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Regular tree languages and quasi orders

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Abstract

Regular languages were characterized as sets closed with respect to monotone well-quasi orders. A similar result is proved here for tree languages. Moreover, families of quasi orders that correspond to positive varieties of tree languages and varieties of finite ordered algebras are characterized.

1 Introduction

Regular languages are characterized by the well-known Myhill–Nerode theorem as those that can be saturated by a congruence, or a right congruence, of finite index defined on the free semigroup over the same alphabet over which the language is defined. A generalization of this result, proved by Ehrenfeucht, Haussler and Rozenberg in [3], characterizes regular languages as closed sets with respect to monotone well-quasi orders. A result analogous to Myhill-Nerode's theorem exists for tree languages, whereas we are going to prove here a characterization of regular tree languages similar to the generalized Myhill–Nerode's theorem from [3].

On the other hand, variety theory establishes correspondences between families of languages, algebras, semigroups and relations. The elementary result of this type is Eilenberg's Variety theorem [4] which was motivated by characterizations of several families of string languages by syntactic monoids or semigroups (see [4, 10]), such as Schützenberger's theorem [12] connecting star-free languages and aperiodic monoids. Eilenberg's theorem has been extended in various directions. We are going to mention here only those that are of the greatest interest for this work. Thérien [16] extended the Eilenberg's correspondence to varieties of congruences on free monoids. Concerning trees and algebras, similar correspondences were established by Steinby [13, 14, 15], Almeida [1], Ésik [5], Ésik and Weil [6]. On the other hand, a correspondence between positive varieties of string languages and varieties of ordered semigroups was established by Pin in [11], and similar results were proved for trees by Ésik [5], and Petković and Salehi in [9]. Motivated by this, and a characterization of regular tree languages established in the first part of the paper, we involve in the correspondence suitable families of quasi orders on term algebras.

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The paper consists of three parts. In Section 2 concepts are introduced and preliminary results given. In Section 3 regular tree languages are characterized by well-quasi orders. In Section 4 varieties of quasi orders are defined and a correspondence between positive varieties of tree languages, varieties of ordered algebras and varieties of quasi orders is established.

2 Preliminaries

A finite set of function symbols is called a *ranked alphabet*. The ranked alphabet Σ will be fixed throughout the paper, and the set of m-ary function symbols from Σ is denoted by Σ_m ($m \geq 0$). A Σ-*algebra* is a structure $\mathcal{A} = (A, \Sigma)$ where A is a set and operations of Σ are interpreted in A, i.e., any $c \in \Sigma_0$ is interpreted by an element $c^{\mathcal{A}} \in A$ and any $f \in \Sigma_m$ ($m > 0$) is interpreted by an m-ary function $f^{\mathcal{A}} : A^m \to A$. Congruences, morphisms, subalgebras, direct products, etc., are defined as usual for algebras (see e.g. [2, 15]).

For a ranked alphabet Σ and a *leaf alphabet* X, the set of ΣX-*trees* $\mathrm{T}_\Sigma(X)$ is the smallest set satisfying

(1) $\Sigma_0 \cup X \subseteq \mathrm{T}_\Sigma(X)$, and

(2) $f(t_1, \ldots, t_m) \in \mathrm{T}_\Sigma(X)$ for all $m > 0$, $f \in \Sigma_m$, $t_1, \ldots, t_m \in \mathrm{T}_\Sigma(X)$.

The ΣX-*term algebra* $\mathcal{T}_\Sigma(X) = (\mathrm{T}_\Sigma(X), \Sigma)$ is determined by

(1) $c^{\mathcal{T}_\Sigma(X)} = c$ for $c \in \Sigma_0$,

(2) $f^{\mathcal{T}_\Sigma(X)}(t_1, \ldots, t_m) = f(t_1, \ldots, t_m)$ for all $m > 0$, $f \in \Sigma_m$ and $t_1, \ldots, t_m \in \mathrm{T}_\Sigma(X)$.

A ΣX-*tree language* is any subset of the ΣX-term algebra. An algebra $\mathcal{A} = (A, \Sigma)$ *recognizes* a tree language $T \subseteq \mathrm{T}_\Sigma(X)$ if there is a morphism $\phi : \mathcal{T}_\Sigma(X) \to \mathcal{A}$ and a subset $F \subseteq A$ such that $T = F\phi^{-1}$. In the case a tree language can be recognized by a finite algebra, it is *regular* or *recognizable*. It is known that a tree language is regular if and only if it is saturated by a congruence of finite index.

Let ξ be a symbol which does not appear in any other alphabet considered here. The set of ΣX-*contexts*, denoted by $\mathrm{C}_\Sigma(X)$, consists of the $\Sigma(X \cup \{\xi\})$-trees in which ξ appears exactly once. For $P, Q \in \mathrm{C}_\Sigma(X)$ and $t \in \mathrm{T}_\Sigma(X)$ the context PQ, the composition of P and Q, is obtained by replacing the special leaf ξ in P with Q, and the term $P(t)$ results from P by replacing ξ with t. Note that $\mathrm{C}_\Sigma(X)$ is a monoid with the composition operation and that $(PQ)(t) = P(Q(t))$ holds for all $P, Q \in \mathrm{C}_\Sigma(X)$, $t \in \mathrm{T}_\Sigma(X)$.

For an algebra $\mathcal{A} = (A, \Sigma)$, an m-ary function symbol $f \in \Sigma_m$ ($m > 0$) and elements $a_1, \ldots, a_m \in A$, the term $f^{\mathcal{A}}(a_1, \ldots, \xi, \ldots, a_m)$ where the new symbol ξ sits in the i-th position, for some $i \leq m$, determines a unary function $A \to A$ defined by $a \mapsto f^{\mathcal{A}}(a_1, \ldots, a, \ldots, a_m)$ which is an *elementary translation* of $\mathcal{A}$. The set of translations of $\mathcal{A}$, denoted by $\mathrm{Tr}(\mathcal{A})$, is the smallest set that contains the identity mapping and elementary translations and is closed under composition of unary functions. The set $\mathrm{Tr}(\mathcal{A})$ equipped with the composition operation is a monoid, called the *translation monoid* of $\mathcal{A}$.

Lemma 1 ([14])**.** *Let $\mathcal{A} = (A, \Sigma)$ and $\mathcal{B} = (B, \Sigma)$ be two algebras, and $\varphi : \mathcal{A} \to \mathcal{B}$ be a morphism. The mapping φ induces a monoid morphism $\mathrm{Tr}(\mathcal{A}) \to \mathrm{Tr}(\mathcal{B})$, $p \mapsto p_\varphi$ such that $p(a)\varphi = p_\varphi(a\varphi)$ for any $a \in A$. Moreover, if φ is an epimorphism then the induced mapping is a monoid epimorphism.*

There is a bijective correspondence between the set of ΣX-contexts $\mathrm{C}_\Sigma(X)$ and translations of term algebra $\mathrm{Tr}(\mathcal{T}_\Sigma(X))$ in a natural way: an elementary context $P = f(t_1, \ldots, \xi, \ldots, t_m)$ corresponds to the translation $P^{\mathcal{T}_\Sigma(X)} = f^{\mathcal{T}_\Sigma(X)}(t_1, \ldots, \xi, \ldots, t_m)$, and the composition of contexts corresponds to the composition of translations.

Let us recall that for a relation ρ defined on a set A, by ρ^{-1} the inverse relation of ρ is denoted, i.e., $a\,\rho^{-1}\,b \Leftrightarrow b\,\rho\,a$ for any $a, b \in A$. Let ρ be a quasi order, i.e., a reflexive and transitive relation, on a set A. Then the relation $\equiv_\rho = \rho \cap \rho^{-1}$ is an equivalence on A and the relation $\leq_\rho$ defined on the factor set $A/{\equiv_\rho}$ by

$$a/{\equiv_\rho} \leq_\rho b/{\equiv_\rho} \Leftrightarrow a\,\rho\,b$$

is an order. The ordered set $(A/{\equiv_\rho}, \leq_\rho)$ is denoted by A/ρ.

