Velocity and Distance of Neighbourhood Sequences

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Abstract

Das et al. [2] defined the notion of periodic neighbourhood sequences. They also introduced a natural ordering relation $\sqsupseteq^*$ for such sequences. Fazekas et al. [4] generalized the concept of neighbourhood sequences, by dropping periodicity. They also extended the ordering to these generalized neighbourhood sequences. The relation $\sqsupseteq^*$ has some unpleasant properties (e.g., it is not a complete ordering). In certain applications it can be useful to compare any two neighbourhood sequences. For this purpose, in the present paper we introduce a norm-like concept, called velocity, for neighbourhood sequences. This concept is in very close connection with the natural ordering relation. We also define a metric for neighbourhood sequences, and investigate its properties.

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1   Introduction

Distance functions are used in many parts of digital geometry. They are usually defined by digital motions, when we can move in the digital space from one point to another, if they are neighbours in some sense. Rosenfeld and Pfaltz [9] introduced two types of motions in $\mathbb{Z}^2$, the cityblock and chessboard motions. The cityblock motion allows only horizontal and vertical steps, while the chessboard motion diagonal movements as well. By these motions Rosenfeld and Pfaltz defined the distances d_4 and d_8, respectively, as the number of steps needed to get from one point to another. To obtain a better approximation of the Euclidean distance they recommended the alternate use of the cityblock and chessboard motions.

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By allowing arbitrary periodic mixture of the cityblock and chessboard motions, Das et al. [2] introduced the concept of periodic neighbourhood sequences, and generalized it to arbitrary dimension. A distance function can be attached to any neighbourhood sequence A by defining the distance of two points as the number of A-steps needed to get from one of them to the other. In [2] the authors provided a criterion to decide when the distance function corresponding to A is a metric. They also introduced a natural ordering relation on the set of periodic neighbourhood sequences in the following way. Given two such sequences A and B, A is "faster" than B, if for every two points the A-distance is less than or equal to the B-distance of these points. The name of this relation expresses that in this case A spreads faster in the digital space than B. Das [1] studied the structure of the set of periodic neighbourhood sequences with respect to this natural ordering.

By dropping the condition of periodicity, Fazekas et al. [4] generalized the concept of neighbourhood sequences. They extended the "faster" relation to these generalized neighbourhood sequences, and investigated its properties. It turned out that this natural ordering has some unpleasant properties. It fails to be a complete ordering on the set of neighbourhood sequences, moreover, the structure obtained is not even a lattice in higher dimension. However, in certain applications it can be useful to compare any two neighbourhood sequences, i.e. to decide which one spreads "faster". For this purpose, in this article we introduce a norm-like concept, called velocity, on the set of neighbourhood sequences, and investigate its properties. This concept has to be introduced in a way to fit the relation "faster", so we need some preliminaries before defining velocity. Further, we define a metric for neighbourhood sequences. In practical applications it is usually sufficient to consider a finite part of a neighbourhood sequence. In our definition of velocity this can be easily reached by assigning zero weight to the elements of the sequence from some point on. On the other hand, the use of general (infinite) neighbourhood sequences provides a more flexible and exact tool in building up the theoretical background. Obviously, the restriction of the concepts to periodic neighbourhood sequences, yields a notion of velocity and metric defined for them. We note that our results are new already in the periodic case.

In this paper we deal with neighbourhood sequences defined on $\mathbb{Z}^n$. However, there can be applications, where the grid points form another kind of structure (e.g., triangular or hexagonal). For a survey on planar grids, see [8]. The concept of neighbourhood sequences can be easily generalized to these grids, see [7] for the cases of triangular and hexagonal grids. The investigations and concepts of the present paper can be extended to these structures, too.

The aim of this article is to introduce velocity and metric on the set of neighbourhood sequences. We do this in a way to fit the algebraic structure of such sequences. We also work out a possible application scheme for broadcasting information. The structure of this paper is as follows. In the second section we give our notation, and provide some properties of the concepts introduced. In the third section we clarify which conditions should be met by the notion of velocity. In Section 4 the concept of velocity is introduced, and some important properties of

this notion are proved. In Section 5 we give some theoretical examples to illustrate the behaviour of velocity, and in the sixth section we show how the theory can be applied for broadcasting information in a general network model. In Section 7 we define a metric on the set of neighbourhood sequences, and study its properties.

