,dS - \alpha(x)\right)dx = \\ &= \lim_{h\to 0} \iiint_{-\infty}^{\infty} \frac{6\alpha(x)}{h^2}\left(\frac{1}{4\pi h^2}\iint_{|\mathfrak{x}|=h} g(x+\mathfrak{x})\,dS - g(x)\right)dx = \\ &= \iiint_{-\infty}^{\infty} \alpha(x)\,\Delta g(x)\,dx. \end{aligned} \tag{A}$$

It follows that $V g(x) \Delta \alpha(x)$ exists. Furthermore

$$\overline{\lim_{h\to 0}} \left| V g(x)\Delta\alpha(x) - \iiint_{-\infty}^{\infty} \frac{6f(x)}{h^2}\left(\frac{1}{4\pi h^2}\iint_{|\mathfrak{x}|=h} \alpha(x+\mathfrak{x})\,dS - \alpha(x)\right)dx\right| \leqq \varepsilon\, V(\alpha).$$

As ε is arbitrarily small,

$$V f(x)\Delta\alpha(x) = \lim_{h\to 0} \iiint_{-\infty}^{\infty} \frac{6f(x)}{h^2}\left(\frac{1}{4\pi h^2}\iint_{|\mathfrak{x}|=h} \alpha(x+\mathfrak{x})\,dS - \alpha(x)\right)dx$$

exists. The fact that

$$V f(x)\Delta\alpha(x) \leqq \max|f(x)|\, V(\alpha)$$

is obvious.

We now proceed to the second part of the theorem. Let $F\{f\}$ be a linear functional and let there be a number K such that

$$F\{f\} \leqq K \max|f(x)|.$$

If f is positive, we make the definition

$$G\{f\} = \underset{0\leqq g\leqq f}{\text{upper bound}}\; F\{g\}$$

and we extend the definition of G linearly to non-positive functions. It is easy to show that G is a well-defined non-negative linear functional, and that

$$G\{f\} \leqq K \max|f(x)|.$$

If f is positive

$$G\{f\} - F\{f\} \geqq 0.$$

Thus every continuous linear functional is the difference between two linear functionals of positive type. We may hence without essential restriction suppose that F is of positive type.

We now form the auxiliary function $\psi_r(x)$, in conformity with the conditions

$$\psi_r(x) = \begin{cases} -\dfrac{1}{4\pi|x|}; & [\,|x| \geqq r\,] \\ \dfrac{2|x|^2 - 3r|x|}{4\pi r^3}; & [\,|x| < r\,]. \end{cases}$$

The functions $\psi_r(x)$ are all negative, and do not increase with decreasing r. Let

$$\alpha_r(x) = F_{\mathfrak{x}}\, \psi_r(x - \mathfrak{x}).$$

We see that all functions $\alpha_r(x)$ are negative, and that the sequence $\alpha_r(x)$ is monotone non-increasing in r. It is easy to show that it follows from the continuity of ψ_r, that $\iiint_R \psi_r(x - \mathfrak{x})\, dx$ can be approximated to uniformly by a finite sum of ψ_r's, if R is any bounded region. From this we can readily conclude that

$$\iiint_R \alpha_r(x)\, dx = F_{\mathfrak{x}} \iiint_R \psi_r(x - \mathfrak{x})\, dx \geqq -\frac{1}{4\pi} F_{\mathfrak{x}} \iiint_R \frac{dx}{|x - \mathfrak{x}|},$$

which is finite. Hence the functions $\alpha_r(x)$ form for $r \to 0$ a monotone non-increasing sequence with bounded integral, and by a familiar theorem from the theory of the Lebesgue integral, have a limit-function $\alpha(x)$, integrable over any R.

The operator F may readily be extended from continuous functions to their limits.[5] If we represent the Laplacian as the limit of a differencequotient, it follows at once from the continuity of F that

$$\Delta \alpha_r(x) = F_{\mathfrak{x}}\, \Delta_x\, \psi_r(x - \mathfrak{x}) = F_{\mathfrak{x}}\, \chi_r(x - \mathfrak{x}),$$

where

$$\chi_r(\mathfrak{b}) = \begin{cases} \dfrac{6|\mathfrak{b}| - 3r}{2\pi r^3 |\mathfrak{b}|}; & [\,|\mathfrak{b}| < r\,] \\ \dfrac{3}{4\pi r^3} & [\,|\mathfrak{b}| = r\,] \\ 0 & [\,|\mathfrak{b}| > r\,] \end{cases}$$

