

# Splitters and Barriers in Open Graphs Having a Perfect Internal Matching\*

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## Abstract

A counterpart of Tutte's Theorem and Berge's formula is proved for open graphs with perfect (maximum) internal matchings. Properties of barriers and factor-critical graphs are studied in the new context, and an efficient algorithm is given to find maximal barriers of graphs having a perfect internal matching.

**Keywords:** graph matchings, splitters, barriers, factor-critical graphs

## 1 Introduction

The concepts “open graph” and “perfect internal matching” have emerged from the study of soliton automata introduced in [11]. In this graph theoretical model for electronic switching at the molecular level, an undirected graph represents the topological structure of a hydrocarbon molecule having an alternating pattern of single and double bonds between its carbon atoms. A soliton is a solitary wave that travels through chains of alternating single and double bonds in small packets, and has the ability to switch each affected bond to its opposite. See [10] for the physico-chemical aspects of soliton switching. Soliton waves are initiated and received at some designated interface points, which are treated as distinguished vertices in the graph model. These vertices are called external, and for convenience it is assumed that a vertex is external iff it has degree one. Hence the notion “open graph”.

The fact that a molecule has an alternating pattern of single and double bonds is captured by requiring that the underlying graph has a matching (e.g. the collection of double bonds) which covers each vertex, with the possible exception of the external ones. The status of any particular external vertex being covered or not by such a “perfect internal” matching changes when a soliton is initiated from

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or received at that vertex. A soliton graph is defined as an open graph having a perfect internal matching. The automaton behavior of soliton graphs arises from the switching capability of the soliton. See [11] for the precise definition of soliton automata.

Even though open graphs and perfect internal matchings have been introduced in [2] with the above specific model in mind, there are other meaningful interpretations of these concepts. Consider, for example, a project on which employees of a company must work in couples. The company employs both full-time and other (e.g. part-time) workers, but its preference is to assign as many full-time employees to the project as possible. In addition, a known compatibility relationship among the employees must be respected, which determines the possible couplings for the job. The underlying graph  $G$  in this case consists of the employees as vertices and the compatibility relation among them as edges. Vertices corresponding to full-time employees are considered internal, whereas the group of other employees constitutes the set of external vertices of  $G$ . The goal is to find a matching  $M$  of  $G$  that covers a maximum number of internal vertices.

Considering the latter interpretation of open graphs and perfect/maximum internal matchings, a splitter is a collection of full-time workers such that no two of these workers can work together by any optimal coupling. A barrier, on the other hand, is a collection  $X$  of full-time workers (internal vertices) such that, when taking out  $X$  from the compatibility graph  $G$ , the number of odd internal components (i.e., connected components consisting of an odd number of internal vertices) in  $G - X$  exceeds the cardinality of  $X$  by the deficiency of  $G$ , which is the number of full-time workers remaining idle by any optimal coupling. Clearly, for collections of full-time workers, being a barrier is a stronger property than being just a splitter.

Regarding the soliton automaton model, it turns out that an internal vertex (carbon atom)  $v$  may belong to a barrier only if  $v$  is “positively inaccessible” for the soliton with respect to any state (perfect internal matching)  $M$ . By this we mean that every  $M$ -alternating path reaching  $v$  (if any) starting from an external vertex will arrive at  $v$  on an  $M$ -negative edge (i.e. single bond). Consequently, whenever the soliton first arrives at  $v$  on edge  $e$ , it must return to  $v$  before quitting, and then leave  $v$  on the same edge  $e$  in the opposite direction. (See the definition of soliton paths/walks in [11].) It follows that a viable soliton graph, in which every carbon atom can be reached by the soliton in an appropriate state, has a unique maximal barrier, namely the set of its inaccessible vertices.

The present paper is a synthesis of the results obtained in [4], [5], and [6] with a special emphasis on barriers. A new shorter proof is given for Tutte’s Theorem for open graphs with perfect internal matchings, and Berge’s Formula is proved as a consequence of this theorem. Barriers are studied in the context of a suitable closure operation, which allows for an analysis of perfect internal matchings in open graphs via perfect matchings of their closures. Maximal splitters in open graphs are compared with maximal barriers in closed graphs, and, using a result from [6], an algorithm is worked out to isolate maximal barriers in linear time.

## 2 Graphs and matchings

By a graph, throughout the paper, we mean a finite undirected graph in the most general sense, with multiple edges and loops allowed. Our notation and terminology follows [14]. For a graph  $G$ ,  $V(G)$  and  $E(G)$  will denote the set of vertices and the set of edges of  $G$ , respectively. An edge  $e = (v_1, v_2)$  in  $E(G)$  connects two vertices  $v_1, v_2 \in V(G)$ , which are called the *endpoints* of  $e$ , and  $e$  is said to be *incident* with  $v_1$  and  $v_2$ . If  $v_1 = v_2$ , then  $e$  is called a *loop* around  $v_1$ . Two edges sharing at least one endpoint are said to be *adjacent* in  $G$ . A subgraph  $G'$  of  $G$  is a graph such that  $V(G') \subseteq V(G)$  and  $E(G') \subseteq E(G)$ . If  $X \subseteq V(G)$  then  $G[X]$  denotes the subgraph of  $G$  for which  $V(G[X]) = X$  and  $E(G[X])$  consists of the edges of  $G$  having both endpoints in  $X$ . The notation  $G - X$  is a shorthand for  $G[V(G) - X]$ .

For a graph  $G$ , the *degree* of a vertex  $v$ , denoted  $d(v)$ , is the number of occurrences of  $v$  as an endpoint of some edge in  $E(G)$ . According to this definition, every loop around  $v$  contributes two occurrences to the count. Vertex  $v$  is called *external* if  $d(v) = 1$ , *internal* if  $d(v) \geq 2$  and *isolated* if  $d(v) = 0$ . *External edges* are those that are incident with at least one external vertex, and an *internal edge* is one that is not external. The sets of external and internal vertices of  $G$  will be denoted by  $\text{Ext}(G)$  and  $\text{Int}(G)$ , respectively. Graph  $G$  is called *open* if  $\text{Ext}(G) \neq \emptyset$ , otherwise  $G$  is *closed*.

A *matching*  $M$  of graph  $G$  is a subset of  $E(G)$  such that no vertex of  $G$  occurs more than once as an endpoint of some edge in  $M$ . As the endpoints of loops count twice, such edges cannot participate in  $M$ . The endpoints of the edges contained in  $M$  are said to be *covered* by  $M$ . A *perfect* (*maximum*) matching of  $G$  is a matching that covers all (respectively, a maximum number of) vertices in  $G$ , and a *perfect internal* (*maximum internal*) matching is one that covers all (respectively, a maximum number of) internal vertices in  $G$ . In this paper we are primarily interested in perfect internal matchings of graphs.

An edge  $e \in E(G)$  is said to be *allowed* (*mandatory*) if  $e$  is contained in some (respectively, all) perfect internal matching(s) of  $G$ . *Forbidden edges* are those that are not allowed. We will also use the term *constant edge* to identify an edge that is either forbidden or mandatory. A *mandatory external vertex* is one that is covered by all perfect internal matchings.

An open graph having a perfect internal matching is called a *soliton graph*. Let  $G$  be a soliton graph, fixed for the rest of this section, and let  $M$  be a perfect internal matching of  $G$ . An edge  $e \in E(G)$  is said to be  *$M$ -positive* ( *$M$ -negative*) if  $e \in M$  (respectively,  $e \notin M$ ). An  *$M$ -alternating path* (*cycle*) in  $G$  is a path (respectively, even-length cycle) stepping on  $M$ -positive and  $M$ -negative edges in an alternating fashion. Let us agree that, if the matching  $M$  is understood or irrelevant in a particular context, then it will not explicitly be indicated in these terms.

An *external alternating path* is one that has an external endpoint. If both endpoints of the path are external, then it is called a *crossing*. An alternating path is *positive* (*negative*) if it is such at its internal endpoints (if any), meaning that the edges incident with those endpoints are positive (respectively, negative). A *positive*

(negative) *alternating fork* is a pair of vertex-disjoint positive (respectively, negative) external alternating paths leading to two distinct internal vertices. Although it sounds somewhat confusing, we still say that these two vertices are *connected* by the fork.

An *alternating unit* is either a crossing or an alternating cycle. *Switching* on an alternating unit amounts to changing the sign of each edge along the unit. It is easy to see that the operation of switching creates a new perfect internal matching for  $G$ . Moreover, as it was proved in [1], every perfect internal matching of  $G$  can be transformed into any other perfect internal matching by switching on a number of pairwise disjoint alternating units. It follows that any edge  $e$  of  $G$  is not constant iff there exists an alternating unit passing through  $e$  with respect to every perfect internal matching of  $G$ .