Let $\preceq$ be a quasi order on an algebra $\mathcal{A} = (A, \Sigma)$, i.e., $\preceq$ is a quasi order on A. Then $\preceq$ is *compatible with* Σ if $a_1 \preceq b_1, \ldots, a_m \preceq b_m$ implies $f^{\mathcal{A}}(a_1, \ldots, a_m) \preceq f^{\mathcal{A}}(b_1, \ldots, b_m)$ for any $f \in \Sigma_m$ $(m > 0)$ and $a_1, \ldots, a_m, b_1, \ldots, b_m \in A$. In case when it is not necessary to emphasize the alphabet Σ, we say that $\preceq$ is a *compatible quasi order* on $\mathcal{A}$.

An *ordered* Σ*-algebra* is a structure $\mathcal{A} = (A, \Sigma, \leqslant)$ where (A, Σ) is a Σ-algebra and $\leqslant$ is an order on A compatible with Σ. Moreover, if a quasi order ρ defined on an algebra $\mathcal{A} = (A, \Sigma, \leqslant)$ is compatible, then $\equiv_\rho$ is a congruence on (A, Σ) and the order factor algebra is $\mathcal{A}/\rho = (A/{\equiv_\rho}, \Sigma, \leq_\rho)$. Compatible quasi orders containing the order of the algebra play on ordered algebras the role of congruences on ordinary algebras. We note that any algebra (A, Σ) in the classical sense is an ordered algebra (A, Σ, Δ_A) in which the order relation is equality.

For a tree language $T \subseteq \mathrm{T}_\Sigma(X)$ the relation (see [9])

$$t \preceq_T s \Leftrightarrow (\forall P \in \mathrm{C}_\Sigma(X))\ (P(s) \in T \Rightarrow P(t) \in T)$$

is a compatible quasi order on $\mathcal{T}_\Sigma(X)$. The corresponding equivalence relation is the well-known *syntactic congruence* of T, denoted by θ_T, and the corresponding order is $\leq_T$. The corresponding factor algebra is the *syntactic ordered algebra* of T, in notation $\mathrm{SOA}(T) = \mathcal{T}_\Sigma(X)/{\preceq_T}$. It is known that a tree language is regular if and only if its syntactic congruence has finite index, i.e., the algebra $\mathrm{SOA}(T)$ is finite. On the other hand, the compatible quasi order $\precsim_T$ is defined on $\mathrm{C}_\Sigma(X)$ by (see [9])

$$P \precsim_T Q \Leftrightarrow (\forall t \in \mathrm{T}_\Sigma(X))\ (\forall R \in \mathrm{C}_\Sigma(X))\ (RQ(t) \in T \Rightarrow RP(t) \in T)$$

and the corresponding equivalence is the m-congruence of T, in notation μ_T, ([15], definition 10.1) defined on $\mathrm{C}_\Sigma(X)$ by

$$P\mu_T Q \Leftrightarrow (\forall t \in \mathrm{T}_\Sigma(X))\ (\forall R \in \mathrm{C}_\Sigma(X))\ (RQ(t) \in T \Leftrightarrow RP(t) \in T).$$

3 Regular tree languages and well-quasi orders

We are going to characterize regular tree languages in terms of well-quasi orders. Motivation for this comes from [3], where a similar result for string languages was given. There are several equivalent ways to define well-quasi orders (see [8]), but we list here only those that we are going to use. A quasi order $\preceq$ defined on a set A is a *well-quasi order* if either of the following conditions is satisfied:

(1) for each infinite sequence $\{x_i\}_{i\in\mathbb{N}}$ of elements of A there exist i and j with $i < j$ such that $x_i \preceq x_j$;
(2) each infinite sequence $\{x_i\}_{i\in\mathbb{N}}$ of elements of A contains an infinite ascending subsequence;
(3) every sequence of $\preceq$-closed subsets of A which is strictly ascending under inclusion is finite.

Recall that a subset H is $\preceq$-*closed* if $a \preceq b$ and $a \in H$ imply $b \in H$.

The following lemma contains some simple properties of well-quasi orders. Parts (a) and (b) are from [3].

Lemma 2.

(a) *If $\rho_1 \subseteq \rho_2$, ρ_1 is a well-quasi order and ρ_2 is a quasi order on A, then ρ_2 is a well-quasi order, too.*
(b) *Let ρ_1 and ρ_2 be well-quasi orders on A_1 and A_2 respectively. Then the transitive closure of $\rho_1 \cup \rho_2$ is a well-quasi order on $A_1 \cup A_2$ and $\rho_1 \times \rho_2$ is a well-quasi order on $A_1 \times A_2$.*
(c) *If ρ_1 and ρ_2 are well-quasi orders on A, then $\rho_1 \cap \rho_2$ is a well-quasi order on A, too.*

Recall that $\rho_1 \times \rho_2$ is defined on $A_1 \times A_2$ by

$$(a_1, a_2)\, \rho_1 \times \rho_2\, (b_1, b_2) \Leftrightarrow a_1\, \rho_1\, b_1 \text{ and } a_2\, \rho_2\, b_2,$$

for $a_1, b_1 \in A_1$ and $a_2, b_2 \in A_2$.

Let ρ be a quasi order on $\mathrm{T}_\Sigma(X)$. Then the relation ρ^C defined on $\mathrm{C}_\Sigma(X)$ by

$$P\rho^C Q \Leftrightarrow (\forall t \in \mathrm{T}_\Sigma(X))\ P(t)\, \rho\, Q(t)$$

is a quasi order *induced* by quasi order ρ. For example, for a tree language $T \subseteq \mathrm{T}_\Sigma(X)$ and the relations defined in Section 2, it can be proved that $\preceq_T^C = \precsim_T$ and $\theta_T^C = \mu_T$.

Theorem 3. *If θ is a congruence on $\mathcal{T}_\Sigma(X)$, then $\mathrm{C}_\Sigma(X)/\theta^C \cong \mathrm{Tr}(\mathcal{T}_\Sigma(X)/\theta)$.*

Proof. Let $\pi : \mathcal{T}_\Sigma(X) \to \mathcal{T}_\Sigma(X)/\theta$ be the natural epimorphism. According to Lemma 1, there is an epimorphism from $\mathrm{C}_\Sigma(X) = \mathrm{Tr}(\mathcal{T}_\Sigma(X))$ to $\mathrm{Tr}(\mathcal{T}_\Sigma(X)/\theta)$ where $P \mapsto P_\pi$ and $P_\pi(t\pi) = (P(t))\pi$ holds for all $P \in \mathrm{C}_\Sigma(X)$ and $t \in \mathrm{T}_\Sigma(X)$. Thus it suffices to prove that the kernel of this epimorphism is θ^C, i.e., that $P_\pi = Q_\pi$ if and only if $P\,\theta^C\,Q$, for any $P, Q \in \mathrm{C}_\Sigma(X)$. Indeed, assume that $P_\pi = Q_\pi$ for some $P, Q \in \mathrm{C}_\Sigma(X)$. Then $P_\pi(t\pi) = Q_\pi(t\pi)$ for every $t\pi \in \mathcal{T}_\Sigma(X)/\theta$, which is equivalent to $(P(t))\pi = (Q(t))\pi$ for every $t \in \mathcal{T}_\Sigma(X)$. This means that $P(t)\,\theta\,Q(t)$ for every $t \in \mathcal{T}_\Sigma(X)$, and so $P\,\theta^C\,Q$. □

A quasi order ρ defined on a set A is of *finite index* if $\equiv_\rho$ is of finite index, i.e., if the set A/ρ is finite. Clearly, such quasi orders are well-quasi orders.

