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On Kleene Algebras of Ternary Co-Relations

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Abstract

In this paper we investigate identities satisfied by a class of algebras made of ternary co-relations – contravariant ("arrow-reversed") analogues of binary relations. These algebras are equipped with the operations of union, co-relational composition, iteration, converse and the empty co-relation and the so-called diagonal co-relation as constants. Our first result is that the converse-free part of the corresponding equational theory consists precisely of Kleenean equations for relations, or, equivalently, for (regular) languages. However, the rest of the equations, involving the symbol of the converse, are relatively axiomatized by involution axioms only, so that the co-relational converse behaves more like the reversal of languages, rather than the relational converse. Actually, the language reversal is explicitely used to prove this result. Therefore, we conclude that co-relations can offer a better framework than relations for the mathematical modeling of formal languages, as well as many other notions from computer science.

1. Introduction

The study of the equational theory of Kleene algebras dates back to sixties, and since then it has a vivid history. However, the term 'Kleene algebra' is of more recent date, while the above equational theory was in the first place considered as the collection of *regular identities*: pairs of regular expressions denoting the same language. It was Redko [23] who proved first that regular identities have no finite base of equational axioms, but that result became available for a larger audience only with the famous booklet of Conway [6] in 1971. Conway's model-theoretic argument is probably the best known proof of Redko's result so far.

What is even more important, Conway's ideas eventually led to further progress in the field. However, the explicite determination of a nontrivial equational base of Kleene algebras had to wait until the last decade, when Krob [19] and Bloom and Ésik [3] solved the problem: the axiomatization from [19] was based on the discovery of a beautiful connection between regular languages and finite groups, while the one in [3] came out from some deep investigations in category theory and

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its applications in computer science (see [2]). These approaches were quite recently unified in [5, 12].

It was realized in the late seventies by relation algebraists that language algebras and Kleene relation algebras are very closely related: they satisfy precisely the same (regular) identities, so that both of these two classes generate the same variety (of Kleene algebras). Moreover, the algebras of regular languages turned out to be just the free Kleene algebras, as proved by Kozen [18] in 1979 (although this result was originally formulated in the context of dynamic algebras).

But when one considers the operations of the converse of relations and the reversal of languages, respectively, the above symmetry between languages and relations is lost. Namely, the involution axioms suffice to capture the equational properties of the reversal of languages [4], while for the converse of relations one should involve an additional identity [13], which *does not* hold for languages. Therefore, relations are not 'good enough' to model the language reversal.

On the other hand, the concept of a co-relation is quite new. Yet, it belongs to the collection of 'co-algebraic' phenomena, which have been studied for some time. Roughly speaking, the main idea is to dualize the notion of an algebra and the main algebraic constructions. The pioneering papers along this line were the ones of Eilenberg and Moore [11] and Kleisli [17], but it was Aczel and Mendler [1] who opened new directions in applying co-algebra in computer science. With this approach at hand, they managed to model (binary) trees, different deterministic and nondeterminstic transition systems, etc. Since then, co-algebraic concepts were widely applied e.g. in object-oriented programming [24] and program verification [14]. For basic notions of co-algebra, see [15, 25].

In 1971, Drbohlav [10] started to investigate co-operations on a set, which one obtains from the notion of an operation by reversing arrows and replacing products by coproducts in the category of sets. Later, this inspired Csákány [9] to introduce clones of co-operations (see also [20]). But as the classical clone theory needs its 'relational part' in order to develop full strength, so the theory of clones of co-operations needs appropriate co-objects as invariants. Hence, Pöschel and Rößiger [22] proposed the concept of a co-relation. While an n-ary relation on X can be thought of as a family of n-ary vectors over X, that is, functions $n \to X$, an *n-ary co-relation on X* is a collection of functions $X \to n$ (n-ary co-vectors on X), which should be imagined just as colourings (partitions) of X in n colours (into n classes).

In [20], the operation of composition was defined for arbitrary co-relations; however, the result of the composition of two n-ary co-relations is again an n-ary co-relation if and only if $n = 3$. Of course, binary co-relations quite clearly correspond to unary relations (subsets). Thus it is natural to expect that the role and importance of binary relations is inherited by ternary co-relations on a set.

In this paper, we consider algebras consisting of ternary co-relations, endowed with the operations of union, composition, iteration (in the sense of the complete union of composition powers), co-relational converse and with two distinguished constants. Our main result is that such algebras generate the same variety as the language algebras equipped with the operations of union, concatenation, Kleene star, reversal and the empty lanugage and the language containing the empty word only, as constants. In particular, it follows that the converse-free reducts of these co-relation algebras are indeed Kleene algebras, justifying the title of the paper. Therefore, we are going to eventually conclude that, from the equational point of view, co-relations model (the operations on) languages better than relations.