2   Notation and basic concepts

In this section, we recall some definitions and notation from [2] and [4] concerning neighbourhood sequences. In what follows, n denotes a positive integer.

Definition 2.1. *Let p be a point in $\mathbb{Z}^n$. The i-th coordinate of p is indicated by $\Pr_i(p)$ ($1 \le i \le n$). Let k be an integer with $0 \le k \le n$. The points p, $q \in \mathbb{Z}^n$ are called k-neighbours, if the following two conditions hold:*

- $|\Pr_i(p) - \Pr_i(q)| \le 1$ $(1 \le i \le n)$,
- $\sum_{i=1}^{n} |\Pr_i(p) - \Pr_i(q)| \le k$.

The sequence $A = (a(i))_{i=1}^{\infty}$, where $1 \le a(i) \le n$ for all $i \in \mathbb{N}$, is called an n-dimensional (shortly nD) neighbourhood sequence. A is periodic, if for some $l \in \mathbb{N}$, $a(i + l) = a(i)$ ($i \in \mathbb{N}$). For every $i \in \mathbb{N}$ and $j = 1, \dots, n$ put

$$a_j(i) = \min(a(i), j) \quad \text{and} \quad f_j^A(i) = \sum_{k=1}^{i} a_j(k).$$

The set of the nD-neighbourhood sequences will be denoted by S_n.

Let p, $q \in \mathbb{Z}^n$, and $A \in S_n$. The point sequence $p = p_0, p_1, \dots, p_m = q$, where p_{i-1} and p_i are $a(i)$-neighbours in $\mathbb{Z}^n$ ($1 \le i \le m$), is called an A-path from p to q of length m. The A-distance $d(p, q; A)$ of p and q is defined as the length of the shortest A-path between them.

A natural partial ordering relation on S_n can be introduced in the following way (see [2] and [4]). For $A, B \in S_n$ we define the relation $\sqsupseteq^*$ by

$$A \sqsupseteq^* B \quad \Leftrightarrow \quad d(p, q; A) \le d(p, q; B) \text{ for all } p, q \in \mathbb{Z}^n.$$

In case of $A \sqsupseteq^* B$ we say that A is faster than B. There is a strong connection between this relation and the values $f_j(i)$, shown by the following result from [4].

Theorem 2.2. *Let $A, B \in S_n$. Then*

$$A \sqsupseteq^* B \quad \Leftrightarrow \quad f_j^A(i) \ge f_j^B(i) \text{ for every } i \in \mathbb{N} \text{ and } j = 1, \dots, n.$$

3   Preliminaries to introduce velocity

By defining velocity, we assign a positive real number to every neighbourhood
sequence. In this section we give some natural conditions, which should be met by
this concept.

(I) *Velocity must be sensitive to the behavior of the sequences in different di-
     mensions.*
     It can happen that a sequence spreads "faster" than another one in higher
     dimensions, but they have the same "speed" in lower dimensional subspaces.
     For example, in 3D the sequences (3,3,3,3,...) and (2,2,2,2,...) have the same
     velocity on the planes $\{x,y\}$, $\{x,z\}$, $\{y,z\}$ defined by the coordinate axes; or
     the sequences (1,3,1,3,...) and (2,2,2,2,...) behave differently in the subspaces
     of $\mathbb{Z}^3$. These features should be reflected in the definition of velocity.

(II) *The elements of the sequences must be weighted with a suitable weight func-
     tion.*
     There are two reasons to establish this condition.   First, it is natural to
     consider the initial elements of the sequences more important than the ele-
     ments which occur later. The second reason comes from theoretical necessity.
     Namely, if we want to take into consideration all elements of the sequences,
     then we have to guarantee the convergence of certain sums or series of the
     (weighted) elements of the sequences.

(III) *Velocity must be defined such that it fits the natural ordering.*
     This condition is very evident: velocity should preserve the ordering $\sqsupseteq^*$. If
     a neighbourhood sequence is faster than another one, its velocity should be
     larger as well. As $\sqsupseteq^*$ is only a partial ordering, the opposite statement cannot
     be true. However, our velocity concept, introduced in the next section, will
     have the nice property that in a certain sense this opposite statement also
     holds (cf. Theorem 4.11).

