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2D Parallel Thinning and Shrinking Based on Sufficient Conditions for Topology Preservation*

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Abstract

Thinning and shrinking algorithms, respectively, are capable of extracting medial lines and topological kernels from digital binary objects in a topology preserving way. These topological algorithms are composed of reduction operations: object points that satisfy some topological and geometrical constraints are removed until stability is reached. In this work we present some new sufficient conditions for topology preserving parallel reductions and fifty-four new 2D parallel thinning and shrinking algorithms that are based on our conditions. The proposed thinning algorithms use five characterizations of endpoints.

Keywords: thinning, shrinking, parallel reductions, digital topology, topology preservation

1 Introduction

Shape- and topological analysis of discrete patterns play an important role in image processing and computer vision [17]. Considering the efficiency, several applications use iterative object reduction, like thinning and shrinking. Reduction algorithms are composed of reduction operations that delete object points.

Thinning is a frequently applied skeletonization technique [17, 18], which provides the relevant geometric and topological properties of the shapes. Thinning algorithms are to produce medial lines from 2-dimensional digital objects in a topology preserving way [8]. They preserve endpoints that provide important geometrical information relative to the shape of the objects.

Shrinking algorithms are to extract topological kernels [6]. A topological kernel is a minimal set of points that is topologically equivalent [8, 9, 16] to the original

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object (i.e., if we remove any further point from it, then the topology is not preserved) [6, 8, 9, 16]. Shrinking algorithms preserve the topology [6, 8, 9, 16] of the objects, however, they do not take the object geometry into consideration.

Parallel thinning and shrinking algorithms are composed of parallel reduction operations which delete a set of object points simultaneously [6, 7]. Note that there exists some "*shrinking to a residue*" algorithms, in which every object is transformed into a single point by eliminating cavities [6]. In this paper our attention is focused on the topology-oriented shrinking, and we consider shrinking algorithms as thinning ones with no endpoint preservation.

All thinning and shrinking algorithms need to preserve the topology [8]. Despite of this topological constraint, Couprie found five existing 2D parallel thinning algorithms that do not satisfy it [5]. In order to verify that a parallel reduction preserves topology, Ronse introduced the minimal non-simple sets in [16], and Kong gave some sufficient conditions [9]. Bertrand introduced the P-simple points [1] and the critical kernels [2] that provide methodologies to design topology preserving parallel thinning algorithms. Bertrand and Couprie proposed various parallel thinning algorithms based on critical kernels [3], and they linked the critical kernels to minimal non-simple sets and P-simple points in [4]. However critical kernels constitute a general framework in the category of abstract complexes in any dimension, designing parallel thinning algorithms working on discrete grids might be difficult. That is why we introduced modified versions of Kong's sufficient conditions [9] and combined them with the known parallel thinning approaches and endpoint characterizations to generate a family of topology preserving thinning and shrinking algorithms [13, 15]. In our opinion, one can implement these algorithms easily.

We use the fundamental concepts of digital topology as reviewed by Kong and Rosenfeld [8].

A 2D (8,4) binary digital picture $\mathcal{P} = (\mathbb{Z}^2, 8, 4, B)$ is a quadruple [8], where $\mathbb{Z}^2$ is the set of 2D discrete points. The elements of $B \subseteq \mathbb{Z}^2$ are the black points, having the value of "1", form the 8-connected objects, while points in $\mathbb{Z}^2 \setminus B$ are white ones, having the value of "0", and are assigned to the 4-connected background and cavities. Let $N_8(p)$ and $N_4(p)$ denote the set of 8- and 4-adjacent points to p, respectively. Furthermore, we use the notations $N_4^*(p)$ and $N_8^*(p)$, where $N_8^*(p) = N_8(p) \setminus \{p\}$ and $N_4^*(p) = N_4(p) \setminus \{p\}$. The lexicographical order relation $\prec$ between two distinct points $p = (p_x, p_y)$ and $q = (q_x, q_y)$ is defined as follows:

$$p \prec q \quad \Leftrightarrow \quad p_y < q_y \vee (p_y = q_y \wedge p_x < q_x).$$

A black point is called a *border point* if it is 4-adjacent to at least one white point. The other black points are called *interior point*s.

There are three major strategies for parallel thinning and shrinking algorithms: fully parallel, subiteration-based, and subfield-based [6, 7, 10, 18]. Németh and Palágyi studied a number of parallel thinning algorithms that are based on some sufficient conditions for topology preservation, and the three conventional types of endpoints were considered [15].

In this paper we introduce some advanced sufficient conditions for topology preservation. Then, we propose forty-five new parallel thinning algorithms and nine shrinking ones that are based on these new conditions. Our thinning algorithms take five endpoint characterizations into consideration.

The rest of this paper is organized as follows. In Section 2, we propose our new sufficient conditions for topology preserving parallel reductions. Section 3 reviews the proposed parallel thinning and shrinking algorithms and presents some illustrative results. In Section 4, some properties of our algorithms are discussed. Finally, we round off the paper with some concluding remarks.

2 Topology Preserving Parallel Reductions

In this section, some results concerning topology preserving parallel reduction operations are reviewed.

A reduction operation may change some black points to white ones, which is referred to as deletion, while white points remain unchanged. A parallel reduction operation deletes a set of black points simultaneously.

A 2D reduction operation does *not* preserve topology [8, 9, 16] if any object in the input picture is split (into several objects) or is completely deleted, any cavity in the input picture is merged with the background or another cavity, or any cavity is created where there was none in the input picture.

A black point is *simple* in a picture if its deletion is a topology preserving reduction [8]. The simplicity of a point is a local property, since it can be decided by investigating its 3×3 neighborhood [7].

Although the deletion of a simple point preserves the topology, simultaneous deletion of a set of simple points may disconnect or eliminate objects, merge cavities with each other or with the background, or create new cavities. To ensure topology preservation for parallel reductions, Kong and Ronse gave some sufficient conditions [9, 16].

Theorem 1. *A parallel reduction operation is topology preserving for* (8,4) *pictures if all of the following conditions hold:*

1. *Only simple points are deleted.*
2. *For any two 4-adjacent points p and q that are deleted, p is simple after deletion of q, or q is simple after p is deleted.*
3. *No black component contained in a 2×2 square is deleted completely.*

Theorem 1 is generally used to verify the topological correctness of the thinning and the shrinking algorithms. Németh and Palágyi proposed alternative sufficient conditions for topology preservation that can be applied to generate deletion conditions for various thinning algorithms [15].