For basics of universal algebra we refer to [21] and for the theory of binary relations to [16]. The same references hold for all undefined notions throughout the paper.

2. Preliminaries

2.1. Kleene algebras

Let X be any set. Consider the following algebra:

$$\mathbf{Rel}(X) = \langle \mathcal{P}(X \times X), \cup, \circ, {}^{\mathrm{rtc}}, \emptyset, \Delta_X \rangle,$$

where $\cup$ is the union, $\circ$ is the composition of relations, ${}^{\mathrm{rtc}}$ is the formation of the reflexive-transitive closure, while Δ_X is the diagonal relation on X. The algebra $\mathbf{Rel}(X)$ is called the *full Kleene algebra of relations on* X. Any algebra which can be embedded into some full Kleene relation algebra is called *representable* (or *standard*) *Kleene algebra*. The variety generated by all algebras $\mathbf{Rel}(X)$ we denote by $\mathcal{KA}$. A *Kleene algebra* is just any member of $\mathcal{KA}$.

Beyond algebras of relations, the most important example of Kleene algebras is the *language algebra* over an alphabet Σ:

$$\mathbf{Lang}(\Sigma) = \langle \mathcal{P}(\Sigma^*), +, \cdot, {}^*, \emptyset, \{\lambda\} \rangle,$$

where Σ^* is the free monoid on Σ (which consists of all words over Σ), $+$ denotes the union, $\cdot$ is the concatenation, * is the Kleene star (iteration), and finally, λ denotes the empty word. The fact that language algebras indeed belong to $\mathcal{KA}$ is a consequence of a more general observation.

Lemma 1. *Let M be any monoid. Then $\mathbf{K}(M) = \langle \mathcal{P}(M), \cup, \cdot, {}^*, \emptyset, \{1\} \rangle$, where $\cdot$ is the complex multiplication, * the generation of a submonoid, and 1 the unit of M, is a Kleene algebra.*

Proof (in outline). Consider the mapping $\xi : \mathcal{P}(M) \to \mathcal{P}(M \times M)$ defined for every $A \subseteq M$ by

$$\xi(A) \doteq \{\langle x, xa \rangle : x \in M, a \in A\} = \bigcup_{a \in A} \varrho_a,$$

where ϱ_a denotes the right translation of the monoid M. It is a routine matter to show that ξ is an embedding of $\mathbf{K}(M)$ into $\mathbf{Rel}(M)$. □

By taking $M = \Sigma^*$, from the above lemma we immediately obtain that $\mathbf{Lang}(\Sigma) = \mathbf{K}(\Sigma^*)$ is a Kleene algebra for any alphabet Σ.

The elements of the subalgebra of $\mathbf{Lang}(\Sigma)$ generated by the languages of the form $\{a\}$, $a \in \Sigma$ (or, equivalently, by all finite languages), are called the *regular languages over* Σ. This subalgebra is denoted by $\mathbf{Reg}(\Sigma)$. Now, the algebras of regular languages have the following remarkable property.

Proposition 2. (Kozen, [18]) $\mathbf{Reg}(\Sigma)$ *is the free Kleene algebra on* Σ, *freely generated by the map* $a \mapsto \{a\}$, $a \in \Sigma$.

Thus, it follows that an identity $p = q$ holds in $\mathcal{KA}$ if and only if the regular expressions p, q represent the same (regular) language. Also, the above proposition implies that if we denote by $\mathcal{L}$ the variety generated by all language algebras $\mathbf{Lang}(\Sigma)$, then $\mathcal{L} = \mathcal{KA}$.

We are not going to state here the well known nonfinite axiomatizability result for $\mathcal{KA}$, due to Redko [23], nor the explicite axiomatizations given by Krob [19] and Bloom and Ésik [3]. However, when one is concerned with Kleene algebras or relations and languages, it is quite customary to consider one more operation. First, we have the natural operation of the *reversal* of languages. If $w = a_1 a_2 \dots a_n \in \Sigma^*$ is a word, we define

$$w^{\vee} = a_n \dots a_2 a_1.$$

Now we have

$$L^{\vee} = \{w^{\vee} : w \in L\}.$$

By adding the operation of the reversal of languages to $\mathbf{Lang}(\Sigma)$ we obtain the algebra $\mathbf{Lang}^{\vee}(\Sigma)$. The variety generated by algebras of this form we denote by $\mathcal{L}^{\vee}$.