4   Assigning velocity to neighbourhood sequences

According to (II), we first give the concept of a weight system, which will be
appropriate in our further investigations.

Definition 4.1. *Let* $n \in \mathbb{N}$. *The set of functions* $\delta_j : \mathbb{N} \to \mathbb{R}$ $(1 \leq j \leq n)$ *is called
a weight system, if the following three conditions hold:*

- $\delta_j(i) \geq 0$ $(1 \leq j \leq n,\ i \in \mathbb{N})$,

- $\sum_{i=1}^{\infty} \delta_j(i) < \infty$ $(1 \leq j \leq n)$,

- δ_j *is monotone decreasing* $(1 \leq j \leq n)$.

In order to meet (I), we introduce the concept of velocity in two steps. First, we
assign an n-tuple to every neighbourhood sequence. The elements of this n-tuple

reflect the "velocity" of the given neighbourhood sequence in the subspaces of $\mathbb{Z}^n$ of dimensions from 1 to n. Then, we define one descriptive velocity value.

Definition 4.2. *Let* $A \in S_n$, *and* δ_j $(1 \leq j \leq n)$ *be a weight system. The j-dimensional velocity of A is defined as*

$$v_j^A = \sum_{i=1}^{\infty} a_j(i)\delta_j(i).$$

Remark 4.3. *Let* T *be the linear space of the bounded real sequences over* $\mathbb{R}$, *and let* δ_j $(1 \leq j \leq n)$ *be a weight system. It is well-known (see e.g., [5]) that for every* j, *if* $\delta_j(i) > 0$ *for all* $i \in \mathbb{N}$ *then with the norm*

$$\|(x_i)_{i=1}^{\infty}\| = \sum_{i=1}^{\infty} |x_i|\delta_j(i) \qquad ((x_i)_{i=1}^{\infty} \in T),$$

T becomes a Banach space. Thus, for any $A = (a(i))_{i=1}^{\infty}$, v_j^A *could be defined as* $v_j^A = \|(a_j(i))_{i=1}^{\infty}\|$.

Remark 4.4. *For every* $A \in S_n$ *we have*

$$\sum_{i=1}^{\infty} \delta_j(i) \leq v_j^A \leq n \sum_{i=1}^{\infty} \delta_j(i).$$

We define the velocity of A by the help of the j-dimensional velocities.

Definition 4.5. *Let* $A \in S_n$. *The velocity of A is given by*

$$v^A = \frac{1}{n} \sum_{j=1}^{n} v_j^A.$$

Remark 4.6. *By the definition of* v_j^A $(j = 1, \ldots, n)$ *and* v^A, *we have that for every* $\varepsilon > 0$ *there exists some* $k_0 \in \mathbb{N}$ *such that for any* $k > k_0$

$$v_j^A - \sum_{i=1}^{k} a_j(i)\delta_j(i) < \varepsilon, \quad (j = 1, \ldots, n),$$

and also

$$v^A - \frac{1}{n} \sum_{j=1}^{n} \sum_{i=1}^{k} a_j(i)\delta_j(i) < \varepsilon.$$

This shows that regardless of the system δ_j, the j-dimensional velocities and the velocity of A is "determined" by the first "few" terms of A. In particular, if we take $\delta_j(i) = 0$ for $1 \leq j \leq n$ with some $i \in \mathbb{N}$, then only the first $i - 1$ terms of A contribute to v^A. This consideration can be important in certain practical applications.

In the next section we analyze the behavior of the velocity with respect to various weight systems. Now, we show how conditions (I), (II) and (III) are met by this velocity concept.

The velocity vector $(v_1^A, v_2^A, \ldots, v_n^A)$, thus also v^A is obviously sensitive to the behavior of the sequence A in subspaces of $\mathbb{Z}^n$ of dimensions from 1 to n. Thus, (I) is satisfied. As we use a weight system to define $(v_1^A, v_2^A, \ldots, v_n^A)$ and v^A, the requirements of (II) are also met. The following theorem verifies that our velocity concept satisfies condition (III), too.

