On the Finiteness of Picture Languages of Synchronous Deterministic Chain Code Picture Systems

Bianca Truthe[*]

Abstract

Chain Code Picture Systems are LINDENMAYER systems over a special alphabet. The strings generated are interpreted as pictures. This leads to Chain Code Picture Languages. In this paper, synchronous deterministic Chain Code Picture Systems (*sD0L* systems) are studied with respect to the finiteness of their picture languages.

First, a hierarchy of abstractions is developed, in which the interpretation of a string as a picture passes through a multilevel process. Second, on the basis of this hierarchy, an algorithm is designed which decides the finiteness or infiniteness of any *sD0L* system in polynomial time.

1 Introduction

Important tasks in the area of picture processing are describing, creating, storing and recognizing pictures. With chain codes FREEMAN provided, in the 1960s, a possibility for describing line graphics [Fre74]. A picture is formed by a sequence of drawing commands that are represented by symbols (letters). A string describes a picture, which is built by the drawing commands of its letters. FREEMAN uses an alphabet $\{0, \ldots, 7\}$, whose elements are interpreted according to the following sketch:
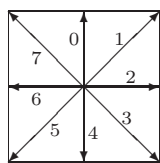
The picture to the right, for example, is generated by the word 1261204153445672606: (For reconstructing start at the tip of the nose.)
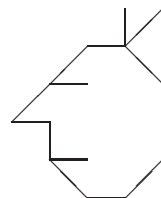
This connection of strings and pictures suggests to search for relations between formal languages and picture sets. For language theoretical considerations the four directions $\{0, 2, 4, 6\}$ are sufficient, because the additional four do not yield

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[*]Fakultät für Informatik, Otto-von-Guericke-Universität Magdeburg, PSF 4120; D-39016 Magdeburg; Germany. E-mail: `truthe@isg.cs.uni-magdeburg.de`

completely different results nor require different methods to prove the decidability of finiteness [DH89].

According to plotter commands, r, u, l, d are written as the directions *right*, *up*, *left*, *down*. With chain codes, patterns like curves, fractals or folklore patterns can be described:

Figure 1: Applications of chain codes

Chain Code Picture Systems are LINDENMAYER systems over chain codes; in this connection, the picture languages generated are of interest.

This paper follows investigations on the decidability of the finiteness of picture languages generated by synchronous Chain Code Picture Systems (*sT0L* systems) presented by Dassow and Hromkovič in [DHr92]. That paper does not say anything about how many pictures are generated in the case of finiteness. During the work on this topic it turned out that synchronous deterministic Chain Code Picture Systems with the synchronization parameter $k = 1$ can generate finite or infinite picture languages, which is in contrast to a statement in [DHr92].

In this paper, conditions are obtained under which such a system generates a finite picture language or an infinite one. For this, a hierarchy of abstractions was developed such that the interpretation of a string as a picture passes through a multilevel process.

On this basis, a complete system of finiteness conditions is obtained such that one can decide, in polynomial time, for any *sD0L* system (with an arbitrary synchronization parameter), whether the picture language generated is finite or infinite, and how many pictures are generated in the case of finiteness.

2 Fundamentals

The finiteness investigations about picture languages of synchronous deterministic Chain Code Picture Systems are based on a hierarchy of abstractions. The lowest level covers the strings over the alphabet $\{\,r, l, u, d\,\}$. Graphs of different levels of abstraction, that represent various interpretations of the strings, are associated with the strings. Such a hierarchy exists for each Chain Code Picture System over the alphabet $\{\,r, l, u, d\,\}$. The lowest level contains the string set generated by the

system. The graph set of the highest level is regarded as the picture language generated by the system.

2.1 Structures over an Alphabet

Let $\mathcal{A} = \{\, r, l, u, d \,\}$ be an alphabet. The set $\mathcal{A}^*$ is the set of all strings (with a finite length) over the alphabet $\mathcal{A}$ that are built by concatenating letters of $\mathcal{A}$. The empty string is symbolized by λ; the set $\mathcal{A}^*$ without the empty string by $\mathcal{A}^+$: $\mathcal{A}^+ = \mathcal{A}^* \setminus \{\lambda\}$. The free structure $(\mathcal{A}^*, \cdot)$ over the alphabet $\mathcal{A}$ with the concatenation operation $\cdot$ is a monoid.

The length $\#\mathsf{w}$ of a string w is the number of letters in w. The set of all strings of length n from $\mathcal{A}^*$ is denoted by $\mathcal{A}^n$. A string $\mathsf{w} \in \mathcal{A}^n$ is composed of letters $w_1, \ldots, w_n$ if not stated otherwise: $\mathsf{w} = w_1 \cdots w_n$. A substring $w_1 \cdots w_i$ $(0 \le i \le n)$ is written as $\overrightarrow{w_i}$ $(\overrightarrow{w_0} = \lambda)$.

For a string $\mathsf{w} \in \mathcal{A}^n$ and a letter $x \in \mathcal{A}$, $\#_x \mathsf{w}$ is the number of occurrences of x in w. For a string $\mathsf{w} \in \mathcal{A}^*$, $[\mathsf{w}]$ is the set of all letters in w:

$$[\mathsf{w}] = \{\, x \mid \#_x \mathsf{w} \ge 1 \,\}.$$

The elements w of $\mathcal{A}^*$ are interpreted as mappings on $\mathbb{Z}^2$:

$$\mathsf{w} : \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2 \quad (\mathsf{w} \in \mathcal{A}^*),$$

which are inductively defined as follows. The atomic mappings r, l, u, d assign to a point $\mathfrak{q} \in \mathbb{Z}^2$ its neighbours:

$$
\begin{array}{llll}
r(\mathfrak{q}) &=& \mathfrak{q} + (1,0) \qquad & l(\mathfrak{q}) &=& \mathfrak{q} - (1,0) \\
u(\mathfrak{q}) &=& \mathfrak{q} + (0,1) & d(\mathfrak{q}) &=& \mathfrak{q} - (0,1)
\end{array}.
$$

The translations $x(\mathfrak{q}) - \mathfrak{q}$ of any point $\mathfrak{q} \in \mathbb{Z}^2$ to its neighbours $x(\mathfrak{q})$ are designated by $\mathfrak{v}_x \in \mathbb{Z}^2$:

$$
\mathfrak{v}_x = \begin{cases}
(1,0), & \text{if } x = r \\
(-1,0), & \text{if } x = l \\
(0,1), & \text{if } x = u \\
(0,-1), & \text{if } x = d
\end{cases}.
$$

The mappings x from $\mathcal{A}$ are translations. Every mapping $x \in \mathcal{A}$ is surjective (the range of values is $\mathbb{Z}^2$), injective (from $x(\mathfrak{p}) = x(\mathfrak{q})$ always follows $\mathfrak{p} = \mathfrak{q}$) and, therefore, bijective (one to one).

Two arbitrary mappings x, y are called disjoint if their function values differ for each argument.

Proposition 1. *Every two different mappings $x, y \in \mathcal{A}$, $(x \ne y)$, never give the same neighbour: $\forall \mathfrak{q} \in \mathbb{Z}^2 : x(\mathfrak{q}) \ne y(\mathfrak{q})$. This means that the mappings in $\mathcal{A}$ are disjoint.*

The empty string corresponds to the identical mapping

$$\lambda : \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2 \quad \text{with } \mathfrak{q} \mapsto \mathfrak{q}.$$

A compound string $\mathsf{vw} \in \mathcal{A}^*$ stands for the concatenated mapping $\mathsf{v} \circ \mathsf{w}$:

$$\mathsf{v} \circ \mathsf{w} : \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2 \quad \text{with } \mathfrak{q} \mapsto \mathsf{w}(\mathsf{v}(\mathfrak{q})).$$

The zero point of the $\mathbb{Z}^2$ is symbolized by $\mathfrak{o}$: $\mathfrak{o} = (0,0)$.

This interpretation of strings as mappings on $\mathbb{Z}^2$ is a homomorphism from the free structure $(\mathcal{A}^*, \cdot)$ in the free structure $(\mathcal{A}^*, \circ)$. Hence $(\mathcal{A}^*, \circ)$ is also a monoid. For each mapping $\mathsf{w} \in \mathcal{A}^*$, an inverse mapping $\mathsf{w}^{-1} \in \mathcal{A}^*$ exists:

- The inverse of the identical mapping is the identical mapping: $\lambda^{-1} = \lambda$.
- The inverses of the atomic mappings are $r^{-1} = l$, $l^{-1} = r$, $u^{-1} = d$, $d^{-1} = u$, because $\mathfrak{v}_x = -\mathfrak{v}_{x^{-1}}$ ($x \in \mathcal{A}$).
- Let $\mathsf{w} = w_1 \circ \cdots \circ w_n$ ($w_i \in \mathcal{A}, i = 1, \ldots, n$) be a concatenated mapping. Then the inverse mapping is $\mathsf{w}^{-1} = w_n^{-1} \circ \cdots \circ w_1^{-1}$.

This result is stated in the following proposition.