Proposition 3. (Bloom, Ésik and Stefanescu, [4]) *The variety* $\mathcal{L}^{\vee}$ *is axiomatized relatively to* $\mathcal{KA}$ *by the involution axioms, that is, by the following identities:*

$$\begin{aligned}
(x+y)^{\vee} &= x^{\vee} + y^{\vee}, && (1)\\
(xy)^{\vee} &= y^{\vee} x^{\vee}, && (2)\\
(x^*)^{\vee} &= (x^{\vee})^*, && (3)\\
(x^{\vee})^{\vee} &= x, && (4)\\
0^{\vee} &= 0, && (5)\\
1^{\vee} &= 1. && (6)
\end{aligned}$$

There is one more way to define an involutorial operation on language algebras, which can be useful in some applications. For an alphabet Σ, let Σ' denote a bijective copy of Σ, $\Sigma' = \{a' : a \in \Sigma\}$. For $w = b_1 b_2 \dots b_n \in (\Sigma \cup \Sigma')^*$ define

$$w' = b_n' \dots b_2' b_1',$$

where for all $1 \leq i \leq n$,

$$b_i' = \begin{cases} a', & b_i = a \in \Sigma, \\ a, & b_i = a' \in \Sigma'. \end{cases}$$

Finally, let $\mathbf{Lang}'(\Sigma, \Sigma')$ be the language algebra $\mathbf{Lang}(\Sigma \cup \Sigma')$ endowed with the unary operation $'$, where, of course, $L' = \{w' : w \in L\}$.

By Proposition 4.3 and Theorem 5.1 of [4], algebras $\mathbf{Lang}'(\Sigma, \Sigma')$ also generate the variety $\mathcal{L}^\vee$.

On the other hand, the operation which extends Kleene algebras of relations is the *converse*:

$$\varrho^\vee = \{\langle y, x\rangle : \langle x, y\rangle \in \varrho\}.$$

By equipping the algebra $\mathbf{Rel}(X)$ with $^\vee$, we obtain the algebra $\mathbf{Rel}^\vee(X)$. All algebras of the latter form generate the variety $\mathcal{KA}^\vee$, which turns out to be a *proper* subvariety of $\mathcal{L}^\vee$.

Proposition 4. (Ésik and Bernátsky, [13]) *The equations (1)–(6) and*

$$x + xx^\vee x = xx^\vee x \tag{7}$$

axiomatize the variety $\mathcal{KA}^\vee$ relatively to $\mathcal{KA}$.

Therefore, we may conclude that the equational properties of the language reversal are not faithfully modeled by the relational converse and hence, it is natural to look after a different setting which would allow to capture those properties, preserving at the same time the Kleenean equations.

2.2. Co-Relations

Clearly, an n-ary relation on X can be thought of as a collection of n-ary vectors over X, that is, functions $n \to X$. Dually, an *n-ary co-relation on X* is a set consisting of *n-ary co-vectors*, i.e. of functions $X \to n$. Of course, the notion of a n-ary co-vector is equivalent to the notion of a colouring of a given set in n colours. In particular, a *ternary co-relation* is a family of functions $X \to 3$. It is convenient to represent a ternary co-vector $f : X \to 3$ through the corresponding partition of X into $A = f^{-1}(0)$, $B = f^{-1}(1)$ and $C = f^{-1}(2)$, so that f is written as $\langle A, B, C\rangle^\nabla$ (we use the symbol $^\nabla$ to indicate that we are not dealing with a ternary *vector* whose elements are A, B, C). In order to introduce a more intuitive (and visualisable) terminology, we are going to call the colours $0, 1, 2$ (i.e. the elements from A, B, C) respectively *red*, *green* and *blue*.

