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Hausdorff Dimension of Univoque Sets

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Dedicated tò Professor Imre Kátai on his 60th birthday

Abstract

In this paper we present the results obtained so far for the determination of the Hausdorff dimension of the univoque set, in number systems with base number greater than 1. The investigation is based on the methods presented in [1] and [2]. We illustrate the theoretical results with interesting examples.
Keywords: Number Theory, Expansions of Numbers, Univoque Sequences

1 Introduction

A lot of interesting problems arose relating to the number systems. A collection of these is presented in D. E. KNUTH's significant book [5].

In this paper we investigate the problem, what the numbers are, the representation of which is unique in a number system. The answer to this question derives immediately, when the base number of the number system is a positive integer. For example, in the decimal system a number has a unique representation if its form is an infinite decimal fraction, except the numbers with a pure nine tail (999...). The integers, the finite decimal fractions and the infinite decimal fractions with the tail 999... have two representations.

$\pi = 3,1415926\ldots$ - only one representation

$3,14 = 3,13999\ldots$ - two representations

Essentially the same is true of other number systems with integer base number, using the pure β tail, where β is the base of the number system.

However, the situation becomes much more complicated when we investigate number systems which have a non-integer base number.

EXAMPLE 1.

Let the base number $\beta = \sqrt{6} + 2 \approx 4,4495$. The digits we can use are $\mathcal{A} = \{0, 1, 2, 3, 4\}$, where the largest digit is the integer part of β. Let $\Theta = \frac{1}{\beta}$. In this

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number system $1 = 4\Theta + 2\Theta^2$, since

$$1 = 4 \cdot \frac{\sqrt{6}-2}{2} + 2 \cdot \left(\frac{\sqrt{6}-2}{2}\right)^2, \text{ with } \Theta = \frac{\sqrt{6}-2}{2}.$$

Let us substitute now in this expansion one Θ^2 with $4\Theta^3 + 2\Theta^4$. This we can do without any restriction, since we apply only usable digits. Then it derives $1 = 4\Theta + 1\Theta^2 + 4\Theta^3 + 2\Theta^4$. Repeating this method we can present infinite different expansions for 1:

$$1 = 0,42_{(\beta)} = 0,4142_{(\beta)} = 0,414142_{(\beta)} = 0,41\ldots 4142_{(\beta)} = 0,4141\ldots_{(\beta)}.$$

Thus, in a general number system numbers can have more than two different expansions. The choice of the numbers with only one expansion seems to be difficult.

2 Expansions of numbers

The methods presented here are based on the work of Z. Daróczy and I. Kátai ([1] and [2]), with new approaches when needed. They have specified the univoque sequences and have presented a method for the computation of the Hausdorff dimension of the univoque set in the cases $1 < \beta \leq 2$, where β is the base of the number system. This is our goal now in general, in an arbitrary number system with base number $\beta > 1$.

As in the example let $\Theta = \frac{1}{\beta}$, and $\mathcal{A} = \{0, 1, \ldots, [\beta]\}$ the set of the usable digits. The set of fractions in this number system is

$$\mathcal{F} = \{x | x = \sum_{n=1}^{\infty} \frac{a_n}{\beta^n} = \sum_{n=1}^{\infty} a_n \Theta^n\},$$

where $a = (a_1, a_2, \ldots) \in \{0, 1, \ldots, [\beta]\}^{\mathrm{N}}$. The smallest element in this set is 0 (with all of the a_i-s $= 0$), and the largest element is

$$L = [\beta]\Theta + [\beta]\Theta^2 + [\beta]\Theta^3 + \ldots = \frac{[\beta]\Theta}{1-\Theta},$$

(with all of the a_i-s $= [\beta]$). Here $L \geq 1$, because from this inequality - substituting L - follows $[\beta] \geq \beta - 1$.

From now on we work only on the set of fractions.

