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Recognizable Tree Series with Discounting

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Abstract

We consider weighted tree automata with discounting over commutative semirings. For their behaviors we establish a Kleene theorem and an MSO-logic characterization. We introduce also weighted Muller tree automata with discounting over the max-plus and the min-plus semirings, and we show their expressive equivalence with two fragments of weighted MSO-sentences.

Keywords: semirings, discounting, weighted tree automata, rational operations on tree series, weighted Muller tree automata, weighted MSO-logic over finite and infinite trees.

1 Introduction

Weighted tree automata over finite trees have been considered by many researchers (cf. [22] for an extended literature) and have contributed in important areas of Computer Science like code selection [3, 20] and monadic second-order evaluations on graphs [33]. Weighted tree automata models are obtained by classical tree automata, top-down or bottom-up, whose transitions are equipped with weights mainly from a semiring. The weights might model resources used for the execution of transitions, the time needed or reliability. For an excellent survey on weighted tree automata we refer the reader to [22] (cf. also [2]).

If we require that weighted tree automata can work also on infinite trees, then clearly the underlying semiring should admit infinite sums and products satisfying special axioms (cf. [32]). *Discounting* is a common strategy to face problems arising on systems with non-terminating behavior, in particular in economic mathematics, in Markov decision processes, and in game theory (cf. [9, 21, 34]). This method was incorporated for weighted automata over infinite words by Droste and Kuske in [14]. More precisely, the authors considered weighted automata over the max-plus and min-plus semirings, acting on infinite words, and employed a discounting parameter which permitted the summation of infinitely many values. In this way, they achieved a Kleene theorem for the infinitary series obtained as the behaviors of their automata. They also considered weighted automata with discounting

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over finite words, and they showed a Kleene-Schützenberger theorem for the series accepted by these automata. In [5, 6] further properties of weighted automata with discounting over infinite words were investigated. A weighted MSO-logic with discounting has been introduced in [16] and a Büchi-type characterization of infinitary recognizable series with discounting has been established. Kuich [27] proved Kleene theorems for weighted automata with discounting acting on finite and infinite words over Conway semirings. Recently, in [17] the authors investigated weighted automata with discounting over semirings and finitely generated graded monoids.

In this paper, we introduce weighted tree automata with discounting, acting on finite trees, and our first goal is a Kleene theorem for their behaviors. Furthermore, we consider a weighted MSO-logic on trees with discounting, and in our second main result we show the expressive equivalence of weighted tree automata with discounting with two fragments of this logic. One of these fragments has a purely syntactic definition provided that the underlying semiring is additively locally finite. Our MSO-logic is a slight modification of the MSO-logic in [18] and goes back to the pioneering work of Droste and Gastin [11] (cf. also [13]) in which weighted logics over semirings were considered for the first time. Very recently, in [19] (cf. also [22]) the authors achieved a purely syntactic description in terms of MSO-logic for weighted tree automata over arbitrary semirings. The discounting method for stochastic tree automata has been also considered in [30, 31].

Infinite trees play a crucial role in practical applications, namely in program optimization [23], and in proving termination of non-deterministic or concurrent programs under any reasonable notion of fairness [25]. Furthermore, tree automata over infinite trees contribute to program synthesis in model checking [38]. All these applications are based on the fundamental fact that every program can be described by an infinite tree (cf. [8, 23, 39]). Weighted Muller tree automata were investigated in [32] but for the underlying semirings special completeness axioms were required. Currently, several tools for model checking are built in a weighted setting, in particular over De Morgan algebras (cf. [4, 7, 24]). Therefore, taking into account the contribution of tree automata to program synthesis [38], we wish to study the extension of these models in a weighted setting for semirings, like max-plus and min-plus, which are already used in practical applications. For this we introduce weighted Muller tree automata with discounting, over the max-plus and the min-plus semirings, and in our third main result we state their characterization in terms of weighted (purely syntactically defined) MSO-logic.

The proofs of our results are similar to the corresponding ones in [15, 18, 32]. Nevertheless, they are more technical because of the involvement of discounting parameters. We present only a few of them which are representative for the discounting techniques. The reader can find detailed proofs in [28]. In the paper, we notify the corresponding results from [15, 18, 32] by e.g. (cf. [15], Lm 4.8).

2 Preliminaries

2.1 Trees

We denote by $\mathbb{N}$ the set of natural numbers and let $\mathbb{N}_+ = \mathbb{N}\backslash\{0\}$. The *prefix relation* $\leq$ over $\mathbb{N}^*$ is a partial order defined in the usual way: for every $w, v \in \mathbb{N}^*$, $w \leq v$ iff there exists $u \in \mathbb{N}^*$ such that $wu = v$. A set $A \subseteq \mathbb{N}^*$ is called *prefix-closed* if $v \in A$ implies $w \in A$ for every $w \leq v$.

A *ranked alphabet* Σ is a pair (Σ, rk_Σ) (simply denoted by Σ) where Σ is a finite set and $rk_\Sigma : \Sigma \to \mathbb{N}$. As usual, we set $\Sigma_k = \{\sigma \in \Sigma \mid rk_\Sigma(\sigma) = k\}$ for every $k \geq 0$, and $\deg(\Sigma) = \max\{k \in \mathbb{N} \mid \Sigma_k \neq \emptyset\}$.

A *tree t over* Σ is a partial mapping $t : \mathbb{N}_+^* \to \Sigma$ such that the domain $dom(t)$ of t is a non-empty prefix-closed set, and for every $w \in dom(t)$ if $t(w) \in \Sigma_k$ $k \geq 0$, then for $i \in \mathbb{N}_+$, $wi \in dom(t)$ iff $1 \leq i \leq k$. The elements of $dom(t)$ are called the *nodes of t*. For every $\sigma \in \Sigma$ we set $dom_\sigma(t) = \{w \in dom(t) \mid t(w) = \sigma\}$ and, for every $A \subseteq \Sigma$ we let $dom_A(t) = \{w \in dom(t) \mid t(w) \in A\}$. A tree t is called *finite* (resp. *infinite)* if its domain is finite (resp. infinite). As usual, we shall denote by T_Σ (resp. T_Σ^ω) the set of all finite (resp. infinite) trees over Σ. Clearly, $T_\Sigma = \emptyset$ iff $\Sigma_0 = \emptyset$.

The set T_Σ can also be inductively defined as the smallest set T such that (i) $\Sigma_0 \subseteq T$ and (ii) if $k \geq 1, \sigma \in \Sigma_k$, and $t_1, \ldots, t_k \in T$, then $\sigma(t_1, \ldots, t_k) \in T$. For every finite set A with $A \cap \Sigma = \emptyset$, we shall write $T_\Sigma(A)$ for the set of finite trees over the ranked alphabet Σ', where $\Sigma'_0 = \Sigma_0 \cup A$ and $\Sigma'_k = \Sigma_k$ for every $k > 0$. The *height* $ht(t)$ of every finite tree $t \in T_\Sigma$ is defined by $ht(t) = \max\{|w| \mid w \in dom(t)\}$ where $|w|$ denotes the length of the word w. Let $a \in \Sigma_0$ and $t \in T_\Sigma$ with $dom_a(t) = \{w_1, \ldots, w_m\}$ such that $w_1 \leq_{lex} \ldots \leq_{lex} w_m$ where $\leq_{lex}$, is the lexicographic order over $\mathbb{N}^*$. Then, for $t_1, \ldots, t_m \in T_\Sigma$ we write $t \cdot_a (t_1, \ldots, t_m)$ for the tree obtained by substituting t_i for a at w_i $(1 \leq i \leq n)$ in t.

> *In the rest of the paper, Σ and Γ will denote an arbitrary ranked alphabet, if not specified otherwise. Moreover, we assume that $\Sigma_0 \neq \emptyset$ and $\Gamma_0 \neq \emptyset$.*

A *relabeling* from Σ to Γ is a surjective mapping $h : \Sigma \to \Gamma$ such that $h(\sigma) \in \Gamma_k$ for every $\sigma \in \Sigma_k$, $k \geq 0$. Then h is extended to a mapping $h : T_\Sigma \to T_\Gamma$ by letting $dom(h(t)) = dom(t)$ and $h(t)(w) = h(t(w))$ for every $t \in T_\Sigma$ and $w \in dom(t)$.

2.2 Semirings

A *semiring* $(K, +, \cdot, 0, 1)$ consists of a set K equipped with two binary operations $+$ and $\cdot$, and two constant elements 0 and 1 such that $(K, +, 0)$ is a commutative monoid, $(K, \cdot, 1)$ is a monoid, multiplication distributes over addition, and $0 \cdot a = a \cdot 0 = 0$ for every $a \in K$. If the operations and the constant elements are understood, then the semiring is simply denoted by K. A semiring K is called *commutative* if the monoid $(K, \cdot, 1)$ is commutative. The second main result of our paper will apply to commutative semirings K which are *additively locally finite*, i.e., such that every finitely generated submonoid of $(K,+,0)$ is finite. For examples of additively locally finite semirings we refer the reader to [16]. In this paper we mainly deal with the additively locally finite semiring *max-plus* (or *arctic*) $\mathbb{R}_{\max} = (\mathbb{R}_+ \cup \{-\infty\}, \max, +, -\infty, 0)$ where $\mathbb{R}_+ = \{r \in \mathbb{R} \mid r \geq 0\}$ and $-\infty + x = -\infty$ for every $x \in \mathbb{R}_+ \cup \{-\infty\}$. Our results remain valid as well, over the additively locally finite semiring *min-plus* (or *tropical*) $(\mathbb{R}_+ \cup \{\infty\}, \min, +, \infty, 0)$ with $\infty + x = \infty$ for every $x \in \mathbb{R}_+ \cup \{\infty\}$.

Let K_1 and K_2 be two semirings. A mapping $f : K_1 \to K_2$ is called a *semiring homomorphism* (or simply a *homomorphism*) if $f(a+b) = f(a)+f(b)$ and $f(a \cdot b) = f(a) \cdot f(b)$ for every $a, b \in K_1$, and $f(0) = 0$ and $f(1) = 1$. A homomorphism $f : K \to K$ is an *endomorphism* of K. The set $End(K)$ of all endomorphisms of K is a monoid with operation the usual composition mapping $\circ$ and unit element the identity mapping on K. If no confusion arises, we shall alternatively denote the operation $\cdot$ of K and the composition operation $\circ$ of $End(K)$ by concatenation.

Example 1. Consider the max-plus semiring $\mathbb{R}_{\max}$ and extend the multiplication $\cdot$ over $\mathbb{R}_+$ by letting $p \cdot (-\infty) = (-\infty) \cdot p = -\infty$ for every $p \in \mathbb{R}_+ \cup \{-\infty\}$. Then the mapping $\overline{p} : \mathbb{R}_{\max} \to \mathbb{R}_{\max}$ $(p \in \mathbb{R}_+)$ given by $x \longmapsto p \cdot x$ is an endomorphism of $\mathbb{R}_{\max}$. Conversely, every endomorphism of $\mathbb{R}_{\max}$ is of this form (cf. [14], Lm. 15).

In the rest of the paper, K will denote an arbitrary commutative semiring if not specified otherwise.

2.3 Discounting

A *discounting over Σ and K* is a family $\Phi = (\Phi_k)_{k \geq 1}$ of mappings $\Phi_k : \Sigma_k \to (End(K))^k$ for $k \geq 1$. For every $\sigma \in \Sigma_k$ $(k \geq 1)$ we shall write $\left(\Phi^1_\sigma, \ldots, \Phi^k_\sigma\right)$ for the k-tuple $\Phi_k(\sigma)$. If no confusion arises with the rank of σ, then we simply denote $\Phi_k(\sigma)$ by Φ_σ. The discounting Φ is alternatively called a Φ-*discounting*. For every $t \in T_\Sigma$ and every $w \in dom(t)$, we define the endomorphism Φ^t_w of K as follows:

$$\Phi^t_w = \begin{cases} id & \text{if } w = \varepsilon \\ \Phi^{i_1}_{t(\varepsilon)} \circ \Phi^{i_2}_{t(i_1)} \circ \ldots \circ \Phi^{i_n}_{t(i_1 \ldots i_{n-1})} & \text{if } w = i_1 \ldots i_n,\ i_1, \ldots, i_n \in \mathbb{N}_+, n > 0 \end{cases}$$

where id is the identity endomorphism of K.

In Sections 3-5, Φ will denote a discounting over Σ and K.

2.4 Tree series

A *formal tree series (*or *tree series* for short) *over Σ and K*, is a mapping $S : T_\Sigma \to K$. As usual we denote by (S,t) the *coefficient* $S(t)$ for every $t \in T_\Sigma$. The *support* of S is the tree language $supp(S) = \{t \in T_\Sigma \mid (S,t) \neq 0\}$. The class of all tree series over Σ and K is denoted by $K\langle\langle T_\Sigma \rangle\rangle$, and the class of *polynomials* (i.e., tree series with finite support) is denoted by $K\langle T_\Sigma \rangle$.