In this paper, we deal with algebras of ternary co-relations of the form

$$\mathbf{cRel}^\sqcup(X) = \langle \mathcal{P}(3^X), \cup, \bullet, ^*, ^\sqcup, \emptyset, \varepsilon_X\rangle$$

(the reduct without $^\sqcup$ is denoted by $\mathbf{cRel}(X)$), where the operations and the constants are defined below. First of all, $\cup$ is simply the set-theoretic union, while the constant ε_X is the co-relation consisting of all green-free colourings of X, that is,

$$\varepsilon_X = \{\langle A, \emptyset, X \setminus A\rangle^\nabla : A \subseteq X\}.$$

The definition of the co-relational composition $\bullet$ is the following:

$$\varrho \bullet \sigma = \{\langle A, B \cup E, F\rangle^\nabla : (\exists C, D \subseteq X)(\langle A, B, C\rangle^\nabla \in \varrho \wedge \langle D, E, F\rangle^\nabla \in \sigma \wedge C = X \setminus D)\}.$$

In other words, two co-vectors can be composed if the blue set of the first one coincides with the non-red part (green+blue) of the second one (or, equivalently, red+green elements of the first are precisely the red elements of the second co-vector). If that is the case, green elements are added together, the red colour is copied from the first and the blue from the second factor.

Since one can define arbitrary unions of co-relations, the unary operation * is just the co-relation analogue of the reflexive-transitive closure of relations, or of the Kleene iteration. If for a ternary co-relation ϱ and $n \geq 1$ we define

$$\varrho^n = \underbrace{\varrho \bullet \ldots \bullet \varrho}_{n}$$

and $\varrho^0 = \varepsilon_X$, then

$$\varrho^\star = \bigcup_{n \geq 0} \varrho^n.$$

Finally, the co-relational converse $^\sqcup$ is defined as the interchanging of the red and the blue colour:

$$\varrho^\sqcup = \{\langle C, B, A\rangle^\nabla : \langle A, B, C\rangle^\nabla \in \varrho\}.$$

In the sequel, we shall need the following fact (whose proof is omitted as being immediate).

Lemma 5. *For any set X, the algebra $\mathbf{cRel}^\sqcup(X)$ satisfies the identities (1)–(6).*

However, note that for all nonempty X, $\mathbf{cRel}^\sqcup(X)$ does not satisfy (7), because for $\varrho = \{\langle \emptyset, X, \emptyset\rangle^\nabla\}$ we have $\varrho \bullet \varrho^\sqcup \bullet \varrho = \emptyset$.

3. The Results

First of all, we prove that the co-relation algebras $\mathbf{cRel}(X)$ are Kleene algebras. Moreover, all such algebras are representable.

Proposition 6. *For any set X, the algebra $\mathbf{cRel}(X)$ is isomorphic to a subalegbra of $\mathbf{Rel}(\mathcal{P}(X))$.*

Proof. Define a mapping $\Theta : \mathcal{P}(3^X) \to \mathcal{P}(\mathcal{P}(X) \times \mathcal{P}(X))$ by

$$\Theta(\varrho) = \{\langle A, A \cup B\rangle : \langle A, B, C\rangle^\nabla \in \varrho\}.$$

It is immediately clear that Θ is injective and completely additive. It remains to prove that for all $\varrho, \sigma \subseteq 3^X$ we have $\Theta(\varrho \bullet \sigma) = \Theta(\varrho) \circ \Theta(\sigma)$ (for then it follows from the the complete additivity that we have $\Theta(\varrho^\star) = (\Theta(\varrho))^{\mathrm{rtc}}$).

Indeed, let $\langle A, B\rangle \in \Theta(\varrho \bullet \sigma)$. Clearly, this is the same as saying that $A \subseteq B$ and $\langle A, B \setminus A, X \setminus B\rangle^\nabla \in \varrho \bullet \sigma$. The latter condition is just equivalent to the existence of

$B_1, B_2, C, D \subseteq X$ such that $C = X \setminus D$, $\langle A, B_1, C\rangle^\nabla \in \varrho$, $\langle D, B_2, X \setminus B\rangle^\nabla \in \sigma$ and $B_1 \cup B_2 = B \setminus A$. Note that from here C can be eliminated, namely $C = X \setminus (A \cup B_1)$, so that we arrive at $\langle A, A \cup B_1\rangle \in \Theta(\varrho)$ and $\langle A \cup B_1, A \cup B_1 \cup B_2\rangle \in \Theta(\sigma)$, where $B_1 \cup B_2 = B \setminus A$. But recall that $A \subseteq B$, so that $\langle A, B\rangle \in \Theta(\varrho \bullet \sigma)$ is the same as $\langle A, A \cup B_1\rangle \in \Theta(\varrho)$ and $\langle A \cup B_1, B\rangle \in \Theta(\sigma)$ for some $B_1 \subseteq B$, i.e. $\langle A, B\rangle \in \Theta(\varrho) \circ \Theta(\sigma)$, which finishes the proof. □

The combined effect of Lemma 5 and the above proposition is just as follows.