Theorem 2. *Let $\mathcal{O}$ be a parallel reduction operation. The operation $\mathcal{O}$ is topology preserving for $(8,4)$ pictures if all of the following conditions hold, for any black point p in picture $(\mathbb{Z}^2, 8, 4, B)$ deleted by $\mathcal{O}$:*

1. *Point p is simple in* $(\mathbb{Z}^2, 8, 4, B)$.
2. *For any simple point* $q \in N_4^*(p) \cap B$, *p is simple in picture* $(\mathbb{Z}^2, 8, 4, B \setminus \{q\})$, *or q is simple in picture* $(\mathbb{Z}^2, 8, 4, B \setminus \{p\})$.
3. *Point p does not coincide with the points marked "$\star$" in the seven black components depicted in Fig.* 1(d)–(j).

By nature of the sufficient conditions of Theorem 2, any parallel reduction operations derived from them can not alter some 2-point wide segments (see Fig. 2). To extract the topological kernel, the algorithm should remove all the simple points, since topological kernels do not contain any simple points. That is why we propose some improved sufficient conditions.

Theorem 3. *Let $\mathcal{O}$ be a parallel reduction operation. The operation $\mathcal{O}$ is topology preserving for* $(8, 4)$ *pictures if all of the following conditions hold, for any black point p in picture* $(\mathbb{Z}^2, 8, 4, B)$ *deleted by $\mathcal{O}$:*

1. *Point p is simple in* $(\mathbb{Z}^2, 8, 4, B)$.
2. *For any simple point* $q \in N_4^*(p) \cap B$, *p is simple in picture* $(\mathbb{Z}^2, 8, 4, B \setminus \{q\})$ *or q is simple in the picture* $(\mathbb{Z}^2, 8, 4, B \setminus \{p\})$, *or* $q \prec p$.
3. *Point p does not coincide with the points marked "$\star$" in the seven black components depicted in Fig.* 1(d)–(j).

Proof. To prove this theorem, it is sufficient to show that all conditions of Theorem 1 are satisfied.

- Condition 1 of Theorem 3 corresponds to Condition 1 of Theorem 1.
- Let p', q' be two 4-adjacent simple points in B such that p' is not simple in $B \setminus \{q'\}$ and q' is not simple in $B \setminus \{p'\}$. If $p' \prec q'$ (hence $q' \not\prec p'$) then set $p = p'$ and $q = q'$ otherwise set $p = p'$ and $q = q'$ by Condition 2 of Theorem 2, p is not deleted by $\mathcal{O}$, thus Condition 2 of Theorem 1 is satisfied, for the pair p', q' is not deleted.
- The black component in Fig. 1(a) is an isolated and non-simple point, hence it can not be deleted by Condition 1 of Theorem 3. The next two black components (see Fig. 1(b) and (c)) are formed by two 4-adjacent black points. One of them (that comes lexicographically first) can not be deleted by Condition 2 of Theorem 3. The remaining seven black components depicted in Fig. 1(d)–(j) can not be deleted completely by Condition 3 of Theorem 3 (the points marked "$\star$" are protected in these cases). Hence, Condition 3 of Theorem 1 holds.

$\square$

Note that the general sufficient conditions of Theorem 3 can be simplified if we consider a given parallel reduction strategy.

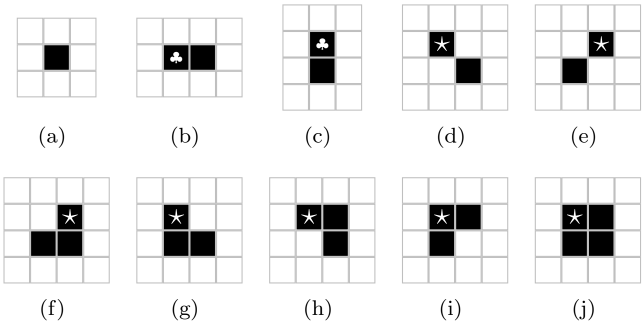


Figure 1: The ten possible black components contained in a 2×2 square. Black points marked "$\star$" are designated to be preserved by Condition 3 of Theorems 2 and 3. Both black components in (b) and (c) must remain unchanged by Theorem 2, but only points marked "$\clubsuit$" are to be preserved by Theorem 3.

3 Thinning and Shrinking Algorithms

In this section, forty-five new parallel thinning algorithms and nine shrinking ones are presented. These topological algorithms are composed of parallel reductions derived from our new sufficient conditions for topology preservation (see Theorem 3).

Thinning algorithms preserve endpoints, some border points that provide relevant geometrical information with respect to the shape of the object. During the shrinking process no endpoint criterion is considered.

Definition 1. *There is no* endpoint of type E0.

To standardize the notations, a shrinking algorithm can be considered a special case of a thinning one, where no endpoint is preserved, hence we use endpoint of type E0 (i.e., the empty set of the endpoints) for it. There exist three conventional types of endpoints E1, E2, and E3 [7].

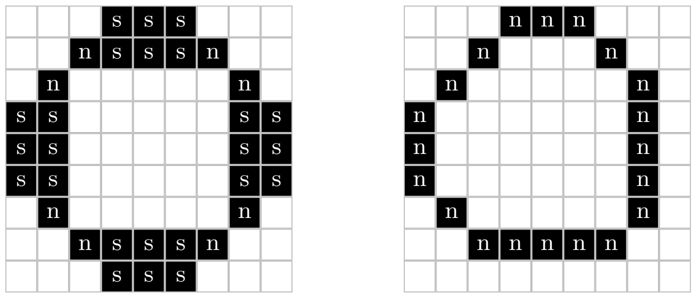
Definition 2. *A border point p is an* endpoint of type E1 *if there is exactly one black point in $N_8^*(p)$.*

Definition 3. *A border point p is an* endpoint of type E2 *if there are at most two 4-adjacent black points in $N_8^*(p)$.*

Definition 4. *A border point p is an* endpoint of type E3 *if there are at most two 8-adjacent black points in $N_8^*(p)$.*

We consider two additional endpoint criteria [3, 12].

Definition 5. *A border point p is an* endpoint of type E4 *if there is no interior point in $N_8^*(p)$.*



(a) (b)

Figure 2: An example to compare the effects of reduction operations derived from Theorems 2 and 3. The points marked "s" are simple, while "n" denotes the non-simple ones. No reduction operation derived from Theorem 2 can remove any point from picture (a), but there are some reduction operations derived from Theorem 3 are capable of removing some additional points (b). Note that there is no simple point in picture (b), hence it is a topological kernel.