Since in our treatment we are particular about external vertices, we do not want to allow that subgraphs of  $G$  possess external vertices other than the ones present in  $G$ . Therefore whenever this happens, and an internal vertex  $v$  becomes external in a subgraph  $G'$  of  $G$ , we shall augment  $G'$  by a looping edge around  $v$ . This augmentation will be understood automatically throughout the paper.

### 3 Perfect matchings vs. perfect internal matchings

There is an easy way to neutralize the concession that external vertices in open graphs need not be covered by perfect internal matchings, without actually withdrawing this privilege. In any open graph  $G$ , attach a loop around each external vertex to obtain a closed graph  $\bar{G}$ . Since loops cannot be part of any matching, the augmentation  $G \mapsto \bar{G}$  will simply cancel the privilege existing in  $G$  by turning every external vertex into an internal one. Thus, perfect matchings of  $G$  can trivially be recaptured as perfect internal matchings of  $\bar{G}$ . This observation will allow us to conveniently refer to results on maximum/perfect matchings without leaving the realm of our current framework dealing with maximum/perfect internal matchings, yet preserving the original scope of these results simply by saying that the objects of consideration are closed graphs.

On the other hand, perfect internal matchings, too, can be studied as ordinary perfect matchings by introducing an appropriate closure operation on graphs.

**Definition 3.1.** The *closure* of graph  $G$  is the closed graph  $G^*$  for which:

- $V(G^*) = V(G)$  if  $|V(G)|$  is even, and
- $V(G^*) = V(G) \cup \{c\}$ ,  $c \notin V(G)$  if  $|V(G)|$  is odd;
- $E(G^*) = E(G) \cup \{(v_1, v_2) \mid v_i \in \text{Ext}(G) \cup \{c\}\}$ .

Intuitively,  $G^*$  is obtained from  $G$  by connecting its external vertices with each other in all possible ways. If  $|V(G)|$  happens to be odd, then a new vertex  $c$  is added to  $G$ , and edges are introduced from  $c$  to all of the external vertices. The edges of  $G^*$  belonging to  $E(G^*) - E(G)$  are called *marginal*, and the vertex  $c$  is referred to as the *collector*. Edges incident with the collector vertex will be called *collector*, too.

Notice that, in the specification of  $E(G^*)$ , it is not required that  $v_1 \neq v_2$ . Consequently, in  $G^*$ , there will be a loop around each external vertex of  $G$ . These loops have no specific role if  $G$  has at least two external vertices, although their introduction as trivial forbidden edges is harmless. If there is only one external vertex in  $G$ , however, the loop is essential to make  $G^*$  closed.

**Proposition 3.1.** *Graph  $G$  has a perfect internal matching iff  $G^*$  has a perfect matching.*

*Proof.* If  $G^*$  has a perfect matching  $M^*$ , then deleting the marginal edges from  $G^*$  and  $M^*$  will leave  $G$  with a perfect internal matching. Conversely, if  $G$  has a perfect internal matching  $M$ , then it is always possible to extend  $M$  to a perfect matching of  $G^*$  by matching up the external vertices of  $G$  not covered by  $M$  in an arbitrary way, using the collector vertex  $c$  if necessary. Obviously, the use of  $c$  is necessary if and only if  $|V(G)|$  is odd.  $\square$

**Lemma 3.1.** *Every  $M$ -alternating crossing of  $G$  gives rise to an  $M^*$ -alternating cycle of  $G^*$  by any extension of  $M$  to a perfect matching  $M^*$ . Conversely, for an arbitrary perfect matching  $M^*$  of  $G^*$ , every  $M^*$ -alternating cycle of  $G^*$  containing at least one marginal edge opens up to a number of alternating crosses with respect to the restriction of  $M^*$  to  $E(G)$  when the marginal edges are deleted from  $G^*$ .*

*Proof.* Straightforward, using the same argument as under Proposition 3.1.  $\square$

**Corollary 3.1.** *For every edge  $e \in E(G)$ ,  $e$  is allowed in  $G$  iff  $e$  is allowed in  $G^*$ .*

*Proof.* Indeed, by Lemma 3.1,  
 $e$  is allowed in  $G$

- iff* there exists a  $M$ -alternating unit through  $e$  in  $G$
- iff* there exists an  $M^*$ -alternating cycle through  $e$  in  $G^*$
- iff*  $e$  is allowed in  $G^*$ .  $\square$

Recall from [14] that a closed graph  $G$  is *elementary* if its allowed edges form a connected subgraph. We shall adopt this definition for open graphs with the additional requirement that the allowed edges must *cover all of the external vertices*.

Based on Corollary 3.1, the following statement was proved in [4].

**Proposition 3.2.** *A connected graph  $G$  is elementary iff  $G^*$  is elementary.*

In general, the subgraph of  $G$  determined by its allowed edges has several connected components, which are called the *elementary components* of  $G$ . An elementary component  $C$  is *external* if it contains external vertices of  $G$ , otherwise  $C$  is *internal*. Notice that an elementary component can be as small as a single external vertex of  $G$ . Such a component is called *degenerate*, and it is the only exception from the general rule that elementary components are elementary graphs. A degenerate external component is the external endpoint of a forbidden external edge. A *mandatory elementary component* is a single mandatory edge  $e \in E(G)$  with a loop around one or both of its endpoints, depending on whether  $e$  is external or

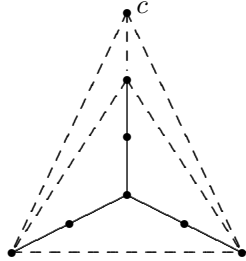


Figure 1: Marginal edges that are forbidden in  $G^*$ .

internal. Note that an edge connecting two external vertices is not mandatory in  $G$ , therefore it is not a mandatory elementary component either.

Observe that if  $v$  is a non-mandatory external vertex and the collector vertex  $c$  is present in  $G^*$ , then the edge  $(v, c)$  cannot be forbidden in  $G^*$ . For, if  $M$  is a perfect internal matching of  $G$  not covering  $v$ , then it is always possible to extend  $M$  to a perfect matching of  $G^*$  by adding the edge  $(v, c)$  first. Consequently, if  $G$  is elementary, then only those marginal edges can become forbidden in  $G^*$  that are different from the collector ones. (An elementary graph  $G$  contains a mandatory external vertex iff  $G$  consists of a single edge with a number of loops attached to one of its endpoints, in which case the collector vertex is not present in  $G^*$ .) Fig. 1 shows a simple example where all these edges are indeed forbidden.

If  $G$  is not elementary, then several of its external elementary components may be amalgamated in  $G^*$ . The internal elementary components of  $G$ , however, will remain intact in  $G^*$ , as every forbidden edge of  $G$  is still forbidden in  $G^*$ . The mandatory external elementary components of  $G$ , too, will remain mandatory in  $G^*$ . We claim that the union of all non-mandatory external elementary components of  $G$ , together with the collector vertex if that is present, forms one elementary component in  $G^*$ , called the *amalgamated* elementary component. Indeed, as we have already seen, every collector edge incident with a non-mandatory external vertex is allowed in  $G^*$ . Similarly, if  $e$  is an edge in  $G^*$  connecting two external vertices of  $G$  belonging to different non-mandatory elementary components, then it is always possible to find a perfect internal matching  $M$  of  $G$  by which the two endpoints of  $e$  are not covered. Then  $M$  can be extended to a perfect matching  $M^*$  of  $G^*$  by putting in the edge  $e$  first, so ensuring that  $e$  becomes allowed in  $G^*$ .

The observations of the previous paragraph are summarized in Theorem 3.1 below, which provides a characterization of the elementary decomposition of  $G^*$ .

**Theorem 3.1.** *The set of elementary components of  $G^*$  consists of:*

- (i) *the internal elementary components of  $G$ ;*
- (ii) *the mandatory external elementary components of  $G$ ;*
- (iii) *the amalgamated elementary component, which is the union of all non-mandatory external elementary components and the collector vertex.*

The closure operation provides a hint toward a new interpretation of our framework dealing with open graphs and perfect/maximum internal matchings. The key observation is that, in our arguments relating perfect internal matchings of  $G$  to perfect matchings of  $G^*$  and vice versa, we did not essentially use the fact that the external vertices have degree 1. The idea works for *any* set of vertices designated as external in  $G$ . The concept arising from this remark is that of a perfect (maximum) matching with a “specified potential defect” explained below.