It follows that

$$\iiint_{-\infty}^{\infty} \chi_r(\mathfrak{b})\, d\mathfrak{b} = 1.$$

Thus if $f(x)$ is a continuous function vanishing at infinity,

[5]) Cf. P. J. Daniell, Annals of Mathematics, 1920.

$$F\{f\}=\lim_{r\to 0} F\left\{\iiint_{-\infty}^{\infty} f(x)\,\chi_r(x-\mathfrak{x})\,dx\right\}=\lim_{r\to 0}\iiint_{-\infty}^{\infty} f(x)\,\Delta\alpha_r(x)\,dx.$$

We have

$$\psi_r(x)=-\frac{1}{4\pi}\iiint_{-\infty}^{\infty}\frac{1}{|\mathfrak{x}|}\,\chi_r(x-\mathfrak{x})\,d\mathfrak{x}.$$

Furthermore

$$\frac{6}{h^2}\left(\frac{1}{4\pi h^2}\iint_{|\mathfrak{x}|=h}\alpha_r(x+\mathfrak{x})\,dS-\alpha_r(x)\right)=$$

$$=F_{\mathfrak{y}}\left\{\frac{6}{h^2}\left(\frac{1}{4\pi h^2}\iint_{|\mathfrak{x}|=h}\psi_r(x-\mathfrak{y}+\mathfrak{x})\,dS-\psi_r(x-\mathfrak{y})\right)\right\}=$$

$$=F_{\mathfrak{y}}\left\{\frac{6}{h^2}\left(\frac{1}{4\pi h^2}\iint_{|\mathfrak{x}|=h}dS\left(-\frac{1}{4\pi}\iiint_{-\infty}^{\infty}\frac{1}{|y|}\,\chi_r(x-\mathfrak{y}+\mathfrak{x}-y)\,dy\right)+\right.\right.$$

$$\left.\left.+\frac{1}{4\pi}\iiint_{-\infty}^{\infty}\frac{1}{|y|}\,\chi_r(x-\mathfrak{y}-y)\,dy\right)\right\}=$$

$$=-\frac{1}{4\pi}F_{\mathfrak{y}}\left\{\frac{6}{h^2}\iiint_{-\infty}^{\infty}\chi_r(x-\mathfrak{y}-y)\,dy\left[\frac{1}{4\pi h^2}\iint_{|\mathfrak{x}|=h}\frac{1}{|y+\mathfrak{x}|}\,d\mathfrak{x}-\frac{1}{|y|}\right]\right\}=$$

$$=F_{\mathfrak{y}}\left\{\iiint_{-\infty}^{\infty}\chi_r(x-\mathfrak{y}-y)\,\chi_h(y)\,dy\right\}.$$

Hence

$$\frac{6}{h^2}\left(\frac{1}{4\pi h^2}\iint_{|\mathfrak{x}|=h}\alpha(x+\mathfrak{x})\,dS-\alpha(x)\right)=$$

$$=\lim_{r\to 0}F_{\mathfrak{y}}\left\{\iiint_{-\infty}^{\infty}\chi_r(x-\mathfrak{y}-y)\,\chi_h(y)\,dy\right\}=F_{\mathfrak{y}}\,\chi_h(x-\mathfrak{y})=\Delta\alpha_h(x).$$

It follows that

$$F\{f\}=\lim_{h\to 0}\iiint_{-\infty}^{\infty}\frac{6f(x)}{h^2}\left(\frac{1}{4\pi h^2}\iint_{|\mathfrak{x}|=h}\alpha(x+\mathfrak{x})\,dS-\alpha(x)\right)dx=\Gamma f(x)\,\Delta\alpha(x).$$

Thus our fundamental theorem is proved.

It remains to show that α is unique. This is equivalent to showing that if α is of limited total variation and

$$\Gamma f(x)\,\Delta\,\alpha(x)=0$$

for every admissible f, then α is harmonic. We have by (A)

$$\iiint_{-\infty}^{\infty} \alpha(x)\, G(x)\, dx = 0,$$

whenever $G(x)$ is a continuous distribution generating a potential vanishing at infinity and itself vanishing at infinity. Hence it follows by considering special types of f (functions f which depend only on the distance from a fixed point) that the mean value of α is constant on concentric spheres. By a simple application of KOEBE's form of the inverse of GAUSS' lemma,[6]) it follows that $\alpha(x)$ is harmonic.