Theorem 4.7. *Let $A, B \in S_n$ with $A \sqsupseteq^* B$, and let δ_j $(j = 1, \ldots, n)$ be a weight system. Then, $v_j^A \geq v_j^B$ for every $j = 1, \ldots, n$.*

Proof. Put $A = (a(i))_{i=1}^{\infty}$ and $B = (b(i))_{i=1}^{\infty}$, and fix some j with $1 \leq j \leq n$. Let $k \in \mathbb{N}$ be arbitrary. Since δ_j is monotone decreasing, we can write

$$\delta_j(k-1) = \delta_j(k) + \varepsilon_j(k-1),$$

$$\delta_j(k-2) = \delta_j(k) + \varepsilon_j(k-1) + \varepsilon_j(k-2),$$

$$\vdots$$

$$\delta_j(1) = \delta_j(k) + \varepsilon_j(k-1) + \varepsilon_j(k-2) + \cdots + \varepsilon_j(1),$$

with $\varepsilon_j(m) \geq 0$ $(m = 1, \ldots, k-1)$. Put $\varepsilon_j(k) = \delta_j(k)$. Using these relations, by a simple calculation we get

$$\sum_{i=1}^{k} a_j(i)\delta_j(i) - \sum_{i=1}^{k} b_j(i)\delta_j(i) = \sum_{m=1}^{k} \varepsilon_j(m) \left(\sum_{h=1}^{m} a_j(h) - \sum_{h=1}^{m} b_j(h) \right). \qquad (\star)$$

Observe that as $\sum_{h=1}^{m} a_j(h) = f_j^A(m)$ and $\sum_{h=1}^{m} b_j(h) = f_j^B(m)$, by Theorem 2.2 $A \sqsupseteq^*$ B implies

$$\sum_{h=1}^{m} a_j(h) \geq \sum_{h=1}^{m} b_j(h)$$

for every m with $1 \leq m \leq k$. As $\varepsilon_j(m) \geq 0$ $(m = 1, \ldots, k)$, $(\star)$ yields

$$\sum_{i=1}^{k} a_j(i)\delta_j(i) \geq \sum_{i=1}^{k} b_j(i)\delta_j(i).$$

By letting $k \to \infty$, we obtain $v_j^A \geq v_j^B$. □

Remark 4.8. *By the definition of the velocity, the above theorem implies that if $A \sqsupseteq^* B$, then $v^A \geq v^B$.*

In the following two remarks we explain why some alternative ways of introducing velocity would not be appropriate.

Remark 4.9. *The monotonity of δ_j is necessary to have Theorem 4.7. Indeed, let $A, B \in S_2$ be defined by*

$$A = (2,\ 1,\ 1,\ 1,\ 1,\dots) \quad and \quad B = (1,\ 2,\ 1,\ 1,\ 1,\dots).$$

Moreover, let δ_1 be arbitrary, and put

$$\delta_2(i) = \left\{ \begin{array}{ll} \frac{1}{4}, & if\ i = 1, \\ \frac{1}{2^{i-1}}, & otherwise. \end{array} \right.$$

Clearly, $A \sqsupseteq^* B$ holds, but $v_2^A = \frac{6}{4}$ and $v_2^B = \frac{7}{4}$. Thus, $v_2^B \geq v_2^A$, and also $v^B \geq v^A$ in this case.

Remark 4.10. *It would be possible to define v_j^A in a more general way. Namely, for any $m \in \mathbb{R}$ with $m > 0$, we could put*

$$v_{j,m}^A = \left(\sum_{i=1}^{\infty} (a_j(i))^m \delta_j(i) \right)^{\frac{1}{m}}.$$

However, on one hand the case $m < 1$ does not seem to be interesting. On the other hand, in case of $m > 1$ it is easy to find sequences $A, B \in S_n$ and a weight system δ_j $(j = 1,\dots,n)$ such that Theorem 4.7, hence condition (III) fails for them.