Corollary 7. *For all X, $\mathbf{cRel}^{\sqcup}(X) \in \mathcal{L}^{\vee}$.*

Now let Σ be an alphabet, $x \in \Sigma$, and let

$$w = a_1 a_2 \dots a_n \in (\Sigma \cup \Sigma')^*$$

be any word (Σ' is, as in the previous section, a bijective copy of Σ). Define a mapping $\psi_w : \Sigma \to \mathcal{P}(3^{\underline{n}})$ (where we use the following notation: $\underline{n} = \{1, 2, \dots, n\}$ and $\underline{0} = \emptyset$) by

$$\psi_w(x) = \{\langle \underline{i-1}, \{i\}, \underline{n} \setminus \underline{i}\rangle^\nabla : \ a_i = x\} \cup \{\langle \underline{n} \setminus \underline{i}, \{i\}, \underline{i-1}\rangle^\nabla : \ a_i = x'\}.$$

Since by Proposition 4.2 from [4] we have that $\mathbf{Lang}'(\Sigma, \Sigma')$ is the free object on Σ in the category of completely idempotent semirings with involution (to which $\mathbf{cRel}^{\sqcup}(X)$ certainly belongs, for all X), the mapping defined above can be extended (by identifying x and $\{x\}$ for all $x \in \Sigma$) to a morphism $\Psi_w : \mathbf{Lang}'(\Sigma, \Sigma') \to \mathbf{cRel}^{\sqcup}(\underline{n})$ (recall that $n = |w|$).

It is not difficult to see that the following assertions hold:

(a) $\Psi_w(\{w\}) = \Psi_w(\{a_1\}) \dots \Psi_w(\{a_n\}) = \psi_w(a_1) \dots \psi_w(a_n) = \{\langle \emptyset, \underline{n}, \emptyset\rangle^\nabla\}$.

(b) $\Psi_w(\{\lambda\}) = \varepsilon_{\underline{n}}$. In particular, $\Psi_\lambda(\{\lambda\}) = \varepsilon_{\underline{0}} = \{\langle \emptyset, \emptyset, \emptyset\rangle^\nabla\}$.

(c) If u is a nonempty subword of w, say $u = a_i \dots a_j$, then, similarly as in (a),

$$\Psi_w(\{u\}) = \{\langle \underline{i-1}, \underline{j} \setminus \underline{i-1}, \underline{n} \setminus \underline{j}\rangle^\nabla\}.$$

Otherwise, $\Psi_w(\{u\}) = \emptyset$.

(d) If L is a language over $\Sigma \cup \Sigma'$, then

$$\Psi_w(L) = \Psi_w(\{u : \ u \text{ is a subword of } w \text{ such that } u \in L\}).$$

Therefore, for any word w, we have the following equivalence:

$$w \in L \iff \langle \emptyset, \underline{n}, \emptyset\rangle^\nabla \in \Psi_w(L). \tag{8}$$

Finally, let

$$\Psi : \mathbf{Lang}'(\Sigma, \Sigma') \to \prod_{w \in (\Sigma \cup \Sigma')^*} \mathbf{cRel}^{\sqcup}(\underline{|w|})$$

be the target tupling of the morphisms Ψ_w, that is, the (unique) function satisfying the condition $\Psi \circ \pi_w = \Psi_w$ for all $w \in (\Sigma \cup \Sigma')^*$ (where π_w is the projection of the above direct product corresponding to w).

Proposition 8. *Ψ is an embedding of completely idempotent semirings with involution (and thus, of Kleene algebras with involution).*

Proof. Since all functions Ψ_w are morphisms of completely idempotent semirings with involution, it suffices to prove that Ψ is injective. But this easily follows from the equivalence (8), because if L_1, L_2 are two different languages over $\Sigma \cup \Sigma'$ and $w_0 \in L_1 \setminus L_2$, $n_0 = |w_0|$, then by (8) we have that $\langle \emptyset, \underline{n_0}, \emptyset \rangle^{\nabla} \in \Psi_{w_0}(L_1) \setminus \Psi_{w_0}(L_2)$. Hence, $\Psi_{w_0}(L_1) \neq \Psi_{w_0}(L_2)$, and so $\Psi(L_1) \neq \Psi(L_2)$. □

As the algebras $\mathbf{Lang}'(\Sigma, \Sigma')$ generate the variety $\mathcal{L}^{\vee}$, we have just proved

Theorem 9. *The variety generated by all algebras of co-relations $\mathbf{cRel}^{\sqcup}(X)$ coincides with $\mathcal{L}^{\vee}$.*

Hence, we may say that the equational behaviour of the language reversal is modeled by ternary co-relations.