For every tree language $L \subseteq T_\Sigma$, the *characteristic series* $1_L \in K\langle\langle T_\Sigma \rangle\rangle$ *of* L *with respect to* K is determined by $(1_L, t) = 1$ if $t \in L$, and $(1_L, t) = 0$ otherwise, for every $t \in T_\Sigma$. Let $S, T \in K\langle\langle T_\Sigma \rangle\rangle$ and $k \in K$. The sum $S+T$, the scalar products kS and Sk, and the Hadamard product $S \odot T$ are defined by $(S+T, t) = (S,t) + (T,t)$, $(kS, t) = k \cdot (S,t)$ and $(Sk, t) = (S,t) \cdot k$, and $(S \odot T, t) = (S,t) \cdot (T,t)$ for every $t \in T_\Sigma$. Clearly, $(K\langle\langle T_\Sigma \rangle\rangle, +, \odot, \mathbf{0}, \mathbf{1})$ and $(K\langle T_\Sigma \rangle, +, \odot, \mathbf{0}, \mathbf{1})$ are commutative semirings, where $\mathbf{0}$ is the tree series (over Σ and K) with all its coefficients being 0, and $\mathbf{1}$ is the tree series (over Σ and K) with all its coefficients being 1.

Let $h : \Sigma \to \Gamma$ be a relabeling. For every tree series $S \in K\langle\langle T_\Sigma \rangle\rangle$ the tree series $h(S) \in K\langle\langle T_\Gamma \rangle\rangle$ is defined for every $s \in T_\Gamma$ by $(h(S), s) = \sum_{t \in h^{-1}(s)} (S, t)$. Similarly, for every $T \in K\langle\langle T_\Gamma \rangle\rangle$ the tree series $h^{-1}(T) \in K\langle\langle T_\Sigma \rangle\rangle$ is determined by $(h^{-1}(T), t) = (T, h(t))$ for every $t \in T_\Sigma$.

3 Φ-recognizable tree series

In this section, we study Φ-recognizable tree series obtained as behaviors of weighted tree automata with Φ-discounting. Intuitively, for every input tree t the weight of every node of t is discounted according to the distance of the node from the root of t; the longer the distance is the greater the grade of discounting is; nodes of the same level get a weight with the same grade of discounting. Our weighted tree automata are bottom-up models without initial distribution. By standard automata constructions, it can be seen that they are equivalent to weighted tree automata with initial distribution. Furthermore, they are equivalent to the corresponding top-down models, with and without terminal distribution (cf. [28]). Firstly, we introduce our weighted tree automata with Φ-discounting and we state normalization results. Then, in Subsection 3.2 we investigate closure properties of Φ-recognizable tree series.

3.1 Weighted tree automata with Φ-discounting

Definition 1. *A* weighted tree automaton with Φ-discounting *(*Φ-wta *for short)* over Σ and K *is a triple* $\mathcal{M} = (Q, wt, ter)$ *where* Q *is the* finite state set, $wt : \bigcup_{k \geq 0} Q^k \times \Sigma_k \times Q \to K$ *is the* mapping assigning weights to the transitions *of the automaton, and* $ter : Q \to K$ *is* the final distribution.

Let $t \in T_\Sigma(Q)$ (without any loss we assume that $\Sigma \cap Q = \emptyset$) and $P \subseteq Q$. A *run of* $\mathcal{M}$ *over* t *using* P is a mapping $r_t : dom(t) \to Q$ such that $r_t(w) = t(w)$ for every $w \in dom_Q(t)$ and $r_t(w) \in P$ for every $w \in dom(t) \setminus (dom_Q(t) \cup \{\varepsilon\})$. The run r_t is called a *q-run* whenever $r_t(\varepsilon) = q$. We shall denote by $R^P_{\mathcal{M}}(t, q)$ the set of all q-runs of $\mathcal{M}$ over t using P, and by $R_{\mathcal{M}}(t, q)$ the set $R^Q_{\mathcal{M}}(t, q)$. Moreover, we let $R_{\mathcal{M}}(t) = \bigcup_{q \in Q} R_{\mathcal{M}}(t, q)$.

The *weight of a run* $r_t \in R^P_{\mathcal{M}}(t,q)$ *at* $w \in dom(t)$ is given by

$$wt\,(r_t, w) = \begin{cases} wt\left((r_t(w1), \ldots, r_t(w.rk_\Sigma(t(w)))), t(w), r_t(w)\right) & \text{if } t\,(w) \in \Sigma_k, k \geq 0 \\ 1 & \text{if } t\,(w) \in Q. \end{cases}$$

The *running* Φ-*weight of* r_t, denoted by $rweight_{\mathcal{M}}\,(r_t)$ (or simply $rweight\,(r_t)$), is the value

$$rweight_{\mathcal{M}}(r_t) = \prod_{w \in dom(t)} \Phi^t_w\left(wt(r_t, w)\right)$$

and the Φ-*weight of* r_t, denoted by $weight_{\mathcal{M}}(r_t)$ (or simply $weight(r_t)$), is given by

$$weight_{\mathcal{M}}(r_t) = rweight_{\mathcal{M}}(r_t) \cdot ter(r_t(\varepsilon)).$$

For every $P \subseteq Q, q \in Q$ we let $\|\mathcal{M}\|\,(P,q)$ be the tree series in $K\,\langle\langle T_\Sigma(Q)\rangle\rangle$ determined by

$$\left(\|\mathcal{M}\|\,(P,q)\,, t\right) = \begin{cases} \sum\limits_{r_t \in R^P_{\mathcal{M}}(t,q)} rweight_{\mathcal{M}}\,(r_t) & \text{if } t \in T_\Sigma\,(Q) \setminus Q \\ 0 & \text{otherwise.} \end{cases}$$

The Φ-*behavior* (or simply *behavior*) of $\mathcal{M}$ is the tree series $\|\mathcal{M}\| \in K\,\langle\langle T_\Sigma\rangle\rangle$ defined for every $t \in T_\Sigma$ by

$$\left(\|\mathcal{M}\|\,, t\right) = \sum_{r_t \in R_{\mathcal{M}}(t)} weight_{\mathcal{M}}(r_t).$$

Clearly,

$$\left(\|\mathcal{M}\|\,, t\right) = \sum_{q \in Q} \left(\|\mathcal{M}\|\,(Q,q)\,, t\right) \cdot ter(q).$$

for every $t \in T_\Sigma$

A tree series $S \in K\,\langle\langle T_\Sigma\rangle\rangle$ is called Φ-*recognizable* if there is a Φ-wta $\mathcal{M}$ over Σ and K such that $S = \|\mathcal{M}\|$. We shall denote by $Rec(\Sigma, K, \Phi)$ the class of all Φ-recognizable tree series over Σ and K. Clearly, if our Φ-discounting employs only the identity mapping on K, i.e, $\Phi_\sigma = (id, \ldots, id)$ for every $\sigma \in \Sigma_k$ $(k \geq 1)$, then $Rec(\Sigma, K, \Phi) = Rec(\Sigma, K)$ the class of recognizable formal tree series over Σ and K (cf. [1, 22]). Two Φ-wta $\mathcal{M}$ and $\mathcal{M}'$ are *equivalent* if $\|\mathcal{M}\| = \|\mathcal{M}'\|$.

Next, we give examples of Φ-recognizable tree series over $\mathbb{R}_{\max}$ obtained as behaviors of deterministic Φ-wta. More precisely, a (Φ-)wta $\mathcal{M} = (Q, wt, ter)$ over Σ and K is called *deterministic* (cf. [2]) if for every $k \geq 0, \sigma \in \Sigma_k$, and $q_1, \ldots, q_k \in Q$ there is at most one $q \in Q$ such that $wt\left((q_1, \ldots, q_k)\,, \sigma, q\right) \neq 0$.

Example 2. Let Σ be a ranked alphabet with $\Sigma_0 = \{a\}, \Sigma_2 = \{\sigma, \gamma\}$, and $\Sigma_3 = \{\delta\}$. We consider the Φ-wta $\mathcal{M} = (\{q\}, wt, ter)$ over Σ and $\mathbb{R}_{\max}$ with its weight assignment mapping defined by $wt((q,q), \sigma, q) = 1$ and $wt(a,q) = wt((q,q), \gamma, q) =$

$wt((q,q,q),\delta,q)=0$. The final distribution is given by $ter(q)=0$. We define a Φ-discounting over Σ and $\mathbb{R}_{\max}$ specified by $\Phi_\sigma=(\overline{1},\overline{1})$, $\Phi_\gamma=(\overline{0},\overline{0})$, and $\Phi_\delta=(\overline{0},\overline{0},\overline{0})$ (cf. Example 1).

Then for every $t\in T_\Sigma$ the coefficient $(\|\mathcal{M}\|,t)$ equals the number of occurrences of σ in the greatest initial σ-subtree of t. One can easily show that there is no deterministic wta without discounting over Σ and $\mathbb{R}_{\max}$ accepting the same tree series.

Example 3. Consider the ranked alphabet Σ of the previous example and let $t\in T_\Sigma$. We say that the *pattern* $\delta(\sigma,\sigma,\delta)$ occurs in t if there are trees $t',s,s_i\in T_\Sigma$ $(1\leq i\leq 7)$ such that $t=t'\cdot_a s$ and $s=\delta\left(\sigma\left(s_1,s_2\right),\sigma\left(s_3,s_4\right),\delta\left(s_5,s_6,s_7\right)\right)$. We construct a deterministic Φ-wta $\mathcal{M}=(Q,wt,ter)$ over Σ and $\mathbb{R}_{\max}$, whose Φ-behavior returns for every input tree $t\in T_\Sigma$ the number of occurrences of the pattern $\delta(\sigma,\sigma,\delta)$ in the greatest initial subtree of t which does not contain any symbol γ. Our Φ-discounting now is given by $\Phi_\sigma=(\overline{1},\overline{1})$, $\Phi_\gamma=(\overline{0},\overline{0})$, and $\Phi_\delta=(\overline{1},\overline{1},\overline{1})$. The Φ-wta $\mathcal{M}$ is determined by $Q=\{q_1,q_2,q_3\}$, $ter\left(q\right)=0$ for every $q\in Q$, and

- $wt\left(a,q_1\right)=0$
- $wt\left(\left(p_1,p_2\right),\sigma,q_2\right)=wt\left(\left(p_1,p_2\right),\gamma,q_1\right)=0$ for every $p_1,p_2\in Q$,
- $wt\left(\left(q_2,q_2,q_3\right),\delta,q_3\right)=1$, and
- $wt\left(\left(p_1,p_2,p_3\right),\delta,q_3\right)=0$ for every $p_1,p_2,p_3\in Q$ with $(p_1,p_2,p_3)\neq(q_2,q_2,q_3)$.

Any other transition is assigned the value $-\infty$. Clearly, $\mathcal{M}$ is deterministic and by standard arguments we can show that $\|\mathcal{M}\|$ cannot be accepted by any deterministic wta without discounting over Σ and $\mathbb{R}_{\max}$.

Next we establish two normalized forms of Φ-wta; they will be used for the proofs of the results in Section 4.

A Φ-wta $\mathcal{M}=(Q,wt,ter)$ is *final weight normalized* (cf. [15], Def. 4.7) if there is one state $q_f\in Q$ such that

- $ter\left(q_f\right)=1$ and, for every $q\in Q$ with $q\neq q_f$, $ter(q)=0$,
- for every $k>0,\sigma\in\Sigma_k,q_1,\ldots,q_k,q\in Q$, if there is an $1\leq i\leq k$ with $q_i=q_f$, then $wt\left(\left(q_1,\ldots,q_k\right),\sigma,q\right)=0$.

In this case we write $\mathcal{M}=\left(Q,wt,q_f\right)$.

Lemma 1. *(cf. [15], Lm 4.8) For every Φ-wta $\mathcal{M}$ there is an equivalent final weight normalized Φ-wta $\mathcal{M}'$. Moreover, $\mathcal{M}'$ can be chosen to have one more state than $\mathcal{M}$.*

Let $a\in\Sigma_0$. A tree series $S\in K\langle\langle T_\Sigma\rangle\rangle$ is called *a-proper* if $(S,a)=0$. We shall denote by $K^a\langle\langle T_\Sigma\rangle\rangle$ the class of all a-proper tree series over Σ and K. Consider a Φ-wta $\mathcal{M}=(Q,wt,ter)$ over Σ and K. We let $I_a=\{q\in Q\mid wt\left(a,q\right)\neq 0\}$, and we call I_a the set of *initial a-states* of $\mathcal{M}$. The Φ-wta $\mathcal{M}$ is called *initial a-state normalized* (cf. [15], Def. 4.10) if there is a state $q_a\in Q$ such that $I_a=\{q_a\}$, $wt\left(a,q_a\right)=1$, and $wt((q_1,\ldots,q_k),\sigma,q_a)=0$ for every $\sigma\in\Sigma\setminus\{a\}$.

Lemma 2. *(cf. [15], Lm. 4.11) Let $\mathcal{M} = (Q, wt, ter)$ be a Φ-wta over Σ and K and $a \in \Sigma_0$. Then there is an initial a-state normalized Φ-wta $\mathcal{M}'$ such that $(\|\mathcal{M}'\|, t) = (\|\mathcal{M}\|, t)$ for every $t \in T_\Sigma \setminus \{a\}$. Moreover, $\mathcal{M}'$ can be chosen to have one more state than $\mathcal{M}$.*

Proof. We construct the Φ-wta $\mathcal{M}' = (Q', wt', ter')$ with $Q' = Q \cup \{q_a\}$. The final distribution ter' is given by $ter'(q_a) = 0$ and by $ter'(q) = ter(q)$ for every $q \in Q$. The weight assignment mapping wt' is defined as follows:

- $wt'((q_1, \ldots, q_k), \sigma, q) =$

$$\sum \left\{ \begin{array}{c} \left(\prod\limits_{\substack{1 \le i \le k \\ q_i = q_a}} \Phi_\sigma^i (wt(a, p_i)) \right) \cdot wt((p_1, \ldots, p_k), \sigma, q) \mid \\ p_1, \ldots, p_k \in Q, \text{ and } p_i = q_i \text{ if } q_i \in Q \end{array} \right\}$$

  for every $k \ge 0, \sigma \in \Sigma_k \setminus \{a\}, q_1, \ldots, q_k \in Q'$, and $q \in Q$

- $wt'((q_1, \ldots, q_k), \sigma, q_a) = 0$ for every $k \ge 0, \sigma \in \Sigma_k \setminus \{a\}$, and $q_1, \ldots, q_k \in Q'$

- $wt'(a, q) = 0$ for every $q \in Q$

- $wt'(a, q_a) = 1$.