For an arbitrary $x \in [0, L]$ we are able to describe at least one sequence $a = (a_1, a_2, \ldots) \in \{0, 1, \ldots, [\beta]\}^{\mathrm{N}}$, which produces the number x, i.e. $x = \sum_{n=1}^{\infty} a_n \Theta^n$. This we can do for example with the β or regular expansion of x. A. Rényi has proved [9], that every number x has a β expansion with $\beta > 1$ as follows:

$$x = \varepsilon_0(x) + \frac{\varepsilon_1(x)}{\beta} + \frac{\varepsilon_2(x)}{\beta^2} + \ldots,$$

where $\varepsilon_0(x) = [x], \varepsilon_1(x) = [\beta(x)], \varepsilon_2(x) = [\beta(\beta(x))], \ldots$ - here $[x]$ denotes the integral part and (x) the fractional part of x. The set $\mathcal{F}$ is closed and bounded (G. A. EDGAR [3]). For our investigation we will use two different expansions, the regular and the quasiregular one. The first is the "restriction" of the β expansion to the set of fractions.

The regular expansion. Let us define the following sequence $\varepsilon_n(x)$ for $x \in [0, L]$, by induction on n:

$\varepsilon_n(x) = j$, if

$$\sum_{i=1}^{n-1} \varepsilon_i(x)\Theta^i + j\Theta^n \leq x$$

where $j \in \mathcal{A}$, but

$$\sum_{i=1}^{n-1} \varepsilon_i(x)\Theta^i + (j+1)\Theta^n > x,$$

or $j+1 > [\beta]$, i.e. we would use a non-usable digit for the expansion. The expansion $x = \varepsilon_1(x)\Theta + \varepsilon_2(x)\Theta^2 + \ldots$ is called the regular expansion of x.

This essentially means, that we choose the largest usable digit in every step. So the expansion $1 = 0,42_{(\beta)}$ is the regular expansion of 1 in the previous example, since $4\Theta < 1$, thus $\varepsilon_1(x) = 4$, and $4\Theta + 2\Theta^2 = 1$, thus $\varepsilon_2(x) = 2$.

The quasiregular expansion. Let us define by induction on n the following sequence $\delta_n(x)$ for $x \in (0, L]$:

$\delta_n(x) = j$, if

$$\sum_{i=1}^{n-1} \delta_i(x)\Theta^i + j\Theta^n < x$$

where $j \in \mathcal{A}$, but

$$\sum_{i=1}^{n-1} \delta_i(x)\Theta^i + (j+1)\Theta^n \geq x,$$

or $j+1 > [\beta]$, i.e. we would use a non-usable digit for the expansion. The expansion $x = \delta_1(x)\Theta + \delta_2(x)\Theta^2 + \ldots$ is called the quasiregular expansion of x.

The quasiregular expansion is always infinite, and if the regular expansion is infinite too, then the two expansions are the same. Comparing with the regular expansion here we choose "almost" the largest usable digit in every step. In the previous example the quasiregular expansion of 1 is $1 = 0,4141\ldots_{(\beta)}$. Here $4\Theta < 1$ so $\delta_1(x) = 4$, $4\Theta + \Theta^2 < 1$ and $4\Theta + 2\Theta^2 = 1$, so $\delta_2(x) = 1$. $4\Theta + \Theta^2 + 4\Theta^3 < 1$, thus $\delta_3(x) = 4$, and eventually $1 = 0,4141\ldots_{(\beta)}$ follows.

In the example we have seen too, that in spite of the "closeness" of the regular and quasiregular expansions, we can find other different infinite expansions among them.

3 Univoque sequences

We call the sequence $\varepsilon \in \{0,1,\ldots,[\beta]\}^{\mathbf{N}}$ univoque (with respect to Θ) if the equation

$$\sum_{n=1}^{\infty} \varepsilon_n \Theta^n = \sum_{n=1}^{\infty} \delta_n \Theta^n$$

is only true in the case $\varepsilon = \delta$, i.e. $\varepsilon_n = \delta_n$, for $n \in \mathbf{N}$ ($\delta \in \{0,1,\ldots,[\beta]\}^{\mathbf{N}}$). In this case the number $\sum_{n=1}^{\infty} \varepsilon_n \Theta^n$ is said to be univoque, too.