Let  $G$  be a graph, and fix  $S \subseteq V(G)$  arbitrarily. A perfect (maximum)  $S$ -matching of  $G$  is a matching  $M$  that covers all (respectively, a maximum number of) vertices in  $S$ . Vertices in  $V(G) - S$  need not, although they may be covered by  $M$ . The set  $\text{Spd}(G) = V(G) - S$  is the specified potential defect of such matchings, which takes over the role of  $\text{Ext}(G)$  in this setting. Although this generalization appears to be substantial for the first sight, a closer look at the definition reveals that it is merely a technical matter. Attach, to each vertex  $v \in \text{Spd}(G)$ , a handle consisting of two adjacent edges leading to a new external vertex  $\hat{v}$ . Furthermore, “protect” the vertices in  $V(G)$  with degree one by attaching a loop around them. Let  $\hat{G}$  denote the resulting graph. Then the restriction of every perfect (maximum) internal matching  $\hat{M}$  of  $\hat{G}$  to  $V(G)$  is a perfect (maximum)  $S$ -matching  $M$  of  $G$  that covers  $v$  iff  $\hat{M}$  covers  $\hat{v}$ . Moreover, the connection  $\hat{M} \mapsto M$  is a one-to-one correspondence. Consequently, in the study of  $S$ -matchings we can always assume, without essential loss of generality, that the vertices belonging to the specified potential defect have degree 1. In this way all substantial results on internal matchings can be rephrased as results on matchings with a specified potential defect in a straightforward way.

The following sections will show that obtaining results on open graphs with perfect/maximum internal matchings from corresponding classical results on closed graphs is by no means a matter of trivial rephrasing, although the results themselves in most cases come as appropriate rewordings of the original statements.

## 4 Tutte’s Theorem and Berge’s Formula

First we restate and prove Tutte’s well-known theorem [15] in terms of perfect internal matchings. Let  $X \subseteq \text{Int}(G)$  be arbitrary, and consider the (connected) components of  $G - X$ . Component  $K$  is called external or internal depending on whether or not  $K$  contains external vertices. An *odd internal* component is an internal one consisting of an odd number of vertices. The number of such components is denoted by  $c_o^{\text{in}}(G, X)$ . If  $M$  is a perfect internal matching of  $G$ , then by the term “vertex  $x \in X$  is *taken* by component  $K$ ” – or, equivalently, “ $K$  takes  $x$ ” – with respect to  $M$  we mean that  $x$  is connected to some vertex in  $K$  by an  $M$ -positive edge.

**Theorem 4.1** (Tutte’s Theorem). *A graph  $G$  has a perfect internal matching iff  $c_o^{\text{in}}(G, X) \leq |X|$ , for all  $X \subseteq \text{Int}(G)$ .*

*Proof.* The “only if” part of the proof is the well-known counting argument: if  $G$

has a perfect internal matching  $M$ , then every odd internal component of  $G - X$  must take a vertex from  $X$  with respect to  $M$ , so that  $c_o^{\text{in}}(G, X) \leq |X|$ . To see the “if” part, consider the closure  $G^*$  of  $G$ , and prove that  $G^*$  has a perfect matching whenever  $c_o^{\text{in}}(G, Y) \leq |Y|$  holds for all  $Y \subseteq \text{Int}(G)$ . Then, by Proposition 3.1,  $G$  will have a perfect internal matching. Using Tutte’s original theorem, it is sufficient to show that  $c_o(G^*, X) \leq |X|$  holds in  $G^*$  for all  $X \subseteq V(G^*)$ , where  $c_o(G^*, X)$  is the number of odd components in  $G^* - X$ .

To avoid unnecessary complications caused by the collector vertex being present in  $G^*$  we can assume, without loss of generality, that  $|V(G)|$  is even. If  $|V(G)|$  were odd, then we would rather “duplicate” an arbitrary external edge  $e$  of  $G$  – that is, introduce a new external vertex with an incident edge adjacent to  $e$  – than bother with the collector vertex when taking the closure. This slight modification is equivalent to introducing an extra edge from the collector vertex to the internal endpoint of one external edge, which preserves the number  $c_o(G^*, X)$  as well as the correspondence between the perfect internal matchings of  $G$  and the perfect matchings of  $G^*$  explained in Proposition 3.1.

Let  $X = Y \cup \{x_1, x_2, \dots, x_k\} \subseteq V(G^*)$  be arbitrary such that  $Y \subseteq \text{Int}(G)$  and  $x_i \in \text{Ext}(G)$ ,  $1 \leq i \leq k$  for some  $k \geq 0$ . We say that a component  $K$  in  $G^* - X$  is owned by  $x_i$  if  $x_i$  is connected to an internal vertex of  $K$ . Component  $K$  is called the joint external component, denoted  $J_X$ , if  $K$  contains an external vertex of  $G$ . Clearly,  $J_X$  is unique, provided that  $k < |\text{Ext}(G)|$ . Observe that each odd component  $K$  of  $G^* - X$  falls in exactly one of the following three pairwise disjoint groups.

- Group  $g_1$ : the odd internal components of  $G - Y$ ;
- Group  $g_2$ : the components owned by the vertices  $x_1, \dots, x_k$ ;
- Group  $g_3$ : the component  $J_X$  by itself, if it exists and is not in group  $g_2$ .

Obviously,  $|g_3| \leq 1$ ,  $|g_2| \leq k$ , and by assumption,  $|g_1| \leq |Y|$ . Thus,

$$c_o(G^*, X) \leq |Y| + k + 1 = |X| + 1.$$

It remains to show that  $c_o(G^*, X) = |X| + 1$  is impossible. This follows from the fact that the parity of  $c_o(G^*, X)$  is the same as that of  $|X|$ . Indeed, on the one hand, the parity of  $|V(G^*)|$  is even. On the other hand,  $|V(G^*)| = |X| + |V(G^* - X)|$ . Concerning  $|V(G^* - X)|$ , an odd (even) number of odd components in  $G^* - X$  contain an odd (even) number of vertices altogether, and any number of even components contribute an even number of vertices to the count. Thus, in order for  $|X|$  and  $|V(G^* - X)|$  to have the same parity it is necessary that  $|X|$  and  $c_o(G^*, X)$  have the same parity, too. This concludes the proof of Theorem 4.1.  $\square$

We are going to use Tutte’s Theorem to derive Berge’s Formula [9] on the deficiency of graphs in our framework. Recall that the *deficiency* of a closed graph  $G$ , denoted  $\text{def}(G)$ , is the number of vertices left uncovered by any maximum matching of  $G$ . Then, according to Berge’s Formula:

$$\text{def}(G) = \max\{c_o(G, X) - |X| \mid X \subseteq V(G)\},$$



where  $c_o(G, X)$  is the number of odd components in  $G - X$ .

For any graph  $G$ , let  $\text{idef}(G)$  denote the *internal deficiency* of  $G$ , that is, the number of internal vertices left uncovered by any maximum internal matching of  $G$ .

**Theorem 4.2** (The Berge Formula). *For any graph  $G$ ,*

$$\text{idef}(G) = \max\{c_o^{\text{in}}(G, X) - |X| \mid X \subseteq \text{Int}(G)\}.$$

*Proof.* If  $G$  is closed, then the statement is equivalent to Berge's original formula. Therefore we can assume that  $G$  is open. We shall follow the idea outlined in [14, Exercise 3.1.16]. Letting

$$\delta'(G) = \max\{c_o^{\text{in}}(G, X) - |X| \mid X \subseteq \text{Int}(G)\},$$

the inequality  $\delta'(G) \leq \text{idef}(G)$  is easily obtained by the standard counting argument seen already in the "only if" part of the proof of Tutte's Theorem. The argument is as follows. If  $M$  is any maximum internal matching, then at most  $|X|$  odd internal components of  $G - X$  can take a vertex from  $X$  with respect to  $M$ . It is therefore inevitable that at least  $c_o^{\text{in}}(G, X) - |X|$  internal vertices of  $G$  remain uncovered by  $M$ .

Now we turn to proving the inequality  $\text{idef}(G) \leq \delta'(G)$ . If  $\delta'(G) = 0$ , then the inequality follows from Theorem 4.1. Assuming that  $\delta'(G) \geq 1$ , construct a new graph  $G'$  from  $G$  by adjoining a set  $H$  of  $\delta'(G)$  new vertices to  $G$ , joining each of these vertices to each internal vertex of  $G$  and also to each other. Furthermore, attach a loop around each vertex in  $H$  to ensure that these vertices become internal in  $G'$ . It is sufficient to prove that  $G'$  has a perfect internal matching. Indeed, if  $M'$  is a perfect internal matching of  $G'$ , then leaving out those edges of  $M'$  which are incident with vertices in  $H$  results in a matching  $M$  of  $G$  that covers at least  $|\text{Int}(G)| - \delta'(G)$  vertices, showing that  $\text{idef}(G) \leq \delta'(G)$ .