Obviously, $\mathcal{M}'$ is initial a-state normalized and q_a is the initial a-state. Observe that for every $t \in T_\Sigma$, $r'_t \in R_{\mathcal{M}'}(t)$, and $w \in dom(t)$, if $t(w) \ne a$ and $r'_t(w) = q_a$, then $weight_{\mathcal{M}'}(r'_t) = 0$. We will show that $(\|\mathcal{M}'\|, t) = (\|\mathcal{M}\|, t)$ for every $t \in T_\Sigma \setminus \{a\}$. Let $t \ne a$. For every $q \in Q$, we define the mapping $v : R_{\mathcal{M}}(t, q) \to R_{\mathcal{M}'}(t, q)$ as follows. For every run $r_t \in R_{\mathcal{M}}(t, q)$ and every $w \in dom(t)$ we put

$$(v(r_t))(w) = \begin{cases} r_t(w) & \text{if } t(w) \ne a \\ q_a & \text{otherwise.} \end{cases}$$

We set $pre_a(t) = \{w \in dom(t) \mid \text{there exists an } i \in \mathbb{N}_+ \text{ such that } t(wi) = a\}$, i.e., the set of all nodes of t which are predecessors of the a-labeled nodes. Let $pre_a(t) = \{w_1, \ldots, w_m\}$. Then for every $1 \le j \le m$, we set $dom_{a,j}(t) = \{i \mid t(w_j i) = a\} = \{i_{j1}, \ldots, i_{jk_j}\}$ (with $i_{j1} < \ldots < i_{jk_j}$) which indicates the set of all a-labeled nodes following w_j. Clearly, $dom_a(t) = \bigcup\limits_{1 \le j \le m} \{w_j i \mid i \in dom_{a,j}(t)\}$.
Finally we define the *a-surrounding* of t to be the set $sur_a(t) = pre_a(t) \cup dom_a(t)$. Let $q \in Q$ and $r'_t \in R_{\mathcal{M}'}(t, q)$ with $r'_t(w) = q_a$ for every $w \in dom_a(t)$. Then we

calculate

$$
\begin{aligned}
&\sum_{r_t \in v^{-1}(r'_t)} \prod_{w \in sur_a(t)} \Phi^t_w \left(wt\left(r_t, w\right)\right) \\
&= \sum_{r_t \in v^{-1}(r'_t)} \prod_{1 \le j \le m} \left(\prod_{i \in dom_{a,j}(t)} \Phi^t_{w_j i}\left(wt\left(r_t, w_j i\right)\right) \right) \cdot \Phi^t_{w_j}\left(wt\left(r_t, w_j\right)\right) \\
&= \sum_{r_t \in v^{-1}(r'_t)} \prod_{1 \le j \le m} \left(\prod_{1 \le l \le k_j} \Phi^t_{w_j i_{jl}}\left(wt\left(a, r_t\left(w_j i_{jl}\right)\right)\right) \right) \\
&\cdot \Phi^t_{w_j} \left(wt \left(\begin{array}{c} \left(\begin{array}{c} r_t\left(w_j 1\right), \ldots, r_t\left(w_j\left(i_{j1}-1\right)\right), r_t\left(w_j i_{j1}\right), \\ r_t\left(w_j\left(i_{j1}+1\right)\right), \ldots, r_t\left(w_j\left(i_{j2}-1\right)\right), r_t\left(w_j i_{j2}\right), \\ \ldots, r_t\left(w_j\left(i_{j(k_j-1)}+1\right)\right), \ldots, r_t\left(w_j\left(i_{jk_j}-1\right)\right), \\ r_t\left(w_j i_{jk_j}\right), r_t\left(w_j\left(i_{jk_j}+1\right)\right), \ldots, r_t\left(w_j \rho_j\right) \end{array} \right), \\ t(w_j), r_t\left(w_j\right) \end{array} \right) \right) \\
&= \prod_{1 \le j \le m} \sum \left\{ \begin{array}{c} \left(\prod_{1 \le l \le k_j} \Phi^t_{w_j i_{jl}}\left(wt\left(a, p_{jl}\right)\right) \right) \cdot \\ \Phi^t_{w_j} \left(wt \left(\begin{array}{c} \left(\begin{array}{c} r'_t\left(w_j 1\right), \ldots, r'_t\left(w_j\left(i_{j1}-1\right)\right), p_{j1}, \\ r'_t\left(w_j\left(i_{j1}+1\right)\right), \ldots, r'_t\left(w_j\left(i_{j2}-1\right)\right), \\ p_{j2}, \ldots, p_{jk_j}, \ldots, r'_t\left(w_j \rho_j\right) \end{array} \right), \\ t(w_j), r'_t\left(w_j\right) \end{array} \right) \right) \\ \mid p_{j1}, \ldots, p_{jk_j} \in Q \end{array} \right\}
\end{aligned}
$$

where for every $1 \le j \le m$ we assume that $rk(t\left(w_j\right)) = \rho_j$. On the other side

$$
\begin{aligned}
&\prod_{w \in sur_a(t)} \Phi^t_w \left(wt'\left(r'_t, w\right)\right) \\
&= \prod_{1 \le j \le m} \left(\Phi^t_{w_j}\left(wt'\left(r'_t, w_j\right)\right) \cdot \prod_{i \in dom_{a,j}(t)} \Phi^t_{w_j i}\left(wt'\left(r'_t, w_j i\right)\right) \right) \\
&= \prod_{1 \le j \le m} \Phi^t_{w_j}\left(wt'\left(r'_t, w_j\right)\right) \\
&= \prod_{1 \le j \le m} \Phi^t_{w_j}\left(wt'\left(\left(r'_t\left(w_j 1\right), \ldots, r'_t\left(w_j \rho_j\right)\right), t(w_j), r'_t\left(w_j\right)\right)\right),
\end{aligned}
$$

which equals

$$\prod_{1\leq j\leq m}\Phi^{t}_{w_j}\left(\sum\left\{\begin{array}{c}\left(\prod\limits_{1\leq l\leq k_j}\Phi^{i_{jl}}_{t(w_j)}\left(wt\left(a,p_{jl}\right)\right)\right)\\ \cdot wt\left(\left(\begin{array}{c}r'_t\left(w_j1\right),\ldots,r'_t\left(w_j\left(i_{j1}-1\right)\right),\\ p_{j1},r'_t\left(w_j\left(i_{j1}+1\right)\right),\ldots,\\ r'_t\left(w_j\left(i_{j2}-1\right)\right),p_{j2},\ldots,p_{jk_j},\ldots,\\ r'_t\left(w_j\rho_j\right)\\ t\left(w_j\right),r'_t\left(w_j\right)\end{array}\right),\right)\\ \mid p_{j1},\ldots,p_{jk_j}\in Q\end{array}\right\}\right)$$

$$=\prod_{1\leq j\leq m}\sum\left\{\begin{array}{c}\left(\prod\limits_{1\leq l\leq k_j}\Phi^{t}_{w_ji_{jl}}\left(wt\left(a,p_{jl}\right)\right)\right)\cdot\\ \Phi^{t}_{w_j}\left(wt\left(\left(\begin{array}{c}r'_t\left(w_j1\right),\ldots,r'_t\left(w_j\left(i_{j1}-1\right)\right),\\ p_{j1},r'_t\left(w_j\left(i_{j1}+1\right)\right),\ldots,\\ r'_t\left(w_j\left(i_{j2}-1\right)\right),p_{j2},\ldots,p_{jk_j},\\ \ldots,r'_t\left(w_j\rho_j\right)\\ t(w_j),r'_t\left(w_j\right)\end{array}\right),\right)\right)\\ \mid p_{j1},\ldots,p_{jk_j}\in Q\end{array}\right\}.$$

Now, we can easily show that $(\|\mathcal{M}\|,t)=(\|\mathcal{M}'\|,t)$. □

3.2 Properties of Φ-recognizable tree series

Proposition 1. *(i) (cf. [18], Lm. 3.3) The class* $Rec(\Sigma,K,\Phi)$ *is closed under sum, scalar product, and Hadamard product.*

(ii) (cf. [18], Lm. 3.4) Let $h:\Sigma\to\Gamma$ *be a relabeling. Furthermore, for the* Φ*-discounting over* Σ *and* K *assume that* $\Phi_\sigma=\Phi_{\sigma'}$ *whenever* $h(\sigma)=h(\sigma')$ *for every* $\sigma,\sigma'\in\Sigma_k,k\geq 1$. *Let* $\Phi'=(\Phi'_k)_{k\geq 1}$ *be the discounting over* Γ *and* K *determined for every* $\gamma\in\Gamma_k$ $(k\geq 1)$ *by* $\Phi'_\gamma=\Phi_\sigma$ *for every* $\sigma\in\Sigma_k$ $(k\geq 1)$ *with* $h(\sigma)=\gamma$. *If* $S\in Rec(\Sigma,K,\Phi)$, *then* $h(S)\in Rec(\Gamma,K,\Phi')$. *Furthermore, if* $T\in Rec(\Gamma,K,\Phi')$, *then* $h^{-1}(T)\in Rec(\Sigma,K,\Phi)$.[1]

(iii) (cf. [18], Lm. 3.3) Let $L\subseteq T_\Sigma$ *be a recognizable tree language. Then* $1_L\in Rec(\Sigma,K,\Phi)$.

A tree series $S\in K\langle\langle T_\Sigma\rangle\rangle$ is called a *recognizable step function* if $S=\sum\limits_{1\leq j\leq n}k_j1_{L_j}$ where $k_j\in K$ and $L_j\subseteq T_\Sigma$ $(1\leq j\leq n$ and $n\in\mathbb{N})$ are recognizable tree languages. By Proposition 1 such a tree series is Φ-recognizable. The class of recognizable tree languages is closed under the Boolean operations, therefore for every recognizable step function $S=\sum\limits_{1\leq j\leq n}k_j1_{L_j}$ we may assume the family $(L_j)_{j\in J}$ to be a partition of T_Σ.

[1]Statement (ii) requires that $\deg(\Sigma)=\deg(\Gamma)$ which is guaranteed by the surjectivity of the relabeling h (cf. definition of relabeling on page 413).

Proposition 2. *(i) (cf. [12]) The class of all recognizable step functions over Σ and K is closed under sum, scalar product, and Hadamard product.*

(ii) Let $h : \Sigma \to \Gamma$ be a relabeling. If $T \in K\langle\langle T_\Gamma\rangle\rangle$ is a recognizable step function, then $h^{-1}(T) \in K\langle\langle T_\Sigma\rangle\rangle$ is also a recognizable step function.

(iii) (cf. [16], Prop. 16) Let K be additively locally finite, $h : \Sigma \to \Gamma$ a relabeling, and $S \in K\langle\langle T_\Sigma\rangle\rangle$ a recognizable step function. Then the tree series $h(S) \in K\langle\langle T_\Gamma\rangle\rangle$ is also a recognizable step function.

4 Φ-rational tree series and a Kleene theorem

In this section we introduce the Φ-rational operations on formal tree series and we show a Kleene theorem for Φ-recognizable tree series.