Proposition 2. *The algebraic structure $(\mathcal{A}^*, \circ)$ is a group.*

The operator $\circ$ is not written if the context shows which operation is meant. For example, $(x_1 x_2)(\mathfrak{o})$ implies that $x_1 x_2$ symbolizes the concatenated mapping $x_1 \circ x_2$, whereas $x_1 x_2$ in $[x_1 x_2]$ represents the compound string $x_1 x_2$.

The mappings ru, ur and ld, dl assign the diagonal neighbours to a point $\mathfrak{q}$:

$$ru(\mathfrak{q}) = ur(\mathfrak{q}) = \mathfrak{q} + (1,1); \quad ld(\mathfrak{q}) = dl(\mathfrak{q}) = \mathfrak{q} - (1,1).$$

These relations are symbolized by $^{-}$ and $^{\perp}$:

x	$\bar{x}$	$x^{\perp}$	$\bar{x}^{\perp}$
r	l	u	d
l	r	d	u
u	d	r	l
d	u	l	r

The mappings $\mathsf{w} \in \mathcal{A}^n$ are translations: $\mathsf{w}(\mathfrak{p} + \mathfrak{q}) = \mathsf{w}(\mathfrak{p}) + \mathfrak{q}$ (can be proved by induction over n). This leads to the following proposition about the correlation between mappings of the zero point.

Proposition 3. *Let v, w be two words of $\mathcal{A}^*$. The mapping of the zero point by the compound mapping $\mathsf{v} \circ \mathsf{w}$ is $(\mathsf{v} \circ \mathsf{w})(\mathfrak{o}) = \mathsf{v}(\mathfrak{o}) + \mathsf{w}(\mathfrak{o})$.*

Proof. The mapping of the zero point by $\mathsf{v} \circ \mathsf{w}$ yields

$$
\begin{aligned}
(\mathsf{v} \circ \mathsf{w})(\mathfrak{o}) &= \mathsf{w}(\mathsf{v}(\mathfrak{o})) \\
&= \mathsf{w}(\mathfrak{o} + \mathsf{v}(\mathfrak{o})) \\
&= \mathsf{w}(\mathfrak{o}) + \mathsf{v}(\mathfrak{o}) \\
&= \mathsf{v}(\mathfrak{o}) + \mathsf{w}(\mathfrak{o}).
\end{aligned}
$$

$\square$

2.2 Graphical Embedding

A grid graph is a graph with the following properties:

– The set of vertices is a subset of $\mathbb{Z}^2$.
– Each edge connects two neighbours $\mathfrak{q} \in \mathbb{Z}^2$ and $x(\mathfrak{q})$ with $x \in \mathcal{A}$.

The position of the vertices is essential; renaming of the vertices does not yield an isomorphic graph. For example, the graphs ⌐ and ⌐ should be considered as non-identical.

For each point $\mathfrak{a} \in \mathbb{Z}^2$, functions exist that assign, to a word $\mathsf{w} \in \mathcal{A}^n$

– the set of vertices $\odot^{\mathfrak{a}}(\mathsf{w}) = \{\ \overrightarrow{w_i}(\mathfrak{a}) \mid i = 0, \ldots n\ \}$,
– the directed grid graph (possibly with multiple edges)

$$g^{\mathfrak{a}}(\mathsf{w}) = \left(\odot^{\mathfrak{a}}(\mathsf{w}), \{\ (\overrightarrow{w_{i-1}}(\mathfrak{a}), \overrightarrow{w_i}(\mathfrak{a}))\ \}_{i=1,\ldots,n} \right),$$

– the simple directed grid graph $s^{\mathfrak{a}}(\mathsf{w})$ of $g^{\mathfrak{a}}(\mathsf{w})$ (without multiple edges),
– the set of edges $\|^{\mathfrak{a}}\mathsf{w}$ of $s^{\mathfrak{a}}(\mathsf{w})$ in a different notation

$$\|^{\mathfrak{a}}\mathsf{w} = \{\ (\overrightarrow{w_{i-1}}(\mathfrak{a}), w_i) \mid i = 1, \ldots, n\ \},$$

– the picture (the shade of $s^{\mathfrak{a}}(\mathsf{w})$)

$$p^{\mathfrak{a}}(\mathsf{w}) = (\odot^{\mathfrak{a}}(\mathsf{w}), \{\ (\overrightarrow{w_{i-1}}(\mathfrak{a}), \overrightarrow{w_i}(\mathfrak{a})), (\overrightarrow{w_i}(\mathfrak{a}), \overrightarrow{w_{i-1}}(\mathfrak{a})) \mid i = 0, \ldots, n\ \}).$$

If the reference point $\mathfrak{a}$ is the zero point, the upper index will be omitted. The set $\|^{\mathfrak{a}}\mathsf{w}$ contains a pair $(\mathfrak{q}, x)$ with $\mathfrak{q} \in \mathbb{Z}^2$ and $x \in \mathcal{A}$ if and only if $(\mathfrak{q}, x(\mathfrak{q}))$ is an edge in the graph $s^{\mathfrak{a}}(\mathsf{w})$ (if $(\mathfrak{q}, \widetilde{\mathfrak{q}})$ is an edge then $x \in \mathcal{A}$ with $x(\mathfrak{q}) = \widetilde{\mathfrak{q}}$ exists uniquely – due to Prop. 1). Thus, the graph $s^{\mathfrak{a}}(\mathsf{w})$ is one to one associated with the set $\|^{\mathfrak{a}}\mathsf{w}$. Throughout this paper, this set is referred to as the edge set of w with respect to $\mathfrak{a}$.

The following example shall demonstrate these correlations:

Example 1. Let $\mathsf{w} = ruullurddrurrulddldr$ be a word from $\mathcal{A}^*$.

If a plotter takes this word as a sequence of elementary commands for drawing, the resulting picture will be:

For investigations on picture languages generated by *sD0L* systems, it is necessary also to know *how* a picture is drawn. To show how a pictured arises, the lines are marked by arrows indicating the drawing direction. Additionally, the grid points are marked (the only points where the direction can change). This leads to the following grid graph (beginning at the zero point). Note that the line from $(1,1)$ upwards is drawn twice.

The vertex set $\odot(\mathsf{w})$ contains all grid points visited:
$\mathfrak{o}$, $r(\mathfrak{o}) = (1,0)$, $ru(\mathfrak{o}) = (1,1)$, $ruu(\mathfrak{o}) = (1,2)$, $\ldots$, $\mathsf{w}(\mathfrak{o}) = (2,0)$.

The directed grid graph $g^{\mathfrak{a}}(\mathsf{w})$ consists of the vertex set $\odot(\mathsf{w})$ and all edges on the 'drawing path': $(\mathfrak{o}, (1,0))$, $((1,0),(1,1))$, $((1,1),(1,2))$, $\ldots$, $((0,1),(1,1))$, $((1,1),(1,2))$, $((1,2),(2,2))$, $\ldots$, $((1,0),(2,0))$. The edge $((1,1),(1,2))$ occurs twice because the underlined letters in $ru\underline{u}llurddr\underline{u}rrulddldr$ both produce this line (because of $ru(\mathfrak{o}) = (1,1) = ruullurddr(\mathfrak{o})$). The edge set $\|\mathsf{w}$ consists of all edges passed as elements of $\mathbb{Z}^2 \times \mathcal{A}$ instead of $\mathbb{Z}^2 \times \mathbb{Z}^2$: $(\mathfrak{o}, r)$, $((1,0), u)$, $((1,1), u)$, $\ldots$, $((1,1), d)$, $((1,0), r)$.

Since the pictures are shades of the simple directed graphs, one immediately notices that two words having the same edge set also represent the same picture. This result is stated in the following proposition, so it can be referred to.

Proposition 4. *If the edge sets $\|\mathsf{v}$ and $\|\mathsf{w}$ of two words $\mathsf{v}, \mathsf{w} \in \mathcal{A}^*$ coincide, so the pictures $p(\mathsf{v})$ and $p(\mathsf{w})$ do as well.*

The following proposition states correlations between concatenating strings and combining graphs.