Corollary 4. *If ρ is a compatible quasi order on $\mathcal{T}_\Sigma(X)$ of finite index, then ρ^C is of finite index as well.*

Proof. According to Theorem 3, $\mathrm{C}_\Sigma(X)/\equiv_{\rho^C}$ has as many elements as $\mathrm{Tr}(\mathcal{T}_\Sigma(X)/\equiv_\rho)$ which is finite since $\mathcal{T}_\Sigma(X)/\equiv_\rho$ is finite. □

We are ready now to prove a tree version of the generalized Myhill–Nerode's theorem (Theorem 3.3 [3]).

Theorem 5. *For a tree language $T \subseteq \mathrm{T}_\Sigma(X)$ the following conditions are equivalent:*

(i) *T is regular;*
(ii) *T is ρ-closed where ρ is a compatible well-quasi order and ρ^C is a well-quasi order too;*
(iii) *T is ρ-closed where ρ is a compatible well-quasi order on $\mathcal{T}_\Sigma(X)$ and there exists a well-quasi order on $\mathrm{C}_\Sigma(X)$ contained in ρ^C.*

Proof. (i)⇒(ii). Since T is regular, the relation θ_T is a congruence of finite index, and hence a compatible well-quasi order. The fact that T is saturated by θ_T implies that T is θ_T-closed. According to Corollary 4 it follows that θ_T^C is of finite index, and so a well-quasi order.

(ii)⇒(iii). This is obvious since ρ^C satisfies the condition.

(iii)⇒(i). Suppose that T is not regular. Then θ_T is not of finite index, and hence there exists an infinite sequence $\{t_i\}_{i\in\mathbb{N}}$ such that $t_i/\theta_T \neq t_j/\theta_T$ whenever $i \neq j$. Since ρ is a well-quasi order there exists an infinite ρ-ascending subsequence of $\{t_i\}_{i\in\mathbb{N}}$. Without losing generality we can assume that $\{t_i\}_{i\in\mathbb{N}}$ itself is ascending, i.e., $t_i\,\rho\,t_j$ whenever $i \leq j$. Using compatibility we get $P(t_i)\rho P(t_j)$ for all $P \in \mathrm{C}_\Sigma(X)$ and $i \leq j$. If $P(t_i) \in T$ then $P(t_j) \in T$ since T is ρ-closed. If we denote by $T.t^{-1}$ the set

$$T.t^{-1} = \{P \in \mathrm{C}_\Sigma(X) \mid P(t) \in T\}$$

then we get $P \in T.t_i^{-1}$ implies $P \in T.t_j^{-1}$, i.e., $T.t_i^{-1} \subseteq T.t_j^{-1}$ when $i \leq j$. Moreover, $t_i/\theta_T \neq t_j/\theta_T$ implies that $T.t_i^{-1} \subset T.t_j^{-1}$ for $i < j$. Therefore the sequence $\{T.t_i^{-1}\}_{i\in\mathbb{N}}$ is infinite.

Let ν be a well-quasi order on $C_\Sigma(X)$ contained in ρ^C. We are going to prove that the set $T.t^{-1}$ is ν-closed for any $t \in T_\Sigma(X)$. Assume that $P\,\nu\,Q$. Since $\nu \subseteq \rho^C$, then $P(t)\,\rho\,Q(t)$ for any $t \in T$. If $P \in T.t^{-1}$ then $P(t) \in T$ and since T is ρ-closed, it follows that $Q(t) \in T$, and so $Q \in T.t^{-1}$.

Finally, we have proved that $\{T.t_i^{-1}\}_{i\in\mathbb{N}}$ is an infinite ascending sequence of ν-closed sets, which contradicts the fact that ν is a well-quasi order. Therefore, T must be regular. □

For a language $T \subseteq T_\Sigma(X)$ the relation $\preceq_T^{-1}$ is the greatest compatible well-quasi order on $\mathcal{T}_\Sigma(X)$ such that T is $\preceq_T^{-1}$-closed. Indeed, if T is ρ-closed for a compatible well-quasi order ρ on $\mathcal{T}_\Sigma(X)$, then from $t_1\,\rho\,t_2$ follows that $P(t_1)\,\rho\,P(t_2)$ for any $P \in C_\Sigma(X)$ and so $P(t_1) \in T$ implies $P(t_2) \in T$, i.e., $t_1 \preceq_T^{-1} t_2$, for any $t_1, t_2 \in T_\Sigma(X)$. Moreover, in case T is a regular language, $\preceq_T^{-1}$ is of finite index and, according to Corollary 4, $(\preceq_T^{-1})^C$ is of finite index too, and thus it is a well-quasi order. Hence, $\preceq_T^{-1}$ is the greatest well-quasi order on $\mathcal{T}_\Sigma(X)$ satisfying condition (ii) of Theorem 5.

Example 6. For a tree $t \in T_\Sigma(X)$, let $\overline{t} \in (\Sigma \cup X)^*$ be the string obtained by reading symbols as they appear in t, i.e., in right Polish notation. Denote by $\leq_e$ the embedding order relation on the free monoid $(\Sigma \cup X)^*$, i.e., the relation defined by $u \leq_e v \Leftrightarrow u = u_1u_2\cdots u_n,\ v = v_0u_1v_1u_2\cdots v_{n-1}u_nv_n$ for $u_1,\ldots,u_n,v_0,v_1,\ldots,v_n \in (\Sigma \cup X)^*$. It is a well order. Let ρ be the relation defined on $T_\Sigma(X)$ by $t_1\,\rho\,t_2 \Leftrightarrow \overline{t_1} \leq_e \overline{t_2}$. It can be proved that ρ is a compatible well-quasi order and ρ^C is a well-quasi order. Thus, every ρ-closed ΣX-language is regular.

4 Varieties of quasi orders

A correspondence between positive varieties of tree languages and varieties of finite ordered algebras has been given in [9]. It is known that in the case of ordinary varieties of (tree) languages and varieties of algebras the corresponding families of relations are varieties of congruences of finite index (see [14]). Results from the previous section, as well as from [9], suggest that families of relations corresponding to positive varieties of languages and varieties of ordered algebras consist of compatible well-quasi orders for which the induced relations on contexts are well-quasi orders. Moreover, the fact that we are dealing only with finite algebras restricts our attention to compatible quasi orders of finite index. According to Corollary 4, their induced quasi orders on contexts are of finite index, too.

Let us recall first necessary concepts and the Positive Variety Theorem from [9].

Let $\mathcal{A} = (A, \Sigma, \leqslant_\mathcal{A})$ and $\mathcal{B} = (B, \Sigma, \leqslant_\mathcal{B})$ be two ordered algebras. The structure $\mathcal{B}$ is an *order subalgebra* of $\mathcal{A}$ if (B, Σ) is a subalgebra of (A, Σ) and $\leqslant_\mathcal{B}$ is the restriction of $\leqslant_\mathcal{A}$ on B. A mapping $\varphi : A \to B$ is an *order morphism* if it is a Σ-morphism, i.e., if $c^\mathcal{A}\varphi = c^\mathcal{B}$ and $f^\mathcal{A}(a_1,\ldots,a_m)\varphi = f^\mathcal{B}(a_1\varphi,\ldots,a_m\varphi)$ for any $c \in \Sigma_0, f \in \Sigma_m$ $(m > 0)$ and $a_1,\ldots,a_m \in A$, and preserves the order, i.e., for any $a, b \in A$ if $a \leqslant_\mathcal{A} b$ then $a\varphi \leqslant_\mathcal{B} b\varphi$. The order morphism φ is an *order epimorphism* if it is surjective, and then $\mathcal{B}$ is an *order image of* $\mathcal{A}$. When φ is bijective and its inverse is also an order morphism, then it is an *order isomorphism*, and $\mathcal{A} \cong \mathcal{B}$ denotes that $\mathcal{A}$ and $\mathcal{B}$ are order isomorphic. The structure $\mathcal{A} \times \mathcal{B} = (A \times B, \Sigma, \leqslant_{\mathcal{A}} \times \leqslant_{\mathcal{B}})$, where $(A \times B, \Sigma)$ is the product of the algebras (A, Σ) and (B, Σ), is the *direct product* of $\mathcal{A}$ and $\mathcal{B}$. A *variety of finite ordered algebras* is a class of finite ordered algebras closed under order subalgebras, order images and direct products.