Let $k \geq 1, \sigma \in \Sigma_k$. The *Φ-top-concatenation with σ* is the operation $\sigma_\Phi : K\langle\langle T_\Sigma\rangle\rangle^k \to K\langle\langle T_\Sigma\rangle\rangle$ on tree series defined for every $S_1, \ldots, S_k \in K\langle\langle T_\Sigma\rangle\rangle$ and $t \in T_\Sigma$ by

$$(\sigma_\Phi(S_1, \ldots, S_k), t) = \begin{cases} \Phi_\sigma^1((S_1, t_1)) \cdot \ldots \cdot \Phi_\sigma^k((S_k, t_k)) & \text{if } t = \sigma(t_1, \ldots, t_k) \\ 0 & \text{otherwise.} \end{cases}$$

Let $S, T \in K\langle\langle T_\Sigma\rangle\rangle$ and $a \in \Sigma_0$. The *(a, Φ)-concatenation of S and T* is the tree series $S \cdot_{a,\Phi} T \in K\langle\langle T_\Sigma\rangle\rangle$ defined for every $t \in T_\Sigma$ by

$$(S \cdot_{a,\Phi} T, t) = \sum_{\substack{s, t_1, \ldots, t_r \in T_\Sigma, t = s \cdot_a (t_1, \ldots, t_r) \\ dom_a(s) = \{w_1, \ldots, w_r\}}} (S, s) \cdot \Phi_{w_1}^t((T, t_1)) \cdot \ldots \cdot \Phi_{w_r}^t((T, t_r)).$$

Proposition 3. *(cf. [15], Lm. 3.3) The (a, Φ)-concatenation of tree series is associative, i.e., for every $S, T, R \in K\langle\langle T_\Sigma\rangle\rangle$ it holds $S \cdot_{a,\Phi} (T \cdot_{a,\Phi} R) = (S \cdot_{a,\Phi} T) \cdot_{a,\Phi} R$.*

Proof. For every $t \in T_\Sigma$ we have

$$\begin{aligned}
&(S \cdot_{a,\Phi} (T \cdot_{a,\Phi} R), t) \\
&= \sum_{\substack{s, t_1, \ldots, t_r \in T_\Sigma, t = s \cdot_a (t_1, \ldots, t_r) \\ dom_a(s) = \{w_1, \ldots, w_r\}}} (S, s) \cdot \prod_{i=1}^{r} \Phi_{w_i}^t((T \cdot_{a,\Phi} R, t_i)) \\
&= \sum_{\substack{s, t_1, \ldots, t_r \in T_\Sigma, t = s \cdot_a (t_1, \ldots, t_r) \\ dom_a(s) = \{w_1, \ldots, w_r\}}} (S, s) \\
&\qquad \cdot \prod_{i=1}^{r} \Phi_{w_i}^t \left(\sum_{\substack{v_i, u_{i1}, \ldots, u_{in_i} \in T_\Sigma, t_i = v_i \cdot_a \left(u_{i1}, \ldots, u_{in_i}\right) \\ dom_a(v_i) = \left\{w_1^i, \ldots, w_{n_i}^i\right\}}} (T, v_i) \cdot \prod_{j_i=1}^{n_i} \Phi_{w_{j_i}^i}^{t_i}((R, u_{ij_i})) \right)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\substack{s,t_1,\ldots,t_r\in T_\Sigma, t=s\cdot_a(t_1,\ldots,t_r)\\ dom_a(s)=\{w_1,\ldots,w_r\}}} (S,s) \\
&\quad \cdot \prod_{i=1}^{r} \left(\begin{array}{c} \sum\limits_{\substack{v_i,u_{i1},\ldots,u_{in_i}\in T_\Sigma, t_i=v_i\cdot_a\left(u_{i1},\ldots,u_{in_i}\right)\\ dom_a(v_i)=\left\{w_1^i,\ldots,w_{n_i}^i\right\}}} \Phi_{w_i}^t\left((T,v_i)\right) \\ \cdot \prod\limits_{j_i=1}^{n_i} \Phi_{w_i}^t \circ \Phi_{w_{j_i}^i}^{t_i}\left((R,u_{ij_i})\right) \end{array} \right) \\
&= \sum_{\substack{s,t_1,\ldots,t_r,v_i,u_{i1},\ldots,u_{in_i}\in T_\Sigma\\ t=s\cdot_a(t_1,\ldots,t_r), t_i=v_i\cdot_a\left(u_{i1},\ldots,u_{in_i}\right)\\ dom_a(s)=\{w_1,\ldots,w_r\}, dom_a(v_i)=\left\{w_1^i,\ldots,w_{n_i}^i\right\}}} (S,s) \\
&\quad \cdot \prod_{i=1}^{r}\left(\Phi_{w_i}^t\left((T,v_i)\right)\cdot \prod_{j_i=1}^{n_i}\Phi_{w_iw_{j_i}^i}^t\left((R,u_{ij_i})\right)\right).
\end{aligned}
$$

On the other side

$$
\begin{aligned}
&((S\cdot_{a,\Phi} T)\cdot_{a,\Phi} R, t) \\
&= \sum_{\substack{v,u_1,\ldots,u_q\in T_\Sigma, t=v\cdot_a(u_1,\ldots,u_q)\\ dom_a(v)=\{w_1,\ldots,w_q\}}} \left((S\cdot_{a,\Phi} T, v)\cdot \prod_{j=1}^{q}\Phi_{w_j}^t\left((R,u_j)\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{\substack{v,u_1,\ldots,u_q\in T_\Sigma, t=v\cdot_a(u_1,\ldots,u_q)\\ dom_a(v)=\{w_1,\ldots,w_q\}}} \left(\begin{array}{c} \sum\limits_{\substack{s,s_1,\ldots,s_r\in T_\Sigma, v=s\cdot_a(s_1,\ldots,s_r)\\ dom_a(s)=\left\{w_1',\ldots,w_r'\right\}}} (S,s)\cdot \prod\limits_{i=1}^{r}\Phi_{w_i'}^v\left((T,s_i)\right) \\ \cdot \prod\limits_{j=1}^{q}\Phi_{w_j}^t\left((R,u_j)\right) \end{array} \right) \\
&= \sum_{\substack{v,u_1,\ldots,u_q,s,s_1,\ldots,s_r\in T_\Sigma\\ t=v\cdot_a(u_1,\ldots,u_q), v=s\cdot_a(s_1,\ldots,s_r)\\ dom_a(v)=\{w_1,\ldots,w_q\}, dom_a(s)=\left\{w_1',\ldots,w_r'\right\}}} (S,s)\cdot \prod_{i=1}^{r}\Phi_{w_i'}^t\left((T,s_i)\right)\cdot \prod_{j=1}^{q}\Phi_{w_j}^t\left((R,u_j)\right).
\end{aligned}
$$

The last equality is true since every node of v is also a node of t. Clearly, there is a one to one correspondence between the two ways of decomposing t. This also implies that the occurred endomorphisms at each node of the corresponding decompositions coincide. Therefore, we get $(S\cdot_{a,\Phi}(T\cdot_{a,\Phi} R), t) = ((S\cdot_{a,\Phi} T)\cdot_{a,\Phi} R, t)$ for every $t\in T_\Sigma$ and thus $S\cdot_{a,\Phi}(T\cdot_{a,\Phi} R) = (S\cdot_{a,\Phi} T)\cdot_{a,\Phi} R$. $\square$

Following [15] we introduce power discounted iterations of tree series. More precisely, let $S \in K\langle\langle T_\Sigma \rangle\rangle$ and $a \in \Sigma_0$. The *nth* (a, Φ)*-iteration of* S is the tree series $S^n_{a,\Phi} \in K\langle\langle T_\Sigma \rangle\rangle$ defined inductively as follows:

(i) $S^0_{a,\Phi} = \mathbf{0}$ and

(ii) $S^{n+1}_{a,\Phi} = S \cdot_{a,\Phi} S^n_{a,\Phi} + 1a$ for every $n \geq 0$.

Lemma 3. *(cf. [15], Lm. 3.10) Let* $S \in K^a\langle\langle T_\Sigma \rangle\rangle$ *and* $t \in T_\Sigma$*. If* $n \geq ht(t) + 1$*, then* $\left(S^{n+1}_{a,\Phi}, t\right) = \left(S^n_{a,\Phi}, t\right)$.

Now we are ready to define the (a, Φ)-Kleene star of a-proper tree series.

Definition 2. *(cf. [15], Def. 3.11) Let* $S \in K^a\langle\langle T_\Sigma \rangle\rangle$*. The* (a, Φ)-Kleene star *(or simply* (a, Φ)-star*) of* S *is a tree series* $S^*_{a,\Phi} \in K\langle\langle T_\Sigma \rangle\rangle$ *which is defined in the following way. For every* $t \in T_\Sigma$ *we set* $\left(S^*_{a,\Phi}, t\right) = \left(S^{ht(t)+1}_{a,\Phi}, t\right)$.

Lemma 4. *(cf. [15], Lm. 3.13) Let* $S \in K^a\langle\langle T_\Sigma \rangle\rangle$*. Then* $S^*_{a,\Phi} = S \cdot_{a,\Phi} S^*_{a,\Phi} + 1a$.

Definition 3. *The set* $Rat\text{-}Exp(\Sigma, K, \Phi)$ *of* Φ-rational expressions *(over* Σ *and* K*) is defined inductively as the smallest set* R *satisfying the following conditions. For every* Φ*-rational expression* $\zeta \in Rat\text{-}Exp(\Sigma, K, \Phi)$ *we define its semantics* $\|\zeta\| \in K\langle\langle T_\Sigma \rangle\rangle$ *simultaneously.*

- *For every* $a \in \Sigma_0$*, the expression* $a \in R$ *and* $\|a\| = 1a$*,*
- *for every* $k \geq 1, \sigma \in \Sigma_k$*, and* $\zeta_1, \ldots, \zeta_k \in R$*, the expression* $\sigma_\Phi(\zeta_1, \ldots, \zeta_k) \in R$ *and* $\|\sigma_\Phi(\zeta_1, \ldots, \zeta_k)\| = \sigma_\Phi(\|\zeta_1\|, \ldots, \|\zeta_k\|)$*,*
- *for every* $\zeta \in R$ *and* $k \in K$*, the expression* $k\zeta \in R$ *and* $\|k\zeta\| = k \cdot \|\zeta\|$*,*
- *for every* $\zeta_1, \zeta_2 \in R$*, the expression* $\zeta_1 + \zeta_2 \in R$ *and* $\|\zeta_1 + \zeta_2\| = \|\zeta_1\| + \|\zeta_2\|$*,*
- *for every* $\zeta_1, \zeta_2 \in R$ *and* $a \in \Sigma_0$*, the expression* $\zeta_1 \cdot_{a,\Phi} \zeta_2 \in R$ *and* $\|\zeta_1 \cdot_{a,\Phi} \zeta_2\| = \|\zeta_1\| \cdot_{a,\Phi} \|\zeta_2\|$*, and*
- *for every* $\zeta \in R$ *and* $a \in \Sigma_0$ *such that* $\|\zeta\|$ *is a-proper, the expression* $\zeta^*_{a,\Phi} \in R$ *and* $\left\|\zeta^*_{a,\Phi}\right\| = \|\zeta\|^*_{a,\Phi}$.

A tree series $S \in K\langle\langle T_\Sigma \rangle\rangle$ is called Φ*-rational over* Σ *and* K if there is a $\zeta \in Rat\text{-}Exp(\Sigma, K, \Phi)$ such that $S = \|\zeta\|$. The class of all Φ-rational tree series over Σ and K is denoted by $Rat(\Sigma, K, \Phi)$. Clearly, the first four conditions in the above definition imply that $K\langle T_\Sigma \rangle \subseteq Rat(\Sigma, K, \Phi)$. Moreover, $Rat(\Sigma, K, \Phi)$ is the smallest subclass of $K\langle\langle T_\Sigma \rangle\rangle$ which has this property and is closed under the Φ-rational operations on tree series.

Next, we wish to establish a Kleene theorem showing the coincidence of Φ-recognizable and Φ-rational tree series. For this, we shall need the subsequent lemma.

Lemma 5. *(cf. [15], Lm. 5.1) Consider a* Φ*-wta* $\mathcal{M} = (Q, wt, ter)$*. Let* $P \subseteq Q, q \in Q$*, and* $p \in Q \setminus P$*. Then*

$$\|\mathcal{M}\| (P \cup \{p\}, q) = \|\mathcal{M}\| (P, q) \cdot_{p,\Phi} \|\mathcal{M}\| (P, p)^*_{p,\Phi} .$$

Let Q be a finite set of nullary symbols with $Q \cap \Sigma = \emptyset$. Then $Rat(\Sigma \cup Q, K, \Phi)$ denotes the class of Φ-rational tree series over $\Sigma \cup Q$ and K defined by Φ-rational expressions from $Rat\text{-}Exp(\Sigma \cup Q, K, \Phi)$. We set

$$Rat(\Sigma + fin, K, \Phi) = \bigcup_{Q \text{ finite}} Rat(\Sigma \cup Q, K, \Phi)$$

and

$$Rec(\Sigma + fin, K, \Phi) = \bigcup_{Q \text{ finite}} Rec(\Sigma \cup Q, K, \Phi).$$

Now, we are ready to prove one half of our Kleene theorem.

Proposition 4. *(cf. [15], Thm. 5.2)* $Rec(\Sigma, K, \Phi) \subseteq Rat(\Sigma + fin, K, \Phi)|_{T_\Sigma}$.

Proof. Let $\mathcal{M} = (Q, wt, q_f)$ be a final weight normalized Φ-wta with $Q = \{q_1, \ldots, q_n\}$. We show that $\|\mathcal{M}\| \in Rat(\Sigma \cup Q, K, \Phi)$. Note that $(\|\mathcal{M}\|, t) = (\|\mathcal{M}\| (Q, q_f), t)$ for every $t \in T_\Sigma$. So

$$\|\mathcal{M}\| = (\ldots((\|\mathcal{M}\| (Q, q_f) \cdot_{\Phi,q_1} \mathbf{0}) \cdot_{\Phi,q_2} \mathbf{0}) \ldots) \cdot_{\Phi,q_n} \mathbf{0}|_{T_\Sigma}.$$

Thus it remains to prove that for every $P \subseteq Q$ and $q \in Q$, the tree series $\|\mathcal{M}\| (P, q) \in Rat(\Sigma \cup Q, K, \Phi)$. To this end, we apply induction on the number of elements of P. Let $P = \emptyset$. For every $k \geq 0, \sigma \in \Sigma_k$, and $p_1, \ldots, p_k \in Q$, we define the run $r^\sigma_{p_1,\ldots,p_k,q} : dom\,(\sigma (p_1, \ldots, p_k)) \to Q$ of $\mathcal{M}$ over $\sigma (p_1, \ldots, p_k)$ using $\emptyset$, such that $r^\sigma_{p_1,\ldots,p_k,q} (\varepsilon) = q$, and $r^\sigma_{p_1,\ldots,p_k,q} (i) = p_i$ for every $1 \leq i \leq k$. Then we have

$$R^\emptyset_\mathcal{M} (t, q) = \begin{cases} \left\{r^\sigma_{p_1,\ldots,p_k,q}\right\} & \text{if } t = \sigma (p_1, \ldots, p_k),\ k \geq 0, \sigma \in \Sigma_k, p_1, \ldots, p_k \in Q \\ \{r_q\} & \text{if } t = q \\ \emptyset & \text{otherwise.} \end{cases}$$

Note that $(\|\mathcal{M}\| (\emptyset, q), q) = 0$ by definition. Thus $supp\,(\|\mathcal{M}\| (\emptyset, q)) \subseteq \Sigma (Q)$ where $\Sigma (Q) = \{\sigma (p_1, \ldots, p_k) \mid k \geq 0, \sigma \in \Sigma_k, p_1, \ldots, p_k \in Q\}$, i.e., $\|\mathcal{M}\| (\emptyset, q)$ is a polynomial, and hence a Φ-rational tree series.