Proposition 5. *The concatenation of strings is associated with a union of vertex sets, directed graphs, edge sets, and pictures: For each point $\mathfrak{a} \in \mathbb{Z}^2$ and any two strings $\mathsf{v}, \mathsf{w} \in \mathcal{A}^*$, one has*

$$
\begin{aligned}
\odot^{\mathfrak{a}}(\mathsf{vw}) &= \odot^{\mathfrak{a}}(\mathsf{v}) \cup \odot^{\mathsf{v}(\mathfrak{a})}(\mathsf{w}), \\
g^{\mathfrak{a}}(\mathsf{vw}) &= g^{\mathfrak{a}}(\mathsf{v}) \cup g^{\mathsf{v}(\mathfrak{a})}(\mathsf{w}), \\
\|^{\mathfrak{a}}\mathsf{vw} &= \|^{\mathfrak{a}}\mathsf{v} \cup \|^{\mathsf{v}(\mathfrak{a})}\mathsf{w}, \\
p^{\mathfrak{a}}(\mathsf{vw}) &= p^{\mathfrak{a}}(\mathsf{v}) \cup p^{\mathsf{v}(\mathfrak{a})}(\mathsf{w}).
\end{aligned}
$$

Proof. Let v be an element of $\mathcal{A}^n$ and w be an element of $\mathcal{A}^m$. Then the union of the vertex sets $\odot^{\mathfrak{a}}(\mathsf{v})$ and $\odot^{\mathsf{v}(\mathfrak{a})}(\mathsf{w})$ is

$$
\begin{aligned}
\odot^{\mathfrak{a}}(\mathsf{v}) \cup \odot^{\mathsf{v}(\mathfrak{a})}(\mathsf{w}) &= \{\, \overrightarrow{v_i}(\mathfrak{a}) \mid i = 0, \ldots, n \,\} \cup \{\, \overrightarrow{w_i}(\mathsf{v}(\mathfrak{a})) \mid i = 0, \ldots, m \,\} \\
&= \{\, \mathfrak{a}, v_1(\mathfrak{a}), \ldots, \mathsf{v}(\mathfrak{a}), w_1(\mathsf{v}(\mathfrak{a})), \ldots, \mathsf{vw}(\mathfrak{a}) \,\} \\
&= \odot^{\mathfrak{a}}(\mathsf{vw}).
\end{aligned}
$$

The union of the other sets can be shown similarly. (Note that the non-simple graphs possibly contain multiple edges.) $\qquad\square$

The sets

- $\mathcal{A}^*$ of strings,
- $\mathcal{G} = \{\, g^{\mathfrak{a}}(\mathsf{w}) \mid \mathsf{w} \in \mathcal{A}^*, \mathfrak{a} \in \mathbb{Z}^2 \,\}$ of directed graphs,
- $\mathcal{S} = \{\, s^{\mathfrak{a}}(\mathsf{w}) \mid \mathsf{w} \in \mathcal{A}^*, \mathfrak{a} \in \mathbb{Z}^2 \,\}$ of simple directed graphs, and
- $\mathcal{P} = \{\, p^{\mathfrak{a}}(\mathsf{w}) \mid \mathsf{w} \in \mathcal{A}^*, \mathfrak{a} \in \mathbb{Z}^2 \,\}$ of pictures
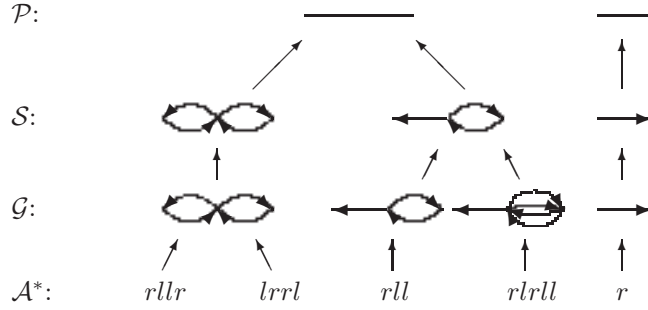
Figure 2: Hierarchy of abstractions

form a hierarchy of different levels of abstraction. A part of this hierarchy is to be seen in Figure 2.

A later derivation of words by a simultanous replacing of letters can be interpreted as a derivation of graphs by replacing an edge by a graph. In deterministic systems, one letter is always replaced by the same word. Thus, one edge is always replaced by the same graph – a derivation of an edge is independent from the number of its occurrences. In non-deterministic systems however, a letter at one position can be replaced by a different word than the same letter at another position. Hence, one occurrence of an edge can be replaced by a different graph than another occurrence of the same edge – the number of occurrences is essential. For this reason, the graphs with multiple edges are kept in the hierarchy, although the simple graphs are sufficient in this paper.

A rectangle R determined by two points $\mathfrak{p} = (p_x, p_y)$ and $\mathfrak{q} = (q_x, q_y)$ is the set of all points $\mathfrak{a} = (a_x, a_y)$ between $\mathfrak{p}$ and $\mathfrak{q}$:

$$R = \left\{ \mathfrak{a} \in \mathbb{Z}^2 \ \middle| \ \begin{matrix} p_x \leq a_x \leq q_x \text{ or } q_x \leq a_x \leq p_x \text{ and} \\ p_y \leq a_y \leq q_y \text{ or } q_y \leq a_y \leq p_y \end{matrix} \right\}.$$

Such a rectangle is written as $R = [\mathfrak{p}, \mathfrak{q}]$. The picture area of a set $S \subseteq \mathbb{Z}^2$, denoted by $\mathfrak{R}_S$, is the smallest rectangle that contains S. By scaling a picture area $\mathfrak{P} = [\mathfrak{p}, \mathfrak{q}]$ by a factor $s \in \mathbb{N}_0$, the picture area $s\mathfrak{P} = \{ s\mathfrak{b} \mid \mathfrak{b} \in \mathfrak{P} \} = [s\mathfrak{p}, s\mathfrak{q}]$ is obtained. The union of two picture areas is not a rectangle in general. The extended union shall give the rectangle covering the normal union:

$$\mathfrak{R}_P \uplus \mathfrak{R}_Q = \mathfrak{R}_{P \cup Q}.$$

Let $\odot^{\mathfrak{a}}(\mathsf{w})$ be the vertex set of a word w with respect to $\mathfrak{a}$. Then, the functions $|\cdot\, _{\mathsf{w}}^{\mathfrak{a}}, \, \_\cdot_{\mathsf{w}}^{\mathfrak{a}}, \, \cdot|_{\mathsf{w}}^{\mathfrak{a}}, \, \overline{\cdot}\,_{\mathsf{w}}^{\mathfrak{a}}$ give the 'border' of $\odot^{\mathfrak{a}}(\mathsf{w})$:

$$|\cdot\, _{\mathsf{w}}^{\mathfrak{a}} = \min \{ x \mid (x, y) \in \odot^{\mathfrak{a}}(\mathsf{w}) \}, \qquad \_\cdot_{\mathsf{w}}^{\mathfrak{a}} = \min \{ y \mid (x, y) \in \odot^{\mathfrak{a}}(\mathsf{w}) \}$$

$$\cdot|_{\mathsf{w}}^{\mathfrak{a}} = \max \{ x \mid (x, y) \in \odot^{\mathfrak{a}}(\mathsf{w}) \}, \qquad \overline{\cdot}\,_{\mathsf{w}}^{\mathfrak{a}} = \max \{ y \mid (x, y) \in \odot^{\mathfrak{a}}(\mathsf{w}) \}$$

The symbols $\llcorner_{\mathsf{w}}^{\mathfrak{a}} = (|\cdot\, _{\mathsf{w}}^{\mathfrak{a}}, \_\cdot_{\mathsf{w}}^{\mathfrak{a}})$ and $\urcorner_{\mathsf{w}}^{\mathfrak{a}} = (\cdot|_{\mathsf{w}}^{\mathfrak{a}}, \overline{\cdot}\,_{\mathsf{w}}^{\mathfrak{a}})$ stand for the lower-left and upper-right corners. The picture area of $\odot^{\mathfrak{a}}(\mathsf{w})$ is denoted by $\square^{\mathfrak{a}}(\mathsf{w}) = [\llcorner_{\mathsf{w}}^{\mathfrak{a}}, \urcorner_{\mathsf{w}}^{\mathfrak{a}}]$.

2.3 Special Endomorphisms

Let κ, μ be two natural numbers, $\kappa, \mu \in \mathbb{N}_0$. An endomorphism h on $\mathcal{A}^*$ is called (κ, μ)-endomorphism if the following conditions are satisfied for each $x \in \mathcal{A}$:
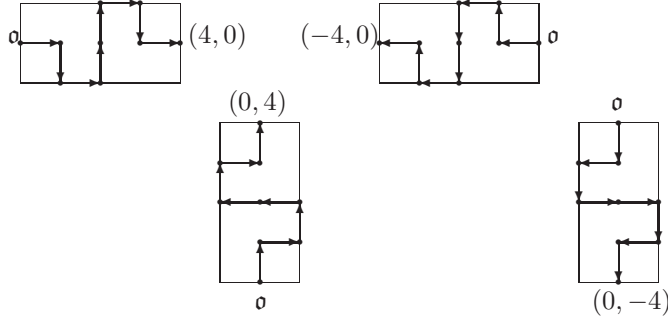
1. $(h(x))(\mathfrak{o}) = \kappa \mathfrak{v}_x$.

2. $\boxdot(h(x)) \subseteq \kappa[\mathfrak{o}, \mathfrak{v}_x] \uplus \mu[\mathfrak{v}_{x^\perp}, \mathfrak{v}_{\bar{x}^\perp}]$.

The following example illustrates this.