Let A and B be arbitrary sets. For a mapping $\phi : A \to B$ and a relation ρ on B the relation $\phi \circ \rho \circ \phi^{-1}$ is defined on A by

$$(a, b) \in \phi \circ \rho \circ \phi^{-1} \Leftrightarrow (a\phi, b\phi) \in \rho.$$

Lemma 7. *For ordered algebras* $\mathcal{A} = (A, \Sigma, \leqslant_{\mathcal{A}})$ *and* $\mathcal{B} = (B, \Sigma, \leqslant_{\mathcal{B}})$ *and order morphism* $\varphi : \mathcal{A} \to \mathcal{B}$*, if* $\preccurlyeq$ *is a compatible quasi order on* $\mathcal{B}$ *containing* $\leqslant_{\mathcal{B}}$*, then the relation* $\varphi \circ \preccurlyeq \circ \varphi^{-1}$ *is a compatible quasi order on* $\mathcal{A}$ *containing* $\leqslant_{\mathcal{A}}$*. Moreover, if* φ *is an order epimorphism then* $\mathcal{A}/(\varphi \circ \preccurlyeq \circ \varphi^{-1}) \cong \mathcal{B}/\preccurlyeq$*.*

Let us recall that for a tree language $T \subseteq \mathrm{T}_\Sigma(X)$, a context $P \in \mathrm{C}_\Sigma(X)$, and a Σ-morphism $\varphi : \mathcal{T}_\Sigma(Y) \to \mathcal{T}_\Sigma(X)$, the inverse translation of T under P is $P^{-1}(T) = \{t \in \mathrm{T}_\Sigma(X) \mid P(t) \in T\}$, and the inverse morphism of T under φ is $T\varphi^{-1} = \{t \in \mathrm{T}_\Sigma(Y) \mid t\varphi \in T\}$ (cf. [14]). An indexed family of recognizable tree languages $\mathscr{V} = \{\mathscr{V}(X)\}$ is a *positive variety of tree languages* if it is closed under positive Boolean operations (intersection and union), inverse translations and inverse morphisms.

Theorem 8 (**Positive Variety Theorem** [9]). *For a positive variety of tree languages* $\mathscr{V}$*, let* $\mathscr{V}^{\mathtt{a}}$ *be the variety of finite ordered algebras generated by syntactic ordered algebras of tree languages in* $\mathscr{V}$*. For a variety of finite ordered algebras* $\mathscr{K}$ *let the indexed family* $\mathscr{K}^{\mathtt{t}} = \{\mathscr{K}^{\mathtt{t}}(X)\}$ *be defined by* $\mathscr{K}^{\mathtt{t}}(X) = \{T \subseteq \mathrm{T}_\Sigma(X) \mid \mathrm{SOA}(T) \in \mathscr{K}\}$*.The mappings* $\mathscr{K} \mapsto \mathscr{K}^{\mathtt{t}}$ *and* $\mathscr{V} \mapsto \mathscr{V}^{\mathtt{a}}$ *are mutually inverse lattice isomorphisms between the class of all varieties of finite ordered algebras and the class of all positive varieties of recognizable tree languages.*

Let us denote by $\mathrm{FQ}(X)$ the set of all compatible quasi orders of finite index defined on $\mathrm{T}_\Sigma(X)$.

Lemma 9. *Let* $\phi : \mathcal{T}_\Sigma(X) \to \mathcal{T}_\Sigma(Y)$ *be a morphism.*

(a) *If* $\rho \in \mathrm{FQ}(Y)$ *then* $\phi \circ \rho \circ \phi^{-1} \in \mathrm{FQ}(X)$*.*
(b) *If* $T \subseteq \mathcal{T}_\Sigma(Y)$ *then*

$$\bigcap_{P \in \mathrm{C}_\Sigma(Y)} \preceq^{-1}_{P^{-1}(T)\phi^{-1}} \subseteq \phi \circ \preceq_T^{-1} \circ \phi^{-1}.$$

Moreover, if T *is regular then the intersection can be taken over a finite subset of* $\mathrm{C}_\Sigma(Y)$*.*

Proof. (a) Clearly $\phi \circ \rho \circ \phi^{-1}$ is reflexive and transitive. Let us prove that it is compatible. Assume $t_1(\phi \circ \rho \circ \phi^{-1})t_2$, i.e., $(t_1\phi)\,\rho\,(t_2\phi)$. Compatibility of ρ implies that $Q(t_1\phi)\,\rho\,Q(t_2\phi)$ for any $Q \in \mathrm{C}_\Sigma(Y)$. In particular, for any $P \in \mathrm{C}_\Sigma(X)$ we have $P_\phi(t_1\phi)\,\rho\,P_\phi(t_2\phi)$, and so $P(t_1)\,(\,\phi\circ\rho\circ\phi^{-1})P(t_2)$.

It remains to prove that $\phi\circ\rho\circ\phi^{-1}$ has a finite index. It is easy to prove that $\equiv_{\phi\circ\rho\circ\phi^{-1}} = \phi\circ\equiv_\rho\circ\phi^{-1}$. Therefore the mapping $t/\equiv_{\phi\circ\rho\circ\phi^{-1}} \mapsto t\phi/\equiv_\rho$ is a well-defined one-to-one mapping. Moreover, it is a bijection onto $\mathcal{T}_\Sigma(X)\phi/\equiv_\rho$. Therefore, $|\mathcal{T}_\Sigma(X)/\equiv_{\phi\circ\rho\circ\phi^{-1}}| = |\mathcal{T}_\Sigma(X)\phi/\equiv_\rho| \le |\mathcal{T}_\Sigma(Y)/\equiv_\rho|$ and this number is finite.

(b) The following proves the claim:

$$\begin{array}{rcl}(t_1,t_2) & \in & \bigcap_{P\in\mathrm{C}_\Sigma(Y)} \preceq^{-1}_{P^{-1}(T)\phi^{-1}} \Leftrightarrow \\ & \Leftrightarrow & (\forall P\in\mathrm{C}_\Sigma(Y))\ t_1 \preceq^{-1}_{P^{-1}(T)\phi^{-1}} t_2 \\ & \Leftrightarrow & (\forall P\in\mathrm{C}_\Sigma(Y))\,(\forall Q\in\mathrm{C}_\Sigma(X)) \\ & & \qquad (Q(t_1)\in P^{-1}(T)\phi^{-1} \Rightarrow Q(t_2)\in P^{-1}(T)\phi^{-1}) \\ & \Rightarrow & (\forall P\in\mathrm{C}_\Sigma(Y))\ (t_1\in P^{-1}(T)\phi^{-1} \Rightarrow t_2\in P^{-1}(T)\phi^{-1}) \\ & \Leftrightarrow & (\forall P\in\mathrm{C}_\Sigma(Y))\ (t_1\phi\in P^{-1}(T) \Rightarrow t_2\phi\in P^{-1}(T)) \\ & \Leftrightarrow & (\forall P\in\mathrm{C}_\Sigma(Y))\ (P(t_1\phi)\in T \Rightarrow P(t_2\phi)\in T) \\ & \Leftrightarrow & (t_1\phi) \preceq^{-1}_T (t_2\phi) \\ & \Leftrightarrow & t_1(\phi\circ \preceq^{-1}_T \circ\phi^{-1})t_2\end{array}$$