Let $\beta(x)$ be such that

$$\frac{1}{4\pi h^2} \iint_{|\mathfrak{r}|=h} \beta(x+\mathfrak{r})\, dS - \beta(x)$$

is non-negative for all x and h. We than call β *subharmonic.* Let us consider a subharmonic function $\beta(x)$ vanishing as $|x| \to \infty$ in such a manner that for all sufficiently large values of R,

$$\frac{d}{dR} \left\{ \frac{1}{R^2} \iint_{|x|=R} \beta(x)\, dS \right\} = O(1/R^2).$$

Then β is of limited total variation.

To show this, let us consider β_h. It is easy to show that this satisfies at infinity the same condition with regard to the radial derivative of its mean as β itself. Let us put

$$V_R = \iiint_{|x| \leq R} \frac{6}{h^2} \left| \frac{1}{4\pi h^2} \iint_{|\mathfrak{r}|=h} \beta(x+\mathfrak{r})\, dS - \beta(x) \right| dx.$$

Then

$$V_R = \iiint_{|x| \leq R} \Delta \beta_h(x)\, dx = \iint_{|x|=R} \frac{\partial \beta_h(x)}{\partial |x|}\, dS =$$

$$= R^2 \frac{d}{dR} \left\{ \frac{1}{R^2} \iint_{|x|=R} \beta_h(x)\, dS \right\} = O(1),$$

uniformly in h. The result is immediately obvious.

The author wishes to follow the lead of the ordinary theory of functions of limited total variation in connecting his work up with the theory of trigonometrical developments. For this he needs

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[6]) P. KOEBE, Herleitung der partiellen Differentialgleichung der Potentialfunktion aus deren Integraleigenschaft, *Sitzungsberichte der Berliner mathematischen Gesellschaft*, 5. Jahrgang, 1906, pp. 39—42.

the three-dimensional analogue of FEJÉR's theorem. The whole point of FEJÉR's theorem consists in finding a positive kernel $K(x-y)$, which has a trigonometric development approximating in form to that of $\sum_1^\infty \cos kx \cos ky$ as $n \to \infty$.

We form the corresponding three-dimensional kernel in the following manner: the FOURIER transform of the function $\varphi_r(x)$ defined by

$$\varphi_r(x) = \begin{cases} 1; & [\,|x| \leqq r\,], \\ 0; & [\,|x| > r\,] \end{cases}$$

is

$$\iiint_{-\infty}^{\infty} \varphi_r(x) e^{ix.u}\, dx = \int_{-r}^{r} \pi(r^2 - \xi^2)\, e^{i|u|\xi}\, d\xi = \frac{4\pi r}{|u|^2}\left(\frac{\sin|u|r}{|u|r} - \cos|u|r\right).$$

Hence the FOURIER transform of

$$\Phi_r(x) = \iiint_{-\infty}^{\infty} \varphi_r(\xi)\, \varphi_r(x-\xi)\, d\xi = \begin{cases} \pi\left(\frac{4r^3}{3} - r^2|x| + \frac{|x|^3}{12}\right) & [|x| \leqq 2r] \\ 0 & [|x| > 2r] \end{cases}$$

is

$$\left[\frac{4\pi r}{|u|^2}\left(\frac{\sin|u|r}{|u|r} - \cos|u|r\right)\right]^2.$$

It follows that if

$$f(u) \sim \iiint_{-\infty}^{\infty} F(x)\, e^{ix.u}\, dx,$$

we have

$$\frac{1}{6\pi r^2} \iiint_{-\infty}^{\infty} f(v)\, \frac{1}{|u-v|4}\left[\frac{\sin|u-v|r}{|u-v|r} - \cos|u-v|r\right]^2 dv =$$

$$= \iiint_{|x| \leqq 2r} F(x)\left(1 - \frac{3|x|}{r} + \frac{|x|^3}{16r^3}\right) e^{ix.u}\, dx.$$

An argument precisely like that used to prove the ordinary FEJÉR's theorem will then show that if $f(u)$ is a continuos function which is summable and of summable square,

$$\lim_{r \to \infty} \iiint_{|x| \leqq 2r} F(x)\left(1 - \frac{3|x|}{r} + \frac{|x|^3}{16r^3}\right) e^{ix.u}\, dx = f(u)$$

If in addition

$$\iiint_{|x| \leqq 2r} |F(x)|^2\, |x^4\, dx = o(r),$$

a simple application of the SCHWARZ inequality will show that

$$\lim_{r\to\infty}\iiint\limits_{|x|\leq 2r} F(x)\left(-\frac{3|x|}{r}+\frac{|x|^3}{16r^3}\right)e^{ix.u}\,dx=0,$$

and hence that

$$\iiint\limits_{-\infty}^{\infty} F(x)\,e^{ix.u}\,dx=f(u)$$

as a convergent infinite integral.