For the induction step, assume that for every $q \in Q$ the tree series $\|\mathcal{M}\| (P, q)$ is Φ-rational over $\Sigma \cup Q$ and K. Let $p \in Q \setminus P$. Then, by Lemma 5 we get that $\|\mathcal{M}\| (P \cup \{p\}, q)$ is Φ-rational over $\Sigma \cup Q$ and K which in turn implies that $\|\mathcal{M}\| \in Rat(\Sigma \cup Q, K, \Phi)$. □

Example 4 (Example 3 continued)**.** We shall construct a Φ-rational expression for the Φ-recognizable tree series $\|\mathcal{M}\|$ of Example 3 on page 417. Consider the next expressions

$\zeta_0 = 0 + a, \quad \zeta_{q_i} = 0 + q_i$ for every $i = 1, 2, 3,$

$\zeta_1 = \left(\max_{p_1, p_2 \in Q} (0 + \gamma_\Phi(p_1, p_2)) \right)^*_{q_1, \Phi}, \quad \zeta_2 = \left(\max_{p_1, p_2 \in Q} (0 + \sigma_\Phi(p_1, p_2)) \right)^*_{q_2, \Phi},$

$\zeta_3 = 1 + \delta_\Phi(q_2, q_2, q_3), \quad \zeta_4 = \max_{\substack{p_1, p_2, p_3 \in Q \\ (p_1, p_2, p_3) \neq (q_2, q_2, q_3)}} (0 + \delta_\Phi(p_1, p_2, p_3))$, and

$\zeta_5 = \max \left((\max(\zeta_3, \zeta_4))^*_{q_3, \Phi}, \zeta_{q_2} \right)$

We have that $\|\mathcal{M}\|$ equals the restriction on T_Σ of the semantics of the Φ-rational expression

$$\left(\left(\left((\zeta_5 \cdot_{q_2, \Phi} \max(\zeta_2, \zeta_{q_1}) \cdot_{q_1, \Phi} \max(\zeta_1, \zeta_{q_3}))^*_{q_3, \Phi} \right) \cdot_{q_1, \Phi} \zeta_0 \right) \cdot_{q_2, \Phi} -\infty \right) \cdot_{q_3, \Phi} -\infty,$$

where we identify $-\infty$ and the constant tree series that takes all trees to $-\infty$.

In the sequel, we establish the inclusion $Rat(\Sigma, K, \Phi) \subseteq Rec(\Sigma, K, \Phi)$. For this, it suffices to show that the class $Rec(\Sigma, K, \Phi)$ contains the tree series $1a$ (for every $a \in \Sigma_0$) and is closed under the Φ-rational operations on tree series.

Lemma 6.

(i) (cf. [15], Lm. 6.1) For every $a \in \Sigma_0$, the tree series $1a \in Rec(\Sigma, K, \Phi)$.

(ii) (cf. [15], Lm. 6.2) The class $Rec(\Sigma, K, \Phi)$ is closed under Φ-top-concatenation.

(iii) (cf. [15], Lm. 6.5) Let $S_1, S_2 \in Rec(\Sigma, K, \Phi)$ and $a \in \Sigma_0$. Then the (a, Φ)-concatenation of S_2 and S_1 is a Φ-recognizable tree series, i.e., $S_2 \cdot_{a, \Phi} S_1 \in Rec(\Sigma, K, \Phi)$.

(iv) (cf. [15], Lm. 6.7) Let $a \in \Sigma_0$ and $S \in K^a \langle\langle T_\Sigma \rangle\rangle$ be Φ-recognizable. Then $S^*_{a, \Phi} \in Rec(\Sigma, K, \Phi)$.

Proof. (iii) Let $\mathcal{M}_1 = (Q_1, wt_1, q_{f_1})$ and $\mathcal{M}_2 = (Q_2, wt_2, q_{f_2})$ be final weight normalized Φ-wta with $\|\mathcal{M}_1\| = S_1$ and $\|\mathcal{M}_2\| = S_2$, and let us assume that $Q_1 \cap Q_2 = \emptyset$. We consider the final weight normalized Φ-wta $\mathcal{M} = (Q, wt, q_{f_2})$ with $Q = (Q_1 \cup Q_2) \setminus \{q_{f_1}\}$. For every $k \geq 0, \sigma \in \Sigma_k, q_1, \ldots, q_k, q \in Q$ we set

$$wt((q_1, \ldots, q_k), \sigma, q)$$
$$= \begin{cases} wt_1((q_1, \ldots, q_k), \sigma, q) & \text{if } q_1, \ldots, q_k, q \in Q_1 \\ wt_1((q_1, \ldots, q_k), \sigma, q_{f_1}) \cdot wt_2(a, q) & \text{if } k \neq 0,\ q_1, \ldots, q_k \in Q_1, \text{and } q \in Q_2 \\ wt_2((q_1, \ldots, q_k), \sigma, q) & \text{if } k \neq 0,\ q_1, \ldots, q_k, q \in Q_2 \\ wt_2(\sigma, q) + wt_1(\sigma, q_{f_1}) \cdot wt_2(a, q) & \text{if } k = 0, \sigma \neq a, \text{ and } q \in Q_2 \\ wt_1(a, q_{f_1}) \cdot wt_2(a, q) & \text{if } k = 0, \sigma = a, \text{ and } q \in Q_2 \\ 0 & \text{otherwise.} \end{cases}$$

Then we can show that $\|\mathcal{M}\| = \|\mathcal{M}_2\| \cdot_{a, \Phi} \|\mathcal{M}_1\|$. □

Theorem 1. *(cf. [15], Thm. 6.8)*

(i) $Rec(\Sigma, K, \Phi)$ *is closed under the* Φ*-rational operations.*

(ii) $Rat(\Sigma, K, \Phi) \subseteq Rec(\Sigma, K, \Phi)$.

Let Q_∞ be an infinite set such that $Q \subseteq Q_\infty$ for every finite set Q. We define the operation $lift_\infty : \bigcup_{Q \text{ finite set}} K\langle\langle T_\Sigma(Q)\rangle\rangle \to K\langle\langle T_\Sigma(Q_\infty)\rangle\rangle$ in the following way. Let Q be a finite set and $S \in K\langle\langle T_\Sigma(Q)\rangle\rangle$. For every $t \in T_\Sigma(Q_\infty)$ we let

$$(lift_\infty(S), t) = \begin{cases} (S, t) & \text{if } t \in T_\Sigma(Q) \\ 0 & \text{otherwise.} \end{cases}$$

Now, we are ready to state our first main result, namely the Kleene theorem for Φ-recognizable tree series.

Theorem 2 (Kleene theorem)**.** *(cf. [15], Thm. 7.1)*

$$lift_\infty(Rec(\Sigma + fin, K, \Phi)) = lift_\infty(Rat(\Sigma + fin, K, \Phi)).$$

Proof. By Theorem 1(ii) we have $Rat(\Sigma, K, \Phi) \subseteq Rec(\Sigma, K, \Phi)$. This implies that $Rat(\Sigma \cup Q, K, \Phi) \subseteq Rec(\Sigma \cup Q, K, \Phi)$, for every finite set Q. Therefore $Rat(\Sigma + fin, K, \Phi) \subseteq Rec(\Sigma + fin, K, \Phi)$ and thus

$$lift_\infty(Rat(\Sigma + fin, K, \Phi)) \subseteq lift_\infty(Rec(\Sigma + fin, K, \Phi)).$$

Conversely, let $S \in lift_\infty(Rec(\Sigma + fin, K, \Phi))$. Then, there is a finite set Q and $S' \in Rec(\Sigma \cup Q, K, \Phi)$ such that $S = lift_\infty(S')$. Then, by the proof of Proposition 4, there is another finite set Q' and a rational expression $\zeta \in Rat\text{-}Exp(\Sigma \cup Q \cup Q', K, \Phi)$ such that $\|\zeta\| \,|_{T_{\Sigma \cup Q}} = S'$ and for every $q \in Q'$, we have $(\|\zeta\|, q) = 0$. Then $lift_\infty(S') = lift_\infty(\|\zeta\|)$, hence $S = lift_\infty(\|\zeta\|) \in lift_\infty(Rat(\Sigma + fin, K, \Phi))$. $\square$

5 Weighted MSO-logic with Φ-discounting over finite trees

In this section, we introduce a weighted monadic second-order logic (abbreviated to weighted MSO-logic) with Φ-discounting over finite trees, and characterize the class $Rec(\Sigma, K, \Phi)$ in terms of this logic. The syntax of our MSO-formulas is the one used in [18] but here we exclude second-order universal quantifiers since we do not need them for the description of our automata. For the semantics of our MSO-formulas, we employ the Φ-discounting. Let us first recall some basic terminology and definitions from [18].

Let $\mathcal{V}$ be a finite set of first and second-order variables. A tree $t \in T_\Sigma$ is represented by the structure $\left(dom(t), edge_1, \ldots, edge_{\deg(\Sigma)}, (label_\sigma)_{\sigma \in \Sigma}\right)$ where for every $w, u \in dom(t)$ and $j \in \{1, \ldots, \deg(\Sigma)\}$, $edge_j(w, u)$ holds true iff $u = wj$ and $label_\sigma(w)$ holds true iff $t(w) = \sigma$. A $(t, \mathcal{V})$-*assignment* ρ is a mapping assigning elements of $dom(t)$ to first order variables from $\mathcal{V}$, and subsets of $dom(t)$ to second-order variables from $\mathcal{V}$. If x is a first order variable and $w \in dom(t)$, then we denote by $\rho[x \to w]$ the $(t, \mathcal{V} \cup \{x\})$-assignment which associates w to x and acts as ρ on $\mathcal{V} \setminus \{x\}$. The notation $\rho[X \to I]$ for a second-order variable X and a set $I \subseteq dom(t)$ has a similar meaning.

In the rest of the paper, $\mathcal{V}$ will denote an arbitrary finite set of first and second-order variables.

Now, we consider the ranked alphabet $\Sigma_{\mathcal{V}} = \Sigma \times \{0,1\}^{\mathcal{V}}$ with $rk_{\Sigma_{\mathcal{V}}}(\sigma, f) = rk_{\Sigma}(\sigma)$ for every $\sigma \in \Sigma$ and $f \in \{0,1\}^{\mathcal{V}}$. For every $(\sigma, f) \in \Sigma_{\mathcal{V}}$ we denote by $(\sigma, f)_1$ and $(\sigma, f)_2$ the symbols σ and f, respectively. A tree $s \in T_{\Sigma_{\mathcal{V}}}$ is called *valid* if for every first order variable $x \in \mathcal{V}$, there is exactly one node w of s such that $(s(w)_2)(x) = 1$. The set of all valid finite trees over $\Sigma_{\mathcal{V}}$ is denoted by $T^v_{\Sigma_{\mathcal{V}}}$. Every valid tree $s \in T_{\Sigma_{\mathcal{V}}}$ corresponds to a pair (t, ρ) where $t \in T_\Sigma$ and ρ is a $(t, \mathcal{V})$-assignment, in the following way. It holds $dom(t) = dom(s)$ and $t(w) = s(w)_1$ for every $w \in dom(s)$, and for every first order variable x, second-order variable X, and every node $w \in dom(s)$, we have $\rho(x) = w$ iff $(s(w)_2)(x) = 1$, and $w \in \rho(X)$ iff $(s(w)_2)(X) = 1$. Then, we say that s and (t, ρ) *correspond to each other*. In the following, we identify every valid tree s with its corresponding pair (t, ρ).

Corollary 1. *The characteristic series $1_{T^v_{\Sigma_{\mathcal{V}}}} : T_{\Sigma_{\mathcal{V}}} \to K$ is Φ-recognizable.*

Let φ be an MSO-formula over trees [36, 37] with $Free(\varphi) \subseteq \mathcal{V}$. As usual we shall write Σ_φ for $\Sigma_{Free(\varphi)}$. For every $(t, \rho) \in T_{\Sigma_{\mathcal{V}}}$ we let $(t, \rho) \models \varphi$ whenever (t, ρ) satisfies φ (cf. [26]). The well-known result of Thatcher and Wright [35], and Doner [10] states that the tree language $\mathcal{L}_{\mathcal{V}}(\varphi) = \{(t, \rho) \in T^v_{\Sigma_{\mathcal{V}}} \mid (t, \rho) \models \varphi\}$ is recognizable; conversely, for every recognizable tree language $L \subseteq T_\Sigma$ there exists an MSO-sentence φ, such that $L = \mathcal{L}(\varphi)$ where $\mathcal{L}(\varphi) = \mathcal{L}_{Free(\varphi)}(\varphi)$.

Next we introduce our weighted MSO-logic with Φ-discounting over trees. For this we extend our Φ-discounting over Σ and K to a discounting over $\Sigma_{\mathcal{V}}$ and K. For simplicity we shall use the same symbol Φ. More precisely, for every $(\sigma, f) \in \Sigma_{\mathcal{V}}$ we set $\Phi_{(\sigma, f)} = \Phi_\sigma$.