On the other hand, it is interesting to see how one obtains Kleene algebras of relations from those of co-relations, provided that we drop the converse operation. It turns out that we do not need the (slightly cumbersome) construction of the direct product: we shall prove that $\mathbf{Rel}(X) \in \mathsf{HS}(\mathbf{cRel}(\omega \times X))$ for all X, i.e. that $\mathbf{Rel}(X)$ is a quotient of a subalgebra of $\mathbf{cRel}(\omega \times X)$. It is worth noting that $\omega \times X$ is just the ω-copower of X (coproduct of ω copies of X) in the category of sets.

First of all, choose a linear order $\leq$ on X, so that $\langle X, \leq \rangle$ is a chain. Further, define a linear order relation $\preceq$ on $\omega \times X$ as follows:

$$\langle k, x_1 \rangle \preceq \langle \ell, x_2 \rangle \text{ if and only if } k < \ell \text{ or } k = \ell,\ x_1 \leq x_2.$$

A ternary co-vector over X (3-colouring of X) of the form $\mathbf{c}_{u,v}^{k,\ell} = \langle A, B, C \rangle^{\nabla}$,

$$\begin{array}{rcl} A & = & \{\langle n, x \rangle : \langle n, x \rangle \preceq \langle k, u \rangle\}, \\ B & = & \{\langle n, x \rangle : \langle k, u \rangle \prec \langle n, x \rangle \preceq \langle \ell, v \rangle\}, \\ C & = & \{\langle n, x \rangle : \langle \ell, v \rangle \prec \langle n, x \rangle\}, \end{array}$$

where $\langle k, u \rangle \preceq \langle \ell, v \rangle$, we call a *cutting* of the set $\omega \times X$. Now for $m \in \omega$ let

$$\chi_{u,v}^{m} = \{\mathbf{c}_{u,v}^{k,\ell} : \ \ell - k = m\}.$$

Note that $\chi_{u,v}^{0}$ is nonempty if and only if $u \leq v$.

A ternary co-relation on $\omega \times X$ is a *closed set of cuttings* if it is representable as a union of co-relations of the form $\chi_{u,v}^{m}$. Alternatively, we can define a closed set of cuttings as a family ϱ of cuttings satisfying, for all $u, v \in X$, the condition

$$(\exists p, q \in \omega)\ \mathbf{c}_{u,v}^{p,q} \in \varrho \ \Rightarrow\ (\forall n \in \omega)\ \mathbf{c}_{u,v}^{n,n+(q-p)} \in \varrho.$$

Finally, we call a ternary co-relation on $\omega \times X$ *good* if it is a union of a closed set of cuttings and a green-free co-relation (that is, a subset of $\varepsilon_{\omega \times X}$) which contains *no* cuttings. The set of all good co-relations on $\omega \times X$ is denoted by $G(X)$.

Lemma 10. *$G(X)$ is the universe of a subalgebra* $\mathbf{G}(X)$ *of* $\mathbf{cRel}(\omega \times X)$.

Proof. First of all, it is clear that $G(X)$ is closed for *arbitrary* unions and that $\emptyset \in G(X)$. Also, $\varepsilon_{\omega\times X} \in G(X)$, because

$$\varepsilon_{\omega\times X} = \left(\bigcup_{x\in X} \chi^0_{x,x}\right) \cup \varepsilon',$$

where ε' is the set of all green-free co-vectors over $\omega \times X$ which are not cuttings. Hence, the lemma will be proved if we show that the composition of two good co-relations remains good.

Therefore, let

$$\begin{aligned}\varrho_1 &= \left(\bigcup_{i\in I} \chi^{m_i}_{u_i,v_i}\right) \cup \varepsilon_1,\\ \varrho_2 &= \left(\bigcup_{j\in J} \chi^{m_j}_{u_j,v_j}\right) \cup \varepsilon_2,\end{aligned}$$

where $\varepsilon_1, \varepsilon_2$ are green-free co-relations containing no cuttings, and I, J are disjoint. Since $\bullet$ is completely distributive over $\cup$ (recall Proposition 6), we have:

$$\varrho_1\bullet\varrho_2 = \left(\bigcup_{\langle i,j\rangle\in I\times J} (\chi^{m_i}_{u_i,v_i} \bullet \chi^{m_j}_{u_j,v_j})\right) \cup \left(\bigcup_{i\in I}(\chi^{m_i}_{u_i,v_i} \bullet \varepsilon_2)\right) \cup \left(\bigcup_{j\in J}(\varepsilon_1 \bullet \chi^{m_j}_{u_j,v_j})\right) \cup (\varepsilon_1\bullet\varepsilon_2).$$