The sequences

$$\underline{0} := (0,0,\ldots), \qquad \underline{[\beta]} := ([\beta],[\beta],\ldots)$$

are univoque, because every other sequence is clearly larger or smaller than these ones, respectively, using lexicographic ordering.

For $\varepsilon \in \{0,1,\ldots,[\beta]\}^{\mathbf{N}}$ let $\overline{\varepsilon} = \underline{[\beta]} - \varepsilon = ([\beta]-\varepsilon_1, [\beta]-\varepsilon_2, \ldots)$, which we will call the complementary sequence. From this it follows that $\overline{\varepsilon} \in \{0,1,\ldots,[\beta]\}^{\mathbf{N}}$, and if $x = \sum_{n=1}^{\infty} \varepsilon_n \Theta^n$, then $\sum_{n=1}^{\infty} \overline{\varepsilon_n} \Theta^n = ([\beta]-\varepsilon_1)\Theta + ([\beta]-\varepsilon_2)\Theta^2 + \ldots = L - x$.

EXAMPLE 1. (CONTINUED)

Are the following sequences univoque in the number system with base number $\beta = \sqrt{6}+2$?

a) The sequence $(3,\ldots,3,3,4,4,\ldots)$.

Here we can substitute the last digit 3 with digit 4, using for example $1 = 0,42_{(\beta)}$. Thus, if we "catch" a digit 4 and a digit 2 from the tail of the sequence, we get $(3,\ldots,3,4,0,2,4,4,\ldots)$, which represents the same number, so it is clearly not univoque. Of course, using other expansions of 1, we get infinitely many different sequences representing the same number: from $1 = 0,4142_{(\beta)}$ it derives $(3,\ldots,3,4,0,3,0,2,4,4,\ldots)$, from $1 = 0,414142_{(\beta)}$ we get $(3,\ldots,3,4,0,3,0,3,0,2,4,4,\ldots)$ etc.

We can work similarly with the complementary sequence $(1,\ldots,1,0,\ldots)$. Here we can substitute the last digit 1 with digit 0, using $1 = 0,42_{(\beta)}$. Thus we get $(1,\ldots,1,0,4,2,0,0,\ldots)$, which represents the same number, and we can produce infinitely many such sequences in the same manner.

b) The sequence $(4,\ldots,4,3,3,\ldots)$.

In this sequence we are not able to change any digit using the equation $1 = 4\Theta + 2\Theta^2$, since we would get in all cases non-usable digits: $(4,\ldots,4,2,3{+}4,3{+}2,3,3,\ldots)$ or $(4,\ldots,4,4,3{-}4,3{-}2,3,3,\ldots)$. The same is true of the complementary sequence, so we conclude, that both of the sequences are univoque.

We have seen, that the expansions of 1 play a very important role in deciding the univoque property. Since $[\beta]\Theta \leq 1$ both in the regular and the quasiregular expansion of 1 surely $\varepsilon_1 = [\beta]$ and $\delta_1 = [\beta]$ if $[\beta] < \beta$.

Now we present the exact theoretical investigation. Some of the results were already presented in [4], these parts we quote briefly, without proofs (Lemma 1., Propositon 1., Theorem 1. and 2.).

By using the regular expansions we are able to decide whether a sequence is univoque or not:

LEMMA 1. *$\varepsilon \in \{0, 1, \dots, [\beta]\}^{\mathrm{N}}$ is univoque with respect to $\Theta \iff \varepsilon$ and $[\beta] - \varepsilon$ are regular.*

(This Lemma is a generalization of Theorem 2.1 in [1].)

