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Note on the Cardinality of some Sets of Clones

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Abstract

All minimal clones containing a three-element grupoid have been determined in [3]. In this paper we solve the problem of the cardinality of the set of clones which contain some of these clones.

1 Notation and Preliminaries

Denote by $\mathbf{N}$ the set $\{1,2,\ldots\}$ of positive integers and for $k,n \in \mathbf{N}$, set $E_k = \{0,1,\ldots,k-1\}$. We say that f is an *i-th projection of arity* n $(1 \leq i \leq n)$ if $f \in P_k^{(n)}$ and f satisfies the identity $f(x_1,\ldots,x_n) \approx x_i$.

For $n,m \geq 1, f \in P_k^{(n)}$ and $g_1,\ldots,g_n \in P_k^{(m)}$, *the superposition of* f and $g_1,\ldots,g_n$, denoted by $f(g_1,\ldots,g_n)$, is defined by $f(g_1,\ldots,g_n)(a_1,\ldots,a_m) = f(g_1(a_1,\ldots,a_m),\ldots,g_n(a_1,\ldots,a_m))$ for all $(a_1,\ldots,a_m) \in E_k^m$. A set

A set C of operations on E_k is called a *clone* if it contains all the projections and is closed under superposition.

For an arbitrary set F of operations on E_k there exists the least clone containing F. This clone is called *the clone generated by* F, and will be denoted by $\langle F\rangle_{\mathrm{CL}}$. Instead of $\langle\{f\}\rangle_{\mathrm{CL}}$ we will write simply $\langle f\rangle_{\mathrm{CL}}$. For a clone C and $n \geq 1$ we denote by $C^{(n)}$ the set of n-ary operations from C.

The clones on E_k form an algebraic lattice $Lat(E_k)$ whose least element is the clone of all projections and whose greatest element is the clone of all operations on E_k. The atoms (dual atoms) of $Lat(E_k)$ are called *minimal (maximal) clones.*

A full description of all clones for $k=2$ was given by Post, for $k=3$ a complete list of all maximal clones was found by Iablonskiĭ and all minimal clones were determined by Csákány.

Let h be a positive integer. A subset ρ of E_k^h (i.e. a set of h-tuples over E_k) is an h-ary *relation on* E_k. An n-ary operation f on E_k *preserves* ρ if for every $h \times n$ matrix $X = [x_{ij}]$ over E_k whose columns are all h-tuples from ρ we have $(f(x_{00},\ldots,x_{0(n-1)}),\ldots,f(x_{(h-1)0},\ldots,x_{(h-1)(n-1)})) \in \rho$. The set of all operations on E_k preserving a given relation ρ is denoted Polρ.

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Let $k = 3$ and let ϕ be a permutation of E_3. To each n-ary function f we assign f^ϕ, called *conjugate* of f, defined by $f^\phi(x_0,\dots,x_{n-1}) = \phi(f(\phi^{-1}(x_0),\dots,\phi^{-1}(x_{n-1}))$. The map $f \to f^\phi$ carries each clone C onto the clone C^ϕ; in particular $\langle f\rangle^\phi_{\mathrm{CL}} = \langle f^\phi\rangle_{\mathrm{CL}}$, and $g \in \langle f\rangle_{\mathrm{CL}}$ implies $g^\phi \in \langle f^\phi\rangle_{\mathrm{CL}}$. We can permute the variables of f as well: for a permutation ψ of E_n put $f_\psi(x_0,\dots,x_{n-1}) = f(x_{\psi(0)},\dots,x_{\psi(n-1)}))$. Remark that always $(f^\phi)_\psi = (f_\psi)^\phi$. Note also that $\langle f_\psi\rangle_{\mathrm{CL}} = \langle f\rangle_{\mathrm{CL}}$ for any ψ. The conjugations and permutations of variables generate a permutation group T_n of order $3n!$ on the set of all n-ary functions on E_3.

A binary idempotent function with Cayley table

	0	1	2
0	0	n_5	n_4
1	n_3	1	n_2
2	n_1	n_0	2

is denoted by b_n, where $n = n_0 + 3n_1 + 3^2n_2 + 3^3n_3 + 3^4n_4 + 3^5n_5$.

It is proved in [3] that every minimal clone on E_3 containing an essential binary operation is a conjugate of exactly one of the following twelve clones: $\langle b_i\rangle_{\mathrm{CL}}$ with $i \in \{0, 8, 10, 11, 16, 17, 26, 33, 35, 68, 178, 624\}$. The following table shows the binary functions on E_3 which generate minimal clones.