Example 2. Let h be an endomorphism on $\mathcal{A}^*$ with $r \mapsto rdruurdr$, $l \mapsto lulddlul$, $u \mapsto urulluru$, and $d \mapsto dldrrdld$. Then one has

$$
\begin{array}{rcccl}
rdruurdr(\mathfrak{o}) &=& 4\mathfrak{v}_r + 2\mathfrak{v}_d + 2\mathfrak{v}_u &=& 4\mathfrak{v}_r, \\
lulddlul(\mathfrak{o}) &=& 4\mathfrak{v}_l + 2\mathfrak{v}_u + 2\mathfrak{v}_d &=& 4\mathfrak{v}_l, \\
urulluru(\mathfrak{o}) &=& 4\mathfrak{v}_u + 2\mathfrak{v}_r + 2\mathfrak{v}_l &=& 4\mathfrak{v}_u, \\
dldrrdld(\mathfrak{o}) &=& 4\mathfrak{v}_d + 2\mathfrak{v}_l + 2\mathfrak{v}_r &=& 4\mathfrak{v}_d.
\end{array}
$$

Hence, the first condition is satisfied. The simple directed graphs of $h(x)$ $(x \in \mathcal{A})$ are:

All points of $\odot(h(x))$ (for each $x \in \mathcal{A}$) lie in the rectangle covering both the lines $[\mathfrak{o}, 4\mathfrak{v}_x]$ and $[\mathfrak{v}_{x^\perp}, \mathfrak{v}_{\bar{x}^\perp}]$. The picture area is $\boxdot(h(x)) = 4[\mathfrak{o}, \mathfrak{v}_x] \uplus [\mathfrak{v}_{x^\perp}, \mathfrak{v}_{\bar{x}^\perp}]$ for $x \in \mathcal{A}$. Thus, h is a $(4, 1)$-endomorphism. The first synchronization condition says where the end point of a drawing lies. The second one causes the pictures to lie in certain rectangles.

The n-ary concatenation of an endomorphism h is written shortly as h^n. Applied to a string $\mathsf{w} \in \mathcal{A}^*$, its result is the n-th derivative of w; written as $\mathsf{w}^{(n)}$ (w', w'', w''' for the first three derivatives, $\mathsf{w}^{(0)} = \mathsf{w}$). The parameter κ defines the length of the derivation picture regarding the respective direction; the derivation is length-contracting in the case of $\kappa < 1$, length-constant in the case of $\kappa = 1$, and length-expanding in the case of $\kappa > 1$. The parameter μ is the width of the derivation picture.

2.4 Chain Code Picture Systems

A synchronous deterministic context free Chain Code Picture System (*sD0L* system) is a triple

$$G = (\mathcal{A}, h, \omega)$$

with the alphabet $\mathcal{A} = \{r, l, u, d\}$, a (κ, μ)-endomorphism h on $\mathcal{A}^*$, and a non-empty start string (axiom) $\omega \in \mathcal{A}^+$.

The picture language P_G generated by an *sD0L* system G is the set of all pictures of derivatives of the axiom ω:

$$P_G = \left\{ p(\omega^{(n)}) \mid n \in \mathbb{N}_0 \right\}.$$

An *sD0L* system is called length-contracting (-constant, -expanding) if the (κ, μ)-endomorphism belonging to it, has this property.

3 Finiteness Investigations

Let $G = (\mathcal{A}, h, \omega)$ be an *sD0L* system with a (κ, μ)-endomorphism h. The n-th derivative $(n \in \mathbb{N})$ of any letter $x \in \mathcal{A}$ maps the zero point $\mathfrak{o}$ to the point $\kappa^n \mathfrak{v}_x$: $x^{(n)}(\mathfrak{o}) = \kappa^n \mathfrak{v}_x$. This can be proved by induction.

Moreover, the first synchronization condition says that $x'(\mathfrak{o}) = \kappa \mathfrak{v}_x$. This means $x'(\mathfrak{o}) = \kappa \mathfrak{v}_x + c \mathfrak{v}_x + c \mathfrak{v}_{\bar{x}} + d \mathfrak{v}_{x^\perp} + d \mathfrak{v}_{\bar{x}^\perp}$ for some natural numbers c, d. Consequently, $x^\perp$ and $\bar{x}^\perp$ have the same numbers of occurrences in the derivative x', and x has κ more occurrences than $\bar{x}$. These observations are summarized in the following proposition.

Proposition 6. *For all $x \in \mathcal{A}$, one has*

1. $x^{(n)}(\mathfrak{o}) = \kappa^n \mathfrak{v}_x$ *for $n \in \mathbb{N}$,*

2. $\#_x x' = \kappa + \#_{\bar{x}} x'$,

3. $\#_{x^\perp} x' = \#_{\bar{x}^\perp} x'$.

The Chain Code Picture Systems are distiguished by their 'length behaviour' (represented by the parameter κ).

3.1 Length-contracting Chain Code Picture Systems

Let $G = (\mathcal{A}, h, \omega)$ be an *sD0L* system with a length-contracting (κ, μ)-endomorphism h. Since $\kappa < 1$ and $\kappa \in \mathbb{N}_0$, κ must be equal to 0.

The second synchronization condition has the effect that the picture area of the derivative x' of a letter $x \in \mathcal{A}$ is a line: $\Box(x') \subseteq \mu[\mathfrak{v}_x^\perp, \mathfrak{v}_{\bar{x}^\perp}]$. Hence, x and $\bar{x}$ do not occur in the string x'.

For any string $\mathsf{w} \in \mathcal{A}^*$, exactly one of the following three cases occurs:

1. $\mathsf{w}' = \lambda$. Then all further derivatives are also empty: $\mathsf{w}^{(n)} = \lambda$, $n \geq 1$.

2. $\mathsf{w}' \neq \lambda$, $\mathsf{w}'' = \lambda$. Then all further derivatives are empty: $\mathsf{w}^{(n)} = \lambda$, $n \geq 2$.

3. $\mathsf{w}' \neq \lambda$, $\mathsf{w}'' \neq \lambda$. Since $\mathsf{w}' \neq \lambda$, some letter $x \in \mathcal{A}$ occurs in w' and also the same number of $\bar{x}$. Hence, the letter set $[\mathsf{w}']$ can be $\{\, r, l \,\}$, $\{\, u, d \,\}$ or $\mathcal{A}$.

Example 3. For example, let h be a $(0, \mu)$-endomorphism with

$$r \mapsto ud, \ u \mapsto rl, \ d \mapsto \lambda \ \text{ and } l \mapsto \lambda.$$

If $\mathsf{w} = r$ then $[\mathsf{w}'] = \{\, u, d \,\}$, if $\mathsf{w} = u$ then $[\mathsf{w}'] = \{\, r, l \,\}$. If $\mathsf{w} = ru$ then the letter set $[\mathsf{w}']$ consists of all letters: $[\mathsf{w}'] = \mathcal{A}$.

If $[\mathsf{w}'] = \{\, r, l \,\}$, the letter set $[\mathsf{w}'']$ is $\{\, u, d \,\}$ (it cannot be empty because $\mathsf{w}'' \neq \lambda$). Similarly, $[\mathsf{w}''] = \{\, r, l \,\}$ if $[\mathsf{w}'] = \{\, u, d \,\}$. If $[\mathsf{w}'] = \mathcal{A}$, some letter $x \in \mathcal{A}$ occurs in the word w together with $x^{\perp}$ or $\bar{x}^{\perp}$. As these letters occur in w' also, w'' consists of the same letters as w'. Summerized, one has

$$[\mathsf{w}''] = \begin{cases} \{u, d\} & \text{if } [\mathsf{w}'] = \{r, l\}, \\ \{r, l\} & \text{if } [\mathsf{w}'] = \{u, d\}, \\ \mathcal{A} & \text{if } [\mathsf{w}'] = \mathcal{A}. \end{cases}$$

An analogous argumentation leads to

$$[\mathsf{w}'''] = \begin{cases} \{r, l\} & \text{if } [\mathsf{w}'] = \{r, l\}, \\ \{u, d\} & \text{if } [\mathsf{w}'] = \{u, d\}, \\ \mathcal{A} & \text{if } [\mathsf{w}'] = \mathcal{A}, \end{cases}$$

that is $[\mathsf{w}'''] = [\mathsf{w}']$. Thus, the letter set of the fourth derivative coincides with that one of the second derivative:

$$[\mathsf{w}^{(4)}] = \bigcup_{x \in [\mathsf{w}''']} [x'] = \bigcup_{x \in [\mathsf{w}']} [x'] = [\mathsf{w}''].$$

From this case distinction one can conclude the following proposition by induction:

Proposition 7. *The letter sets from the second derivative of a string $\mathsf{w} \in \mathcal{A}^*$ on are either empty or alternate beginning with the first derivative:*

$$[\mathsf{w}''] = \emptyset \implies [\mathsf{w}^{(n)}] = \emptyset \quad (n \geq 2),$$

$$[\mathsf{w}''] \neq \emptyset \implies [\mathsf{w}^{(2n-1)}] = [\mathsf{w}'] \wedge [\mathsf{w}^{(2n)}] = [\mathsf{w}''] \quad (n \geq 2).$$