It is easy to see that the following holds:

$$\chi^k_{u,v} \bullet \chi^\ell_{z,t} = \begin{cases} \chi^{k+\ell}_{u,t}, & v = z,\\ \emptyset, & v \neq z.\end{cases}$$

Also, note that a co-vector which is a cutting can be composed (from the left or from the right) with a green-free co-vector only if the latter is a cutting, too (because the blue part of the considered green-free co-vector must coincide either with the green+blue part, or with the blue part of the cutting which it is composed with). Thus for all $u, v \in X$ and $m \in \omega$ we have

$$\chi^m_{u,v} \bullet \varepsilon_2 = \varepsilon_1 \bullet \chi^m_{u,v} = \emptyset.$$

Finally, it is not difficult to see that the composition of two green-free co-relations coincides with their intersection (because two green-free co-vectors can be composed if and only if they are equal), so

$$\varepsilon_1 \bullet \varepsilon_2 = \varepsilon_1 \cap \varepsilon_2,$$

which is a green-free co-relation containing no cuttings. We conclude that $\varrho_1 \bullet \varrho_2$ is a good co-relation. $\square$

Now define a relation $\equiv$ on $G(X)$ by

$\varrho_1 \equiv \varrho_2$ if and only if $(\forall u, v \in X)((\exists k \in \omega)\ \chi^k_{u,v} \subseteq \varrho_1 \ \Leftrightarrow\ (\exists \ell \in \omega)\ \chi^\ell_{u,v} \subseteq \varrho_2)$.

Proposition 11. *The relation $\equiv$ is a congruence of* $\mathbf{G}(X)$ *and*

$$\frac{\mathbf{G}(X)}{\equiv} \cong \mathbf{Rel}(X).$$

Proof. Define a mapping $\Upsilon : G(X) \to \mathcal{P}(X \times X)$ by

$$\Upsilon(\varrho) = \{\langle u, v\rangle :\ (\exists k \in \omega)\ \chi^k_{u,v} \subseteq \varrho\}.$$

It is obvious that the kernel of Υ coincides with $\equiv$. Thus it remains to prove that Υ is a surjective morphism of complete semirings.

First, for $\sigma \subseteq X \times X$ define

$$\varrho_\sigma = \bigcup_{\langle u,v\rangle \in \sigma} \chi^1_{u,v}.$$

According to the above definition, $\Upsilon(\varrho_\sigma) = \sigma$. Hence, Υ is surjective.

Now we prove that Υ is completely additive. We have:

$$\Upsilon(\textstyle\bigcup_{i\in I} \varrho_i) = \{\langle u, v\rangle :\ (\exists k \in \omega)\ \chi^k_{u,v} \subseteq \bigcup_{i\in I} \varrho_i\}.$$

But all the co-relations ϱ_i are good, which means that if $\mathbf{c}^{p,q}_{u,v} \in \varrho_i$, then $\chi^{q-p}_{u,v} \subseteq \varrho_i$. Therefore, if $\varrho_i \cap \chi^k_{u,v} \neq \emptyset$, then $\chi^k_{u,v} \subseteq \varrho_i$, and so $\chi^k_{u,v} \subseteq \bigcup_{i\in I} \varrho_i$ implies $\chi^k_{u,v} \subseteq \varrho_{i_0}$ for some $i_0 \in I$. Clearly, the converse of the latter conclusion is true, which amounts to say that $\Upsilon(\bigcup_{i\in I} \varrho_i) = \bigcup_{i\in I} \Upsilon(\varrho_i)$.

Finally, let $\varrho_1 = \theta_1 \cup \varepsilon_1$ and $\varrho_2 = \theta_2 \cup \varepsilon_2$ be two good co-relations, where θ_1, θ_2 are closed sets of cuttings and $\varepsilon_1, \varepsilon_2$ are green-free co-relations containing no cuttings. As seen in the previous lemma, we have

$$\varrho_1 \bullet \varrho_2 = (\theta_1 \bullet \theta_2) \cup (\varepsilon_1 \bullet \varepsilon_2).$$