$xy \to$	00	01	02	10	11	12	20	21	22
b_0	0	0	0	0	1	0	0	0	2
b_8	0	0	0	0	1	0	2	2	2
b_{10}	0	0	0	0	1	1	0	1	2
b_{11}	0	0	0	0	1	1	0	2	2
b_{16}	0	0	0	0	1	1	2	1	2
b_{17}	0	0	0	0	1	1	2	2	2
b_{26}	0	0	0	0	1	2	2	2	2
b_{33}	0	0	0	1	1	0	2	0	2
b_{35}	0	0	0	1	1	0	2	2	2
b_{68}	0	0	0	2	1	1	1	2	2
b_{178}	0	0	2	0	1	1	2	1	2
b_{624}	0	2	1	2	1	0	1	0	2

2 Results

Theorem 2.1 *The cardinality of the set of clones on E_3 containing a conjugate of $\langle b_j\rangle_{\mathrm{CL}}, j \in \{0, 8, 11, 17, 33, 35\}$ is continuum.*

Proof. The proof is based on the operations of Janov–Mučnik.

We shall define a countable set of operations F and an operation g so that for all $f \in F$, $f \notin \langle (F \setminus \{f\}) \cup \{g\}\rangle_{\mathrm{CL}}$. This implies that for each $G, H \subseteq F$, from $G \neq H$ it follows $\langle G \cup \{g\}\rangle_{\mathrm{CL}} \neq \langle H \cup \{g\}\rangle_{\mathrm{CL}}$. In this way we get a set of distinct clones of a continuum cardinality.

For $i = 1, \ldots, m$ denote by $\mathbf{e}_i$ the m-tuples $(1, \ldots, 1, 2, 1, \ldots, 1)$ with 2, at the i-th place. Let $A_m = \{\mathbf{e}_1, \ldots, \mathbf{e}_m\}$.

For $m > 2$, consider the m-ary operation f_m (Janov–Mučnik,[5]) which takes the value 1 on A_m and 0 otherwise.

Modifying an idea which is attributed to Rónyai in [1], we define the relations $\rho_m \subseteq E_3^m$ on E_3 for $m > 2 : \rho_m = A_m \cup B_m$, where $B_m = \{(b_1, \ldots, b_m) | b_j = 0$ for some $j, 1 \leq j \leq n\}$.

In what follows we prove that for each $i \neq m$ and $j \in \{0, 8, 11, 17, 33, 35\}$, f_i and b_j preserve ρ_m while f_m does not.

Let $X = [x_{ij}]$ be the $m \times m$ matrix with $x_{11} = \ldots x_{mm} = 2$ and $x_{ij} = 1$ otherwise.The i-th column of X is $\mathbf{e}_i \in \rho_m (i = 1, \ldots, m)$ while the values of f_m on the rows of X form $(f_m(\mathbf{e}_1), \ldots, f_m(\mathbf{e}_m))^T = (1, \ldots, 1)^T \notin \rho$. Hence, $f_m \notin Pol \rho_m$.

Suppose to the contrary that f_i doesn't preserve ρ_m for some $i \neq m$. Then there is an $m \times i$ matrix X with all columns in ρ_m and with rows $\mathbf{a}_1, \ldots, \mathbf{a}_m$ such that $\mathbf{b} := (f_i(\mathbf{a}_1), \ldots, f_i(\mathbf{a}_m))^T \notin \rho$. Since $im f_i = \{0, 1\}$ and $B_m \subset \rho_m$ clearly $\mathbf{b} = (1, \ldots, 1)^T$. By the definition of f_i there exist $1 \leq j_1, \ldots, j_m \leq i$ such that $\mathbf{a}_k = \mathbf{e}_{j_k}$ for all $k = 1, \ldots, m$. If $j_k = j_l$ for some $1 \leq k < l \leq m$ then the j_k-th column of X contains at least two 2s and so does not belong to ρ_m. As $i \neq m$ we can choose $k \in \{1, \ldots, i\} \setminus \{j_1, \ldots, j_m\}$. Clearly, the k-th column of X is $(1, \ldots, 1)^T \notin \rho_m$.