As one can easily see, it can happen that with some weight system δ_j, $v_j^A \geq v_j^B$ for every $j = 1,\dots,n$, but $A \sqsupseteq^* B$ does not hold. However, in some sense we can reverse Theorem 4.7, even in case of positive weight systems, i.e. when $\delta_j(i) > 0$ for all $i \in \mathbb{N}$, $j = 1,\dots,n$. More precisely, we have

Theorem 4.11. *Let $A, B \in S_n$. If for any positive weight system δ_j $(1 \leq j \leq n)$, $v_j^A \geq v_j^B$ holds for all $j = 1,\dots,n$, then $A \sqsupseteq^* B$.*

Proof. Let $k \in \mathbb{N}$ be arbitrary, and for every j with $1 \leq j \leq n$ set

$$\delta_j^{(k)}(i) = \left\{ \begin{array}{ll} 1, & if\ i \leq k, \\ \frac{1}{2^{i-k}n}, & otherwise. \end{array} \right.$$

Clearly, the system $\delta_j^{(k)}$ $(j = 1,\dots,n)$ is a weight system. Thus, by our assumptions we have

$$\sum_{i=1}^{\infty} a_j(i)\delta_j^{(k)}(i) = v_j^A \geq v_j^B = \sum_{i=1}^{\infty} b_j(i)\delta_j^{(k)}(i).$$

Hence, for every $j = 1,\dots,n$

$$\sum_{i=1}^{k} a_j(k) + \sum_{h=k+1}^{\infty} \frac{a_j(h)}{2^{h-k}n} \geq \sum_{i=1}^{k} b_j(k) + \sum_{h=k+1}^{\infty} \frac{b_j(h)}{2^{h-k}n}$$

holds. Replacing $\sum_{i=1}^{k} a_j(k)$ and $\sum_{i=1}^{k} b_j(k)$ by $f_j^A(k)$ and $f_j^B(k)$, respectively, we get

$$f_j^A(k) - f_j^B(k) \geq \sum_{h=k+1}^{\infty} \frac{b_j(h)}{2^{h-k}n} - \sum_{h=k+1}^{\infty} \frac{a_j(h)}{2^{h-k}n} \geq \frac{1}{n} - 1.$$

Since $f_j^A(k) - f_j^B(k)$ is an integer, we may infer that

$$f_j^A(k) - f_j^B(k) \geq 0.$$

By Theorem 2.2 the proof is complete.   □

Remark 4.12. *It can be easily verified that the condition $v_j^A \geq v_j^B$ for all $j = 1, \ldots, n$ cannot be replaced by $v^A \geq v^B$.*

5   Examples of weight systems

In this section we give examples of weight systems, and analyze the behavior of the velocity concept. We investigate exponentially decreasing systems, and calculate the velocity of some concrete sequences with respect to different weight systems.

Let $c > 1$, and put

$$\delta_j(i) = \frac{1}{c^{i-1}} \text{ for every } j = 1, \ldots, n \text{ and } i \in \mathbb{N}.$$

Obviously, δ_j is a weight system with

$$\sum_{i=1}^{\infty} \delta_j(i) = \frac{c}{c-1} \quad (j = 1, \ldots, n).$$

Consider the nD-neighbourhood sequences

$$A = (h, 1, 1, 1, 1, \ldots) \text{ and } B = (1, n, n, n, n, \ldots), \text{ where } 2 \leq h \leq n.$$

Then

$$v^A = v_j^A = \frac{1}{c-1} + h \text{ and } v^B = v_j^B = \frac{n}{c-1} + 1 \quad (j = 1, \ldots, n).$$

Clearly, the sequences A and B cannot be compared by the ordering $\sqsupseteq^*$. We show how the relation between the velocity values of A and B change according to the choice of the parameter c.

First, suppose that $c > n$. Then, we have

$$v^A = v_j^A = \frac{1}{c-1} + h \geq \frac{1}{c-1} + 2 = \frac{c}{c-1} + 1 > \frac{n}{c-1} + 1 = v_j^B = v^B.$$

In general, using this weight system we obtain a very strong condition, namely that $v^A > v^B$ if and only if A precedes B lexicographically.

Now, let $c = 2$. In this case we have

$$v^A = v_j^A = 1 + h \le 1 + n = v_j^B = v^B,$$

with equality only for $h = n$.

Finally, set $c < 2$. Now, by a simple calculation, we get $v^B = v_j^B > v_j^A = v^A$.