By Lemma 1., in order to decide whether a sequence is univoque or not we need the regular expansion of the sequence and that of the complementary sequence. To establish that an expansion producing a number less than 1 is regular, we use the result of W. PARRY [8]. By reformulating his results according to our notations we get the following

PARRY CONDITION.

$$b = (b_1, b_2, \dots) \text{ is regular representing a number less than } 1 \iff$$

$$(b_n, b_{n+1}, \dots) < (\ell_1, \ell_2, \dots) \ \forall \ n \geq 1,$$

where $0 \leq x = b_1\Theta + b_2\Theta^2 + \dots < 1$ with $b \in \{0, 1, \dots, [\beta]\}^{\mathrm{N}}$, and $(\ell_1, \ell_2, \dots)$ is the coefficient sequence of the quasiregular expansion of 1.

Using only the Parry condition we are not able to decide the univoque or regular property of a sequence representing a number in the interval $[1, L]$. To accomplish this we shall use other Propositions [4], and eventually it follows, that the further regular sequences are in the form $([\beta], \dots, [\beta], b_1, b_2, \dots)$ where the sequence $(b_1, b_2, \dots)$ is regular representing a number less than 1.

The set of the univoque numbers. Let

$$H = \{x = \sum_{n=1}^{\infty} \varepsilon_n \Theta^n | x \in [0, L] \text{ and } \varepsilon \text{ univoque with respect to } \Theta\}$$

be the set of the univoque numbers of the interval $[0, L]$, and similarly H^* and H_1 the set of the univoque numbers of the intervals $[\Theta, 1)$, $[0, 1)$ respectively.

PROPOSITION 1. *We have*

$$H_1 = \{0\} \cup \bigcup_{n=0}^{\infty} \Theta^n H^*.$$

By Lemma 1., the location of the univoque numbers in the interval $[0, L]$ is symmetrical, and from Proposition 1., it is self-similar. Thus, if $L < 2$, then from the univoque numbers of the interval $[0, 1)$ by reflection we can get the univoque numbers in $[1, L]$, and so eventually the univoque numbers of the whole interval $[0, L]$.

$L = \frac{[\beta]\Theta}{1-\Theta} < \frac{1}{1-\Theta}$, and the fraction on the right side is less than 2 if $1 - \Theta \geq \frac{1}{2}$, i.e. if $\beta \geq 2$. This will be assumed in the sequel, since the properties of the univoque set in the cases $1 < \beta \leq 2$ are already well-known ([1],[2]).

Thus, we can specify all univoque numbers (the set H) if we know the univoque numbers in the interval $[\Theta, 1)$, i.e. the set H^*.

Breaking down the problem into two cases. Let us notice now that

$$\frac{1}{K+1} < \Theta \leq \frac{1}{K}, \text{ i.e. } K = [\beta].$$

Clearly, in this interval there exists a Θ_K for which $K\Theta_K + {\Theta_K}^2 = 1$, since for all Θ in this interval $K\Theta \leq 1$ but $(K+1)\Theta > 1$. The value of this number is

$$\Theta_K = \frac{-K + \sqrt{K^2+4}}{2},$$

and if we use the notation $\beta_K = \frac{1}{\Theta_K}$, then $\beta_K = K + \Theta_K$.

The case when the fraction part is larger than Θ_K $(K+1 > \beta > K + \Theta_K)$ will be called from now on the "big case", and the case when the fraction part is smaller than Θ_K $(K \leq \beta \leq K + \Theta_K)$ the "small case".

4 Univoque sequences in the small case

Let

$$Z = \{z = \varepsilon_1\Theta + \varepsilon_2\Theta^2 + \varepsilon_3\Theta^3 + \ldots | 1 \leq \varepsilon_i \leq K-1\}.$$

THEOREM 1. *All elements of Z are univoque numbers.*

However, there are also other univoque sequences. According to our former investigation, the univoque sequences are the following:

a) The sequences $\underline{[\beta]} = \underline{K}$ and $\underline{0}$,

b) the sequences of type $[\beta]\ldots[\beta]b_ib_{i+1}\ldots$ and $0\ldots0b_ib_{i+1}\ldots$, where for the tail $b = b_ib_{i+1}\ldots$

$$\bar{t}_1\bar{t}_2\ldots < \sigma^j(b) < t_1t_2\ldots$$

is true of all $j = 0, 1, \ldots$ (σ is the shift operator).