Definition 6. *A border point p is an* endpoint of type E5 *if there is no interior point in $N_4^*(p)$.*

Let $\mathcal{E}i$ denote the set of endpoints of type Ei ($i = 1, \ldots, 5$) for an arbitrarily chosen binary image. It is easy to see that

$$\mathcal{E}1 \subset \mathcal{E}2 \subset \mathcal{E}3 \subset \mathcal{E}4 \subset \mathcal{E}5.$$

Some examples of these characterizations of endpoints are depicted in Fig. 3.

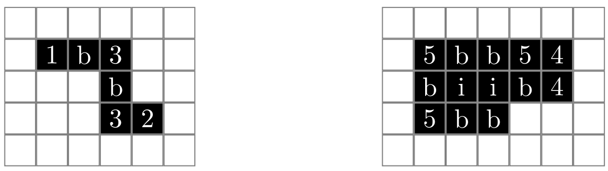


Figure 3: Examples of endpoints. Points marked "k" are endpoints of type Ei ($k = 1, \ldots, 5$; $i = k, \ldots, 5$). Points marked "i" are interior points, while points marked "b" are border points that are not endpoints.

It is easy to see that all points in all possible black components contained in a 2×2 square are endpoints of types E4 and E5, since there is no interior point in these components (see Fig. 1). Consequently, Condition 3 of Theorem 3 can be omitted in the case of parallel reductions that preserve endpoints of type E4 or E5. Hence, we can state the following:

Theorem 4. *Let $\mathcal{O}^\varepsilon$ be a parallel reduction operation that preserves endpoints of type ε, where $\varepsilon \in \{\mathrm{E0}, \ldots, \mathrm{E5}\}$. The operation $\mathcal{O}^\varepsilon$ is topology preserving for $(8, 4)$*

pictures if all of the following conditions hold, for any black point p in picture $(\mathbb{Z}^2, 8, 4, B)$ deleted by $\mathcal{O}^\varepsilon$:

1. *Point p is simple and not an endpoint of type ε.*
2. *For any simple point $q \in N_4^*(p) \cap B$ that is not an endpoint of type ε, p is simple in $(\mathbb{Z}^2, 8, 4, B \setminus \{q\})$ or q is simple in $(\mathbb{Z}^2, 8, 4, B \setminus \{p\})$, or $q \prec p$.*
3. *Depending on a given endpoint characterization ε, the following conditions are to be satisfied:*
   - *If $\varepsilon = \mathrm{E0}$, then point p does not coincide with the point marked "$\star$" depicted in Fig.* 1(d)-(j).
   - *If $\varepsilon = \mathrm{E1}$, then point p does not coincide with the point marked "$\star$" depicted in Fig.* 1(f)-(j).
   - *If $\varepsilon \in \{\mathrm{E2}, \mathrm{E3}\}$, then point p does not coincide with the point marked "$\star$" depicted in Fig.* 1(j).

Proof. Conditions 1, 2, and 3 fulfill the Conditions 1, 2, and 3 of Theorem 3, respectively. In the case of Conditions 3, black points in Fig. 1(d) and (e) are endpoints of type E1. In Fig. 1(d) – (i), there is at least one E2 or E3 endpoint in the component, hence it is not deletable completely. If $\varepsilon \in \{\mathrm{E4}, \mathrm{E5}\}$, then no black component contained in a 2×2 square can be deleted completely by $\mathcal{O}^\varepsilon$ since all of their elements are endpoints to be preserved.

□

In the rest of this section thinning and shrinking algorithms composed of parallel reduction operations that satisfy Theorem 4 are reported. The properties of these algorithms are discussed in Section 4.

The proposed algorithms were tested on objects of different shapes. Here we can present their results superimposed on just one 120×45 picture with 2572 object points, see Figs. 4, 6, 7, 11, 12. The pairs of numbers in parentheses are the count of object points in the produced pictures and the parallel speed (i.e., the number of the required parallel reduction operations [7]).

3.1 Fully Parallel Algorithms

In fully parallel algorithms, the same parallel reduction operation is applied in each iteration step [7].

The general scheme of the fully parallel thinning algorithms FP-ε using endpoint of type ε are sketched by Alg. 1 ($\varepsilon \in \{\mathrm{E0}, \dots, \mathrm{E5}\}$).

The FP-ε-deletable points are defined as follows:

Definition 7. *Let $\varepsilon \in \{\mathrm{E0}, \dots, \mathrm{E5}\}$ be a characterization of endpoints. Black point p is* FP-ε-deletable *if all the conditions of Theorem 4 hold.*

Algorithm 1 Algorithm FP-ε

Input: picture $(\mathbb{Z}^2,8,4,\mathrm{X})$
Output: picture $(\mathbb{Z}^2,8,4,\mathrm{Y})$
$\mathrm{Y} = \mathrm{X}$
repeat
    $D = \{p \mid p \text{ is FP-}\varepsilon\text{-deletable in Y}\}$
    $\mathrm{Y} = \mathrm{Y} \setminus D$
until $D = \emptyset$

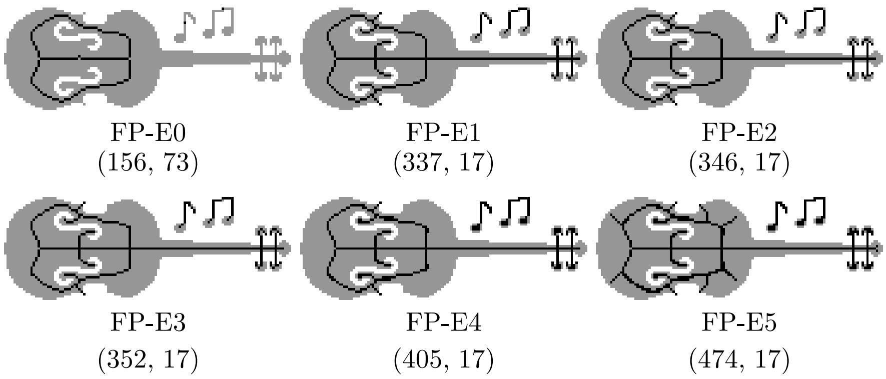


Figure 4: A topological kernel and five medial lines produced by the proposed fully parallel algorithms.

Figure 4 presents illustrative examples for topological kernels and medial lines produced by algorithms FP-ε ($\varepsilon \in \{\mathrm{E0}, \ldots, \mathrm{E5}\}$).

It can readily be seen that deletable points of the proposed fully parallel algorithms (see Def. 7) are derived directly from conditions of Theorem 4. Hence, all of the six algorithms are topology preserving.