Let us define a relation ν on $\mathrm{C}_\Sigma(Y)$ by $P\,\nu\,Q \Leftrightarrow P^{-1}(T) = Q^{-1}(T)$. Clearly, ν is an equivalence and $\mu_T \subseteq \nu$. In case T is regular μ_T has finite index, and hence ν has finite index. Therefore, there can be only finitely many different sets of the form $P^{-1}(T)$. □

A family $\mathscr{R} = \{\mathscr{R}(X)\}$, where $\mathscr{R}(X)$ is a set of compatible quasi orders on $\mathcal{T}_\Sigma(X)$ of finite index, is a *variety of quasi orders* if

(1) $\rho_1,\rho_2 \in \mathscr{R}(X)$ then $\rho_1\cap\rho_2 \in \mathscr{R}(X)$ for any X;
(2) $\rho_1 \subseteq \rho_2$ and $\rho_1 \in \mathscr{R}(X)$ then $\rho_2 \in \mathscr{R}(X)$ for any X;
(3) $\phi : \mathcal{T}_\Sigma(X) \to \mathcal{T}_\Sigma(Y)$ is a morphism and $\rho \in \mathscr{R}(Y)$ then $\phi\circ\rho\circ\phi^{-1} \in \mathscr{R}(X)$.

In other words, $\mathscr{R}(X)$ is a filter of the lattice $\mathrm{FQ}(X)$ satisfying condition (3).

Lemma 10. *Let $\mathscr{V} = \{\mathscr{V}(X)\}$ be a positive variety of tree languages. Let $\mathscr{V}^{\mathfrak{r}}(X)$ be the filter in the lattice $\mathrm{FQ}(X)$ generated by the set $\{\preceq^{-1}_T \mid T \in \mathscr{V}(X)\}$. Then $\mathscr{V}^{\mathfrak{r}} = \{\mathscr{V}^{\mathfrak{r}}(X)\}$ is a variety of quasi orders.*

Proof. Conditions (1) and (2) from the definition of varieties of quasi orders are fulfilled by the way $\mathscr{V}^{\mathfrak{r}}$ is defined. Assume that $\rho \in \mathscr{V}^{\mathfrak{r}}(Y)$ and $\phi : \mathcal{T}_\Sigma(X) \to \mathcal{T}_\Sigma(Y)$ is a morphism. Since $\rho \in \mathscr{V}^{\mathfrak{r}}(Y)$ there are languages $T_1,\dots,T_n \in \mathscr{V}(Y)$, $n\in\mathbb{N}$, such that $\cap_{k=1}^n \preceq^{-1}_{T_k} \subseteq \rho$. For a language $T_k \in \mathscr{V}(Y)$ and any $P \in \mathrm{C}_\Sigma(Y)$ we have that $P^{-1}(T_k) \in \mathscr{V}(Y)$, and then $P^{-1}(T_k)\phi^{-1} \in \mathscr{V}(X)$. This implies that $\preceq^{-1}_{P^{-1}(T_k)\phi^{-1}} \in \mathscr{V}^{\mathfrak{r}}(X)$. Since T_k is regular, the family $\{P^{-1}(T_k)\phi^{-1} \in \mathscr{V}(X) \mid P \in \mathrm{C}_\Sigma(Y)\}$ is finite. Therefore, $\phi\circ \preceq^{-1}_{T_k} \circ\phi^{-1} \in \mathscr{V}^{\mathfrak{r}}(X)$ according to Lemma 9. Now from $\cap_{k=1}^n \preceq^{-1}_{T_k} \subseteq \rho$ follows that $\cap_{k=1}^n(\phi\circ \preceq^{-1}_{T_k} \circ\phi^{-1}) \subseteq \phi\circ\rho\circ\phi^{-1}$, and so $\phi\circ\rho\circ\phi^{-1} \in \mathscr{V}^{\mathfrak{r}}(X)$. □

Lemma 11. *Let $\mathscr{R} = \{\mathscr{R}(X)\}$ be a variety of quasi orders. Let us denote $\mathscr{R}^{\mathtt{t}}(X) = \{T \subseteq \mathrm{T}_\Sigma(X) \mid \preceq_T^{-1} \in \mathscr{R}(X)\}$. Then $\mathscr{R}^{\mathtt{t}} = \{\mathscr{R}^{\mathtt{t}}(X)\}$ is a positive variety of tree languages.*

Proof. According to Theorem 5 it follows that languages belonging to the family are regular. From $\preceq_{T_1}^{-1} \cap \preceq_{T_2}^{-1} \subseteq \preceq_{T_1 \cap T_2}^{-1}$ and $\preceq_{T_1}^{-1} \cap \preceq_{T_2}^{-1} \subseteq \preceq_{T_1 \cup T_2}^{-1}$ it follows that $\mathscr{R}^{\mathtt{t}}(X)$ is closed for positive Boolean operations. Similarly, $\preceq_T^{-1} \subseteq \preceq_{P^{-1}(T)}^{-1}$ implies closure for quotients. Finally, if $\phi : \mathcal{T}_\Sigma(X) \to \mathcal{T}_\Sigma(Y)$ is a morphism and $T \in \mathscr{R}^{\mathtt{t}}(Y)$ then $\preceq_T^{-1} \in \mathscr{R}(Y)$, and so $\phi \circ \preceq_T^{-1} \circ \phi^{-1} \in \mathscr{R}(X)$. It is easy to prove that $\phi \circ \preceq_T^{-1} \circ \phi^{-1} \subseteq \preceq_{T\phi^{-1}}^{-1}$, which further implies $\preceq_{T\phi^{-1}}^{-1} \in \mathscr{R}(X)$, and hence $T\phi^{-1} \in \mathscr{R}^{\mathtt{t}}(X)$. □

Lemma 12. *For positive varieties of tree languages $\mathscr{V} = \{\mathscr{V}(X)\}$, $\mathscr{V}_1 = \{\mathscr{V}_1(X)\}$ and $\mathscr{V}_2 = \{\mathscr{V}_2(X)\}$, and varieties of quasi orders $\mathscr{R} = \{\mathscr{R}(X)\}$, $\mathscr{R}_1 = \{\mathscr{R}_1(X)\}$ and $\mathscr{R}_2 = \{\mathscr{R}_2(X)\}$, the following hold:*

(a) $\mathscr{V} = \mathscr{V}^{\mathtt{rt}}$;
(b) $\mathscr{R} = \mathscr{R}^{\mathtt{tr}}$;
(c) $\mathscr{V}_1 \subseteq \mathscr{V}_2$ *implies* $\mathscr{V}_1^{\mathtt{r}} \subseteq \mathscr{V}_2^{\mathtt{r}}$;
(d) $\mathscr{R}_1 \subseteq \mathscr{R}_2$ *implies* $\mathscr{R}_1^{\mathtt{t}} \subseteq \mathscr{R}_2^{\mathtt{t}}$.