If $b_j, j \in \{0, 8, 11, 17, 33, 35\}$ does not preserve ρ then there exist $\mathbf{a}, \mathbf{b} \in \rho$ such that $(b_j(a_1, b_1), \ldots, b_j(a_m, b_m)) \notin \rho$, i.e. $(b_j(a_1, b_1), \ldots, b_j(a_m, b_m)) \in \{1, 2\}^m \setminus A_m$. It follows that $((b_j(a_1, b_1), \ldots, b_j(a_m, b_m)) = \mathbf{a}$ since $b_j(a_l, b_l) = 1$ implies $a_l = 1$ and $b_j(a_l, b_l) = 2$ implies $a_l = 2$. So, we get a contradiction.

The set of clones of the form $\langle G \cup \{b_0, b_8, b_{11}, b_{17}, b_{33}, b_{35}\}\rangle_{\text{CL}}$, $G \subseteq \{f_2, f_3, \ldots\}$ has a continuum cardinality. □

Theorem 2.2 *The cardinality of the set of clones on E_3 containing a conjugate of $\langle b_j \rangle_{\text{CL}}, j \in \{10, 16, 26, 68\}$ is at least $\aleph_0$.*

Proof.

Let $\{0, 1, 2\} = \{p, q, r\}$, and for $i = 1, \ldots, m$ denote by $\mathbf{e}_i$ the m-tuples $(p, \ldots, p, r, p, \ldots, p)$ with r at the i-th place. Let $A_m = \{\mathbf{e}_1, \ldots, \mathbf{e}_m\}$.

For $m > 2$, consider the m-ary operation f_m (similar to the Janov–Mučnik operations :

$$f_m(\mathbf{x}) = \begin{cases} p & if \ x \in A_m, \\ q & otherwise \end{cases} .$$

Define the following relations $\rho_m \subseteq E_3^m$ on E_3 for $m > 2 : \rho_m = E_3^m \setminus \{(p, \ldots, p)\}$.

In what follows we prove that f_i preserves ρ_m if and only if $i > m$.

Suppose to the contrary that f_i doesn't preserve ρ_m for some $i > m$. Then there is an $m \times i$ matrix X with all columns in ρ_m and with rows $\mathbf{a}_1, \ldots, \mathbf{a}_m$ such that $\mathbf{b} := (f_i(\mathbf{a}_1), \ldots, f_i(\mathbf{a}_m))^T \notin \rho$, i.e. $\mathbf{b} = (p, \ldots, p)^T$. By the definition of f_i, $\mathbf{a}_k = \mathbf{e}_{j_k}, 1 \leq j_k \leq i$, for all $k = 1, \ldots, m$. Since $i > m$, $i - m + 1$ column has to be equal $(p, \ldots, p)$, which gives a contradiction.

Let $i \leq m$ and $X = [x_{ij}]$ be the $m \times i$ matrix with $x_{lj} = p$ if $l \neq j$ and $x_{jj} = r$ for $j \in \{1, \ldots, i-1\}, l \in \{1, \ldots, m\}, x_{1i} = \ldots = x_{(i-1)i} = p$ and $x_{ii} = \ldots = x_{mi} = r$. The values of f_i on the rows of X form $(p, \ldots, p) \notin \rho$.

We shall prove that b_{10} and b_{16} preserve ρ_m with $r = 1, p = 2$ and $q = 0$; b_{26} preserves ρ_m with $p = 1, q = 0$, and $r = 2$; and b_{68} preserves ρ_m with $p = 0, q = 2$, and $r = 1$.

Suppose to the contrary that $b_j, j \in \{10, 16, 26, 68\}$ does not preserve ρ_m. Then, there is an $m \times 2$ matrix with both columns in ρ_m such that $(b_j(x_1, y_1), \ldots, b_j(x_m, y_m)) = (p, \ldots, p)$. Therefore by the definition of b_j clearly $x_l = p, l \in \{1, \ldots, m\}$ for each $j \in \{10, 16, 26, 68\}$. Thus, the first column of X is $(p, \ldots, p)^T \notin \rho$, a contradiction.

So, we proved that for each $j \in \{10, 16, 26, 68\}$, the set $\{\bigcup_{m>2} f_m\}$ satisfies $\langle \bigcup_{i>m} \{f_m\} \cup \{b_j\} \rangle_{\mathrm{CL}} \supset \langle \bigcup_{i>m+1} \{f_m\} \cup \{b_j\} \rangle_{\mathrm{CL}} \supset \langle \bigcup_{i>m+2} \{f_m\} \cup \{b_j\} \rangle_{\mathrm{CL}} \ldots$, proving that there are at least $\aleph_0$ clones containing $\langle b_j \rangle_{\mathrm{CL}}$. □

It is still an open problem to determine a cardinality of the set of clones that contain a clone generated by b_{178} and b_{624}.

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