So we can represent the whole univoque set from Z as follows:

$$H = \{0\} \cup \{L\} \cup Z \cup \bigcup_{j=1}^{\infty} \Theta^j Z \cup \bigcup_{j=1}^{\infty} (K\Theta + K\Theta^2 + \ldots + K\Theta^j + \Theta^j Z).$$

The Hausdorff dimension of the set H. The Hausdorff dimensions of the sets $\Theta^j Z$ and $K\Theta + \ldots + K\Theta^j + \Theta^j Z$ $(j = 1, 2, \ldots)$ are clearly the same as the dimension of the set Z. Thus, the Hausdorff dimension of the whole set H equals the dimension of the base set Z.

To compute the dimension of the set Z, we first specify the self-similarity dimension, and after this we check the fulfilment of the open set condition, which guarantees that the Hausdorff dimension equals the self-similarity dimension, according to the method presented by G. A. EDGAR [3].

THEOREM 2. *The Hausdorff dimension of the univoque set is*

$$\dim H = \frac{\log(K-1)}{\log\beta}.$$

REMARK. To represent the univoque sequences we can use a graphic model. We build a directed graph, the nodes of which are the usable digits in the number system, and draw an edge from the node a to b if the digit b is allowable (in a univoque sequence) after digit a. We label the edges by Θ. Thus, we get a directed graph called Mauldin-Williams graph [3]. Wandering over all the digits of the graph, we can construct all univoque sequences. The fulfilment of the open set condition guarantees, that the self-similarity dimension of the graph is the same as the Hausdorff dimension of the set H. The graph model is a useful means to demonstrate the univoque sequences, but it is not absolutely necessary.

The number system with base number $1+\sqrt{2}$.
In this number system $[\beta]=2$, $\mathcal{A}=\{0,1,2\}$, $\Theta=\sqrt{2}-1$,

$$L=\frac{2\Theta}{1-\Theta}=\frac{2\sqrt{2}-2}{2-\sqrt{2}}=\sqrt{2}.$$

Since $2\Theta+\Theta^2=1$, the sequences belonging to the regular and the quasiregular expansion of 1 are 21 and $(20)^\infty$, respectively. This number system belongs to the small case, thus the univoque sequences are the following:

a) The sequences $\underline{2}$ and $\underline{0}$,

b) the sequences of type $2\ldots211\ldots$ and $0\ldots011\ldots$.

Let us denote the set of the univoque numbers beginning with i with H_i, where $i=0,1,2$. The set of all univoque numbers is $H=H_0\cup H_1\cup H_2$, with

$$H_0=(0+\Theta H_0)\cup(0+\Theta H_1),$$

$$H_1=(\Theta+\Theta H_1),$$

$$H_2=(2\Theta+\Theta H_1)\cup(2\Theta+\Theta H_2).$$

The structure of the univoque set is representable by the Mauldin-Williams graph shown on Figure 1.

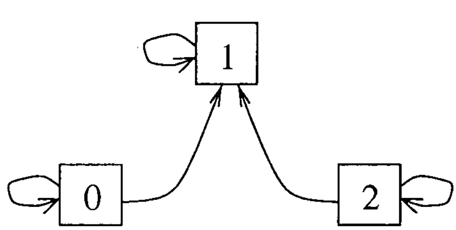

Figure 1: The Mauldin-Williams graph in the number system $1+\sqrt{2}$.

Since

$$H_1 = \{(1)^\infty\} = \{\frac{\sqrt{2}}{2}\},$$

the set H contains only countably many elements. Thus, its self-similarity and Hausdorff dimension is 0, according to Theorem 2.

The set H approximately has the form [1] shown on Figure 2.