Proof. (a) The inclusion $\mathscr{V} \subseteq \mathscr{V}^{\mathtt{rt}}$ is obvious. Assume now that $T \in \mathscr{V}^{\mathtt{rt}}(X)$. Then $\preceq_T^{-1} \in \mathscr{V}^{\mathtt{r}}(X)$. This means that there are languages $T_1, \ldots, T_n \in \mathscr{V}(X)$, $n \in \mathbb{N}$, such that $\cap_{k=1}^n \preceq_{T_k}^{-1} \subseteq \preceq_T^{-1}$, which implies that $\mathrm{SOA}(T)$ is an order image of an order subalgebra of $\mathrm{SOA}(T_1) \times \cdots \times \mathrm{SOA}(T_n)$. Now $\mathrm{SOA}(T_1), \ldots, \mathrm{SOA}(T_n) \in \mathscr{V}^{\mathtt{a}}$ and $\mathscr{V}^{\mathtt{a}}$ is a variety of ordered algebras, which implies that $\mathrm{SOA}(T) \in \mathscr{V}^{\mathtt{a}}$, and hence $T \in \mathscr{V}^{\mathtt{at}}(X) = \mathscr{V}(X)$, according to Theorem 8.

(b) It is easy to check that $\mathscr{R}^{\mathtt{tr}} \subseteq \mathscr{R}$. Consider now $\rho \in \mathscr{R}(X)$. Since ρ has finite index, there are finitely many ρ-closed sets. Let $T_1, \ldots, T_n$, $n \in \mathbb{N}$, be all of them. We are going to prove that $\cap_{k=1}^n \preceq_{T_k}^{-1} \subseteq \rho$. Assume that $t, s \in \mathrm{T}_\Sigma(X)$ are such that $t \,\rho\, s$ does not hold. Then the set $\{t' \in \mathrm{T}_\Sigma(X) \mid t \,\rho\, t'\}$ is ρ-closed and hence equal to some T_i, and so $t \preceq_{T_i}^{-1} s$ does not hold, i.e., $(t, s) \notin \cap_{k=1}^n \preceq_{T_k}^{-1}$. On the other hand, $\rho \subseteq \preceq_{T_k}^{-1}$ for every $k \in \{1, \ldots, n\}$ since T_k is ρ-closed and $\preceq_{T_k}^{-1}$ is the greatest such well-quasi order. Therefore, $\preceq_{T_k}^{-1} \in \mathscr{R}(X)$ which implies $T_k \in \mathscr{R}^{\mathtt{t}}(X)$, this further gives $\preceq_{T_k}^{-1} \in \mathscr{R}^{\mathtt{tr}}(X)$, which finally, together with $\cap_{k=1}^n \preceq_{T_k}^{-1} \subseteq \rho$, implies $\rho \in \mathscr{R}^{\mathtt{tr}}(X)$.

(c) and (d) are obvious. □

Summing up the results from Lemmas 10, 11, 12 we get the following variety theorem.

Theorem 13. *For a positive variety of tree languages $\mathscr{V} = \{\mathscr{V}(X)\}$, let $\mathscr{V}^{\mathtt{r}}(X)$ be the filter of the lattice $\mathrm{FQ}(X)$ generated by the set*

$$\{\preceq_T^{-1} \mid T \in \mathscr{V}(X)\}.$$

On the other hand, for a variety of quasi orders $\mathscr{R} = \{\mathscr{R}(X)\}$, let us denote

$$\mathscr{R}^{\mathtt{t}}(X) = \{T \subseteq \mathrm{T}_\Sigma(X) \mid \preceq_T^{-1} \in \mathscr{R}(X)\}.$$

The mappings $\mathscr{V} \mapsto \mathscr{V}^{\mathtt{r}} = \{\mathscr{V}^{\mathtt{r}}(X)\}$ *and* $\mathscr{R} \mapsto \mathscr{R}^{\mathtt{t}} = \{\mathscr{R}^{\mathtt{t}}(X)\}$ *are mutually inverse lattice isomorphisms between the lattices of all positive varieties of tree languages and all varieties of quasi orders.*

The next theorem establishes a similar result for varieties of finite ordered algebras and varieties of quasi orders. First we need to prove several lemmas.

Lemma 14. *Let* $\mathscr{K}$ *be a variety of finite ordered* Σ*-algebras. Let* $\mathscr{K}^{\mathtt{r}}(X) = \{\rho \in \mathrm{FQ}(X) \mid \mathcal{T}_\Sigma(X)/\rho \in \mathscr{K}\}$*. Then* $\mathscr{K}^{\mathtt{r}} = \{\mathscr{K}^{\mathtt{r}}(X)\}$ *is a variety of quasi orders.*

Proof. Let $\rho_1, \rho_2 \in \mathscr{K}^{\mathtt{r}}(X)$. Then $\mathcal{T}_\Sigma(X)/(\rho_1 \cap \rho_2)$ is an order image of an order subalgebra of $\mathcal{T}_\Sigma(X)/\rho_1 \times \mathcal{T}_\Sigma(X)/\rho_2$, and hence $\mathcal{T}_\Sigma(X)/\rho_1, \mathcal{T}_\Sigma(X)/\rho_2 \in \mathscr{K}$ imply $\mathcal{T}_\Sigma(X)/(\rho_1 \cap \rho_2) \in \mathscr{K}$, what means $\rho_1 \cap \rho_2 \in \mathscr{V}^{\mathtt{r}}(X)$. Similarly, if $\rho_1 \in \mathscr{K}^{\mathtt{r}}(X)$ and $\rho_1 \subseteq \rho_2$ then $\mathcal{T}_\Sigma(X)/\rho_2$ is an order image of $\mathcal{T}_\Sigma(X)/\rho_1 \in \mathscr{K}$, and so $\mathcal{T}_\Sigma(X)/\rho_2 \in \mathscr{K}$, which implies $\rho_2 \in \mathscr{K}^{\mathtt{r}}(X)$.

Consider now $\rho \in \mathscr{K}^{\mathtt{r}}(Y)$ and a morphism $\phi : \mathcal{T}_\Sigma(X) \to \mathcal{T}_\Sigma(Y)$. The mapping $\psi : \mathcal{T}_\Sigma(X)/(\phi \circ \rho \circ \phi^{-1}) \to \mathcal{T}_\Sigma(Y)/\rho$ defined by $t/(\phi\circ \equiv_\rho \circ\phi^{-1}) \mapsto (t\phi)/\equiv_\rho$ is an order isomorphism from $\mathcal{T}_\Sigma(X)/(\phi \circ \rho \circ \phi^{-1})$ to $\mathcal{T}_\Sigma(X)\phi/\rho$, which is an order subalgebra of $\mathcal{T}_\Sigma(Y)/\rho$. Therefore, $\mathcal{T}_\Sigma(Y)/\rho \in \mathscr{K}$ implies $\mathcal{T}_\Sigma(X)/(\phi\circ\rho\circ\phi^{-1}) \in \mathscr{K}$, and so $\phi \circ \rho \circ \phi^{-1} \in \mathscr{K}^{\mathtt{r}}(X)$. □

Lemma 15. *Let* $\mathscr{R} = \{\mathscr{R}(X)\}$ *be a variety of quasi orders. Let* $\mathscr{R}^{\mathsf{a}}$ *be the set of all ordered* Σ*-algebras* $\mathcal{A}$ *such that* $\mathcal{A} \cong \mathcal{T}_\Sigma(X)/\rho$ *for some* X *and* $\rho \in \mathscr{R}(X)$*. Then* $\mathscr{R}^{\mathsf{a}}$ *is a variety of finite ordered algebras.*

Proof. Let us notice first that for any order algebra $\mathcal{A} \cong \mathcal{T}_\Sigma(X)/\rho$ for some alphabet X and a compatible quasi order ρ, there exists an epimorphism $\phi : \mathcal{T}_\Sigma(X) \to \mathcal{A}$ such that $\rho = \phi\circ\leq_{\mathcal{A}}\circ\phi^{-1}$, where $\leq_{\mathcal{A}}$ is the order of $\mathcal{A}$. Indeed, if $\pi : \mathcal{T}_\Sigma(X) \to \mathcal{T}_\Sigma(X)/\rho$ is the natural epimorphism defined by $t \mapsto t/\equiv_\rho$, and $\psi : \mathcal{T}_\Sigma(X)/\rho \to \mathcal{A}$ is an order isomorphism, then $\pi\psi : \mathcal{T}_\Sigma(X) \to \mathcal{A}$ is an epimorphism and $\rho = (\pi\psi)\circ \leq_{\mathcal{A}} \circ(\pi\psi)^{-1}$.