Summarizing, using such a weight system, we can get rid of the (sometimes excessive) importance of the first "few" elements of a neighbourhood sequence. Especially, for every $k \in \mathbb{N}$, by choosing a suitable c, we can have $v_j^B > v_j^A$ for the nD-sequences

$$A = (\underbrace{n, n, \ldots, n}_{k}, 1, 1, 1, \ldots) \quad \text{and} \quad B = (\underbrace{1, 1, \ldots, 1}_{k}, n, n, n, \ldots).$$

On the other hand, by the appropriate choice of c we can give large significance to the first "few" elements of the neighbourhood sequences, ignoring their later elements.

By choosing other (e.g., polinomially decreasing) types of weight systems we can have different properties. The weight system should be chosen appropriately for the actual application, as we can see from a practical example given in the following section.

6 An application for broadcasting information

In this section we give an application scheme of neighbourhood sequences and velocity in a network model, where the members of the network are the points of $\mathbb{Z}^2$. As we mentioned in the introduction, neighbourhood sequences and velocity can be introduced also for other types of grids. Hence, this application scheme could be used in such cases, too.

Our network model can be considered as a variant of the Manhattan Street Network (MSN) introduced by Maxemchuk in [6]. In our scheme, the clients are connected with horizontal/vertical and diagonal edges (see Figure 1). There is an information source at the center (origin) which broadcasts information to the other members of the network. The system is based on priority, that is if a client is "closer" to the origin than another one, it has greater priority, and receives the information earlier. We can think of subscription systems for instance, where clients pay different fees according to their positions with respect to the information source.

It is worth indexing the clients by their "reachability" from the origin. For this purpose, if a client sits on the point $(x, y) \in \mathbb{Z}^2$, then its index will be given by the first few (significant) elements of the slowest neighbourhood sequence A, for which $d((x, y), (0, 0); A)$ is minimal. The clients with the same index have equal priority, so they should pay the same fee (especially, clients indexed by "1" have the greatest priority).
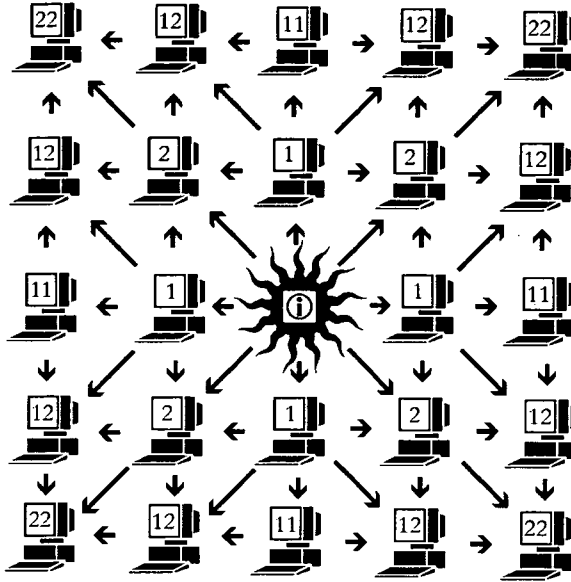
Figure 1: 2D priority-based model for broadcasting information

In this model, we use 2D-neighbourhood sequences to deliver the information to the clients. Suppose that the cost of broadcasting information decreases with the number of 2-s in the chosen neighbourhood sequence. The most expensive sequence is $(2, 2, \dots)$, while the cheapest one is $(1, 1, \dots)$. Knowing the importance of the information, we have to choose one of the cheapest sequences, which is still "fast" enough. That is, we take a neighbourhood sequence, whose velocity fits the importance of the information to be sent. According to the size of the network, it is sufficient to consider the first "few" terms of the sequences (i.e. to work with a weight system in which $\delta_j(N) = 0$ for $j = 1, \dots, n$ with an appropriate $N \in \mathbb{N}$).

By choosing different weight systems, we can increase or decrease the initial priority of the clients in the network. If we do not take much care of the clients residing far from the source, we need to choose a weight system, which decreases rapidly. In the opposite case we can take a very slowly decreasing weight system.