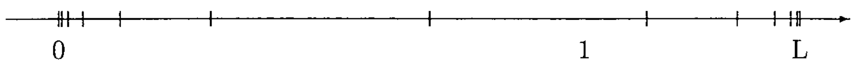

Figure 2: The approximate form of the set H

REMARK. The univoque set is very similar to this in all of the cases, when $2 < \beta \leq \beta_2 = 1 + \sqrt{2}$, since the univoque sequences have the same form. These base numbers give the simplest univoque sets investigating number systems with $\beta > 2$.

5 Univoque sequences in the big case

In the big case the structure of the univoque set is much more complicated than in the small case. Usually, the Mauldin-Williams graph of the set is not strongly connected, it contains more strongly connected parts. We build up this graph in the same manner, as before (using the Parry condition). The nodes of it represent the allowable digits (or sequence parts) in the univoque sequences (because of the complicated structure it is possible, that in the representation we have to use sequence parts - see the following example). From the node $i_1 i_2 \dots i_n$ there is an edge to the node $j_1 j_2 \dots j_n$, if $i_2 i_3 \dots i_n = j_1 j_2 \dots j_{n-1}$, and the sequence part $i_1 i_2 \dots i_n j_n$ is allowed in a univoque sequence.

THEOREM 3. *The Hausdorff dimension of the univoque set in the big case is the same, as the largest self similarity dimension of the strongly connected parts.*

PROOF. Let us assume, that the graph representing the univoque set has m strongly connected parts, $\mathcal{D}^{(1)}, \mathcal{D}^{(2)}, \dots, \mathcal{D}^{(m)}$. The graph part $\mathcal{D}^{(1)}$ has a unique self-similarity dimension, let us denote this by $\dim_s \mathcal{D}^{(1)}$.

a) First we prove, that the self-similarity dimension of $\mathcal{D}^{(1)}$ is the same, as the Hausdorff dimension of $\mathcal{D}^{(1)}$ (which we denote by $\dim \mathcal{D}^{(1)}$).

To do this we have the check the open set condition. We prove this part generally, for an arbitrary (strongly) connected graph part. Let us consider for all nodes v the set I_v on the number line, which contains the picture of the sequences beginning with v. This is a closed interval. Intervals according to different nodes can not

[1]To draw Figure 2 and Figure 4 we have used the computer algebra software Maple [7]. Maple is a registered trademark of Waterloo Maple Inc.

have an intersection. Let us assume to the contrary, that $I_{i_1 i_2 \dots i_n} \cap I_{j_1 j_2 \dots j_n} \neq \emptyset$. Since the nodes are different, we can find different digits in the same position: $i_k \neq j_k$. It can be assumed, that for example $i_k = j_k + 1$, since if it is not possible, then clearly larger difference can not exist, too. The smallest element in the first interval is

$$i_1 \cdot \Theta + i_2 \cdot \Theta^2 + \ldots + i_k \cdot \Theta^k + 0 \cdot \Theta^{k+1} + 0 \cdot \Theta^{k+2} + \ldots$$

and the largest in the second must be smaller than

$$i_1 \cdot \Theta + i_2 \cdot \Theta^2 + \ldots + (i_k - 1) \cdot \Theta^k + \ell_1 \cdot \Theta^{k+1} + \ell_2 \cdot \Theta^{k+2} + \ldots + \ell_p \cdot \Theta^{k+p} + \ell_1 \cdot \Theta^{k+p+1} + \ldots$$

where in the following the digits $\ell_1 \ell_2 \ldots \ell_p$ are repeated, which are the coeffients in the quasiregular expansion of 1 (this result follows from the Parry condition). Thus, the intervals can not have an intersection. Choosing open intervals "little bit larger" than these closed intervals there is still no intersection, so eventually $\dim_s \mathcal{D}^{(1)} = \dim \mathcal{D}^{(1)}$.

b) As in a) we have $\dim \mathcal{D}^{(2)} = \dim_s \mathcal{D}^{(2)}$. The set $\mathcal{D}^{(1)} \cup \mathcal{D}^{(2)}$ is situated in the graph representing the whole univoque set in such manner, that we complete the two strongly connected parts with the through leading edges and nodes. Now, using the results of R. D. MAULDIN and S. C. WILLIAMS [6], we deduce that

$$\dim(\mathcal{D}^{(1)} \cup \mathcal{D}^{(2)}) = \max\{\dim_s(\mathcal{D}^{(1)}), \dim_s(\mathcal{D}^{(2)})\}.$$