3.2 Subiteration-based Algorithms

In subiteration-based (or frequently referred to as directional) thinning algorithms, an iteration step is decomposed into k successive parallel reduction operations according to the k deletion directions. If direction d is the current deletion direction, then a set of d-border points can be deleted by the parallel reduction operation assigned to it [7].

A black point p is an N-border point if point p_N (see Fig. 5) is white. The W-, S-, E-border points can be defined similarly. In addition, a black point p is an NE-border point if p_N or p_E is white. Considering another pairs of directions, we can likewise talk about NW-, SW-, SE-border points (see Fig. 5).

Let ε be a type of endpoint ($\varepsilon \in \{\mathrm{E0}, \ldots, \mathrm{E5}\}$) and $Q = \langle d_1, \ldots, d_k \rangle$ be a sequence of the deletion directions. For existing 2-subiteration algorithms, the

p_{NW}	p_N	p_{NE}
p_W	p	p_E
p_{SW}	p_S	p_{SE}

Figure 5: Notations for the 3×3 neighborhood of point p.

two deletion directions NE and SW are generally applied [7, 10, 18]. Of course, applying the intermediate directions SE and NW, we get algorithms SI-$\langle NW, SE\rangle$-ε. In the case of the existing 4-subiteration algorithms, cardinal deletion directions N, E, S, and W are considered [7, 10, 18].

Note that these subiteration-based algorithms with intermediate deletion directions produce distorted results for symmetric objects. In order to improve the medialness property of the 2-subiteration algorithms, we propose a new 4-subiteration scheme with the sequence of intermediate deletion directions $\langle NE, SW, NW, SE\rangle$. Note that this scheme is capable of removing the two outmost layers from the objects at an iteration step.

Directional thinning algorithms using endpoint of type ε and a sequence of deletion directions Q are sketched by algorithm SI-$\langle Q\rangle$-ε (see Alg. 2), ($\varepsilon \in \{\mathrm{E0}, \ldots, \mathrm{E5}\}$; $Q = \langle NE, SW\rangle$, $\langle NW, SE\rangle$, $\langle N, E, S, W\rangle$, $\langle NE, SW, NW, SE\rangle$).

Algorithm 2 Algorithm SI-$\langle Q\rangle$-ε

Input: picture ($\mathbb{Z}^2$,8,4,X)
Output: picture ($\mathbb{Z}^2$,8,4,Y)
Y = X
repeat
  $D = \emptyset$
  for all $d \in Q$ **do**
    $D_d = \{p \mid p$ SI-d-ε-deletable in Y $\}$
    $\mathrm{Y} = \mathrm{Y} \setminus D_d$
    $D = D \cup D_d$
  end for
until $D = \emptyset$

The SI-d-ε-deletable points are defined as follows:

Definition 8. *Black point p is called* SI-d-ε-deletable *if all the following conditions hold ($\varepsilon \in \{\mathrm{E0}, \ldots, \mathrm{E5}\}$; $d \in \{N, E, S, W, NE, SW, NW, SE\}$):*

1. *Point p is a d-border point, simple, and not an endpoint of type ε,*

2. *For any simple d-border point $q \in N_4^*(p)$ that is not an endpoint of type ε, p is simple in $N_8^*(p) \setminus \{q\}$ or q is simple in $N_8^*(q) \setminus \{p\}$, or $q \prec p$,*

3. *Depending on a given endpoint characterization ε, the following conditions are to be satisfied:*

   a) *If* $\varepsilon =$ E0, *then the following cases have to be taken into consideration:*
      - *if* $d = NE$, *then* p *does not coincide with the point marked "$\star$" depicted in Fig.* 1(d), (e), (f), (h), *and* (i),
      - *if* $d = SW$, *then* p *does not coincide with the point marked "$\star$" depicted in Fig.* 1(d), (e), (f), (g), *and* (i),
      - *if* $d = NW$, *then* p *does not coincide with the point marked "$\star$" depicted in Fig.* 1(d), (e), (g), (h), *and* (i),
      - *if* $d = SE$, *then* p *does not coincide with the point marked "$\star$" depicted in Fig.* 1(d), (e), (f), (g), *and* (h),
      - *if* $d \in \{N, E, S, W\}$, *then* p *does not coincide with the point marked "$\star$" depicted in Fig.* 1(d) *and* (e).

   b) *If* $\varepsilon =$ E1, *then the following cases have to be checked:*
      - *if* $d = NE$, *then* p *does not coincide with the point marked "$\star$" depicted in Fig.* 1(f), (h), (i),
      - *if* $d = SW$, *then* p *does not coincide with the point marked "$\star$" depicted in Fig.* 1(f), (g), (i),
      - *if* $d = NW$, *then* p *does not coincide with the point marked "$\star$" depicted in Fig.* 1(g), (h), (i),
      - *if* $d = SE$, *then* p *does not coincide with the point marked "$\star$" depicted in Fig.* 1(f), (g), (h).

Some topological kernels and medial lines proposed 2- and 4-subiteration algorithms are presented in Figs. 6 and 7, respectively.

It can readily be seen that deletable points of the proposed subiteration-based algorithms (see Def. 8) are derived from the conditions of Theorem 4. Hence, all of the twenty-four algorithms are topology preserving.

3.3 Subfield-based Algorithms

Subfield-based algorithms partition the digital space into k subfields. During an iteration step, the subfields are alternatively activated, and a set of border points in the active subfield can be deleted by a parallel reduction operation [7].

The existing subfield-based thinning algorithms partition the 2-dimensional digital space into two and four subfields, see Fig. 8. In the case of k subfields, the i-th subfield denoted by $S_k(i)$ is defined as follows ($k = 2, 4$; $i = 0, \ldots, k-1$):

$$S_2(i) = \{p = (x, y) \mid (x + y) \equiv i \pmod 2\}, \tag{1}$$

$$S_4(i) = \{p = (x, y) \mid 2 \cdot (y \bmod 2) + (x \bmod 2) = i\}. \tag{2}$$

It is easy to see that there is no 4-adjacent pair of points in the same subfield, hence Theorem 4 can be simplified in the following way.