This network model can be easily extended to higher dimensions. In this case, we can take more advantage of the behavior of neighbourhood sequences in lower dimensional subspaces. For example, in $\mathbb{Z}^3$, if we know in advance that a special type of information is important only for a group of clients, we can place them onto or close to the (x, y), (y, z) and (x, z) planes. Thus, for the distribution of this special kind of information we can choose quite a cheap neighbourhood sequence, which consists of mainly 1 and 2 values. To have a similar possibility in 2D, we have to put such clients near the coordinate axes, and use sequences containing mostly 1-s. For a known higher dimensional generalization of MSN, see [3].

Finally, we note that our application scheme may be used at any other areas which are based on regular network topologies. Such areas are e.g., chip architecture design, or parallel processing.

7 Metric space of the neighbourhood sequences

We introduce a metric on the set of neighbourhood sequences in a similar fashion as we did it for velocity. From this point on, we assume that for every weight system $\Delta = \{\delta_j \mid j = 1, \ldots, n\}$ we have $\delta_n(i) > 0$ $(i \in \mathbb{N})$, unless we state the contrary.

Definition 7.1. *Let* $\Delta = \{\delta_j \mid j = 1, \ldots, n\}$ *be a weight system and* $A, B \in S_n$. *The distance* ϱ_Δ *of these sequences is defined by*

$$\varrho_\Delta(A, B) = \frac{1}{n} \sum_{j=1}^{n} \sum_{i=1}^{\infty} |a_j(i) - b_j(i)| \, \delta_j(i).$$

Remark 7.2. *One can easily verify that in case of any weight system* Δ, *the function* ϱ_Δ *is a metric on* S_n. *At this point we make use of our assumption that* $\delta_n(i) > 0$ *for every* $i \in \mathbb{N}$.

Remark 7.3. *The metric space* (S_n, ϱ_Δ) *is bounded. Its diameter is*

$$diam(S_n, \varrho_\Delta) = \varrho_\Delta(\ (1, \ 1, \ \ldots), \ (n, \ n, \ \ldots)\ ) = \frac{n-1}{n} \sum_{j=1}^{n} \sum_{i=1}^{\infty} \delta_j(i).$$

Remark 7.4. *In practical applications usually only "finite" sequences are need-ed. Thus, it may be useful to consider those sequences identical which agree in their first "few" elements. This can be done in the following way. Let* $N \in \mathbb{N}$, *and* $\Delta = \{\delta_j \mid j = 1, \ldots, n\}$ *be a weight system such that* $\delta_j(N + 1) = 0$ *for all* j *(including* $j = n$*), but* $\delta_n(N) > 0$. *Consider two neighbourhood sequences equivalent if and only if their first* N *elements coincide. Then Definition 7.1 provides a metric on the classes of* S_n *induced by this equivalence relation.*

In what follows, we establish some useful and interesting properties of the metric spaces (S_n, ϱ_Δ).

Theorem 7.5. *For any weight system* Δ, (S_n, ϱ_Δ) *is a complete metric space.*

Proof. Let Δ be any weight system. We prove the theorem by showing that every Cauchy sequence in S_n has a limit. We actually construct this limit sequence in the proof. Let $(A_k)_{k=1}^{\infty}$ be a Cauchy sequence in (S_n, ϱ_Δ), and let $m \in \mathbb{N}$. By the Cauchy-property of $(A_k)_{k=1}^{\infty}$, there exists some $k_0 \in \mathbb{N}$ such that for every $k_1, k_2 > k_0$, $\varrho_\Delta(A_{k_1}, A_{k_2}) < \delta_n(m)$. Hence the first m elements of the neighbour-hood sequences A_{k_1} and A_{k_2} are identical. Define the sequence A in the following way. For every $m \in \mathbb{N}$ choose a $k_0 \in \mathbb{N}$, such that the m-th elements of A_{k_1} and A_{k_2} with $k_1, k_2 \geq k_0$ are equal. Let $a(m)$ be this element, and put $A = (a(m))_{m=1}^{\infty}$. Clearly, A is well defined. By the construction of A we immediately get that $\lim_{k \to \infty} A_k = A$. $\qquad \square$

A sequence $(A_k)_{k=1}^\infty$ is monotone increasing (resp. decreasing), if $A_{i+1} \sqsupseteq^* A_i$ (resp. $A_i \sqsupseteq^* A_{i+1}$) holds for every $i \in \mathbb{N}$.