A similar correlation can be found for edge sets. Let $\mathsf{w}^{(n)}$ be the n-th derivative of a string $\mathsf{w} \in \mathcal{A}^*$:

$$\mathsf{w}^{(n)} = x_1 \cdots x_l \quad (x_i \in \mathcal{A}, i = 1, \ldots, l).$$

The edge set of $\mathsf{w}^{(n+1)}$ is, according to Proposition 5,

$$\|\mathsf{w}^{(n+1)} = \|x_1' \cup \|^{x_1'(\circ)} x_2' \cup \cdots \cup \|^{(x_1' \cdots x_{l-1}')(\circ)} x_l'.$$

The Propositions 3 and 6 imply that $(x'_1 \cdots x'_i)(\mathfrak{o}) = x'_1(\mathfrak{o}) + \cdots + x'_i(\mathfrak{o}) = \mathfrak{o}$ for $i = 1, \ldots, l$. Hence, the edge set of $\mathsf{w}^{(n+1)}$ is

$$\|\mathsf{w}^{(n+1)} = \bigcup_{i=1}^{l} \|x'_i = \bigcup_{x \in [\mathsf{w}^{(n)}]} \|x' \quad (\text{if } \hat{x} = \check{x} \text{ then } \hat{x}' = \check{x}' \text{ and } \|\hat{x}' = \|\check{x}').$$

According to Proposition 7, one obtains $\|\mathsf{w}^{(n)} = \emptyset$ if $\mathsf{w}'' = \lambda$ and, in the case that $\mathsf{w}'' \neq \lambda$:

$$\|\mathsf{w}^{(2n)} = \bigcup_{x \in [\mathsf{w}^{(2n-1)}]} \|x' = \bigcup_{x \in [\mathsf{w}']} \|x' = \|\mathsf{w}'',$$

$$\|\mathsf{w}^{(2n+1)} = \bigcup_{x \in [\mathsf{w}^{(2n)}]} \|x' = \bigcup_{x \in [\mathsf{w}'']} \|x' = \|\mathsf{w}''',$$

for $n \geq 2$.

Thus, if $\mathsf{w}'' = \lambda$ then the pictures of $\mathsf{w}^{(n)}$ $(n \geq 2)$ consist of the zero point only. If $\mathsf{w}'' \neq \lambda$ then the pictures of even derivations coincide from the second derivation on; those of odd derivations from the third one on (because of Prop. 4). After the third derivation of a word $\mathsf{w} \in \mathcal{A}^*$, no new picture arises.

Theorem 1. *Let* $G = (\mathcal{A}, h, \omega)$ *be a length-contracting sD0L system. The picture language generated is*

$$P_G = \{\, p(\omega), p(\omega'), p(\omega''), p(\omega''') \,\}.$$

The following example shows a length-contracting sD0L system together with its picture language.

Example 4. Let $G = (\mathcal{A}, h, r)$ be an sD0L system with a $(0, \mu)$-endomorphism h with $r \mapsto ud$, $l \mapsto du$, $u \mapsto rl$, and $d \mapsto \lambda$. The words generated by G are r, ud, rl, $uddu$, $(rl)^2$, $(uddu)^2$, $(rl)^4$, $(uddu)^4$, etc. The corresponding pictures are

━, │, ━, │, ━, │, ━, │, etc.

Since a new picture does not occur, the picture language is $\left\{\, ━, │, │ \,\right\}$.

3.2 Length-expanding Chain Code Picture Systems

Let $G = (\mathcal{A}, h, \omega)$ be a length-expanding sD0L system. Since the (κ, μ)-endomorphism h is length-expanding, κ is greater than 1.

Let $x \in \mathcal{A}$ be the initial letter of ω (since $\omega \in \mathcal{A}^+$ it has at least one letter): $\omega = x\mathsf{w}$ ($\mathsf{w} \in \mathcal{A}^*$). The n-th derivative is $\omega^{(n)} = x^{(n)}\mathsf{w}^{(n)}$, and $x^{(n)}(\mathfrak{o})$ is a vertex of every graph of $\omega^{(n)}$:

$$x^{(n)}(\mathfrak{o}) \in \odot(\omega^{(n)}), \quad n \in \mathbb{N}_0.$$

Furthermore, let X_ω be the set of all points $x^{(n)}(\mathfrak{o})$, V_ω the union of the vertex sets $\odot(\omega^{(n)})$, and P_ω the set of all pictures $p(\omega^{(n)})$, $n \in \mathbb{N}_0$:

$$X_\omega = \left\{ x^{(n)}(\mathfrak{o}) \mid n \in \mathbb{N}_0 \right\} = \left\{ k^n \mathfrak{v}_x \mid n \in \mathbb{N}_0 \right\} \quad \text{(Prop. 6)},$$

$$V_\omega = \bigcup_{n \in \mathbb{N}_0} \odot(\omega^{(n)}),$$

$$P_\omega = \left\{ p(\omega^{(n)}) \mid n \in \mathbb{N}_0 \right\}.$$

The set X_ω is infinite because $\kappa > 1$ and $\mathfrak{v}_x \neq \mathfrak{o}$. Every point of X_ω also occurs in V_ω: $X_\omega \subseteq V_\omega$; thus the set V_ω is also infinite. Each vertex set $\odot(\omega^{(n)})$, $n \in \mathbb{N}_0$, is finite; hence, there are infinitely many different ones in the union V_ω. If the vertex sets of two strings $\mathsf{u}, \mathsf{v} \in \mathcal{A}^*$ differ, so the pictures do as well. Hence, among the pictures $p(\omega^{(n)})$ with $n \in \mathbb{N}_0$, there are infinitely many different ones: $|P_\omega| = \infty$.

Theorem 2. *For every length-expanding sD0L system $G = (\mathcal{A}, h, \omega)$, the picture language P_G generated by G is infinite.*

The following example shows a length-expanding *sD0L* system.

Example 5. Let $G = (\mathcal{A}, h, ruld)$ be a length-expanding *sD0L* system with the $(4, 1)$-endomorphism h given in Example 2. The first pictures generated are drawn below (up to the third derivation).
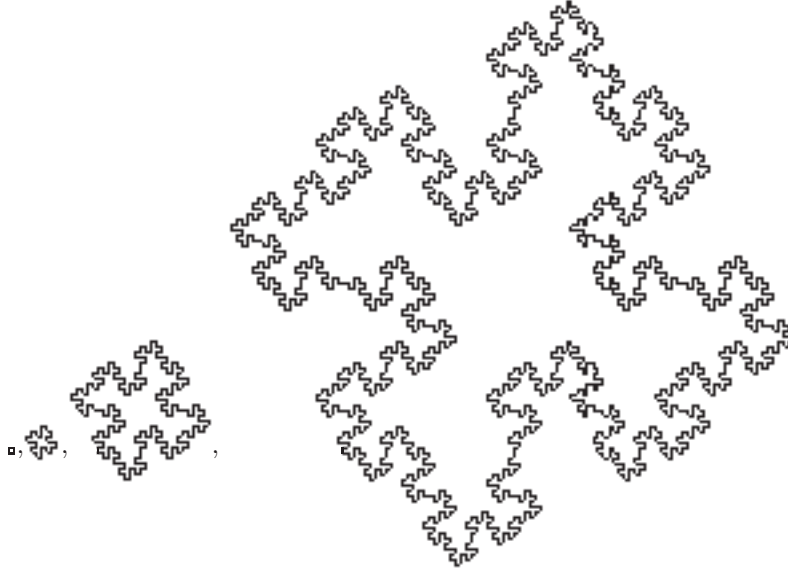
3.3 Length-constant Chain Code Picture Languages

In contrast to the previous situations, among the length-constant *sD0L* systems, there are those with a finite picture language as well as those with an infinite one. The following example shows an *sD0L* system with a finite picture language and a similar *sD0L* system with an infinite picture language.

Example 6. Let h be a $(1, \mu)$-endomorphism with $r \mapsto rud$, $u \mapsto uldru$, $d \mapsto d$, and $l \mapsto l$. Then $G = (\mathcal{A}, h, r)$ is a length-constant *sD0L* system. The simple directed graph of the axiom is ⟶, of the first derivative , and of the second derivative . The graphs of the later derivatives are the same: . Hence, the picture language generated is finite: $P_G = \left\{ \underline{}, \underline{\lrcorner}, \square \right\}$. Now, change h such that $l \mapsto lrl$. Then, the simple directed graph of the third derivative is not the same as that one of the second derivative, but . A new r-edge arose that will be replaced (in the next step) by its derivative; thus, the fourth derivative has the

graph . A new r-edge occurs in every third derivative (6th, 9th, etc.). Hence, the picture sizes increase; the picture language is infinite.

The example above shows that, in the case of $\kappa = 1$, further investigations are needed to find out when an *sD0L* system generates a finite picture language and when it does not.

Several examples lead to the supposition that the difference between the edge sets of the second and third derivatives indicates the finiteness of the picture language: If they do not differ, the picture language is finite; if there is a difference, then the language is infinite. This supposition will be confirmed and proved.

Let $G = (\mathcal{A}, h, \omega)$ be a length-constant *sD0L* system. The next proposition extends the first statement of Proposition 6 to words.