We use Theorem 4.1 to show that  $G'$  has a perfect internal matching. Let  $X \subseteq \text{Int}(G')$  be arbitrary, and concentrate on the set of components in  $G' - X$ . If  $X = \emptyset$ , then this set consists of a single external component. (Remember that  $G$  is open, and  $H \neq \emptyset$ .) Thus,  $c_o^{\text{in}}(G', X) = 0$ . If  $|X| \geq 1$  and  $H \not\subseteq X$ , then  $G' - X$  has at most one internal component, so that  $c_o^{\text{in}}(G', X) \leq 1 \leq |X|$ . If, however,  $X = H \cup Y$ , then the components of  $G' - X$  coincide with those of  $G - Y$ . Thus,

$$c_o^{\text{in}}(G', X) = c_o^{\text{in}}(G, Y) \leq \delta'(G) + |Y| = |H| + |Y| = |X|.$$

The statement now follows from Tutte's Theorem. □

Another fundamental theorem in matching theory is the Gallai-Edmonds Structure Theorem ([12],[13]). The main idea of this theorem is to decompose a closed graph  $G$  into three sets of vertices as follows.

- $D(G)$ : vertices not covered by at least one maximum matching of  $G$ ;
- $A(G)$ : vertices in  $V(G) - D(G)$  adjacent to at least one vertex in  $D(G)$ ;

- $C(G) = V(G) - A(G) - D(G)$ .

The five statements of the theorem are listed below. To explain statements (a) and (d), a closed graph  $G$  is called *factor-critical* if  $G - v$  has a perfect matching for every  $v \in V(G)$ . In this case, a *near-perfect matching* of  $G$  is one that covers all vertices but one. Clearly, every factor-critical graph is connected and has an odd number of vertices.

- (a) The components of the subgraph induced by  $D(G)$  are factor-critical.
- (b) The subgraph induced by  $C(G)$  has a perfect matching.
- (c) The bipartite graph obtained from  $G$  by deleting the vertices of  $C(G)$  and the edges spanned by  $A(G)$  and by contracting each component of  $D(G)$  to a single vertex has positive surplus (as viewed from  $A(G)$ ).
- (d) If  $M$  is any maximum matching of  $G$ , it contains a near-perfect matching of each component of (the graph induced by)  $D(G)$ , a perfect matching of  $C(G)$ , and matches all vertices of  $A(G)$  with vertices in distinct components of  $D(G)$ .
- (e)  $\text{def}(G) = c(D(G)) - |A(G)|$ , where  $c(D(G))$  denotes the number of components in  $G[D(G)]$ .

The counterpart of the Gallai-Edmonds Structure Theorem for maximum internal matchings was proved in [3]. Not surprisingly, the difference between the statement of the original theorem and that of its counterpart is of a rewording nature, which can be summarized as follows:

- the set  $D(G)$ , as well as  $A(G)$ , is a subset of  $\text{Int}(G)$ ;
- the subgraph induced by  $C(G)$ , which will contain all the external vertices, has a perfect internal matching;
- in general, the words “perfect matching” and “maximum matching” are replaced by “perfect internal matching” and “maximum internal matching”, respectively;
- in statement (e) above,  $\text{def}(G)$  is replaced by  $\text{idef}(G)$ .

In the light of the Gallai-Edmonds Theorem one can easily argue about the deficiency of the graph  $G^*$ . In general, it cannot be expected that  $\text{def}(G^*) = \text{idef}(G)$  holds. Equation of these two deficiencies could only be guaranteed if the decision whether to add a collector vertex to  $V(G)$  or not depended on the number of vertices in  $C(G)$  rather than  $V(G)$ . This follows immediately from statements (d) and (e) above. If the parity of  $|V(G)|$  and  $|C(G)|$  is the same, then  $C(G^*) = (C(G))^*$ , so that  $\text{def}(G^*) = \text{idef}(G)$ . Otherwise  $C(G^*) \neq (C(G))^*$  and  $\text{def}(G^*) = \text{idef}(G) + 1$ , because the closure of  $G$  in this case is implemented through an incorrect closure of  $C(G)$ , and the discrepancy caused by the missing or unjustified collector vertex in the latter closure contributes +1 to the overall deficiency.

## 5 Splitters, barriers, and the canonical partition of elementary graphs

Recall from [14] that a *barrier* of a closed graph  $G$  is a set  $X \subseteq V(G)$  for which the maximum is reached in Berge's formula. We extend this definition to open graphs in the following natural way.

**Definition 5.1.** A *barrier* of graph  $G$  is a set  $X \subseteq \text{Int}(G)$  for which  $|X| = c_0^{\text{in}}(G, X) - \text{idef}(G)$ .

Let  $X$  be a barrier in graph  $G$  (open or closed). It is evident that, for every  $x \in X$ ,  $X - \{x\}$  is a barrier in  $G - x$ . Moreover, the (internal) deficiency of  $G - x$  is one greater than that of  $G$ . Consequently,  $X \subseteq C(G) \cup A(G)$ , according to the Gallai-Edmonds decomposition of  $G$ . As it was proved in [14, Theorem 3.3.15],  $A(G)$  is the intersection of all (inclusionwise) maximal barriers in a closed graph  $G$ . For the reader's convenience we repeat the leading argument of this proof here, without assuming that  $G$  is closed.

**Theorem 5.1.** *The set  $A(G)$  is contained in every maximal barrier of  $G$ .*

*Proof.* Let  $X$  be any maximal barrier. We claim that  $A(G - X) = \emptyset$ . For, if  $A(G - X)$  were not empty, then  $X \cup A(G - X)$  would be a barrier properly containing  $X$ . It is also easy to see that, for any vertex  $u \in A(G)$ ,  $A(G - u) = A(G) - \{u\}$ , and for any  $u \in C(G)$ ,  $A(G - u) \supseteq A(G)$ . (See [14, Lemma 3.2.2] for these statements in closed graphs.) Thus,  $A(G) \subseteq A(G - X) \cup X$ , so that  $A(G - X) = \emptyset$  implies  $A(G) \subseteq X$ .  $\square$

**Corollary 5.1.** *For every maximal barrier  $X$  of  $G$ ,  $X = A(G) \cup Y$ , where  $Y$  is a maximal barrier of  $C(G)$ .*

*Proof.* Evident by Theorem 5.1, since  $A(G)$  separates  $C(G)$  from  $D(G)$ .  $\square$

Our concern in this paper is with maximal barriers of graphs. Therefore, on the basis of Corollary 5.1, we can restrict our attention to graphs having a perfect internal matching. Let  $G$  be a graph (open or closed) having a perfect internal matching, fixed for the rest of this paper. For two internal vertices  $u$  and  $v$  of  $G$ , we say that  $u$  and  $v$  *attract* (*repel*) each other if an extra edge  $e = (u, v)$  becomes allowed (respectively, forbidden) in the graph  $G + e$ . The binary relation of two vertices repelling each other is denoted by  $\sim$ . The following simple statement was proved in [4].

**Lemma 5.1.** *Two internal vertices  $u$  and  $v$  of  $G$  attract each other iff  $u$  and  $v$  can be connected by a positive alternating path or fork with respect to every perfect internal matching of  $G$ .*

A vertex  $v \in \text{Int}(G)$  is called *accessible* with respect to a perfect internal matching  $M$  if there exists a positive external  $M$ -alternating path leading to  $v$ . It was proved in [4] that a vertex  $v$  is accessible with respect to some perfect internal

matching of  $G$  iff  $v$  is accessible with respect to all perfect internal matchings of  $G$ . It is therefore meaningful to say that vertex  $v$  is accessible in  $G$  without specifying the matching  $M$ . Vertex  $v$  is *inaccessible* if it is not accessible. An edge  $e \in E(G)$  is called *viable* if at least one endpoint of  $e$  is accessible. Otherwise  $e$  is said to be *impervious*. See also [11] for an equivalent definition of impervious edges.

**Definition 5.2.** A set  $X \subseteq \text{Int}(G)$  is a *splitter* if every two vertices of  $X$  repel each other in  $G$ . Splitter  $X$  is *inaccessible* if all of its vertices are such.

The concept maximal splitter (maximal inaccessible splitter) is meant inclusion-wise. Notice that, in this way, a maximal inaccessible splitter is not necessarily a maximal splitter.