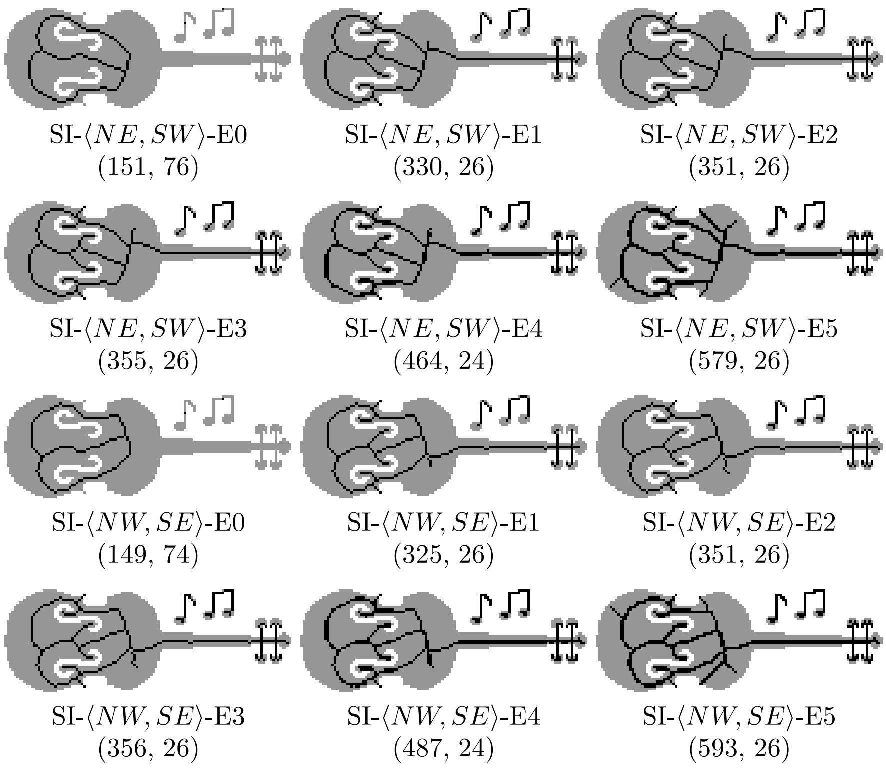


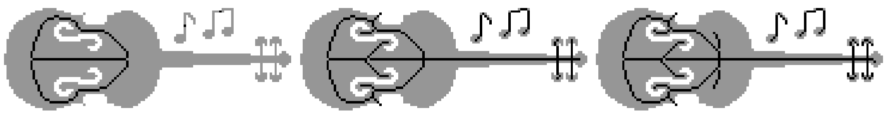
Figure 6: Two topological kernels and ten medial lines produced by the proposed 2-subiteration algorithms.

Theorem 5. *Let a picture $\mathcal{P} = (\mathbb{Z}^2, 8, 4, B)$ be partitioned into k subfields, and let ε be a type of endpoints ($\varepsilon \in$ E0, ..., E5). Let $p \in B$ be a black point in the active subfield $S_k(i)$, and $\mathcal{O}^{\varepsilon}_{SF_k}$ be a parallel reduction operation such that $\mathcal{O}^{\varepsilon}_{SF_k}$ deletes p. The parallel reduction operation $\mathcal{O}^{\varepsilon}_{SF_k}$ is topology preserving if all of the following conditions hold:*

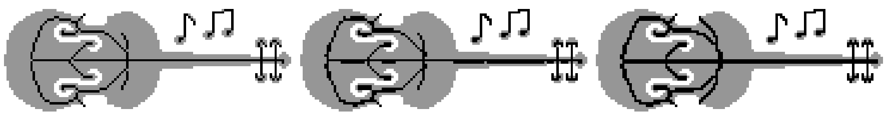
1. *Point $p \in S_k(i) \cap B$ is simple in $\mathcal{P}$ and not an endpoint of type ε.*

2. *If $k = 2$ and $\varepsilon =$ E0, then p does not coincide with the point marked "$\star$" depicted in Fig.* 1(d) *and* (e).

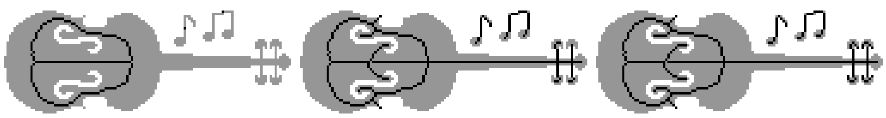
Proof. The proof of this theorem is very easy, since it is sufficient to see that it is a special case of Theorem 3. If Condition 1 of Theorem 5 is fulfilled, then Condition 1 of Theorem 1 is satisfied. Condition 2 of Theorem 3 is not necessary to be checked, since there is no 4-adjacent pair of points in the same subfield, and operation $\mathcal{O}^{\varepsilon}_{SF_k}$ can delete a set of points in the active subfield. Finally, it is obvious that only two black components depicted in Fig. 1(d) and (e) contain simple points belonging to



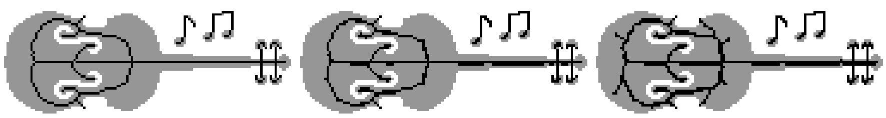
SI-$\langle N, E, S, W \rangle$-E0 (148, 104) SI-$\langle N, E, S, W \rangle$-E1 (325, 48) SI-$\langle N, E, S, W \rangle$-E2 (356, 48)



SI-$\langle N, E, S, W \rangle$-E3 (365, 48) SI-$\langle N, E, S, W \rangle$-E4 (509, 52) SI-$\langle N, E, S, W \rangle$-E5 (639, 48)



SI-$\langle NE, SW, NW, SE \rangle$-E0 (158, 76) SI-$\langle NE, SW, NW, SE \rangle$-E1 (339, 28) SI-$\langle NE, SW, NW, SE \rangle$-E2 (356, 28)



SI-$\langle NE, SW, NW, SE \rangle$-E3 (360, 28) SI-$\langle NE, SW, NW, SE \rangle$-E4 (462, 28) SI-$\langle NE, SW, NW, SE \rangle$-E5 (589, 28)

Figure 7: Two topological kernels and ten medial lines produced by the proposed 4-subiteration algorithms.

the same subfield if $k = 2$. Therefore, if Condition 2 of Theorem 5 holds, then Condition 3 of Theorem 1 is satisfied. $\square$

Our subfield-based thinning algorithms SF-k-ε ($k = 2, 4$; $\varepsilon \in \{\mathrm{E0}, \ldots, \mathrm{E5}\}$) derived from Theorem 5 are sketched by Alg. 3.

In the case of subfield-based algorithms, deletable points are defined as follows.