Proposition 8. *If h is a $(1, \mu)$-endomorphism, then $\mathsf{w}^{(n)}(\mathfrak{o}) = \mathsf{w}(\mathfrak{o})$ for every string $\mathsf{w} \in \mathcal{A}^*$ and every derivation step $n \in \mathbb{N}_0$.*

Proof. Let $\mathsf{w} \in \mathcal{A}^l$ be a word $w_1 \cdots w_l$. Then, one can conclude the following equations:

$$
\begin{aligned}
\mathsf{w}^{(n)}(\mathfrak{o}) &= (w_1^{(n)} \cdots w_l^{(n)})(\mathfrak{o}) \\
&= w_1^{(n)}(\mathfrak{o}) \ldots w_l^{(n)}(\mathfrak{o}) && \text{(Prop. 3)} \\
&= \mathfrak{v}_{w_1} + \cdots + \mathfrak{v}_{w_l} && \text{(Prop. 6)} \\
&= w_1(\mathfrak{o}) + \cdots + w_l(\mathfrak{o}) \\
&= \mathsf{w}(\mathfrak{o}).
\end{aligned}
$$

This proves the proposition. $\qquad\square$

Let $\mathsf{w} \in \mathcal{A}^*$ be a word. The operator $\|\cdot$ gives the edge set $\|\mathsf{w}$ of w. Applying the $(1, \mu)$-endomorphism h for n times produces the n-th derivative $\mathsf{w}^{(n)}$; its edge set is $\|\mathsf{w}^{(n)}$. The following proposition shows how the edge set $\|\mathsf{w}^{(n)}$ can be obtained from the edge set $\|\mathsf{w}$.

$$
\begin{array}{ccc}
\mathsf{w} & \xrightarrow{\|\cdot} & \|\mathsf{w} \\
\downarrow h & & \Downarrow ? \\
\mathsf{w}^{(n)} & \xrightarrow{\|\cdot} & \|\mathsf{w}^{(n)}
\end{array}
$$

Proposition 9. *The edge set $\|\mathsf{w}^{(n)}$ of the n-th derivative of a string $\mathsf{w} \in \mathcal{A}^*$ is the union of the n-th derivatives of all edges in $\|\mathsf{w}$:*

$$
\|\mathsf{w}^{(n)} = \bigcup_{(\mathsf{q},x)\in\|\mathsf{w}} \|^{\mathsf{q}}x^{(n)} \quad \mathsf{w} \in \mathcal{A}^*, \ n \in \mathbb{N}_0.
$$

Proof. Let $\mathsf{w} \in \mathcal{A}^l$ be the string $w_1 \cdots w_l$. Then the n-th derivative of w is

$$
\mathsf{w}^{(n)} = w_1^{(n)} \cdots w_l^{(n)}.
$$

The edge set of w is

$$
\begin{aligned}
\|\mathsf{w} &= \|w_1 \cup \|^{\overrightarrow{w_1}(\mathsf{o})}w_2 \cup \ldots \cup \|^{\overrightarrow{w_{l-1}}(\mathsf{o})}w_l && \text{(Prop. 5)} \\
&= \{\,(\mathsf{o}, w_1)\,\} \cup \{\,(\overrightarrow{w_1}(\mathsf{o}), w_2)\,\} \cup \cdots \cup \{\,(\overrightarrow{w_{l-1}}(\mathsf{o}), w_l)\,\} \\
&= \{\,(\mathsf{o}, w_1), (\overrightarrow{w_1}(\mathsf{o}), w_2), \ldots (\overrightarrow{w_{l-1}}(\mathsf{o}), w_l)\,\},
\end{aligned}
$$

the edge set of $\mathsf{w}^{(n)}$ is

$$
\begin{aligned}
\|\mathsf{w}^{(n)} &= \|w_1^{(n)} \cup \|^{\overrightarrow{w_1}^{(n)}(\mathsf{o})}w_2^{(n)} \cup \cdots \cup \|^{\overrightarrow{w_{l-1}}^{(n)}(\mathsf{o})}w_l^{(n)} && \text{(Prop. 5)} \\
&= \|w_1^{(n)} \cup \|^{\overrightarrow{w_1}(\mathsf{o})}w_2^{(n)} \cup \cdots \cup \|^{\overrightarrow{w_{l-1}}(\mathsf{o})}w_l^{(n)} && \text{(Prop. 8)} \\
&= \bigcup_{(\mathsf{q},x)\in\|\mathsf{w}} \|^{\mathsf{q}}x^{(n)},
\end{aligned}
$$

which proves the proposition. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The simple directed graph of the n-th derivative of a word $\mathsf{w} \in \mathcal{A}^*$ arises from the simple directed graph of w by replacing each x-edge $(\mathsf{q}, x) \in \|\mathsf{w}$ by the simple directed graph of the n-th derivative of x, beginning at the point q.

Example 7. Let h be a $(1, \mu)$-endomorphism with $r \mapsto dru$, $u \mapsto rul$, $l \mapsto uld$, and $d \mapsto ldr$. The simple directed graphs of the second atomic derivatives are shown below (those of the first derivatives are inserted grey coloured):
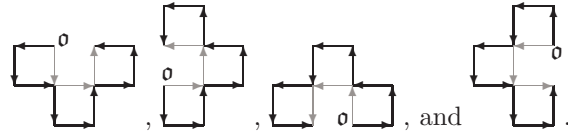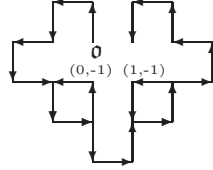
Let w be the first derivative of r. Its edge set $\|dru$ is

$$\{\,(\mathfrak{o},d),((0,-1),r),((1,-1),u))\,\}\,.$$

The edge set of the second derivative of w consists of all edges of the edge sets of the second derivatives of d with respect to $\mathfrak{o}$, of r with respect to $(0,-1)$, and of u with respect to $(1,-1)$: $\|\mathsf{w}'' = \|^{\mathfrak{o}}d'' \cup \|^{(0,-1)}r'' \cup \|^{(1,-1)}u''$. These edge sets are (shown as simple directed graphs):
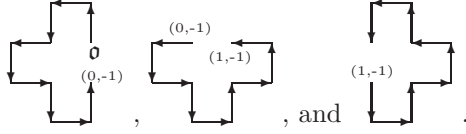
Hence, the the simple directed graph of w'' is

.

Since w is the first derivative of r, this graph should be the same as that one of the third derivative of r – and is as the following sequence of derivative graphs shows.
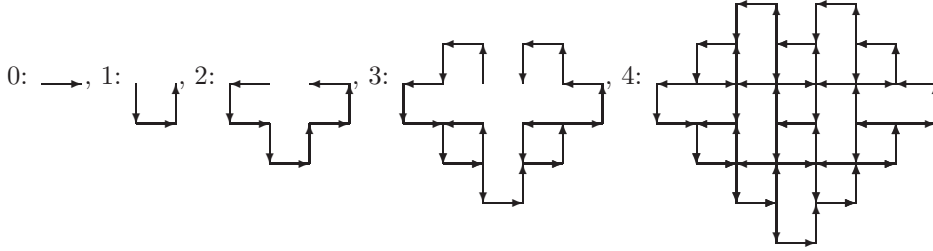
   With the help of the proposition above, the following proposition about the stability of edge sets can be proved.

Proposition 10. *If the edge set of a string $\mathsf{w} \in \mathcal{A}^*$ coincides with that of its derivative w', then it coincides with the edge set of every higher derivative:*

$$\|\mathsf{w} = \|\mathsf{w}' \implies \|\mathsf{w} = \|\mathsf{w}^{(n)}, \quad \mathsf{w} \in \mathcal{A}^*,\ n \in \mathbb{N}.$$

Proof. The proof is carried out by induction. Suppose, that $\|\mathsf{w} = \|\mathsf{w}^{(i)}$ for $1 \le i \le n$. Then the edge set $\|\mathsf{w}^{(n+1)}$ is

$$
\begin{aligned}
\|\mathsf{w}^{(n+1)} &= \bigcup_{(\mathsf{q},x)\in\|\mathsf{w}^{(n)}} \|^{\mathsf{q}}x' && \text{(Prop. 9)}\\
&= \bigcup_{(\mathsf{q},x)\in\|\mathsf{w}} \|^{\mathsf{q}}x' && \text{(inductional assumption)}\\
&= \|\mathsf{w}' && \text{(Prop. 9)}\\
&= \|\mathsf{w} && \text{(inductional assumption)}
\end{aligned}
$$

from which the proposition follows. $\qquad\qquad\square$

With the Propositions 10 and 4, the first supposition (see page 65) is proved. So, it is stated in a lemma.

Lemma 1. *Let $G = (\mathcal{A}, h, \omega)$ be a length-constant sD0L system. If the edge sets of the second and third derivatives of the axiom ω coincide, then the picture language generated consists of the pictures up to the second derivative at most. That is shortly written as*

$$\|\omega'' = \|\omega''' \Longrightarrow P_G = \{\, p(\omega), p(\omega'), p(\omega'') \,\}\,.$$

In the first example of this section (p. 65), one can observe that if the edge sets of the second and third derivatives of the axiom ω do not coincide, then at least one x-edge exists in the second derivative which is later replaced by a graph that contains another x-edge. The next propostion gives an even stronger restriction.