For any set  $X \subseteq \text{Int}(G)$ , let  $G_X$  be the graph obtained from  $G$  by connecting, with an extra edge, each vertex in  $X$  with all internal vertices of  $G$ , provided that this edge does not already exist in  $G$ . If  $G = G_X$ , then we say that  $G$  is  *$X$ -complete*.

**Lemma 5.2.** For every  $X \subseteq \text{Int}(G)$ ,  $X$  is a splitter in  $G$  iff  $X$  is a splitter in  $G_X$ .

*Proof.* Let  $G_e$  be the graph obtained from  $G$  by adding just one edge  $e$  towards constructing  $G_X$ . In order to prove the lemma it is sufficient to show that if  $X$  is a splitter in  $G$ , then it is one in  $G_e$  as well. Assume, to the contrary, that  $X$  is a splitter in  $G$ , yet, two vertices  $x, y \in X$  attract each other in  $G_e$ . Let  $M$  be any perfect internal matching of  $G_e$  that is also a perfect internal matching of  $G$ , i.e., one by which the edge  $e$  is negative. By Lemma 5.1,  $x$  and  $y$  can be connected by a positive  $M$ -alternating path or fork  $\beta$ . Leaving out the edge  $e$ ,  $\beta$  splits into several subpaths. Since one endpoint of  $e$  is in  $X$ , it is inevitable that one of these subpaths becomes a positive  $M$ -alternating path connecting two vertices in  $X$ , or two of them constitute a positive  $M$ -alternating fork connecting two such vertices. Either way, this contradicts  $X$  being a splitter in  $G$ .  $\square$

We claim that any two distinct vertices  $u, v$  in a barrier  $X$  of  $G$  repel each other. Indeed, assume that the edge  $e = (u, v)$  is part of some perfect internal matching  $M$  of  $G + e$ . Since  $|X| = c_o^{\text{in}}(G, X) = c_o^{\text{in}}(G_e, X)$ , at least two odd components of  $G_e - X$  could not take a vertex from  $X$  with respect to  $M$ ; a contradiction. Thus, every barrier is a splitter. The converse of this statement is not true, however, as shown by the graph of Fig. 2. It is also clear by Corollary 3.1 that a set  $X \subseteq \text{Int}(G)$  is a splitter in  $G$  iff  $X$  is a splitter in  $G^*$ . As we shall see, splitters are in close relationship with extreme sets of vertices. Recall from [14] that a set of vertices  $X$  in a closed graph  $G$  is *extreme* if  $\text{def}(G - X) = \text{def}(G) + |X|$ .

**Proposition 5.1.** A set  $X \subseteq V(G)$  of a closed graph  $G$  having a perfect matching is a splitter iff  $X$  is extreme.

*Proof.* By Lemma 5.2 we can assume, without loss of generality, that  $G$  is  $X$ -complete. If  $X$  is extreme, then  $G$  cannot have a perfect matching containing an edge in  $X \times X$ . Indeed, if  $M$  was such a matching, then the restriction of  $M$  to  $G - X$  would cover more than  $|V(G)| - |X|$  vertices. Thus,  $X$  is a splitter.

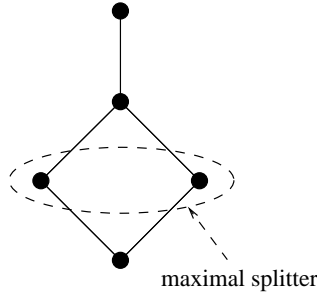


Figure 2: A maximal splitter that is not a barrier

Now let  $X$  be a splitter. Obviously,  $\text{def}(G - X) \leq |X|$ . Assume, by way of contradiction, that  $\text{def}(G - X) = |X| - k$  for some  $k > 0$ , and let  $M$  be any maximum matching of  $G - X$ . Couple up each vertex in  $G - X$  not covered by  $M$  with an arbitrary vertex in  $X$ , and extend  $M$  by these edges to form a matching  $\bar{M}$  in  $G$ . (Remember that  $G$  is  $X$ -complete.) Observe that  $k$  must be odd, otherwise  $\bar{M}$  could further be extended to a perfect matching of  $G$  containing  $k/2$  edges from  $X \times X$ , contradicting the fact that  $X$  is a splitter. On the other hand,  $k$  cannot be odd, for  $|V(G)|$  is even.  $\square$

**Proposition 5.2.** *A set  $X \subseteq \text{Int}(G)$  is a barrier in  $G^*$  iff  $|X| = c_o^{\text{in}}(G, X)$  or  $|X| = c_o^{\text{in}}(G, X) + 1$ .*

*Proof.* Using the trick of duplicating an external edge  $e$  of  $G$  when  $|V(G)|$  is odd, as seen under the proof of Tutte’s Theorem, we can assume, without loss of generality, that  $|V(G)|$  is even. The reason is that  $X$  is a barrier in  $G^*$  iff  $X$  is one in the closure of the augmented graph  $G_e$ . Consider the components in  $G - X$  and those in  $G^* - X$ . The only difference between these two groups is that the external components in  $G - X$  are joined to form one component  $J_X$  in  $G^* - X$ . Assume that  $J_X$  is even. Then, clearly,  $X$  is a barrier in  $G^*$  iff  $|X| = c_o^{\text{in}}(G, X)$ . On the other hand, if  $J_X$  is odd, then  $X$  is a barrier in  $G^*$  iff  $|X| = c_o^{\text{in}}(G, X) + 1$ . Moreover, if  $|X| = c_o^{\text{in}}(G, X)$ , then  $J_X$  must be even, and if  $|X| = c_o^{\text{in}}(G, X) + 1$ , then  $J_X$  must be odd.  $\square$

**Corollary 5.2.** *Every barrier of  $G$  is also a barrier in  $G^*$ .*

**Theorem 5.2.** *Every maximal splitter of  $G$  is a barrier in  $G^*$ .*

*Proof.* We can again assume, without loss of generality, that  $|V(G)|$  is even. Let  $Y$  be a maximal splitter of  $G$ . By Proposition 5.2 it is enough to prove that  $|Y| = c_o^{\text{in}}(G, Y)$  or  $|Y| = c_o^{\text{in}}(G, Y) + 1$ . Since  $Y$  is also a splitter in  $G^*$ , it is extreme in that graph according to Proposition 5.1. Thus, by [14, Lemma 3.3.8],  $Y$  can be extended to a maximal barrier  $X$  of  $G^*$ . Clearly,  $X - Y = \{x_1, \dots, x_k\} \subseteq \text{Ext}(G)$ , because  $Y$  is maximal. Concentrate on the odd components of  $G^* - X$ , and observe

that the situation is analogous to the one analyzed in the proof of Tutte's Theorem. Each of these components falls in one of the three groups specified there. Note that the number of odd components in  $G^* - X$  can reach the barrier level  $|X| = |Y| + k$  only if the size of group  $g_1$  is at least  $|Y| - 1$ , that is,  $c_o^{\text{in}}(G, Y) \geq |Y| - 1$ . On the other hand,  $c_o^{\text{in}}(G, Y) \leq |Y|$  is guaranteed by Tutte's Theorem. The proof is now complete.  $\square$

**Proposition 5.3.** *No barrier exists in an elementary soliton graph  $G$ , other than the empty set.*

*Proof.* The empty set is trivially a barrier in all soliton graphs. By way of contradiction, assume that  $X \subseteq V(G)$  is a non-empty barrier. Since  $c_o^{\text{in}}(G, X) = |X|$ , each vertex in  $X$  must be taken by an appropriate odd internal component of  $G - X$  with respect to any perfect internal matching  $M$  of  $G$ . Consequently, all edges connecting  $X$  to other components of  $G - X$  are forbidden. This implies that either the allowed edges of  $G$  do not form a connected subgraph, or, when they do, none of the external vertices of  $G$  are covered by them; a contradiction.  $\square$

**Corollary 5.3.** *For every maximal splitter  $X$  of an elementary soliton graph  $G$  having at least one internal vertex,  $c_o^{\text{in}}(G, X) = |X| - 1$ .*

*Proof.* By Proposition 5.2 and Theorem 5.2 we know that either  $c_o^{\text{in}}(G, X) = |X| - 1$  or  $c_o^{\text{in}}(G, X) = |X|$ . The latter equation is ruled out, however, due to Proposition 5.3.  $\square$

The proof of the following statement uses the exact same argument that was introduced under Proposition 5.3.