Definition 9. *Black point p is* SF-k-i-ε-deletable *if the following conditions hold,* ($k = 2, 4$; $i = 0, \ldots, k-1$; $\varepsilon \in \{\mathrm{E0}, \ldots, \mathrm{E5}\}$)*:*

1. *Point p is simple in subfield $S_k(i)$ and not an endpoint of type ε,*
2. *If $k = 2$ and $\varepsilon = E0$, then p does not coincide with the points marked "$\star$" in Fig.* 1(d) *and* 1(e).

Some topological kernels and medial lines produced by our subfield-based algorithms are presented in Figs. 11 and 12.

0	1	0	1	0
1	0	1	0	1
0	1	0	1	0
1	0	1	0	1
0	1	0	1	0

(a) 2 subfields

0	1	0	1	0
2	3	2	3	2
0	1	0	1	0
2	3	2	3	2
0	1	0	1	0

(b) 4 subfields

Figure 8: Partitions of $\mathbb{Z}^2$ into two (a) and four (b) subfields. For the k-subfield case, the points marked i are in the subfield $S_k(i)$ ($k = 2, 4$, $i = 0, \ldots, k-1$).

Algorithm 3 Algorithm SF-k-ε

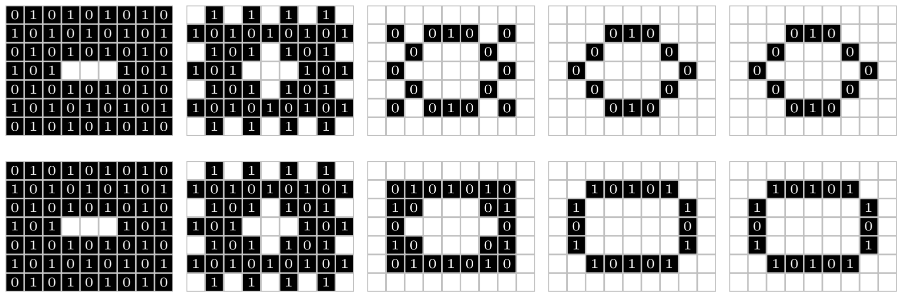
```
Input: picture (Z^2,8,4,X)
Output: picture (Z^2,8,4,Y)
Y = X
repeat
   D = ∅
   for i = 0 to k − 1 do
      D_i = {p | p is SF-k-i-ε-deletable in Y}
      Y = Y \ D_i
      D = D ∪ D_i
   end for
until D = ∅
```

A drawback of the conventional subfield-based thinning algorithms is that they may produce several unwanted skeletal branches or the thinning process can be blocked, since endpoints preserved in a subiteration inhibit the deletion of some further points in the next iteration steps. In order to overcome this problem, we proposed a new scheme with iteration-level endpoint checking [14, 15]. According to this strategy, endpoints are marked in the beginning of each iteration step. In addition, this new strategy allows deletion of a set of border points, which were in the outmost layer in the beginning of the current iteration step.

Our new subfield-based scheme produces different residues in the case of shrinking algorithms as well, see Fig. 9. It can be stated that the new strategy produces less unwanted side branches than the conventional subfield-based thinning scheme (see Figs. 10, 11, and 12).

Our subfield-based thinning algorithms SF-k-IL-ε ($k = 2, 4$; $\varepsilon \in \{\mathrm{E0}, \ldots, \mathrm{E5}\}$) with iteration-level endpoint checking are sketched by Alg. 4.

It can be seen that deletable points of the proposed subfield-based algorithms (see Def. 9) are derived directly from conditions of Theorem 4. Hence, all of the twenty-four algorithms are topology preserving.



initial object $S_2(0)$ $S_2(1)$ $S_2(0)$ $S_2(1)$

Figure 9: Two iteration steps of the conventional 2-subfield shrinking algorithm SF-2-E0 (upper row) and algorithm SF-2-IL-E0 with iteration-level endpoint checking (lower row). Subfield $S_2(i)$ is activated in the i-th phase ($i = 0, 1$). The numbers indicate the subfield indices.

Algorithm 4 Algorithm SF-k-IL-ε

Input: picture ($\mathbb{Z}^2$,8,4,X)
Output: picture ($\mathbb{Z}^2$,8,4,Y)
Y = X
repeat
    $D = \emptyset$
    $E = \{p \mid p$ is a border point but not an endpoint of type ε in ($\mathbb{Z}^2$,8,4,Y) $\}$
    for $i = 0$ to $k - 1$ **do**
        $D_i = \{p \mid p$ is SF-k-i-E0-deletable in $E \cap S_k(i)\}$
        Y = Y $\setminus D_i$
        $D = D \cup D_i$
    end for
until $D = \emptyset$

4 Discussion

This section is to discuss some important properties of the proposed algorithms.

Definition 10. *A result of a thinning algorithm is minimal if it does not contain any simple point except the type of endpoint taken into consideration.*

Definition 11. *A result of a shrinking algorithm is minimal if it there is no simple point in it.*

Proposition 1. *The results of the proposed fully parallel algorithms FP-ε are minimal for any pictures ($\varepsilon \in \{$E0, …, E5$\}$).*

Proof. Let us suppose that a black and non-end point p remains simple after the last iteration step.