Definition 4. *The set $MSO(\Sigma, K)$ of all formulas of the* weighted MSO-logic with Φ-discounting over Σ and K *on finite trees is defined to be the smallest set F such that*

- *F contains all atomic formulas $k, label_\sigma(x), edge_i(x, y), x \in X$ and the negations $\neg label_\sigma(x), \neg edge_i(x, y), \neg(x \in X)$, and*
- *if $\varphi, \psi \in F$, then also $\varphi \vee \psi, \varphi \wedge \psi, \exists x \centerdot \varphi, \exists X \centerdot \varphi, \forall x \centerdot \varphi \in F$,*

where $k \in K$, $\sigma \in \Sigma$, $1 \leq i \leq \deg(\Sigma)$, x, y are first order variables, and X is a second-order variable.

Next we define the semantics of the formulas in $MSO(\Sigma, K)$ as tree series in $K \langle\langle T_{\Sigma_{\mathcal{V}}} \rangle\rangle$. As in the word case [16], we employ the Φ-discounting only in the semantics of first order universal quantifications.

Definition 5. *Let $\varphi \in MSO(\Sigma, K)$ with $Free(\varphi) \subseteq \mathcal{V}$. The* Φ-semantics *of φ is a tree series $\|\varphi\|_{\mathcal{V}} \in K\langle\langle T_{\Sigma_{\mathcal{V}}}\rangle\rangle$ defined as follows. Let $s \in T_{\Sigma_{\mathcal{V}}}$. If s is not a valid tree, then $(\|\varphi\|_{\mathcal{V}}, s) = 0$. Otherwise, let ρ be a $(t, \mathcal{V})$-assignment such that s and (t, ρ) correspond to each other. Then, we inductively define $(\|\varphi\|_{\mathcal{V}}, s) \in K$ as follows:*

- $(\|k\|_{\mathcal{V}}, s) = k$
- $(\|label_\sigma(x)\|_{\mathcal{V}}, s) = \begin{cases} 1 & \text{if } t(\rho(x)) = \sigma \\ 0 & \text{otherwise} \end{cases}$
- $(\|edge_i(x,y)\|_{\mathcal{V}}, s) = \begin{cases} 1 & \text{if } \rho(y) = \rho(x)i \\ 0 & \text{otherwise} \end{cases}$
- $(\|x \in X\|_{\mathcal{V}}, s) = \begin{cases} 1 & \text{if } \rho(x) \in \rho(X) \\ 0 & \text{otherwise} \end{cases}$
- $(\|\neg\varphi\|_{\mathcal{V}}, s) = \begin{cases} 1 & \text{if } (\|\varphi\|_{\mathcal{V}}, s) = 0 \\ 0 & \text{if } (\|\varphi\|_{\mathcal{V}}, s) = 1 \end{cases}$, *provided that φ is of the form $label_\sigma(x)$, $edge_i(x,y)$, or $x \in X$*
- $(\|\varphi \vee \psi\|_{\mathcal{V}}, s) = (\|\varphi\|_{\mathcal{V}}, s) + (\|\psi\|_{\mathcal{V}}, s)$
- $(\|\varphi \wedge \psi\|_{\mathcal{V}}, s) = (\|\varphi\|_{\mathcal{V}}, s) \cdot (\|\psi\|_{\mathcal{V}}, s)$
- $(\|\exists x \,.\, \varphi\|_{\mathcal{V}}, s) = \sum_{w \in dom(t)} \left(\|\varphi\|_{\mathcal{V}\cup\{x\}}, s[x \to w]\right)$
- $(\|\exists X \,.\, \varphi\|_{\mathcal{V}}, s) = \sum_{I \subseteq dom(t)} \left(\|\varphi\|_{\mathcal{V}\cup\{X\}}, s[X \to I]\right)$
- $(\|\forall x \,.\, \varphi\|_{\mathcal{V}}, s) = \prod_{w \in dom(t)} \Phi_w^t \left(\left(\|\varphi\|_{\mathcal{V}\cup\{x\}}, s[x \to w]\right)\right)$.

We shall simply write $\|\varphi\|$ for $\|\varphi\|_{Free(\varphi)}$. If φ has no free variables, i.e., if it is a sentence, then $\|\varphi\| \in K\langle\langle T_\Sigma\rangle\rangle$. One should observe that the Φ-semantics $\|\varphi\|_{\mathcal{V}}$ of every formula $\varphi \in MSO(\Sigma, K)$ is defined according to a finite set of variables $\mathcal{V}$ containing $Free(\varphi)$. Actually, this is not an essential restriction as it is announced in the subsequent proposition.

Proposition 5. *(cf. [11], Prop. 3.3) Let $\varphi \in MSO(\Sigma, K)$ with $Free(\varphi) \subseteq \mathcal{V}$. Then*

$$(\|\varphi\|_{\mathcal{V}}, s) = \left(\|\varphi\|, s|_{Free(\varphi)}\right)$$

for every $s \in T^v_{\Sigma_{\mathcal{V}}}$. Moreover, the tree series $\|\varphi\|$ is Φ-recognizable (resp. a recognizable step function) over Σ_φ iff $\|\varphi\|_{\mathcal{V}}$ is Φ-recognizable (resp. a recognizable step function) over $\Sigma_{\mathcal{V}}$.

Definition 6. *(i) A formula $\varphi \in MSO(\Sigma, K)$ is called* restricted *if whenever φ contains a universal first order quantification $\forall x \,.\, \psi$, then $\|\psi\|$ is a recognizable step function.*

(ii) A formula $\varphi \in MSO(\Sigma, K)$ is called almost existential *if whenever φ contains a universal first order quantification $\forall x \,.\, \psi$ and ψ contains a universal first order quantification $\forall y . \psi'$, then ψ' is composed from conjunctions of negations of atomic formulas of the form $edge_i(z, z')$, where $1 \leq i \leq \deg(\Sigma)$.*

We denote by $RMSO(\Sigma, K)$ the class of all restricted formulas of $MSO(\Sigma, K)$, and by $REMSO(\Sigma, K)$ the class of all restricted existential $MSO(\Sigma, K)$-formulas, i.e., formulas of the form $\exists X_1 \ldots \exists X_n \,.\, \psi$ with $\psi \in RMSO(\Sigma, K)$ containing no set quantification. Furthermore, we let $AEMSO(\Sigma, K)$ for the class of all almost existential formulas of $MSO(\Sigma, K)$. A tree series $S \in K\langle\langle T_\Sigma \rangle\rangle$ is called *RMSO-Φ-definable* (resp. *REMSO-Φ-definable*, *AEMSO-Φ-definable*) if there is a sentence $\varphi \in RMSO(\Sigma, K)$ (resp. $\varphi \in REMSO(\Sigma, K)$, $\varphi \in AEMSO(\Sigma, K)$) such that $S = \|\varphi\|$. We let $r\text{-}Mso(\Sigma, K, \Phi)$ (resp. $er\text{-}Mso(\Sigma, K, \Phi)$, $ae\text{-}Mso(\Sigma, K, \Phi)$) comprise all RMSO-Φ-definable (resp. REMSO-Φ-definable, AEMSO-Φ-definable) tree series over Σ and K.

Our second main result is the following.

Theorem 3. *(i) $Rec(\Sigma, K, \Phi) = r\text{-}Mso(\Sigma, K, \Phi) = er\text{-}Mso(\Sigma, K, \Phi)$.*

(ii) If K is additively locally finite, then $Rec(\Sigma, K, \Phi) = ae\text{-}Mso(\Sigma, K, \Phi)$.

For the proof, we firstly show by induction on the structure of formulas φ that $r\text{-}Mso(\Sigma, K, \Phi) \subseteq Rec(\Sigma, K, \Phi)$, and whenever K is additively locally finite, then $ae\text{-}Mso(\Sigma, K, \Phi) \subseteq Rec(\Sigma, K, \Phi)$. This is incorporated in the subsequent lemma.

Lemma 7. *Let $\varphi, \psi \in MSO(\Sigma, K)$. Then*

(i) (cf. [18], Lm. 5.2) if φ is an atomic formula or the negation of an atomic formula, then $\|\varphi\|$ is a recognizable step function,

(ii) (cf. [18], Lm. 5.3, and [16], Lm. 13) if $\|\varphi\|, \|\psi\|$ are Φ-recognizable (resp. recognizable step functions), then $\|\varphi \vee \psi\|$ and $\|\varphi \wedge \psi\|$ are Φ-recognizable (resp. recognizable step functions),

(iii) (cf. [18], Lm. 5.4) if $\|\varphi\|$ is Φ-recognizable, then $\|\exists x \,.\, \varphi\|$ and $\|\exists X \,.\, \varphi\|$ are Φ-recognizable,

(iv) if K is additively locally finite and $\|\varphi\|$ is a recognizable step function, then $\|\exists x \,.\, \varphi\|$ and $\|\exists X \,.\, \varphi\|$ are recognizable step functions,

(v) if $\|\varphi\|$ is a recognizable step function, then $\|\forall x \,.\, \varphi\|$ is Φ-recognizable, and

(vi) if $\|\varphi\| = 1_L$, where $L \subseteq T^v_{\Sigma_\varphi}$ is a recognizable tree language, then $\|\forall x \,.\, \varphi\|$ is a recognizable step function.

Proof. (iv) We follow the proof of Lemma 17 in [16] using our Proposition 2(iii) on page 421.

(v) Let $\mathcal{W} = Free(\varphi) \cup \{x\}$ and $\mathcal{V} = Free(\forall x \,.\, \varphi) = \mathcal{W} \setminus \{x\}$. By Proposition 5 (in case $x \notin Free(\varphi)$) let $\|\varphi\|_{\mathcal{W}} = \sum_{j=1}^{n} k_j 1_{L_j}$, where $k_j \in K$ and $L_j \subseteq T^v_{\Sigma_{\mathcal{W}}}$ are recognizable tree languages ($1 \leq j \leq n$). Furthermore, we assume that the family $(L_j)_{1\leq j\leq n}$ is a partition of $T^v_{\Sigma_{\mathcal{W}}}$.

Let $\widetilde{\Sigma} = \Sigma \times \{1, \ldots, n\}$ be the ranked alphabet with $rk_{\widetilde{\Sigma}}((\sigma, j)) = rk_{\Sigma}(\sigma)$ for every $(\sigma, j) \in \widetilde{\Sigma}$. Every tree $s \in T^v_{\widetilde{\Sigma}_{\mathcal{V}}}$ can be written as a triple (t, v, ρ) where $(t, \rho) \in T^v_{\Sigma_{\mathcal{V}}}$, $dom(t) = dom(s)$, and v is a mapping $v : dom(s) \to \{1, \ldots, n\}$ determined by $v(w) = j$ whenever $s(w) = (\sigma, j, f)$ for some $\sigma \in \Sigma$ and $f \in \{0,1\}^{\mathcal{V}}$. Conversely, every such triple (t, v, ρ) corresponds to a tree $s \in T^v_{\widetilde{\Sigma}_{\mathcal{V}}}$. Hence in the sequel, we write the elements of $T^v_{\widetilde{\Sigma}_{\mathcal{V}}}$ in the form (t, v, ρ). Let $\widetilde{L}$ be the set of all trees $(t, v, \rho) \in T^v_{\widetilde{\Sigma}_{\mathcal{V}}}$ such that for every $w \in dom(t)$ and $1 \leq j \leq n$ if $v(w) = j$, then $(t, \rho[x \to w]) \in L_j$.

Since $(L_j)_{1\leq j\leq n}$ is a partition of $T^v_{\Sigma_{\mathcal{W}}}$, for every $(t, \rho) \in T^v_{\Sigma_{\mathcal{V}}}$ there is a unique $v : dom(t) \to \{1, \ldots, n\}$ such that $(t, v, \rho) \in \widetilde{L}$. By Lemma 5.5 in [18], we get that $\widetilde{L}$ is recognizable. Let $\widetilde{\mathcal{M}} = \left(Q, \widetilde{\Sigma}_{\mathcal{V}}, \delta, F\right)$ be a deterministic bottom-up tree automaton accepting $\widetilde{L}$. We consider the Φ-wta $\mathcal{M} = (Q, wt, ter)$ over $\widetilde{\Sigma}_{\mathcal{V}}$ and K with weight assignment mapping wt defined for every $m \geq 0, (\sigma, j, f) \in \left(\widetilde{\Sigma}_{\mathcal{V}}\right)_m$, and $q_1, \ldots, q_m, q \in Q$ by

$$wt((q_1, \ldots, q_m), (\sigma, j, f), q) = \begin{cases} k_j & \text{if } \delta_{(\sigma,j,f)}(q_1, \ldots, q_m) = q \\ 0 & \text{otherwise.} \end{cases}$$

The final distribution ter is determined by $ter(q) = 1$ if $q \in F$, and $ter(q) = 0$ otherwise for every $q \in Q$.