**Proposition 5.4.** *Every barrier  $X$  of  $G$  is an inaccessible splitter.*

*Proof.* As it has been noticed earlier, every barrier is a splitter. It is therefore sufficient to prove that every vertex of  $X$  is inaccessible. Let  $M$  be an arbitrary perfect internal matching of  $G$ . Since  $X$  is a barrier, every vertex  $v \in X$  is taken by some odd internal component of  $G - X$  with respect to  $M$ . Consequently, any alternating path starting out from  $v$  on an  $M$ -positive edge is locked forever inside the subgraph of  $G$  determined by the odd internal components of  $G - X$  plus  $X$ . In other words,  $v$  is inaccessible.  $\square$

It is well-known (cf. [14]) that the collection of maximal barriers in a closed elementary graph  $G$  forms a partition of  $V(G)$ , called the *canonical partition* of  $G$ . Canonical partition is established in open graphs in the same way, using maximal splitters rather than barriers.

**Theorem 5.3.** *The collection of maximal splitters in an elementary graph  $G$  forms a partition of  $\text{Int}(G)$ .*

*Proof.* Let  $\mathcal{P} = \{X_1, \dots, X_n\}$  be the collection of maximal splitters in  $G$ . By Theorem 5.2, each  $X_i$  ( $1 \leq i \leq n$ ) can be extended to a maximal barrier  $X_i^*$  of  $G^*$ . Since  $X_i^* \setminus X_i$  may only contain external vertices of  $G$  for every  $1 \leq i \leq n$ , it follows that  $\mathcal{P}$  is the restriction of the canonical partition of  $G^*$  to  $\text{Int}(G)$ . Thus,  $\mathcal{P}$  is a partition itself.  $\square$

Theorem 5.3 above states that the relation  $\sim$  of two internal vertices repelling each other is an equivalence of  $\text{Int}(G)$ , provided that  $G$  is elementary. This fact was first observed in [1]. If  $G$  is not elementary, then  $\sim$  fails to be transitive in general. It is an important question, however, if the restriction of  $\sim$  to a concrete non-degenerate elementary component  $C$  of  $G$ , denoted  $\sim|_C$ , is still an equivalence, and if so, does it coincide with canonical equivalence in  $C$  alone? As it was pointed out in [4],  $\sim|_C$  can be specified as canonical equivalence in the elementary graph  $C_h$ , which is obtained from  $C$  by adding the so called “hidden edges”. A hidden edge  $(u, v)$  in  $C$  between two distinct internal vertices arises from a negative alternating path or fork  $\alpha$  connecting  $u$  and  $v$  with respect to any perfect internal matching  $M$  of  $G$ , such that no vertex of  $\alpha$ , other than its two endpoints  $u$  and  $v$ , lies in  $C$ . Following [14], if  $\alpha$  is a path, then it is called a *negative (M-)ear* to  $C$ . Clearly, all hidden edges are forbidden both in  $G$  and  $C_h$ , but their presence affects canonical equivalence in  $C_h$  in such a way that it will eventually coincide with  $\sim|_C$ . See Fig. 3 for two hidden edges, one in elementary component  $C_1$ , and the other in  $C_4$ . Notice that the two vertices in  $C_1$  not connected by the hidden edge fall in the same canonical class in  $C_1$ , but different canonical classes according to  $(C_1)_h$ . This holds for the elementary component  $C_4$  as well. Let us agree that, in the future, by a canonical class of  $C$  we shall in fact mean one of  $C_h$ .

## 6 Finer structure of maximal splitters and barriers

Recall that a closed graph  $G$  is *factor-critical* if  $G - v$  has a perfect matching for every  $v \in V(G)$ . We shall adopt this definition word by word for open graphs, assuming of course that  $v \in \text{Int}(G)$ , and requiring that  $G - v$  has a perfect internal matching. We also require that  $G$  be connected, because, unlike for closed graphs, this property does not come as a consequence. The following simple result is quoted from [5].

**Proposition 6.1.** *A connected open graph  $G$  is factor-critical iff  $G$  has a perfect internal matching and every internal vertex in  $G$  is accessible.*

**Corollary 6.1.** *No barrier exists in factor-critical open graphs, other than the empty set.*

*Proof.* Immediate by Propositions 5.4 and 6.1. Notice that the statement trivially holds for closed graphs as well, even though such graphs do not have a perfect (internal) matching. The reason is that a factor-critical closed graph  $G$  is the

single component of  $D(G)$  by itself, and all barriers lie completely in  $A(G) \cup C(G)$ . (See the Gallai-Edmonds decomposition of graphs.)  $\square$

Using factor-critical graphs, the following characterization of maximal splitters was obtained in [5]. Recall that a component  $K$  is *degenerate* if  $K$  consists of a single external vertex of  $G$ .

**Theorem 6.1.** *For a set  $X$  of internal vertices of a soliton graph  $G$ , the following two statements are equivalent.*

- (i) *The set  $X$  is a maximal splitter.*
- (ii) *Each non-degenerate component of  $G - X$  is factor-critical such that*
  - (iia)  $|X| = c_o^{\text{in}}(G, X) + 1$ , *or*
  - (iib)  $|X| = c_o^{\text{in}}(G, X)$  *with every external component of  $G - X$  being degenerate.*

*Furthermore, condition (iib) holds in (ii) above iff  $X$  is inaccessible.*

The structure of elementary components in a soliton graph  $G$  has been analysed in [4]. To summarize the main results of this analysis, we first need to review some of the key concepts introduced in that paper. The reader can obtain a good understanding of these concepts by following the definitions to come on Fig. 3.

An elementary component of  $G$  is *viable* if it does not contain impervious allowed edges. (Recall that an edge  $e$  is impervious if both endpoints of  $e$  are inaccessible.) In Fig. 3, all elementary components, with the exception of  $C_7$ , are viable. A viable internal elementary component  $C$  is *one-way* if all external alternating paths (with respect to any perfect internal matching  $M$ ) enter  $C$  in vertices belonging to the same canonical class of  $C$ . This unique class, as well as the vertices belonging to this class, are called *principal*. Furthermore, every non-degenerate external elementary component is considered a priori one-way (with no principal canonical class, of course). In Fig. 3, elementary components  $C_1, C_4$ , and  $C_6$  are one-way internal, with their principal vertices encircled. A viable elementary component is *two-way* if it is not one-way. An *impervious* elementary component is one that is not viable.

We say that elementary component  $C'$  is *two-way accessible* from component  $C$  with respect to any (or all) perfect internal matching(s)  $M$ , in notation  $C\rho C'$ , if  $C'$  is covered by a negative ( $M$ -)ear to  $C$ . The ear itself might be closed, meaning that its two endpoints are the same. It is required, though, that if  $C$  is one-way and internal, then the endpoints of this ear *are not* in the principal canonical class of  $C$ . As it was shown in [4], the two-way accessible relationship is matching invariant. In Fig. 3,  $C_2$  is two-way accessible from  $C_1$ ,  $C_3$  from  $C_2$ , and  $C_5$  from  $C_4$ . (But  $C_3$  is not two-way accessible from  $C_1$ , and  $C_2, C_3, C_4, C_5$  are not two-way accessible from  $C_6$ , even though there exists a negative closed ear originating from the principal vertex of  $C_6$  that covers all four of these components.) It was also proved in [4] that the transitive closure of the two-way accessible relationship between elementary components is asymmetric.

A *family* of elementary components in  $G$  is a block of the partition induced by the smallest equivalence relation containing  $\rho$ . A family  $\mathcal{F}$  is called external if it contains an external elementary component, otherwise  $\mathcal{F}$  is internal. Family  $\mathcal{F}$



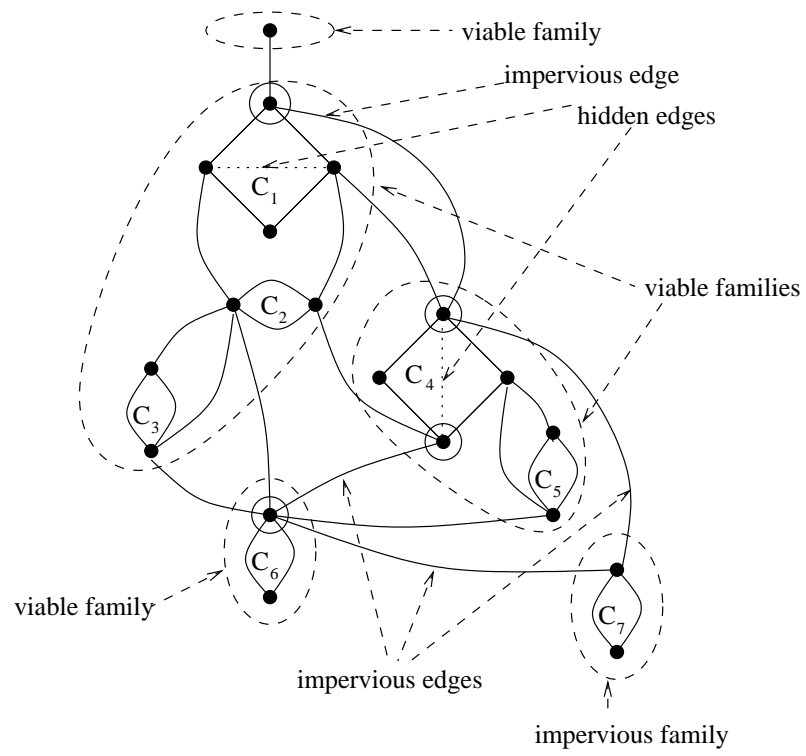


Figure 3: The structure of elementary components in a soliton graph

is *viable* if every elementary component in  $\mathcal{F}$  is such. Otherwise the family  $\mathcal{F}$  is *impervious*. A soliton graph  $G$  is *viable* if all of its families are such. The graph of Fig. 3 has five families, four of which are viable. The only external family is a stand-alone degenerate external elementary component.