In the sequel we consider the graph part containing the sets $\mathcal{D}^{(1)}$ and $\mathcal{D}^{(2)}$ as one component, and we add the set $\mathcal{D}^{(3)}$ etc. Finally we get for the Hausdorff dimension of the whole graph $\mathcal{G}$:

$$\dim \mathcal{G} = \max(\dim_s \mathcal{D}^{(1)}, \dim_s \mathcal{D}^{(2)}, \ldots, \dim_s \mathcal{D}^{(m)}),$$

and since the open set condition is satisfied, this is the Hausdorff dimension of the set H.

□

The number system with base number $\frac{3(26+6\sqrt{33})^{\frac{1}{3}}}{(26+6\sqrt{33})^{\frac{2}{3}}-8-(26+6\sqrt{33})^{\frac{1}{3}}}$.

In this number system [2] $\beta \approx 3,3830$, thus $\mathcal{A} = \{0, 1, 2, 3\}$. The regular expansion of 1 is $1 = 3\Theta + \Theta^2 + \Theta^3$, and the quasiregular form is $1 = 0,310310\ldots$. Thus, the univoque sequences can not contain the following parts:

$$(3,3), (0,0), (3,2), (0,1), (3,1,3), (0,2,0), (3,1,2), (0,2,1), (3,1,1), (0,2,2).$$

The possible parts are the following:

$$(3,1,0), (3,0,3), (3,0,2), (2,3,1), (2,3,0), (2,2,3), (2,2,2), (2,2,1), (2,2,0),$$

[2]The base number of the number system is the reciprocal value of the real solution of the equation $1 = 3\Theta + \Theta^2 + \Theta^3$

$$(2,1,3),(2,1,2),(2,1,1),(2,1,0),(2,0,3),(2,0,2),$$

and their complementers:

$$(0,2,3),(0,3,0),(0,3,1),(1,0,2),(1,0,3),(1,1,0),(1,1,1),(1,1,2),(1,1,3),$$

$$(1,2,0),(1,2,1),(1,2,2),(1,2,3),(1,3,0),(1,3,1).$$

Now if we would like to denote the possible parts unambigous, then we have to use three digits in a node. Similarly as before, we will use the sets $H_{i_1 i_2 i_3}$, and we can write the set equations, for example:

$$H_{023} = (0 + \Theta H_{231}) \cup (0 + \Theta H_{230})$$

$$H_{102} = \Theta + \Theta H_{023}$$

$$H_{111} = (\Theta + \Theta H_{110}) \cup (\Theta + \Theta H_{111}) \cup (\Theta + \Theta H_{112}) \cup (\Theta + \Theta H_{113})$$

$$\dots$$

To save place, we have omitted the further equations, but it is an easy exercise to write those ones, too. The structure of the univoque set is representable with the Mauldin-Williams graph shown on Figure 3. The graph contains two strongly connected parts. To indicate this, we have separated the nodes. The nodes which have "through leading" role are shown alone, and their edges indicate the connections.

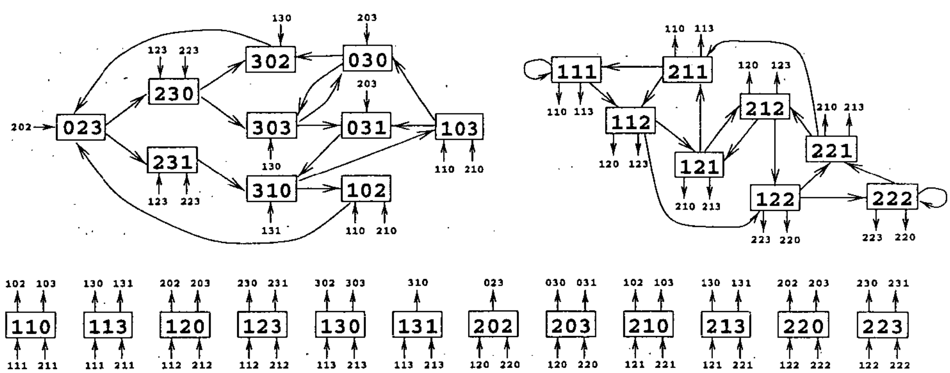

Figure 3: Mauldin-Williams graph. (See text for details.)