Now we have the following chain of equivalences:

$$\begin{aligned}
\langle u, v\rangle \in \Upsilon(\varrho_1 \bullet \varrho_2) \quad &\Leftrightarrow \quad (\exists k \in \omega)\ \chi^k_{u,v} \subseteq \varrho_1 \bullet \varrho_2 \\
&\Leftrightarrow \quad (\exists k \in \omega)\ \chi^k_{u,v} \subseteq \theta_1 \bullet \theta_2 \\
&\Leftrightarrow \quad (\exists z \in X)(\exists p, q \in \omega)(\chi^p_{u,z} \subseteq \theta_1 \subseteq \varrho_1 \ \wedge\ \chi^q_{z,v} \subseteq \theta_2 \subseteq \varrho_2) \\
&\Leftrightarrow \quad (\exists z \in X)(\langle u, z\rangle \in \Upsilon(\varrho_1) \ \wedge\ \langle z, v\rangle \in \Upsilon(\varrho_2)) \\
&\Leftrightarrow \quad \langle u, v\rangle \in \Upsilon(\varrho_1) \circ \Upsilon(\varrho_2)
\end{aligned}$$

So, $\Upsilon(\varrho_1 \bullet \varrho_2) = \Upsilon(\varrho_1) \circ \Upsilon(\varrho_2)$, and the proposition is proved. □

Finally, it is well known that any direct product of full Kleene relation algebras (possibly with converse) is a represetable Kleene algebra. Namely, such a direct product (say, of $\mathbf{Rel}^\vee(X_i)$, $i \in I$) can be embedded into the full Kleene algebra of the relations on $\coprod_{i\in I} X_i$, the coproduct (disjoint union) of the base sets X_i. In the last assertion of this paper, we note that the direct product of co-relation algebras $\mathbf{cRel}^\sqcup(X_i)$ can be in a similar fashion represented by co-relations.

Proposition 12. *Any direct product of algebras of the form* $\mathbf{cRel}^{\sqcup}(X)$ *is embeddable into an algebra of that form. More precisely, the direct product of algebras* $\mathbf{cRel}^{\sqcup}(X_i)$, $i \in I$, *is isomorphic to a subalgebra of* $\mathbf{cRel}^{\sqcup}(\coprod_{i\in I} X_i)$, *where* $\coprod_{i\in I} X_i$ *denotes the coproduct of the sets* X_i.

Proof (in outline). In order to relax the notation, we may assume that the sets X_i are already disjoint and argue that $\prod_{i\in I}\mathbf{cRel}^{\sqcup}(X_i)$ is embeddable into $\mathbf{cRel}^{\sqcup}(X)$, where $X = \bigcup_{i\in I} X_i$. Consider the mapping $\varphi : \prod_{i\in I}\mathcal{P}(3^{X_i}) \to \mathcal{P}(3^X)$ given by

$$\varphi(\langle \varrho_i : i \in I\rangle) = \bigcup_{i\in I} \varrho_i.$$

One shows in a routine way that φ is an injecitve morphism of complete semirings with involution. □

Therefore, the embedding Ψ from Proposition 8 composed with the embedding φ from the above proposition gives an embedding of the language algebra $\mathbf{Lang}'(\Sigma, \Sigma')$ into $\mathbf{cRel}^{\sqcup}(S)$, where S is a set of cardinality $|\Sigma| + \aleph_0$.

4. An Open Problem

The algebras $\mathbf{cRel}^{\sqcup}(X)$, whose identities were investigated in this paper, turned up as categorical duals of Kleene relation algebras (with converse). However, we can consider another kind of co-relation algebras which arise from the analogy with relation algebras of Tarski (by droping the operation of iteration and taking all of the Boolean operations):

$$\mathbf{cR}(X) = \langle \mathcal{P}(3^X), \cup, \cap, ^-, \emptyset, 3^X, \bullet, ^{\sqcup}, \varepsilon_X\rangle.$$

It is well known (Monk, 1964) that the variety generated by the corresponding relation algebras is not finitely axiomatizable. Also, several explicite axiomatizations are known. Here we raise the question whether the same is true for the variety determined by algebras of the form $\mathbf{cR}(X)$. First of all, it would be interesting to give any nontrivial equational axiomatization for this variety (or any other description of its equational theory). Of course, we have proved in the present paper that the equations of co-relation algebras $\mathbf{cR}(X)$ not involving $\cap, ^-, 3^X$, are just those of idempotent unitary semirings with involution. However, the equations of relation and co-relation algebras which contain the above symbols are *not* equal, since the famous *Tarski identity*:

$$(x^{\vee} \circ (\overline{x \circ y})) \cap y = \emptyset,$$

does not hold for co-relations (see [20]).

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