Theorem 7.6. *Every monotone increasing or decreasing sequence $(A_k)_{k=1}^\infty$, with $A_k \in S_n$ ($k \in \mathbb{N}$) is convergent.*

Proof. As in the previous proof, we construct the limit of $(A_k)_{k=1}^\infty$. We may assume that $(A_k)_{k=1}^\infty$ is monotone increasing, the proof in the other case is similar.

Put $A_k = (a^{(k)}(i))_{i=1}^\infty$. As $(A_k)_{k=1}^\infty$ is increasing, so is $(a^{(k)}(1))_{k=1}^\infty$. As $n \geq a^{(k)}(1)$ ($k \in \mathbb{N}$), there exists some $k_0 \in \mathbb{N}$ such that for any $k_1, k_2 \geq k_0$ we have $a^{(k_1)}(1) = a^{(k_2)}(1)$. Put $a(1) = a^{(k_0)}(1)$. Suppose that $a(i)$ is already given for $i \leq m$, and define $a(m+1)$ in the following way. Choose $t_0 \in \mathbb{N}$ such that for $t_1, t_2 \geq t_0$ and $1 \leq i \leq m$, $a^{(t_1)}(i) = a^{(t_2)}(i)$ holds. Since $(A_k)_{k=t_0}^\infty$ is increasing, so is the sequence $(a^{(k)}(m+1))_{k=t_0}^\infty$. As $n \geq a^{(k)}(m+1)$ for every $k \in \mathbb{N}$, there exists some $s_0 \in \mathbb{N}$ such that for any $s_1, s_2 \geq s_0$ we have $a^{(s_1)}(m+1) = a^{(s_2)}(m+1)$. Put $a(m+1) = a^{(s_0)}(m+1)$.

From the construction of $A = (a(m))_{m=1}^\infty$ it is clear that for every $m \in \mathbb{N}$ there exists some $k_0 \in \mathbb{N}$, such that if $k \geq k_0$, the first m elements of A_k and A coincide. Thus, $\lim_{k \to \infty} A_k = A$, and the theorem follows. $\qquad\square$

The next result shows that the Bolzano-Weierstrass theorem is true in the constructed metric spaces.

Proposition 7.7. *For any weight system Δ, every subset of (S_n, ϱ_Δ) of infinite cardinality has an accumulation point.*

Proof. Let H be an infinite subset of S_n. We construct an accumulation point of H. Let $a(1)$ be a number which is the first element of infinitely many sequences in H. Suppose that $a(i)$ with $i \leq m$ is already defined. Let $a(m+1)$ be a number which is the $(m+1)$-th element of infinitely many such sequences in H, whose first m elements are $a(1), a(2), \ldots, a(m)$. Put $A = (a(m))_{m=1}^\infty$. Clearly, A is an accumulation point of H. $\qquad\square$

Periodic neighbourhood sequences can play important role in certain applications. Our last result shows that they form a dense subset of (S_n, ϱ_Δ). As the set of periodic neighbourhood sequences is countable, this also yields that (S_n, ϱ_Δ) is a separable metric space.

Theorem 7.8. *For any weight system Δ, the set of periodic neighbourhood sequences is dense in (S_n, ϱ_Δ).*

Proof. Let $A \in S_n$ and $\varepsilon > 0$. By the definition of ϱ_Δ there exists some $k_0 \in \mathbb{N}$, such that if the first k_0 elements of $B \in S_n$ is the same as those of A, then $\varrho_\Delta(A, B) < \varepsilon$ holds. So put $b(i) = a(i \bmod k_0)$ ($i \in \mathbb{N}$), and $B = (b(i))_{i=1}^\infty$. Clearly, B is periodic and $\varrho_\Delta(A, B) < \varepsilon$, thus the proof is complete. $\qquad\square$

8 Conclusion

In this paper, we introduce velocity and metric for the set of neighbourhood sequences. We show that these notions fit well the structure of such sequences. By their help we can compare neighbourhood sequences more precisely, than using only the natural partial ordering relation. We also work out a possible application scheme for broadcasting information.

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