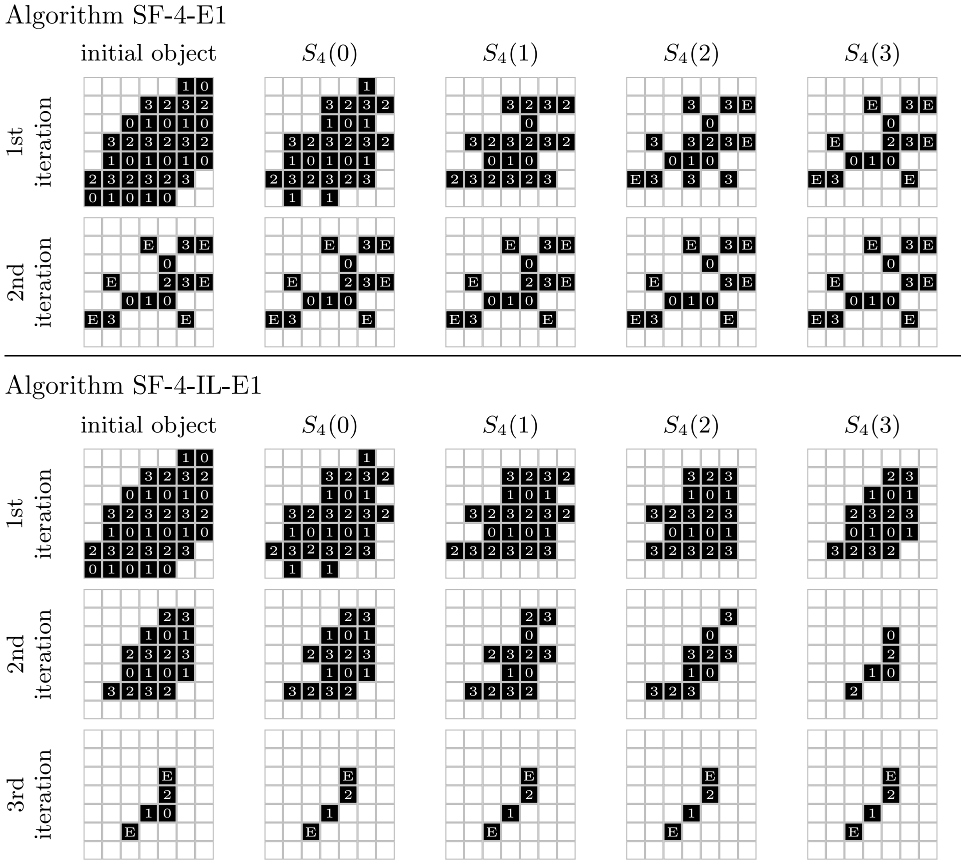


Figure 10: Phases of algorithms SF-4-E1 and SF-4-IL-E1. Subfield $S_4(i)$ is activated in the i-th phase ($i = 0, 1, 2, 3$). Points marked E are the E1 endpoints to be preserved and numbers indicate the subfield indices.

Let $\mathcal{R}$ be the set of all remaing simple points that are not endpoints in the output picture. There is a $p \in \mathcal{R}$ such that, for any $q \in \mathcal{R} \setminus \{p\}$, $q \prec p$. This means that p satisfies all conditions of Theorem 4:

- each point in $\mathcal{R}$ satisfies Condition 1 of Theorem 4,
- since $q \prec p$ for each $q \in \mathcal{R} \setminus \{p\}$, thus p satisfies Conditions 2 and 3 of Theorem 4.

Therefore, p is FP-ε-deletable. We come to a contradiction with our assumption, as the algorithm should have deleted p in the last iteration.

□

Proposition 2. *The results of the proposed subiteration-based algorithms SI-$\langle Q \rangle$-ε*

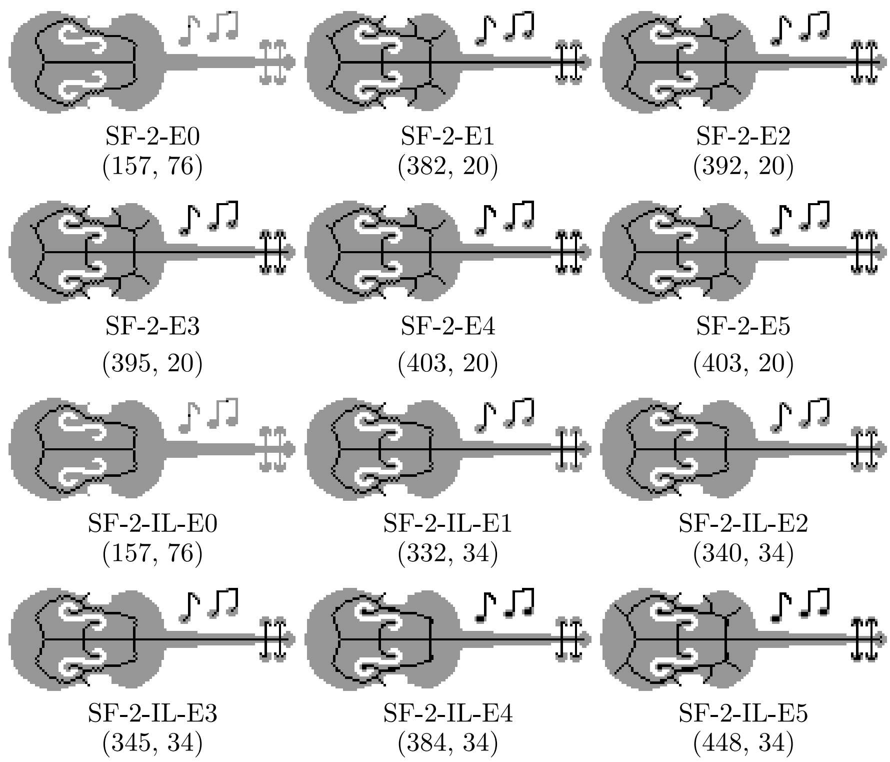


Figure 11: Two topological kernels and ten topological kernels produced by the proposed 2-subfield algorithms.

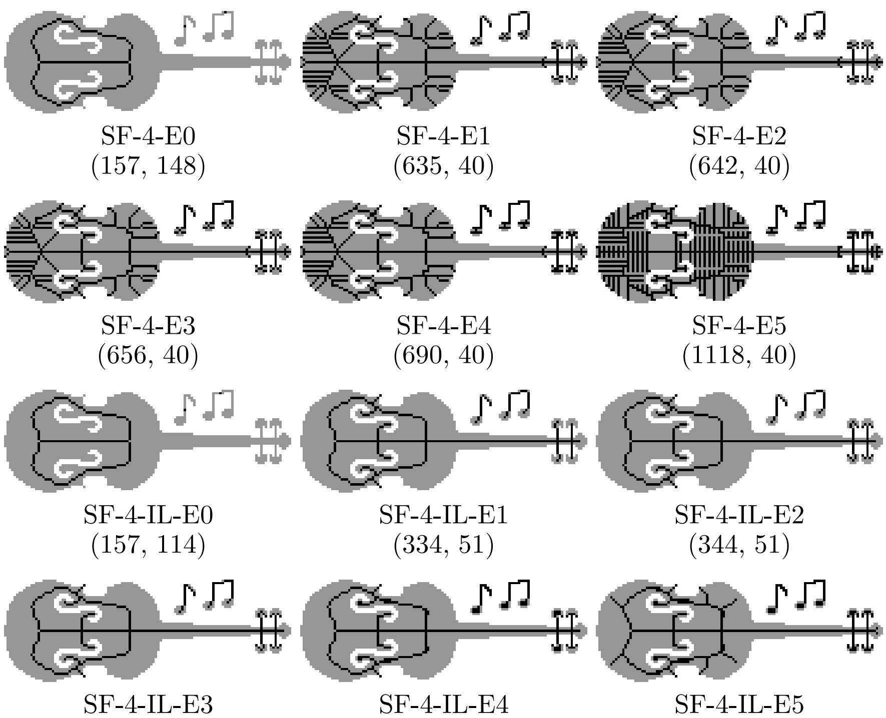
are minimal for any picture ($Q \in \{\langle NE, SW\rangle, \langle NW, SE\rangle, \langle N, E, S, W\rangle, \langle NE, SW, NW, SE\rangle\}$; $\varepsilon \in \{\mathrm{E0}, \ldots, \mathrm{E5}\}$).