Consider now $\mathcal{A} \in \mathscr{R}^{\mathsf{a}}$. Then there exists an alphabet X and $\rho \in \mathscr{R}(X)$ such that $\mathcal{A} \cong \mathcal{T}_\Sigma(X)/\rho$, and let $\phi : \mathcal{T}_\Sigma(X) \to \mathcal{A}$ be an order epimorphism such that $\rho = \phi\circ \leq_{\mathcal{A}} \circ\phi^{-1}$.

Let $\mathcal{B}$ be an order subalgebra of $\mathcal{A}$. Then there exists a finitely generated order subalgebra $\mathcal{C}$ of $\mathcal{T}_\Sigma(X)$ such that $\mathcal{B}$ is the order image of $\mathcal{C}$ under epimorphism ϕ. Let Y be a finite alphabet such that there exists an order epimorphism $\psi : \mathcal{T}_\Sigma(Y) \to \mathcal{C}$. Therefore, the mapping $\psi\phi : \mathcal{T}_\Sigma(Y) \to \mathcal{B}$ is an order epimorphism and $\mathcal{B} \cong \mathcal{T}_\Sigma(Y)/((\psi\phi)\circ(\leq_{\mathcal{B}})\circ(\psi\phi)^{-1})$ where $\leq_{\mathcal{B}}$ is the restriction of $\leq_{\mathcal{A}}$ on B. It is easy to check that $\mathcal{B} \cong \mathcal{T}_\Sigma(Y)/((\psi\phi) \circ (\leq_{\mathcal{B}}) \circ (\psi\phi)^{-1}) = \mathcal{T}_\Sigma(Y)/((\psi\phi)\circ \leq_{\mathcal{A}} \circ(\psi\phi)^{-1})$. Now $\mathcal{A} \in \mathscr{R}^{\mathsf{a}}$ implies $\phi\circ \leq_{\mathcal{A}} \circ\phi^{-1} = \rho \in \mathscr{R}(X)$, what further implies $(\psi\phi)\circ \leq_{\mathcal{A}} \circ(\psi\phi)^{-1} = \psi \circ (\phi\circ \leq_{\mathcal{A}} \circ\phi^{-1}) \circ \psi^{-1} \in \mathscr{R}(Y)$. Therefore, $\mathcal{B} \cong \mathcal{T}_\Sigma(Y)/((\psi\phi)\circ \leq_{\mathcal{B}} \circ(\psi\phi)^{-1}) \in \mathscr{R}^{\mathsf{a}}$.

Assume now that $\mathcal{B}$ is an order image of $\mathcal{A}$ and let $\psi : \mathcal{A} \to \mathcal{B}$ be the order epimorphism. Then $\phi\psi : \mathcal{T}_\Sigma(X) \to \mathcal{B}$ is an order epimorphism. If $\leq_{\mathcal{B}}$ is the order of $\mathcal{B}$, then $\mathcal{B} \cong \mathcal{T}_\Sigma(X)/((\phi\psi)\circ \leq_{\mathcal{B}} \circ(\phi\psi)^{-1})$. From the fact that ψ is an order morphism, it follows that $\leq_{\mathcal{A}} \subseteq \psi \circ \leq_{\mathcal{B}} \circ \psi^{-1}$. This further implies $\rho = \phi \circ \leq_{\mathcal{A}} \circ \phi^{-1} \subseteq \phi \circ \psi \circ \leq_{\mathcal{B}} \circ \psi^{-1} \circ \phi^{-1} \in \mathscr{R}(X)$, and so $\mathcal{B} \in \mathscr{R}^{\mathtt{a}}$.

Consider now two ordered algebras $\mathcal{A}_1, \mathcal{A}_2 \in \mathscr{R}^{\mathtt{a}}$. Let $\leq_1, \leq_2$ be their orders respectively, and X_1 and X_2 alphabets for which there are quasi orders $\rho_1 \in \mathscr{R}(X_1)$ and $\rho_2 \in \mathscr{R}(X_2)$ such that $\mathcal{A}_1 \cong \mathcal{T}_\Sigma(X_1)/\rho_1$ and $\mathcal{A}_2 \cong \mathcal{T}_\Sigma(X_2)/\rho_2$, respectively. Denote by $\pi_1 : \mathcal{T}_\Sigma(X_1) \to \mathcal{A}_1$ and $\pi_2 : \mathcal{T}_\Sigma(X_2) \to \mathcal{A}_2$, respectively, order epimorphisms such that $\rho_1 = \pi_1 \circ \leq_1 \circ \pi_1^{-1}$ and $\rho_2 = \pi_2 \circ \leq_2 \circ \pi_2^{-1}$. Let Y be a finite alphabet such that there is an epimorphism $\psi : \mathcal{T}_\Sigma(Y) \to \mathcal{T}_\Sigma(X_1) \times \mathcal{T}_\Sigma(X_2)$, and let $\psi_1 : \mathcal{T}_\Sigma(Y) \to \mathcal{T}_\Sigma(X_1)$ and $\psi_2 : \mathcal{T}_\Sigma(Y) \to \mathcal{T}_\Sigma(X_2)$ be the projection mappings of ψ. Then the mapping $\Phi : \mathcal{T}_\Sigma(Y) \to \mathcal{A}_1 \times \mathcal{A}_2$ whose projection mappings are $\Phi_1 = \psi_1\pi_1$ and $\Phi_2 = \psi_2\pi_2$ is an order epimorphism and $\mathcal{A}_1 \times \mathcal{A}_2 \cong \mathcal{T}_\Sigma(Y)/(\Phi \circ (\leq_1 \times \leq_2) \circ \Phi^{-1})$. It can be easily checked that $\Phi \circ (\leq_1 \times \leq_2) \circ \Phi^{-1} = (\Phi_1 \circ \leq_1 \circ \Phi_1^{-1}) \cap (\Phi_2 \circ \leq_2 \circ \Phi_2^{-1})$. Now $\Phi_1 \circ \leq_1 \circ \Phi_1^{-1} = \psi_1 \circ \pi_1 \circ \leq_1 \circ \pi_1^{-1} \circ \psi_1^{-1} = \psi_1 \circ \rho_1 \circ \psi_1^{-1} \in \mathscr{R}(Y)$ since $\rho_1 \in \mathscr{R}(X_1)$. Similarly, $\Phi_2 \circ \leq_2 \circ \Phi_2^{-1} \in \mathscr{R}(Y)$, and hence $\Phi \circ (\leq_1 \times \leq_2) \circ \Phi^{-1} \in \mathscr{R}(Y)$ what implies $\mathcal{A} \times \mathcal{B} \in \mathscr{R}^{\mathtt{a}}$.

Therefore, $\mathscr{R}^{\mathtt{a}}$ is a variety of finite ordered algebras. □

Lemma 16. *For varieties of finite ordered algebras $\mathscr{K}$, $\mathscr{K}_1$ and $\mathscr{K}_2$, and varieties of quasi orders $\mathscr{R} = \{\mathscr{R}(X)\}$, $\mathscr{R}_1 = \{\mathscr{R}_1(X)\}$ and $\mathscr{R}_2 = \{\mathscr{R}_2(X)\}$, the following hold:*

(a) $\mathscr{K} = \mathscr{K}^{\mathtt{ra}}$;
(b) $\mathscr{R} = \mathscr{R}^{\mathtt{ar}}$;
(c) $\mathscr{K}_1 \subseteq \mathscr{K}_2$ *implies* $\mathscr{K}_1^{\mathtt{r}} \subseteq \mathscr{K}_2^{\mathtt{r}}$;
(d) $\mathscr{R}_1 \subseteq \mathscr{R}_2$ *implies* $\mathscr{R}_1^{\mathtt{a}} \subseteq \mathscr{R}_2^{\mathtt{a}}$.