Since $\widetilde{\mathcal{M}}$ is deterministic, for every $(t, v, \rho) \in T^v_{\widetilde{\Sigma}_{\mathcal{V}}}$ there is at most one run $r_{(t,v,\rho)}$ of $\widetilde{\mathcal{M}}$ over (t, v, ρ). Moreover, since $\left|\widetilde{\mathcal{M}}\right| = \widetilde{L}$ we get

$$(\|\mathcal{M}\|, (t, v, \rho)) = \begin{cases} \prod\limits_{w\in dom((t,v,\rho))} \Phi_w^{(t,v,\rho)}\left(wt\left(r_{(t,v,\rho)}, w\right)\right) & \text{if } (t, v, \rho) \in \widetilde{L} \\ 0 & \text{otherwise.} \end{cases}$$

Let $(t, v, \rho) \in \widetilde{L}$. For every $w \in dom(t)$ with $v(w) = j$, we have $wt\left(r_{(t,v,\rho)}, w\right) = k_j$, and $(t, \rho[x \to w]) \in L_j$ which in turn implies that $\left(\|\varphi\|_{\mathcal{V}\cup\{x\}}, (t, \rho[x \to w])\right) = k_j$. We consider the relabeling $h : \widetilde{\Sigma}_{\mathcal{V}} \to \Sigma_{\mathcal{V}}$ by $h((\sigma, j, f)) = (\sigma, f)$ for every $(\sigma, j, f) \in \widetilde{\Sigma}_{\mathcal{V}}$. Then for every $(t, \rho) \in T^v_{\Sigma_{\mathcal{V}}}$,

$$\begin{aligned} (h(\|\mathcal{M}\|), (t, \rho)) &= \sum_{(t,v,\rho)\in h^{-1}((t,\rho))} (\|\mathcal{M}\|, (t, v, \rho)) = (\|\mathcal{M}\|, (t, v, \rho)) \\ &\quad (\text{where } (t, v, \rho) \in \widetilde{L}) \\ &= \prod_{w\in dom((t,v,\rho))} \Phi_w^{(t,v,\rho)}\left(wt\left(r_{(t,v,\rho)}, w\right)\right) \\ &= \prod_{w\in dom(t)} \Phi_w^{t}\left(\left(\|\varphi\|_{\mathcal{V}\cup\{x\}}, (t, \rho[x \to w])\right)\right) = (\|\forall x \centerdot \varphi\|, (t, \rho)). \end{aligned}$$

Therefore, $\|\forall x \,.\, \varphi\| = h(\|\mathcal{M}\|)$ which by Proposition 1 on page 420 is Φ-recognizable. $\square$

Proposition 6.

- *r-Mso$(\Sigma, K, \Phi) \subseteq$ Rec(Σ, K, Φ).*
- *If K is additively locally finite, then ae-Mso$(\Sigma, K, \Phi) \subseteq$ Rec(Σ, K, Φ).*

Proposition 7. *Rec$(\Sigma, K, \Phi) \subseteq$ er-Mso$(\Sigma, K, \Phi) \cap$ ae-Mso(Σ, K, Φ).*

Proof of Theorem 3. It is immediate by Propositions 6 and 7. $\square$

6 Weighted Muller tree automata with Φ-discounting

In this section, we investigate weighted Muller tree automata with Φ-discounting acting on infinite trees. The underlying semiring is the max-plus semiring $\mathbb{R}_{\max} = (\mathbb{R}_+ \cup \{-\infty\}, \sup, +, -\infty, 0)$, where we consider sup instead of max since we need to compute over infinite trees. Our results can be applied to the min-plus semiring $(\mathbb{R}_+ \cup \{\infty\}, \inf, +, \infty, 0)$ as well. A weighted Muller tree automaton with Φ-discounting computes the weight of a run (of an input infinite tree) by applying the Φ-discounting over $\mathbb{R}_{\max}$. By considering suitable endomorphisms for Φ, we do not require any completeness axioms (for the sum operation) in $\mathbb{R}_{\max}$ (cf. [32]). For a study on weighted Muller automata with Φ-discounting over infinite words cf. [16].

An *infinitary tree series S over Σ and* $\mathbb{R}_{\max}$ is a mapping $S : T_\Sigma^\omega \to \mathbb{R}_{\max}$. The class of all infinitary tree series over Σ and $\mathbb{R}_{\max}$ is denoted by $\mathbb{R}_{\max}\langle\langle T_\Sigma^\omega\rangle\rangle$. Let $S \in \mathbb{R}_{\max}\langle\langle T_\Sigma^\omega\rangle\rangle$. The *image $Im(S)$ of S* is the set $Im(S) = \{k \in \mathbb{R}_+ \cup \{-\infty\} \mid \exists t \in T_\Sigma^\omega$ with $(S,t) = k\}$. We say that S has *bounded image* if there is an $m \in \mathbb{R}_+$ such that $k \leq m$ for every $k \in Im(S)$. Consider an infinitary tree language $L \subseteq T_\Sigma^\omega$. The *characteristic series* $0_L : T_\Sigma^\omega \to \mathbb{R}_{\max}$ *of L* is defined in a similar way as for finitary tree languages. Furthermore, for $S, T \in \mathbb{R}_{\max}\langle\langle T_\Sigma^\omega\rangle\rangle$ and $k \in \mathbb{R}_{\max}$, the sum, the scalar product, and the Hadamard product are now written as $\max(S,T), k + S$, and $S + T$, respectively, and defined in the obvious way.

Let $h : \Sigma \to \Gamma$ be a relabeling. Then h is extended to a mapping $h : T_\Sigma^\omega \to T_\Gamma^\omega$ such that $dom(h(t)) = dom(t)$ and $h(t)(w) = h(t(w))$ for every $t \in T_\Sigma^\omega$ and $w \in dom(t)$. Moreover, h can be extended to a partial mapping $h : \mathbb{R}_{\max}\langle\langle T_\Sigma^\omega\rangle\rangle \to \mathbb{R}_{\max}\langle\langle T_\Gamma^\omega\rangle\rangle$ in the following way. For every $S \in \mathbb{R}_{\max}\langle\langle T_\Sigma^\omega\rangle\rangle$ with bounded image, we define the series $h(S) \in \mathbb{R}_{\max}\langle\langle T_\Gamma^\omega\rangle\rangle$ by $(h(S), s) = \sup\{(S,t) \mid t \in h^{-1}(s)\}$ for every $s \in T_\Gamma^\omega$. Furthermore, for every $T \in \mathbb{R}_{\max}\langle\langle T_\Gamma^\omega\rangle\rangle$, the series $h^{-1}(T) \in \mathbb{R}_{\max}\langle\langle T_\Sigma^\omega\rangle\rangle$ is determined by $\left(h^{-1}(T), t\right) = (T, h(t))$ for every $t \in T_\Sigma^\omega$.

Let $\Phi = (\Phi_k)_{k\geq 1}$ be a discounting over Σ and $\mathbb{R}_{\max}$. Recall (cf. [14], and Example 1 on page 414) that every endomorphism of $\mathbb{R}_{\max}$ is of the form $\overline{p}$:

$\mathbb{R}_{\max} \to \mathbb{R}_{\max}$ where $p \in \mathbb{R}_+$ and $x \longmapsto p \cdot x$ for every $x \in \mathbb{R}_+ \cup \{-\infty\}$ with the convention that $p \cdot (-\infty) = (-\infty) \cdot p = -\infty$ for every $p \in \mathbb{R}_+ \cup \{-\infty\}$. For our Φ-discounting here, we require for every $k \geq 1, \sigma \in \Sigma_k$ that $\Phi_\sigma^i = \overline{p_\sigma^i}$ with $0 \leq p_\sigma^i < 1$. Then, we simply write $\Phi_\sigma = \overline{p_\sigma} = \left(\overline{p_\sigma^1}, \ldots, \overline{p_\sigma^k}\right)$ for every $k \geq 1$ and $\sigma \in \Sigma_k$. Furthermore, for every $t \in T_\Sigma^\omega$ and every $w \in dom(t)$ we write $\overline{p_w^t}$ for Φ_w^t where

$$p_w^t = \begin{cases} 1 & \text{if } w = \varepsilon \\ p_{t(\varepsilon)}^{i_1} \cdot p_{t(i_1)}^{i_2} \cdot \ldots \cdot p_{t(i_1 \ldots i_{n-1})}^{i_n} & \text{if } w = i_1 \ldots i_n \text{ with } i_1, \ldots, i_n \in \mathbb{N}_+, n > 0. \end{cases}$$

We let $m_\Phi = \max\left\{p_\sigma^i \mid k \geq 1, \sigma \in \Sigma_k, \text{ and } 1 \leq i \leq k\right\}$. In the sequel, we shall use also the concatenation notation for the multiplication in $\mathbb{R}_+ \cup \{-\infty\}$.

Definition 7. *A* weighted Muller tree automaton with Φ-discounting *(*Φ-wmta *for short)* over Σ and $\mathbb{R}_{\max}$ *is a quadruple* $\mathcal{M} = (Q, in, wt, \mathcal{F})$*, where* Q *is the* finite state set*,* $in : Q \to \mathbb{R}_{\max}$ *is the* initial distribution*,* $wt : \bigcup_{k \geq 0} Q \times \Sigma_k \times Q^k \to \mathbb{R}_{\max}$ *is the* mapping assigning weights to the transitions *of the automaton, and* $\mathcal{F} \subseteq \mathcal{P}(Q)$ *is the* family of final states sets*.*

Let $t \in T_\Sigma^\omega$. A *run of* $\mathcal{M}$ *over* t is a mapping $r_t : dom(t) \to Q$. The *weight of* r_t *at* $w \in dom(t)$ is the value

$$wt(r_t, w) = wt(r_t(w), t(w), (r_t(w1), \ldots, r_t(w.rk_\Sigma(t(w))))).$$

The Φ-*weight* (or simply *weight*) *of* r_t, which is denoted by $weight_\mathcal{M}(r_t)$ (or simply $weight(r_t)$), is defined by

$$weight_\mathcal{M}(r_t) = in(r_t(\varepsilon)) + \sum_{w \in dom(t)} \frac{p_w^t}{\deg(\Sigma)^{|w|}} \cdot wt(r_t, w).$$

One should observe that in comparison to the finitary case, here we divide every summand of the infinite sum with a power of $\deg(\Sigma)$. This is needed to achieve the convergence of the infinite sum. Indeed, let $M =$
$\max\left\{wt(\tau) \mid \tau \in \bigcup_{k \geq 0} Q \times \Sigma_k \times Q^k\right\}$. Then we have

$$\begin{aligned} \sum_{w \in dom(t)} \frac{p_w^t}{\deg(\Sigma)^{|w|}} \cdot wt(r_t, w) &\leq M \cdot \sum_{n \geq 0} \sum_{\substack{w \in dom(t) \\ |w| = n}} \frac{m_\Phi^{|w|}}{\deg(\Sigma)^{|w|}} \\ &\leq M \cdot \sum_{n \geq 0} \deg(\Sigma)^n \frac{m_\Phi^n}{\deg(\Sigma)^n} = M \cdot \frac{1}{1 - m_\Phi}. \end{aligned}$$

Every infinite prefix-closed chain $\pi \subseteq dom(t)$ is called an *infinite path of* t. The run r_t is called *successful* if for every infinite path π of t, the set of states that appear infinitely often along π, constitutes a final state set. We shall denote by $R_{\mathcal{M}}(t)$ the set of all runs of $\mathcal{M}$ over t, and by $R^{suc}_{\mathcal{M}}(t)$ the set of all successful runs in $R_{\mathcal{M}}(t)$.

The Φ-*behavior* (or simply *behavior*) *of* $\mathcal{M}$ is the infinitary tree series $\|\mathcal{M}\| \in \mathbb{R}_{\max}\langle\langle T^{\omega}_{\Sigma}\rangle\rangle$ whose coefficients are determined for every $t \in T^{\omega}_{\Sigma}$ by

$$(\|\mathcal{M}\|, t) = \sup\{(weight_{\mathcal{M}}(r_t)) \mid r_t \in R^{suc}_{\mathcal{M}}(t)\}.$$

Clearly, this supremum exists in $\mathbb{R}_{\max}$ since the values $weight_{\mathcal{M}}(r_t)$ are bounded by $N + M \cdot \frac{1}{1-m_\Phi}$, where $N = \max\{in(q) \mid q \in Q\}$.

A tree series $S \in \mathbb{R}_{\max}\langle\langle T^{\omega}_{\Sigma}\rangle\rangle$ is said to be (Φ,ω)-*recognizable* (or Φ-*Muller recognizable*) if there exists a Φ-wmta $\mathcal{M}$ over Σ and $\mathbb{R}_{\max}$ such that $S = \|\mathcal{M}\|$. The family of all (Φ,ω)-recognizable tree series over Σ and $\mathbb{R}_{\max}$ is denoted by ω-$Rec(\Sigma, \mathbb{R}_{\max}, \Phi)$. Clearly, every (Φ,ω)-recognizable tree series $S \in \mathbb{R}_{\max}\langle\langle T^{\omega}_{\Sigma}\rangle\rangle$ has bounded image. The next proposition collects closure properties of (Φ,ω)-recognizable tree series.