To specify the self-similarity dimension, for the strongly connected graph parts we get the following equation systems:

The first graph part

$$q^s_{023} = \lambda \cdot q^s_{230} + \lambda \cdot q^s_{231}$$

$$q^s_{030} = \lambda \cdot q^s_{302} + \lambda \cdot q^s_{303} \qquad = q^s_{230}$$

$$q^s_{031} = \lambda \cdot q^s_{310} \qquad = q^s_{231}$$

$$
\begin{aligned}
q^s_{102} &= \lambda \cdot q^s_{023} && = q^s_{302} \\
q^s_{103} &= \lambda \cdot q^s_{030} + \lambda \cdot q^s_{031} && = q^s_{303} \\
q^s_{310} &= \lambda \cdot q^s_{102} + \lambda \cdot q^s_{103},
\end{aligned}
$$

where q_{ijk} is the Perron number belonging to the node ijk, s is the self-similarity dimension of the graph part, and $\lambda = \Theta^s$. After repeated substitution

$$
\begin{aligned}
q^s_{023} &= \lambda \cdot q^s_{030} + \lambda^2 \cdot q^s_{030} \\
q^s_{030} &= \lambda^2 \cdot q^s_{023} + \lambda \cdot q^s_{023}
\end{aligned}
$$

We can choose without any restriction one of the Perron numbers ([3]), let for example $q_{023} = 1$. Thus, from the last equations

$$
\begin{aligned}
1 &= \lambda \cdot q^s_{030} + \lambda^2 \cdot q^s_{030} \text{ and} \\
q^s_{030} &= \lambda^2 + \lambda.
\end{aligned}
$$

From these two equations $1 = \lambda^2 + 2\lambda^3 + \lambda^4 = (\lambda + \lambda^2)^2$. The solution of $\lambda^2 + \lambda - 1 = 0$ is

$$\lambda = \frac{\sqrt{5} - 1}{2},$$

so the dimension is

$$s = \frac{\log(\frac{\sqrt{5}+1}{2})}{\log\beta} \approx 0,3948.$$

The second graph part

$$
\begin{aligned}
q^s_{111} &= \lambda \cdot q^s_{111} + \lambda \cdot q^s_{112} && = q^s_{211} \\
q^s_{112} &= \lambda \cdot q^s_{121} + \lambda \cdot q^s_{122} && = q^s_{212} \\
q^s_{121} &= \lambda \cdot q^s_{211} + \lambda \cdot q^s_{212} && = q^s_{221} \\
q^s_{122} &= \lambda \cdot q^s_{221} + \lambda \cdot q^s_{222} && = q^s_{222}
\end{aligned}
$$

After repeated substitution

$$q^s_{111} = \lambda \cdot q^s_{111} + \lambda \cdot q^s_{112} \qquad = q^s_{112}$$

From this $q^s_{111} = 2\lambda \cdot q^s_{111}$, and $\lambda = \frac{1}{2}$.
Thus, the dimension is

$$s = \frac{\log 2}{\log\beta} \approx 0,5687,$$

and the dimension of the whole graph is eventually the dimension of the second graph part. The open set criterion is now satisfied, so this is the Hausdorff dimension of the whole univoque set, which approximately has the form shown on Figure 4.

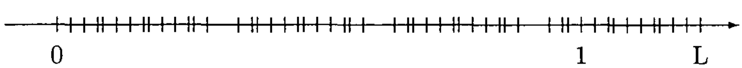

Figure 4: The approximate form of the set H

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