Proposition 11. *If the edge sets of the second and third derivatives of the axiom ω do not coincide, then there exists a letter $x \in [\omega'']$ such that one of the first three derivation edge sets $\|x'$, $\|x''$ or $\|x'''$ contains an edge different from $(\mathfrak{o}, x)$.*

Proof. The statement of the proposition is equivalent to the following statement. If, for each letter $x \in [\omega'']$, the edge sets of the first three derivatives of x do not contain any x-edge different from $(\mathfrak{o}, x)$ then the edge sets of ω'' and ω''' coincide:

$$(\forall x \in [\omega''] : \|_x x = \|_x x' = \|_x x'' = \|_x x''') \Longrightarrow \|\omega'' = \|\omega'''.$$

This statement will be proved now.

For all letters $x \in [\omega'']$, let $\|_x x = \|_x x' = \|_x x'' = \|_x x'''$. If $(\mathfrak{q}, y)$ is an edge of any $\|x^{(i)}$, then $(\mathfrak{q}, y)$ is also an edge of $\|x^{(i+1)}$ (because of $\|_y y = \|_y y'$). Hence, each edge set includes those of lower derivatives, that is $\|x \subseteq \|x' \subseteq \|x'' \subseteq \|x'''$.

In order to conclude that $\|\omega'' = \|\omega'''$, the inclusion $\|x'' \subseteq \|x'''$ must be an equation. In the sequel, the inverted inclusion ($\|x''' \subseteq \|x''$) will be shown. The case distinction used follows from Proposition 6.

1. $[x'] = \{\, x \,\}$, hence $[x^{(n)}] = \{\, x \,\}$ for all natural numbers n. Especially, x'' is equal to x''', and also $\|x'' = \|x'''$.

2. $[x'] = \{\, x, \bar{x} \,\}$.

   (a) $[x''] = [x']$, hence $[x^{(n)}] = [x']$ for all natural numbers n. Because of the second synchronization condition, the letters x and $\bar{x}$ alternate in x'' and in x'''. That means for the edge sets

   $$\|x'' = \|x \cup \|^{x(\mathfrak{o})}\bar{x} \cup \|x \cup \|^{x(\mathfrak{o})}\bar{x} \cup \cdots \|x \qquad \text{(Prop. 5)}$$
   $$= \|x'''.$$

   (b) $[x''] = \mathcal{A}$. The edge set $\|x'$ consists of the edges $(\mathfrak{o}, x)$ and $(x(\mathfrak{o}), \bar{x})$ (due to the second synchronization condition). The edge set $\|x''$ does not contain any other x-edge nor $\bar{x}$-edge (the edge $(\mathfrak{o}, x)$ can produce

the edges of $\|x'$ only; the edge $(x(\mathbf{o}), \bar{x})$ can produce $(\mathbf{o}, x)$ – due to $\|_x x = \|_x x''$ – and $(x(\mathbf{o}), \bar{x})$ – due to $\|_{\bar{x}} \bar{x} = \|_{\bar{x}} \bar{x}'$ – only). Hence, the $x^{\perp}$- and $\bar{x}^{\perp}$-edges occur pairwise (between the same points):

$$(\mathfrak{q}, x^{\perp}) \in \|x'' \iff (x^{\perp}(\mathfrak{q}), \bar{x}^{\perp}) \in \|x''.$$

The set $\|_x x'''$ consists of $(\mathbf{o}, x)$ only. The set $\|_{\bar{x}} x'''$ consists of $(x(\mathbf{o}), \bar{x})$ only (due to $\|_{\bar{x}} \bar{x} = \|_{\bar{x}} \bar{x}''$). The $x^{\perp}$- and $\bar{x}^{\perp}$-edges do not produce new x-, $\bar{x}$-edges (because of the same reasons), thus, the $x^{\perp}$- and $\bar{x}^{\perp}$-edges occur pairwise in $\|x'''$; $x^{\perp}$ does not produce a new $x^{\perp}$-edge, hence, nor a new $\bar{x}^{\perp}$-edge. Because of the same reason for $\bar{x}^{\perp}$, there is not an edge in $\|x'''$ which is not in $\|x''$.

3. $[x'] = \{\, x, x^{\perp}, \bar{x}^{\perp} \,\}$.

   (a) $[x''] = [x']$, hence $[x^{(n)}] = [x']$ for all natural numbers n. As there are no $\bar{x}$-edges in the edge sets of any derivative, the edge set does not change from the first to the second derivative: $\|x' = \|x''$. Because of Proposition 10, it follows $\|x'' = \|x'''$.

   (b) $[x''] = \mathcal{A}$. This means, $\bar{x}$ occurs in the derivative of $x^{\perp}$ or $\bar{x}^{\perp}$. As above, the $x^{\perp}$- and $\bar{x}^{\perp}$-edges occur pairwise in $\|x'$. With each $\bar{x}$-edge, also an x-edge is produced. If there are more than one $x^{\perp}$- and $\bar{x}^{\perp}$-edges, then they have different positions with respect to the edge $(\mathbf{o}, x)$ and then another x-edge will appear. This is a contradiction. Hence, there are one $x^{\perp}$- and one $\bar{x}^{\perp}$-edge only. The following cases are the only possibilities (the graphs given to the right shall illustrate the case of $x = r$):

      i. $\|x' = \{\, (\mathbf{o}, x^{\perp}), (x^{\perp}(\mathbf{o}), \bar{x}^{\perp}), (\mathbf{o}, x) \,\}$

      ii. $\|x' = \{\, (\mathbf{o}, x), (x(\mathbf{o}), x^{\perp}), ((xx^{\perp})(\mathbf{o}), \bar{x}^{\perp}) \,\}$

      iii. $\|x' = \{\, (\mathbf{o}, \bar{x}^{\perp}), (\bar{x}^{\perp}(\mathbf{o}), x^{\perp}), (\mathbf{o}, x) \,\}$

      iv. $\|x' = \{\, (\mathbf{o}, x), (x(\mathbf{o}), \bar{x}^{\perp}), ((x\bar{x}^{\perp})(\mathbf{o}), x^{\perp}) \,\}$

   In $\|x''$, there are not any new x-, $x^{\perp}$-, $\bar{x}^{\perp}$-edges. Hence, the new $\bar{x}$-edge is $(x(\mathbf{o}), \bar{x})$ (but then $\|x'''$ is the same as $\|x''$, because it does not get any new edge), or the new $\bar{x}$-edge is $((xx^{\perp})(\mathbf{o}), \bar{x})$ (in the cases i. and ii.) or $((x\bar{x}^{\perp})(\mathbf{o}), \bar{x})$ (in the cases iii. and iv.). Then a new vertex $\mathfrak{e}$ appears:

      i. $\mathfrak{e} = (xx^{\perp})(\mathbf{o})$

      ii. $\mathfrak{e} = x^{\perp}(\mathbf{o})$

      iii. $\mathfrak{e} = (x\bar{x}^{\perp})(\mathbf{o})$

      iv. $\mathfrak{e} = \bar{x}^{\perp}(\mathbf{o})$

   Since $\mathfrak{e}$ is neither $\mathbf{o}$ nor $x(\mathbf{o})$, there is another edge in $\|x''$ that is not in the set $\|x'$:

   i. $(x(\mathfrak{o}), x^\perp) \in \|x''$

   ii. $(x^\perp(\mathfrak{o}), \bar{x}^\perp) \in \|x''$

   iii. $(x(\mathfrak{o}), \bar{x}^\perp) \in \|x''$

   iv. $(\bar{x}^\perp(\mathfrak{o}), x^\perp) \in \|x''$

   This implies that the $\bar{x}$-edge is produced by $\bar{x}^\perp$ in the cases i. and iv., and by $x^\perp$ in the other cases. Both do not produce further $\bar{x}$-edges (otherwise they would produce a new x-edge or new $x^\perp$-, $\bar{x}^\perp$-edges). As the $\bar{x}$-edge cannot produce any new edge, the set $\|x'''$ has no additional edges: $\|x''' \subseteq \|x''$.

4. $[x'] = \mathcal{A}$. Let v, w, y be different letters of the set $\mathcal{A} \setminus \{x\}$. The x- and v-edges produced by deriving v in x' are not new (they are in $\|x'$ already). Possibly, a new w-edge occurs. The x-, v-, and w-edges produced by deriving w in x'' are in $\|x''$ already. Similar to the previous case, also all arising y-edges are in $\|x''$. Hence, all edges of $\|x'''$ are elements of $\|x''$.