In proving this theorem, we can replace the continuity of f by the weaker condition

$$\lim_{h\to 0}\frac{1}{4\pi h^2}\iint\limits_{|\mathfrak{x}|=h} f(x+\mathfrak{x})\,dS=f(x).$$

If f has a bounded Laplacian, we have indeed

$$\lim_{h\to 0}\left[\frac{1}{4\pi h^2}\iint\limits_{|\mathfrak{x}|=h} f(x+\mathfrak{x})\,dS-f(x)\right]=O(1/h^2).$$

An intermediate condition is

$$\lim_{h\to 0}\left[\frac{1}{4\pi h^2}\iint\limits_{|\mathfrak{x}|=h} f(x+\mathfrak{x})\,dS-f(x)\right]=o(1/h) \qquad \text{(C)}$$

uniformly in x.

In the study of functions of limited total variation, the author[7]) has found the notion of the quadratic variation of a function very useful. The quadratic variation of $f(x)$ is

$$\lim_{h\to 0}\int_{-\infty}^{\infty}\frac{1}{h}\,|f(x+h)-f(x)|^2\,dx.$$

An analogous expression in the three-dimensional case is

$$Q_1(f)=\lim_{h\to 0}\iiint\limits_{-\infty}^{\infty}\frac{6}{h^3}\left|\frac{1}{4\pi h^2}\iint\limits_{|\mathfrak{x}|=h} f(x+\mathfrak{x})\,dS-f(x)\right|^2 dx.$$

If

$$f(x)\sim\iiint\limits_{-\infty}^{\infty} F(u)\,e^{iu.x}\,du,$$

[7]) N. WIENER, The quadratic variation of a function and its Fourier coefficients, *Journal of Mathematics and Physics of the Massachusetts Institute of Technology*, 3, pp. 72—94.

we have

$$Q_1(f) = \lim_{h \to 0} \frac{6}{h^3} \iiint_{-\infty}^{\infty} \left\{ \frac{\sin h|u|}{h|u|} - 1 \right\}^2 |F(u)|^2 \, du$$

It is easy to see that (C) implies that $Q_1(f) = 0$. Under these conditions, we see at once that

$$\lim_{h \to 0} h \iiint_{|u| < \frac{1}{h}} |u|^4 |F(u)|^2 \, du = 0.$$

It is thus clear that *if $f(x)$ is a function of limited total variation, and is summable and of summable square, and if condition* (C) *is fulfilled, then the* Fourier *integral of f converges uniformly to f.*

The expression $Q_1(f)$ is one of the hierarchy of expressions

$$Q_1(f) = \lim_{h \to 0} \frac{6}{h^3} \iiint_{-\infty}^{\infty} \left| \frac{1}{4\pi h^2} \iint_{|\mathfrak{x}|=h} f(x+\mathfrak{x}) \, dS - f(x) \right|^2 dx;$$

$$Q_2(f) = \lim_{h \to 0} \frac{6}{h^2} \iiint_{-\infty}^{\infty} \left| \frac{1}{4\pi h^2} \iint_{|\mathfrak{x}|=h} f(x+\mathfrak{x}) \, dS - f(x) \right|^2 dx;$$

$$Q_3(f) = \lim_{h \to 0} \frac{6}{h} \iiint_{-\infty}^{\infty} \left| \frac{1}{4\pi h^2} \iint_{|\mathfrak{x}|=h} f(x+\mathfrak{x}) \, dS - f(x) \right|^2 dx.$$

If f is of limited total variation, it is easy to show that $Q_1(f)$ is finite, and if the distribution generating f contains point charges, Q_2 and Q_3 are infinite. If the charges of the charge distribution generating f are distributed continuously on a smooth curve Q_3 is infinite If the charges of the charge distribution generating f are distributed continuously on a smooth surface, all three are finite. We can thus regard the functions of limited total variation for which Q_2 and Q_1 respectively are finite as the natural generalizations of finite line and finite surface distributions, respectively.

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