Proof. The proof goes similarly, as in the fully parallel case. Let us suppose that a black point p remains simple after the last iteration step.

- If $\varepsilon = \mathrm{E0}$, then if p is a d-border point (considering any deletion direction d) and there is no other d-border simple point in $N_4^*(p)$, then p is deletable. Otherwise, let $q \in N_4^*(p)$ be a simple d-border point. Then, p or q can be deletable according to Condition 2 of Theorem 4, when the lexicographically first remains simple after the deletion of the other one. Both of these two cases lead to a contradiction.
- If $\varepsilon \in \{\mathrm{E1}, \ldots, \mathrm{E5}\}$, then p must fulfill the considered endpoint criterion, otherwise it should have been deleted.

$\square$

Proposition 3. *The results of the proposed subfield-based reduction algorithms SF-k-ε, SF-k-IL-ε are minimal for any pictures* ($k = 2, 4; \varepsilon \in \{\mathrm{E0}, \ldots, \mathrm{E5}\}$).



SF-4-E0 (157, 148) | SF-4-E1 (635, 40) | SF-4-E2 (642, 40)

SF-4-E3 (656, 40) | SF-4-E4 (690, 40) | SF-4-E5 (1118, 40)

SF-4-IL-E0 (157, 114) | SF-4-IL-E1 (334, 51) | SF-4-IL-E2 (344, 51)

SF-4-IL-E3 (347, 51) | SF-4-IL-E4 (392, 51) | SF-4-IL-E5 (448, 51)

Figure 12: Two topological kernels and ten medial lines produced by the proposed 4-subfield algorithms.

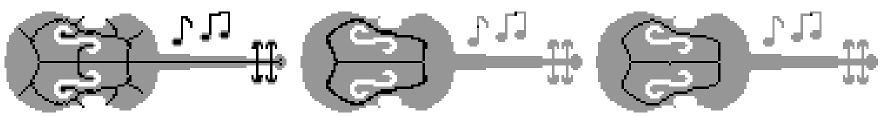
Proof. If the 2-dimensional digital space is partitioned into 2 or 4 subfields as presented in Fig. 8, then it is easy to see that no 4-adjacent pair of simple points belongs to the same subfield. Hence, it is evident that if two simple points are 4-adjacent in the picture, then either of them can be deleted when the subfield is activated. If two 4-adjacent points remain simple after the final iteration step, then they must be endpoints of type E1, E2, E3, E4, or E5. □

To summarize the properties of the presented algorithms, we state the followings:

- All the fifty-four algorithms are different from each other (see Figs. 4, 6, 7, 11, 12).
- All thinning algorithms with endpoint characterizations E4 and E5 may produce 2-point wide medial curves.
- The 2-subiteration algorithms (i.e., SI-$\langle NE, SW\rangle$-ε and SI-$\langle NW, SE\rangle$-ε; $\varepsilon \in \{$E0,..., E5$\}$) may produce a "distorted" asymmetric medial curves for symmetric objects (see Fig. 6).

- The 4-subiteration algorithms (i.e., SI-$\langle NE, SW, NW, SE\rangle$-$\varepsilon$ and SI-$\langle N, E, S, W\rangle$-$\varepsilon$; $\varepsilon \in$ {E0, ...,E5}) can produce "almost symmetric" results for symmetric objects.
- The 4-subfield thinning algorithms SF-4-ε ($\varepsilon \in \{E1, \ldots, E5\}$) may produce numerous unwanted side branches. (That is why we proposed the thinning scheme with iteration-level endpoint checking.)
- Subfield-based algorithms SF-k-IL-ε (k=2, 4; $\varepsilon \in$ {E0,...,E5}) with iteration-level endpoint checking produce much less unwanted side branches than algorithms SF-k-ε (k=2,4; $\varepsilon \in$ {E0, ..., E5}) that use the conventional scheme (see Figs. 11 and 12).
- The fully parallel algorithms FP-ε ($\varepsilon \in$ {E0, ..., E5}) require the least numbers of parallel reductions.
- The 4-subiteration algorithms that are taken the intermediate deletion direction into consideration (i.e., SI-$\langle NE, SW, NW, SE\rangle$-$\varepsilon$; $\varepsilon \in$ {E0,...,E5}) require less numbers of parallel reductions than the 4-subiteration ones considering the cardinal deletion directions (i.e., SI-$\langle N, W, S, E\rangle$-$\varepsilon$; $\varepsilon \in$ {E0, ..., E5}).
- The 2-subfield algorithms (i.e., SF-2-ε and SF-2-IL-ε; $\varepsilon \in$ {E0, ..., E5}) are require less numbers of parallel reductions than the 4-subfield ones (i.e., SF-4-ε and SF-4-IL-ε; $\varepsilon \in$ {E0, ..., E5}).

In order to illustrate that our algorithms differ from the algorithms based on critical kernels, Fig. 13 presents three skeletons produced by algorithms AK^2, MK^2, and NK^2 [3].



AK^2	MK^2	NK^2
(448, 17)	(214, 17)	(155, 72)

Figure 13: Skeletons produced by algorithms AK^2, MK^2, and NK^2 [3].

Unfortunately, there is no room to present here more examples, hence we invite the reader to visit the website at
`http://www.inf.u-szeged.hu/~gnemeth/localweb/skeleton_alg2d.php`,
where skeletons produced by various existing algorithms are also presented.

5 Conclusions

This paper presents fifty-four topological algorithms for extracting skeleton-like shape features (i.e., topological kernels and medial lines) from binary objects. The major contributions of this work are:

- We proposed new sufficient conditions for topology preserving parallel reductions that are suitable for generating deletion rules for parallel topological algorithms.
- Fifty-four variations for parallel thinning and shrinking algorithms were constructed (each algorithm differ from the other ones). Deletion rules of the proposed algorithms were not given by matching templates (as it is usual), they were derived from our conditions.
- We introduced the 4-subiteration scheme with intermediate deletion directions, and the iteration-level endpoint checking in subfield-based algorithms.
- We proved that all the fifty-four algorithms produce minimal results for any pictures.

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