There are no other cases (s. Prop. 6). Every case yields that $\|x''' \subseteq \|x''$. Together with the inclusion $\|x'' \subseteq \|x'''$, one obtains that $\|x'' = \|x'''$. Hence, the edge sets, with respect to any point $\mathfrak{a} \in \mathbb{Z}^2$, coincide: $\|^{\mathfrak{a}}x'' = \|^{\mathfrak{a}}x'''$. Let $w_1, \ldots, w_n$ be the letters of ω: $\omega = w_1 \cdots w_n$. For the edge set of the second derivative of ω, the considerations above yield

$$\|\omega'' = \|w_1'' \cup \|^{\overrightarrow{w_1''(\mathfrak{o})}}w_2'' \cup \cdots \cup \|^{\overrightarrow{w_{n-1}''(\mathfrak{o})}}w_n'' \qquad \text{(Prop. 5)}$$

$$= \|w_1'' \cup \|^{\overrightarrow{w_1'''(\mathfrak{o})}}w_2'' \cup \cdots \cup \|^{\overrightarrow{w_{n-1}'''(\mathfrak{o})}}w_n'' \qquad \text{(Prop. 8)}$$

$$= \|w_1''' \cup \|^{\overrightarrow{w_1'''(\mathfrak{o})}}w_2''' \cup \cdots \cup \|^{\overrightarrow{w_{n-1}'''(\mathfrak{o})}}w_n''' \qquad (\|^{\mathfrak{a}}x'' = \|^{\mathfrak{a}}x''')$$

$$= \|\omega''' \qquad \text{(Prop. 5)}.$$

This proves the proposition. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

In the sequel, consider ω such that $\|_x x \neq \|_x x^{(l)}$ for a letter $x \in [\omega'']$ and a derivation step $l \in \{1, 2, 3\}$. The edge set $\|x$ consists of the edge $(\mathfrak{o}, x)$. Another x-edge is in the graph of the l-th derivative of x: $(\mathfrak{q}, x) \in \|x^{(l)}$ with $\mathfrak{q} \neq \mathfrak{o}$. According to Proposition 9, one obtains that $\|^{\mathfrak{q}}x^{(l)} \subseteq \|x^{(2l)}$, which means that all edges of the l-th derivative are in the $2l$-th derivative displaced by the point $\mathfrak{q}$. Thus, it is especially $(\mathfrak{q} + \mathfrak{q}, x) \in \|x^{(2l)}$. Induction leads to $(n\mathfrak{q}, x) \in \|x^{(nl)}$ in general.

Consider $\omega'' = \mathsf{v}x\tilde{\mathsf{v}}$, then the edge sets of every l-th derivative of ω'' are

$$\|\omega^{(nl+2)} = \|\mathsf{v}^{(nl)} \cup \|^{\mathsf{v}^{(nl)}(\mathfrak{o})}x^{(nl)} \cup \|^{(\mathsf{v}x)^{(nl)}(\mathfrak{o})}\tilde{\mathsf{v}}^{(nl)}$$

$$= \|\mathsf{v}^{(nl)} \cup \|^{\mathsf{v}(\mathfrak{o})}x^{(nl)} \cup \|^{(\mathsf{v}x)(\mathfrak{o})}\tilde{\mathsf{v}}^{(nl)} \qquad \text{(Prop. 8)}.$$

The edge $(n\mathfrak{q} + \mathsf{v}(\mathfrak{o}), x)$ occurs in every set $\|^{\mathsf{v}(\mathfrak{o})}x^{(nl)}$ with $n \in \mathbb{N}_0$. Hence, for each $n \in \mathbb{N}_0$, the edge $(n\mathfrak{q} + \mathsf{v}(\mathfrak{o}), x)$ is also an element of the set $\|\omega^{(nl+2)}$. Thus, the vertex $n\mathfrak{q} + \mathsf{v}(\mathfrak{o})$ is an element of the vertex set $\odot(\omega^{(nl+2)})$.

Let X_ω be the set of all vertices $n\mathfrak{q} + \mathsf{v}(\mathfrak{o})$, V_ω the union of the vertex sets $\odot(\omega^{(nl+2)})$, and P_ω the set of all pictures $p(\omega^{(nl+2)})$ over all $n \in \mathbb{N}_0$:

$$X_\omega = \{\, n\mathfrak{q} + \mathsf{v}(\mathfrak{o}) \mid n \in \mathbb{N}_0 \,\},$$
$$V_\omega = \bigcup_{n \in \mathbb{N}_0} \odot(\omega^{(nl+2)}),$$
$$P_\omega = \left\{\, p(\omega^{(nl+2)}) \mid n \in \mathbb{N}_0 \,\right\}.$$

The set X_ω is infinite because $\mathfrak{q} \neq \mathfrak{o}$. Each point in X_ω also occurs in V_ω; thus V_ω is also infinite. Every vertex set $\odot(\omega^{(nl+2)})$ with $n \in \mathbb{N}_0$ is finite; hence, there are infinitely many different ones in the union V_ω. If the vertex sets of two strings $\mathsf{x}, \mathsf{y} \in \mathcal{A}^*$ differ, then the pictures are also different. Hence, P_ω contains infinitely many different pictures. This result is summerized in the next proposition.

Proposition 12. *If there exists a letter $x \in [\omega'']$ such that one of the first three derivation edge sets $\|x'$, $\|x''$ or $\|x'''$ contains an edge different from $(\mathfrak{o}, x)$, then the picture language generated is infinite.*

Together with the Proposition 11, this leads immediately to the following lemma.

Lemma 2. *Let $G = (\mathcal{A}, h, \omega)$ be a length-constant sD0L system. If the edge sets of the second and third derivatives of the axiom ω do not coincide, then the picture language generated is infinite.*

Hence, the second supposition on page 65 is confirmed and proved. The Lemmas 1 and 2 together state that the difference of the edge sets of the second and third derivatives of the axiom is a necessary and sufficient criterion of the finiteness. This result is summarized in the next theorem.

Theorem 3. *Let $G = (\mathcal{A}, h, \omega)$ be a length-constant sD0L system. The picture language P_G generated by G is finite if and only if the edge sets of the second and third derivatives of the axiom ω coincide.*

From an algorithmic point of view, the case $\mu = 0$ must be emphasized.

According to the second synchronization condition, the edge set $\|^{\mathfrak{a}}x^{(n)}$ of each derivative of any letter $x \in \mathcal{A}$ with respect to the point $\mathfrak{a} \in \mathbb{Z}^2$ consists of the edge $(\mathfrak{a}, \mathfrak{a} + \mathfrak{v}_x)$. Hence, the pictures of all derivatives of ω coincide (Prop. 4).

Theorem 4. *Let $G = (\mathcal{A}, h, \omega)$ be an sD0L system with a $(1,0)$-endomorphism h. Then the picture language P_G generated by G is a singleton set, that is*

$$P_G = \{\, p(\omega) \,\}.$$

This means that the finiteness can be stated immediately in the case $\mu = 0$ (without further investigation of the system).

4   Conclusion and Future Work

The paper investigates synchronous deterministic Chain Code Picture Systems with respect to the finiteness of their picture languages.

Let $G = (\mathcal{A}, h, \omega)$ be an *sD0L* system with a (κ, μ)-endomorphism h. The synchronization parameter κ defines a division of the *sD0L* systems in length-contracting ($\kappa < 1$), length-constant ($\kappa = 1$), and length-expanding ($\kappa > 1$) systems.

The following table summarizes the results:

$$\kappa < 1: \quad P_G = \{\, p(\omega), p(\omega'), p(\omega''), p(\omega''') \,\}$$

$$\kappa > 1: \quad |P_G| = \infty$$

$$k = 1: \quad \mu = 0 \Longrightarrow P_G = \{\, p(\omega) \,\}$$

$$s(\omega'') = s(\omega''') \Longrightarrow P_G = \{\, p(\omega), p(\omega'), p(\omega'') \,\}$$

$$s(\omega'') \neq s(\omega''') \Longrightarrow |P_G| = \infty$$

If the picture language generated by an *sD0L* system is finite, then it consists of four elements at most. In addition, an algorithm is given that decides for any *sD0L* system G, whether the picture language P_G generated is finite or not.

The decision about the finiteness of the picture language of a given *sD0L* system $G = (\mathcal{A}, h, \omega)$ can be made immediately (without further investigation of the system) if the (κ, μ)-endomorphism h is length-contracting or length-expanding or length-constant with $\mu = 0$. Otherwise, the start string must be derived three times, and the edge sets of the second and third derivatives must be checked for equality. This time effort is cubic in the lengths of the replacement strings.

For synchronous deterministic Chain Code Picture Systems $G = (\mathcal{A}, h, \omega)$, the finiteness and infiniteness are decidable in time $\mathcal{O}(pn^3)$, where $p = \#\omega$ is the length of the start string ω and $n = \max\{\, \#h(x) \mid x \in \mathcal{A} \,\}$ is the maximum length of the replacement strings.

Future work will address the finiteness of picture languages generated by non-deterministic *s0L* systems and tabled systems such as *sDT0L* systems and *sT0L* systems. The deterministic systems generate four pictures at most in the case of finiteness. For applications however, system are desirable that generate a large but a finite picture language.

Acknowledgements

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