Proof. It is easy to check (a), (c), (d) and the inclusion $\mathscr{R}(X) \subseteq \mathscr{R}^{\mathtt{ar}}(X)$ for any X.

Consider $\rho \in \mathscr{R}^{\mathtt{ar}}(X)$. Then $\mathcal{A} = \mathcal{T}_\Sigma(X)/\rho \in \mathscr{R}^{\mathtt{a}}$, which further implies that $\mathcal{A} \cong \mathcal{T}_\Sigma(Y)/\mu$ for some alphabet Y and $\mu \in \mathscr{R}(Y)$. Let $\phi : \mathcal{T}_\Sigma(X) \to \mathcal{A}$ and $\psi : \mathcal{T}_\Sigma(Y) \to \mathcal{A}$ be order epimorphisms such that $\rho = \phi \circ \leq_{\mathcal{A}} \circ \phi^{-1}$ and $\mu = \psi \circ \leq_{\mathcal{A}} \circ \psi^{-1}$, where $\leq_{\mathcal{A}}$ is the order of $\mathcal{A}$. Let us define the morphism $\Phi : \mathcal{T}_\Sigma(X) \to \mathcal{T}_\Sigma(Y)$ so that $x\Phi \in x\phi\psi^{-1}$ for any $x \in X$. Then $\phi = \Phi\psi$ and so $\phi \circ \leq_{\mathcal{A}} \circ \phi^{-1} = (\Phi\psi) \circ \leq_{\mathcal{A}} \circ (\Phi\psi)^{-1}$, i.e., $\rho = \Phi \circ \mu \circ \Phi^{-1} \in \mathscr{R}(X)$ since $\mu \in \mathscr{R}(Y)$. □

As a corollary of Lemmas 14, 15, 16 we get the following variety theorem for algebras and relations.

Theorem 17. *For a variety of finite ordered Σ-algebras $\mathscr{K}$, let us define*

$$\mathscr{K}^{\mathtt{r}}(X) = \{\rho \in \mathrm{FQ}(X) \mid \mathcal{T}_\Sigma(X)/\rho \in \mathscr{K}\}.$$

For a variety of quasi orders $\mathscr{R} = \{\mathscr{R}(X)\}$, let $\mathscr{R}^{\mathtt{a}}$ be the set of all ordered Σ-algebras $\mathcal{A}$ such that $\mathcal{A} \cong \mathcal{T}_\Sigma(X)/\rho$ for some alphabet X and $\rho \in \mathscr{R}(X)$.
The mappings $\mathscr{K} \mapsto \mathscr{K}^{\mathtt{r}} = \{\mathscr{K}^{\mathtt{r}}(X)\}$ and $\mathscr{R} \mapsto \mathscr{R}^{\mathtt{a}}$ are mutually inverse lattice isomorphisms between the lattices of all varieties of finite ordered algebras and all varieties of quasi orders.

The correspondences established here are similar to those used in [14] between varieties of tree languages, varieties of finite algebras and varieties of finite congruences. However, in [14] the variety of algebras assigned to a variety of finite congruences was generated by a family which resembles our family $\mathscr{R}^{\mathsf{a}}$, and it has been shown here that the family already forms a variety of finite ordered algebras.

Example 18. Ordered nilpotent algebras and cofinite tree language were introduced in [9]. Namely, an ordered algebra $\mathcal{A} = (A, \Sigma, \leqslant)$ is *ordered n-nilpotent*, $n \in \mathbb{N}$, if $p_1 \cdots p_n(a) \leqslant b$ holds for all $a, b \in A$ and non-trivial translations $p_1, \ldots, p_n$ of $\mathcal{A}$, and it is *ordered nilpotent* if it is ordered n-nilpotent for some $n \in \mathbb{N}$. A non-empty tree language $T \subseteq \mathrm{T}_\Sigma(X)$ is *cofinite* if its complement $\mathrm{T}_\Sigma(X) \setminus T$ is finite. The family of cofinite tree languages for all leaf alphabets X is a positive variety of tree languages and finite ordered nilpotent algebras form the corresponding variety of finite ordered algebras. Let ρ_n, $n \in \mathbb{N}$, be the relation on $\mathrm{T}_\Sigma(X)$ defined by

$$t \,\rho_n s \Leftrightarrow \mathrm{hg}(s) \geq n \text{ or } t = s$$

where $\mathrm{hg}(s)$ is the height of s. It is easy to show that ρ_n is a compatible quasi order of finite index for every $n \in \mathbb{N}$, and a tree language T is cofinite if and only if $\rho_n \subseteq \preceq_T^{-1}$ for some $n \in \mathbb{N}$. Therefore, the corresponding variety of quasi orders is $\mathscr{R} = \{\mathscr{R}(X)\}$, where $\mathscr{R}(X)$ is the filter of $\mathrm{FQ}(X)$ generated by $\{\rho_n \,|\, n \in \mathbb{N}\}$.

Example 19. Symbolic algebras and symbolic tree languages were introduced in [9]. An algebra $\mathcal{A} = (A, \Sigma, \leq_{\mathcal{A}})$ is *symbolic* if it satisfies the following: for every $f, g \in \Sigma$ and $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, a \in A$, where boldface letters stand for appropriately long sequences of elements from A:

$$f^{\mathcal{A}}(\mathbf{a}, f^{\mathcal{A}}(\mathbf{a}, a, \mathbf{b}), \mathbf{b}) = f^{\mathcal{A}}(\mathbf{a}, a, \mathbf{b});$$
$$f^{\mathcal{A}}(\mathbf{a}, g^{\mathcal{A}}(\mathbf{c}, a, \mathbf{d}), \mathbf{b}) = g^{\mathcal{A}}(\mathbf{c}, f^{\mathcal{A}}(\mathbf{a}, a, \mathbf{b}), \mathbf{d});$$
$$f^{\mathcal{A}}(\mathbf{a}, a, \mathbf{b}) \leq_{\mathcal{A}} a.$$

For a tree $t \in \mathrm{T}_\Sigma(X)$, the *contents* $\mathrm{c}(t)$ of t is the set of symbols from $\Sigma \cup X$ which appear in t. For a subset $Z \subseteq \Sigma \cup X$, the tree language $T(Z)$ consists of all trees which contain at least one appearance of each symbol from Z. A tree language $T \subseteq \mathrm{T}_\Sigma(X)$ is *symbolic* if it is a union of tree languages of the form $T(Z)$ for some subsets $Z \subseteq \Sigma \cup X$. It was shown in [9] that symbolic tree languages form a positive variety of tree languages, symbolic algebras form a variety of finite ordered algebras and that the positive variety of symbolic tree languages corresponds to this variety of ordered algebras. It can be easily proved that the relation ρ defined on $\mathrm{T}_\Sigma(X)$ by

$$t \,\rho\, s \Leftrightarrow c(t) \subseteq c(s)$$

is a compatible quasi order of finite index, and a tree language T is symbolic if and only if $\rho \subseteq \preceq_T^{-1}$. Therefore, the variety of quasi orders corresponding to the classes of symbolic tree languages and symbolic algebras consists of filters of $\mathrm{FQ}(X)$ generated by ρ, i.e., $\mathscr{R}(X) = \{\sigma \in \mathrm{FQ}(X) \,|\, \rho \subseteq \sigma\}$.

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