Proposition 8. *(i) (cf. [32], Prop. 8) The class* ω-$Rec(\Sigma, \mathbb{R}_{\max}, \Phi)$ *is closed under sum, scalar product, and Hadamard product.*

(ii) (cf. [32], Prop. 9) Let $h : \Sigma \to \Gamma$ *be a relabeling. Furthermore, for the* Φ-*discounting over* Σ *and* $\mathbb{R}_{\max}$ *we assume that* $\overline{p_\sigma} = \overline{p_{\sigma'}}$ *whenever* $h(\sigma) = h(\sigma')$ *for every* $\sigma, \sigma' \in \Sigma_k$*, and* $k \geq 1$*. Let* $\Phi' = (\Phi'_k)_{k\geq 1}$ *be the discounting over* Γ *and* $\mathbb{R}_{\max}$ *determined for every* $\gamma \in \Gamma_k$ $(k \geq 1)$ *by* $\overline{p'_\gamma} = \overline{p_\sigma}$ *for every* $\sigma \in \Sigma_k$ $(k \geq 1)$ *with* $h(\sigma) = \gamma$*. If* $S \in \omega$-$Rec(\Sigma, \mathbb{R}_{\max}, \Phi)$*, then* $h(S) \in \omega$-$Rec(\Gamma, \mathbb{R}_{\max}, \Phi')$*. Furthermore, if* $T \in \omega$-$Rec(\Gamma, \mathbb{R}_{\max}, \Phi')$*, then* $h^{-1}(T) \in \omega$-$Rec(\Sigma, \mathbb{R}_{\max}, \Phi)$*.*

(iii) (cf. [32], Prop. 10) Let $L \subseteq T^{\omega}_{\Sigma}$ *be an* ω*-recognizable tree language. Then* $0_L \in \omega$-$Rec(\Sigma, \mathbb{R}_{\max}, \Phi)$*.*

An infinitary tree series $S \in \mathbb{R}_{\max}\langle\langle T^{\omega}_{\Sigma}\rangle\rangle$ is called an ω-*recognizable step function* (or *Muller recognizable step function*) if $S = \max\limits_{1\leq j\leq n}\left(k_j + 0_{L_j}\right)$ where $k_j \in \mathbb{R}_+ \cup \{-\infty\}$ and L_j is an ω-recognizable tree language for every $1 \leq j \leq n$.

Proposition 9. *(i) (cf. [32], Prop. 11) The class of* ω*-recognizable step functions over* Σ *and* $\mathbb{R}_{\max}$ *is closed under sum, scalar product, and Hadamard product.*

(ii) (cf. [32], Prop. 12) Let $h : \Sigma \to \Gamma$ *be a relabeling. Then* $h : \mathbb{R}_{\max}\langle\langle T^{\omega}_{\Sigma}\rangle\rangle \to \mathbb{R}_{\max}\langle\langle T^{\omega}_{\Gamma}\rangle\rangle$ *and* $h^{-1} : \mathbb{R}_{\max}\langle\langle T^{\omega}_{\Gamma}\rangle\rangle \to \mathbb{R}_{\max}\langle\langle T^{\omega}_{\Sigma}\rangle\rangle$ *preserve* ω*-recognizable step functions.*

7 Weighted MSO-logic with Φ-discounting over infinite trees

In this section we deal with weighted MSO-logic with Φ-discounting over the semiring $\mathbb{R}_{\max}$, and we interpret the semantics of weighted MSO-formulas as formal series over infinite trees. In our logic here, we inlude the atomic formula $x = y$ and its negation as well as second-order universal quantifiers (cf. [32]).

Every infinite tree $t \in T_\Sigma^\omega$ is represented by the structure $(dom(t), edge_1, \ldots, edge_{\deg(\Sigma)}, (label_\sigma)_{\sigma\in\Sigma})$. The notions of a $(t, \mathcal{V})$-assignment and the set $T_{\Sigma_\mathcal{V}}^{\omega,v}$ of all valid infinite trees over $\Sigma_\mathcal{V}$ are defined as in the case of finite trees. Similarly, every valid tree $s \in T_{\Sigma_\mathcal{V}}^{\omega,v}$ corresponds to a pair (t, ρ) where $t \in T_\Sigma^\omega$ and ρ is a $(t, \mathcal{V})$-assignment. The infinitary tree language $T_{\Sigma_\mathcal{V}}^{\omega,v}$ is ω-recognizable (cf. [32]), and thus the characteristic series $0_{T_{\Sigma_\mathcal{V}}^{\omega,v}} \in \mathbb{R}_{\max}\left\langle\left\langle T_{\Sigma_\mathcal{V}}^\omega\right\rangle\right\rangle$ is (Φ, ω)-recognizable.

Let φ be an MSO-formula [36, 37] over trees. Then for $Free(\varphi) \subseteq \mathcal{V}$, the well-known result of Rabin [29] states that the tree language $\mathcal{L}_\mathcal{V}^\omega(\varphi) = \{(t, \rho) \in T_{\Sigma_\mathcal{V}}^{\omega,v} \mid (t, \rho) \models \varphi\}$ is ω-recognizable; conversely, for every ω-recognizable tree language $L \subseteq T_\Sigma^\omega$ there exists an MSO-sentence φ, such that $L = \mathcal{L}^\omega(\varphi)$, where we simply write $\mathcal{L}^\omega(\varphi)$ for $\mathcal{L}_{Free(\varphi)}^\omega(\varphi)$.

Definition 8. *The set $MSO(\Sigma, \mathbb{R}_{\max})$ of all formulas of the weighted MSO-logic with Φ-discounting over Σ and $\mathbb{R}_{\max}$ on infinite trees is defined to be the smallest set F such that*

- *F contains all atomic formulas $k, label_\sigma(x), edge_i(x, y), x = y, x \in X$ and the negations $\neg label_\sigma(x), \neg edge_i(x, y), \neg(x = y), \neg(x \in X)$,*
- *if $\varphi, \psi \in F$, then $\varphi \vee \psi, \varphi \wedge \psi, \exists x \centerdot \varphi, \exists X \centerdot \varphi, \forall x \centerdot \varphi \in F$, and if φ does not contain any constant $k \in \mathbb{R}_+ \setminus \{0\}$, then $\forall X \centerdot \varphi \in F$*

where $k \in \mathbb{R}_+ \cup \{-\infty\}$, $\sigma \in \Sigma$, $1 \leq i \leq \deg(\Sigma)$, x, y are first order variables, and X is a second-order variable.

Next we define the semantics of the formulas in $MSO(\mathbb{R}_{\max}, \Sigma)$ as infinitary tree series in $\mathbb{R}_{\max}\left\langle\left\langle T_{\Sigma_\mathcal{V}}^\omega\right\rangle\right\rangle$.

Definition 9. *Let $\varphi \in MSO(\Sigma, \mathbb{R}_{\max})$ and $Free(\varphi) \subseteq \mathcal{V}$. The* Φ-semantics *of φ is an infinitary tree series $\|\varphi\|_\mathcal{V} \in \mathbb{R}_{\max}\left\langle\left\langle T_{\Sigma_\mathcal{V}}^\omega\right\rangle\right\rangle$ defined as follows. Let $s \in T_{\Sigma_\mathcal{V}}^\omega$. If s is not a valid tree, then $(\|\varphi\|_\mathcal{V}, s) = -\infty$. Otherwise, let ρ be a $(t, \mathcal{V})$-assignment such that s and (t, ρ) correspond to each other. Then, we inductively define $(\|\varphi\|_\mathcal{V}, s)$ as in Definition 5, where $K = \mathbb{R}_{\max}$, except for the formulas $x = y$, $\forall x \centerdot \varphi$, and $\forall X \centerdot \varphi$ where we set*

- $(\|x = y\|_\mathcal{V}, s) = \begin{cases} 0 & \textit{if } \rho(x) = \rho(y) \\ -\infty & \textit{otherwise} \end{cases}$

- $(\|\forall x \centerdot \varphi\|_\mathcal{V}, s) = \sum\limits_{w \in dom(t)} \frac{p_w^t}{\deg(\Sigma)^{|w|}} \left(\|\varphi\|_{\mathcal{V}\cup\{x\}}, s[x \to w]\right).$

- $(\|\forall X \centerdot \varphi\|_{\mathcal{V}}, s) = \sum_{I \subseteq dom(t)} \Big(\|\varphi\|_{\mathcal{V} \cup \{X\}}, s[X \to I]\Big)$.

Note that in the above definition of the semantics, the sums and products in Definition 5 on page 428 are replaced respectively, by suprema and sums in $\mathbb{R}_{\max}$. Moreover, in case of $\forall X \centerdot \varphi$ the infinite sum is well-defined, because by definition the semantics of φ takes on only the values 0 and $-\infty$. We shall simply write $\|\varphi\|$ for $\|\varphi\|_{Free(\varphi)}$. As in the case of finitary tree series we can show the subsequent result.

Proposition 10. *Let $\varphi \in MSO(\Sigma, \mathbb{R}_{\max})$ with $Free(\varphi) \subseteq \mathcal{V}$. Then*

$$(\|\varphi\|_{\mathcal{V}}, s) = \big(\|\varphi\|, s|_{Free(\varphi)}\big)$$

for every $s \in T^{\omega,v}_{\Sigma_{\mathcal{V}}}$. Moreover, the tree series $\|\varphi\|$ is (Φ, ω)-recognizable (resp. an ω-recognizable step function) over Σ_φ iff $\|\varphi\|_{\mathcal{V}}$ is (Φ, ω)-recognizable (resp. an ω-recognizable step function) over $\Sigma_{\mathcal{V}}$.

Definition 10. *A formula $\varphi \in MSO(\Sigma, \mathbb{R}_{\max})$ is called* restricted *if whenever φ contains a universal first order quantification $\forall x \centerdot \psi$, then $\|\psi\|$ is an ω-recognizable step function.*

Definition 11. *A formula $\varphi \in MSO(\Sigma, \mathbb{R}_{\max})$ is called* incomplete universal *if whenever φ contains a subformula $\forall x \centerdot \psi$ such that ψ contains universal quantifiers, then ψ cannot contain any constant $k \in \mathbb{R}_+ \setminus \{0\}$.*

We denote by $RMSO(\Sigma, \mathbb{R}_{\max})$ (resp. $IUMSO(\Sigma, \mathbb{R}_{\max})$) the class of all restricted (resp. incomplete universal) formulas of $MSO(\Sigma, \mathbb{R}_{\max})$. A tree series $S \in \mathbb{R}_{\max}\langle\langle T^\omega_\Sigma \rangle\rangle$ is called *RMSO-Φ-definable* (resp. *IUMSO-Φ-definable*) if there is a sentence $\varphi \in RMSO(\Sigma, \mathbb{R}_{\max})$ (resp. $\varphi \in IUMSO(\Sigma, \mathbb{R}_{\max})$) such that $S = \|\varphi\|$. We denote by ω-*r*-*Mso*$(\Sigma, \mathbb{R}_{\max}, \Phi)$ (resp. ω-*iu*-*Mso*$(\Sigma, \mathbb{R}_{\max}, \Phi)$) the class of all RMSO-Φ-definable (resp. IUMSO-Φ-definable) infinitary tree series.

The main result of this section is the subsequent Rabin-type theorem.

Theorem 4. *ω-Rec$(\Sigma, \mathbb{R}_{\max}, \Phi) = \omega$-r-Mso$(\Sigma, \mathbb{R}_{\max}, \Phi) = \omega$-iu-Mso$(\Sigma, \mathbb{R}_{\max}, \Phi)$.*

First, using induction on the structure of formulas, we state the inclusions ω-*r*-*Mso*$(\Sigma, \mathbb{R}_{\max}, \Phi) \subseteq \omega$-*Rec*$(\Sigma, \mathbb{R}_{\max}, \Phi)$ and ω-*iu*-*Mso*$(\Sigma, \mathbb{R}_{\max}, \Phi) \subseteq \omega$-*Rec*$(\Sigma, \mathbb{R}_{\max}, \Phi)$.

Lemma 8. *Let $\varphi, \psi \in MSO(\Sigma, \mathbb{R}_{\max})$. Then*

(i) (cf. [32], Lm. 22) if φ is an atomic formula or the negation of an atomic formula, then $\|\varphi\|$ is an ω-recognizable step function,

(ii) (cf. [32], Lm. 23) if $\|\varphi\|, \|\psi\|$ are (Φ, ω)-recognizable (resp. ω-recognizable step functions), then $\|\varphi \vee \psi\|$ and $\|\varphi \wedge \psi\|$ are (Φ, ω)-recognizable (resp. ω-recognizable step functions),

(iii) *(cf. [32], Lm. 24) if $\|\varphi\|$ is (Φ, ω)-recognizable (resp. an ω-recognizable step function), then $\|\exists x \centerdot \varphi\|$ and $\|\exists X \centerdot \varphi\|$ are (Φ, ω)-recognizable (resp. ω-recognizable step functions),*

(iv) *(cf. [32], Lm. 25) if $\|\varphi\|$ is an ω-recognizable step function, then $\|\forall x \centerdot \varphi\|$ is (Φ, ω)-recognizable, and*

(v) *(cf. [32], Lm. 26) if $\|\varphi\| = 0_L$ where the tree language $L \subseteq T^{\omega,v}_{\Sigma_\varphi}$ is ω-recognizable, then $\|\forall X \centerdot \varphi\|$ is an ω-recognizable step function.*

Therefore, we get

Proposition 11.

- $\omega\text{-}r\text{-}Mso(\Sigma, \mathbb{R}_{\max}, \Phi) \subseteq \omega\text{-}Rec(\Sigma, \mathbb{R}_{\max}, \Phi)$.
- $\omega\text{-}iu\text{-}Mso(\Sigma, \mathbb{R}_{\max}, \Phi) \subseteq \omega\text{-}Rec(\Sigma, \mathbb{R}_{\max}, \Phi)$.

Conversely, following the proof of Proposition 29 in [32] we state

Proposition 12.

$$\omega\text{-}Rec(\Sigma, \mathbb{R}_{\max}, \Phi) \subseteq \omega\text{-}r\text{-}Mso(\Sigma, \mathbb{R}_{\max}, \Phi) \cap \omega\text{-}iu\text{-}Mso(\Sigma, \mathbb{R}_{\max}, \Phi).$$

Proof of Theorem 4. It is immediate by Propositions 11 and 12. □

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