The first group of results obtained in [4] on the structure of elementary components of  $G$  can now be stated as follows.

**Theorem 6.2.** *Each viable family of  $G$  contains a unique one-way elementary component, called the root of the family. Each internal vertex in every member of the family, except for the principal vertices of the root, is accessible. The principal vertices themselves are inaccessible, but all other vertices are only accessible through them.*

For two distinct viable families  $\mathcal{F}_1$  and  $\mathcal{F}_2$ ,  $\mathcal{F}_2$  is said to *follow*  $\mathcal{F}_1$ , in notation  $\mathcal{F}_1 \mapsto \mathcal{F}_2$ , if there exists an edge in  $G$  connecting any non-principal vertex in  $\mathcal{F}_1$  with a principal vertex of the root of  $\mathcal{F}_2$ . The reflexive and transitive closure of  $\mapsto$  is denoted by  $\mapsto^*$ . The second group of results in [4] characterizes the edge-connections between members inside one viable family, and those between two different families.

**Theorem 6.3.** *The following three statements hold for the families of any soliton graph  $G$ .*

1. *An edge  $e$  inside a viable family  $\mathcal{F}$  is impervious iff both endpoints of  $e$  are in the principal canonical class of the root. Every forbidden edge  $e$  connecting two different elementary components in  $\mathcal{F}$  is part of a negative ear to some member  $C \in \mathcal{F}$ .*
2. *For every edge  $e$  connecting a viable family  $\mathcal{F}_1$  to any other family (viable or not)  $\mathcal{F}_2$ , at least one endpoint of  $e$  is principal in  $\mathcal{F}_1$  or  $\mathcal{F}_2$ . If the endpoint of  $e$  in  $\mathcal{F}_1$  is not principal, then  $\mathcal{F}_2$  is viable and it follows  $\mathcal{F}_1$ .*
3. *The relation  $\mapsto^*$  is a partial order between viable families, by which the external families are maximal elements. This relation reflects the order in which families are reachable by alternating paths starting from external vertices.*

In the light of Theorem 6.2 and Proposition 6.1 it is immediate that a soliton graph  $G$  is factor-critical iff  $G$  consists of a single non-degenerate external family. Thus, Corollary 6.1 is in fact a generalization of Proposition 5.3. The following theorem is a further generalization along this line.

**Theorem 6.4.** *Every viable soliton graph has a unique maximal barrier, which is the collection of its inaccessible vertices.*

*Proof.* By Theorem 6.2 and Proposition 5.4 it is sufficient to show that the set  $P$  of all principal vertices of a viable soliton graph  $G$  is a barrier. Let  $\mathcal{F}$  be an arbitrary internal family of  $G$  with root  $C$ , and let  $X_C$  be the set of principal vertices in  $C$ . By Theorem 6.3, the principal vertices of the families that follow  $\mathcal{F}$  separate  $\mathcal{F}$  from all the families that are below  $\mathcal{F}$  in the Hasse diagram of the partial order  $\mapsto^*$ . Similarly, the vertices  $X_C$  separate all other families from  $\mathcal{F}$ . Thus, we can

concentrate on the family  $\mathcal{F}$  alone as a closed graph, and prove that  $X_C$  is a barrier in that graph. Doing this for all internal families will then prove that  $P$  is a barrier in  $G$ .

As we have already seen,  $X_C$  is a canonical class of  $C_h$  (remember the extra hidden edges being present in  $C_h$ ), therefore a maximal barrier in that graph. Let  $K$  be an odd component of  $\mathcal{F} - X_C$ , and consider an arbitrary elementary component  $D$  of  $G$  present as a subgraph in  $K - C$ . As  $D$  is a two-way member of family  $\mathcal{F}$ , it can be reached by a cascade of negative ears originating from the root  $C$ . Since all hidden edges are present in  $C_h$ , and the graph  $K - C$  — being essentially a group of interconnected two-way elementary components of  $\mathcal{F}$  — has an even number of vertices, the restriction of  $K$  to  $C$  defines an odd component of  $C_h - X_C$ . Moreover, this correspondence between the odd components of  $\mathcal{F} - X_C$  and those of  $C_h - X_C$  is one-to-one. Consequently, since  $X_C$  is a barrier in  $C_h$ , it must be one in  $\mathcal{F}$  as well.  $\square$

**Corollary 6.2.** *Every maximal inaccessible splitter of  $G$  is a barrier.*

*Proof.* Let  $v(G)$  be the subgraph of  $G$  determined by its viable families. We first prove that each principal vertex  $u$  of  $G$  repels each internal vertex  $w$  lying in  $G - v(G)$ . Let  $M$  be a perfect internal matching of  $G$  and suppose, by way of contradiction, that there exists a positive  $M$ -alternating path  $p$  connecting  $u$  and  $w$ . Furthermore, starting from  $u$  let  $z$  denote the last vertex of  $p$  which belongs to  $v(G)$ . It is clear that the subpath of  $p$  from  $u$  to  $z$  is positive at its both ends. However, by Theorem 6.3,  $z$  is also a principal vertex, consequently Theorem 6.4 implies that  $u \sim z$ , which is a contradiction.

By the previous paragraph, every maximal inaccessible splitter  $X$  of  $G$  is the union of the set  $S$  of principal vertices in  $G$  and a maximal splitter  $Y$  in  $G - v(G)$ . Since  $G - v(G)$  is a closed graph,  $Y$  is a barrier in that graph by Theorem 5.2. Also,  $S$  is a barrier in  $v(G)$ . By Theorem 6.3, all edges connecting  $v(G)$  to  $G - v(G)$  originate from vertices in  $S$ , which implies that  $X = S \cup Y$  is a barrier in  $G$ .  $\square$

**Corollary 6.3.** *A set  $X \subseteq \text{Int}(G)$  is a maximal barrier in  $G$  iff  $X$  is a maximal inaccessible splitter.*

*Proof.* Immediate by Proposition 5.4 and Corollary 6.2.  $\square$

If  $G$  is closed, then Corollary 6.3 says that maximal splitters coincide with maximal barriers in  $G$ . This result follows already from Theorem 5.3, considering that  $G^* = G$  for closed graphs.

On the basis of Corollaries 6.2 and 6.3 we can outline a simple procedure to find one random maximal barrier in a soliton graph  $G$ . The procedure uses a global set variable  $B$ , the contents of which is initially empty.

*Step 1.* Isolate the subgraph  $v(G)$  consisting of the viable families of  $G$ , and add the principal (inaccessible) vertices of  $v(G)$  to  $B$ .

*Step 2.* If  $G = v(G)$ , then terminate. Otherwise, in the remainder graph  $G - v(G)$  – which is now closed – attach an external edge to an arbitrary vertex  $u$  to obtain a soliton graph  $G_u$ . Set  $G := G_u$ , and goto *Step 1*.

Clearly, the maximal barriers of  $G_u$  coincide with the ones of  $G - v(G)$  containing vertex  $u$ . (Note that  $u$  is trivially inaccessible in  $G_u$ .) Therefore the procedure above is capable of finding any maximal barrier of  $G$  by choosing the vertex  $u$  in *Step 2* in an appropriate way. It was proved in [6] that *Step 1* of the above procedure takes linear time in terms of the number of edges in  $v(G)$ , provided that a perfect internal matching  $M$  has previously been found for  $G$ . Also notice that, if  $G$  is closed, then choosing a vertex  $u \in V(G)$  to be part of a maximal barrier is equivalent to turning  $G$  into a soliton graph by attaching an external edge to  $u$  before applying the above procedure. Thus, we have proved the following result.

**Theorem 6.5.** *A random maximal barrier of  $G$  (open or closed) can be found in linear time, provided that a perfect internal matching has previously been constructed for  $G$ .*

We wish to emphasize that the above procedure can only be used to find a *random* maximal barrier of  $G$  in linear time. Finding e.g. a *maximum size* barrier  $X$  is a much more complicated issue, which would have to be addressed in a different way. See [7, 8]. Our contribution in this regard concerns only the implications of adding one particular vertex to  $X$ .

Let  $G$  be a viable soliton graph and  $X$  be its maximal barrier. The question arises how  $X$  can be extended to a maximal splitter  $Y$  of  $G$ . By Theorem 6.1 we know that a proper extension exists iff  $G$  has non-degenerate external families. Then an obvious way to construct  $Y$  is to add an arbitrary maximal splitter of any (one) non-degenerate external family to  $X$ . Observe that this is the only way to achieve the goal, since any two internal vertices belonging to different external families attract each other.

Another interesting issue is to relate the maximal barriers of  $G^*$  to the maximal splitters of  $G$ . Notice that the restriction  $X$  of a maximal barrier  $X^*$  in  $G^*$  to  $\text{Int}(G)$  need not be maximal as a splitter in  $G$ . For example, if  $G$  has a single external family  $\mathcal{F}$  with the root of this family being a mandatory edge, then  $X^*$  might contain the external endpoint  $v$  of this edge (as the only external vertex in  $G$ ). Since  $\mathcal{F}$  is a factor-critical graph,  $v$  must be the only vertex from  $\mathcal{F}$  present in  $X^*$ . Vertex  $v$ , however, could be replaced by any other maximal splitter of  $\mathcal{F}$  in  $X^*$ , so that the result would be a maximal splitter of  $G$  as well as a maximal barrier of  $G^*$ . Clearly, the restriction  $X$  of the original  $X^*$  is a maximal barrier of  $G$  in this case. Another example of this nature manifests itself when all vertices in  $X^*$  belonging to the amalgamated elementary component  $A$  of  $G^*$  are from  $\text{Ext}(G)$ , but  $A$  does contain vertices in  $\text{Int}(G)$ . Despite these examples, we still have the following positive result.

**Theorem 6.6.** *Let  $X$  be a maximal splitter in soliton graph  $G$ . Then  $X$  is either a maximal barrier in  $G$ , or it can be extended in a unique way to a maximal barrier  $X^*$  of  $G^*$ .*

The proof is preceded by a preliminary observation.

**Lemma 6.1.** *Let  $u$  and  $v$  be distinct vertices of a closed graph  $G$  such that  $u \sim v$ . Assume, furthermore, that there exists a perfect matching  $M$  in  $G$  and an  $M$ -alternating path  $p$  connecting  $u$  with  $v$  in such a way that  $p$  is  $M$ -positive at its  $v$  end. Then, for an arbitrary vertex  $z$ ,  $z \sim v$  only if  $z \sim u$ .*

*Proof.* Suppose, on the contrary, that for some vertex  $z \in V(G)$  with  $z \sim v$  there exists a positive  $M$ -alternating path  $p'$  connecting  $z$  and  $u$ . Starting from  $v$ , let  $w$  denote the first vertex of  $p$  which is also on  $p'$ . The prefix of  $p'$  from  $z$  to  $w$ , joined with the section of  $p$  from  $w$  to  $v$  then becomes a path, which cannot be  $M$ -alternating, for the positivity of this path would contradict  $z \sim v$ . But then the section of  $p$  from  $v$  to  $w$ , continued with the suffix of  $p'$  from  $w$  to  $u$  does form a positive  $M$ -alternating path, which contradicts  $v \sim u$ .  $\square$

*Proof. (of Theorem 6.6)* By Theorem 6.1 we can assume that  $X$  contains an accessible vertex  $v$ . Indeed, if this is not the case, then  $X$  is inaccessible, so that  $|X| = c_o^{\text{in}}(G, X)$ , meaning that  $X$  is a barrier in  $G$ . We show that  $X$  can be extended in a unique way to a barrier of  $G^*$ . Clearly,  $X^* - X \subseteq \text{Ext}(G)$ . Consider the graph  $G^*$  as the underlying graph in Lemma 6.1, and notice that for any mandatory external vertex  $u \in \text{Ext}(G)$ , either  $u \not\sim v$ , or  $u$  and  $v$  satisfy the conditions of Lemma 6.1 with a suitable alternating path  $p$  that starts out from  $v$  on a negative marginal edge. In either case, Lemma 6.1 implies that  $u \in X^*$  iff  $u$  is present in all maximal barriers of  $G^*$  containing  $v$ .

Now let  $u \in \text{Ext}(G) \cup \{c\}$  be in the amalgamated elementary component of  $G^*$ . It is again true that there exists an alternating path  $p$  with respect to some perfect matching of  $G^*$  connecting  $u$  and  $v$  in such a way that  $p$  is positive at its  $v$  end. This is trivial if  $v$  is accessible from  $u$  in  $G$ , but even if  $v$  is accessible from some other external vertex  $u'$  in  $G$  through path  $p'$ , this path  $p'$  can be augmented by one or two marginal edges to obtain a suitable path  $p$  in  $G^*$  starting already from  $u$ . If  $u \not\sim v$ , then  $u$  cannot be present in any maximal barrier containing  $v$ . On the other hand, if  $u \sim v$ , then Lemma 6.1 applies and  $u \in X^*$  iff  $u$  is present in all maximal barriers containing  $v$ .  $\square$

## 7 Conclusion

We have proved a counterpart of Tutte's Theorem and Berge's Formula for open graphs with perfect (maximum) internal matchings. We have also provided a comparison between barriers in open and closed graphs, and studied the finer structure of maximal splitters and barriers on the basis of earlier results.

An algorithm has been given to find the maximal barriers of an open or closed graph. This algorithm isolates a random maximal barrier in linear time, provided that a perfect internal matching has previously been found for the graph. Maximal splitters of open graphs have been extended to maximal barriers of their closures, and it was proved that this extension is unique, unless the maximal splitter in hand is already a maximal barrier of the original graph.

## References

- [1] M. Bartha, E. Gombás, On graphs with perfect internal matchings, *Acta Cybernetica* **12** (1995), 111-124.
- [2] M. Bartha, E. Gombás, A structure theorem for maximum internal matchings in graphs, *Information Processing Letters* **40** (1991), 289-294.
- [3] M. Bartha, The Gallai-Edmonds algebra of graphs, *Congressus Numerantium* **123** (1997), 205–219.
- [4] M. Bartha, M. Krész, Structuring the elementary components of graphs having a perfect internal matching, *Theoretical Computer Science* **299** (2003), 179–210.
- [5] M. Bartha, M. Krész, Tutte type theorems in graphs having a perfect internal matching, *Information Processing Letters* **91** (2004), 277–284.
- [6] M. Bartha, M. Krész, Isolating the families of soliton graphs, *Pure Mathematics and Applications* **13** (2002), 49–62.
- [7] D. Bauer, H. J. Broersma, A. Morgana, E. Schmeichel, Tutte sets in graphs I: Maximal tutte sets and D-graphs, *Journal of Graph Theory* **55** (2007), 343–358.
- [8] D. Bauer, H.J. Broersma, N. Kahl, A. Morgana, E. Schmeichel and T. Surowiec, Tutte sets in graphs II: The complexity of finding maximum Tutte sets, *Discrete Applied Mathematics* **155** (2007), 1336-1343
- [9] C. Berge, Sur le couplage maximum d'un graphe, *C. R. Acad. Sci. Paris Sér. I Math.* **247** (1958), 258–259.
- [10] F. L. Carter, Conformational switching at the molecular level, in *Molecular Electronic Devices* (F. L. Carter ed.), Marcel Dekker, Inc., New York, 1982, pp. 51–72.
- [11] J. Dassow, H. Jürgensen, Soliton automata, *J. Comput. System Sci.* **40** (1990), 154-181.
- [12] J. Edmonds, Paths, trees and flowers, *Canad. J. Math.* **17** (1965), 449–467.
- [13] T. Gallai, Maximale Systeme unabhängiger Kanten, *Magyar Tud. Akad. Mat. Kutató Int. Közl.* **9** (1964), 401–413.
- [14] L. Lovász, M. D. Plummer, *Matching Theory*, North Holland, Amsterdam, 1986.
- [15] W. T. Tutte, The factorization of linear graphs, *J. London Math. Soc.* **22** (1947), 107–111.

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