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Università di Pisa

Dipartimento di Matematica

Two applications of the Theory of Currents

Dottorato di Ricerca in Matematica

Tesi per il conseguimento del titolo

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Contents

Introduction

In this thesis we examine two very different problems in Geometric Measure Theory, whose common point is a substantial use of the Theory of Currents as a tool for proofs.

The first part deals with the differentiability of Lipschitz functions. We want to find an adapted version of Rademacher theorem, valid for every Radon measure μ on $\mathbb{R}^d$. Namely, given a Radon measure μ on $\mathbb{R}^d$, we find a map S mapping a point $x \in \mathbb{R}^d$ into a linear subspace $S(x)$ of $\mathbb{R}^d$ with the following property: every Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$, is differentiable along the vector space $S(x)$ at μ-almost every $x \in \mathbb{R}^d$ (i.e. the restriction of f to the affine subspace $x + S(x)$ is differentiable at x for μ-almost every x). We prove also that the map S is maximal with respect to the previous property, in a very strong sense: there exists a Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$ which is non-differentiable at μ-almost every $x \in \mathbb{R}^d$ along any line that is not a vector subspace of $S(x)$. The map S is defined through a property of the measures that we call 1-decomposability, which means being equal to an integral of 1-rectifiable measures. We find a strict correlation between measures that are 1-decomposable and normal 1-currents, whose linearity properties are essential in the proof of the differentiability result.

In the second part we look for a formulation of the Steiner tree problem as a minimization problem in an abstract class of objects, with nice compactness properties. Steiner tree problem consists in finding a connected set of minimal 1-dimensional measure containing a given set of finitely many points. It turns out that a family of 1-dimensional currents with coefficients in a group with certain properties provides the correct tool to establish an equivalence between the Steiner problem and a mass minimization problem. By this we mean that it is easy to obtain the solutions of the mass minimization problem from the solutions of the Steiner problem and viceversa. The representation given for the class of currents in consideration allows us to state a calibration principle and therefore to prove the (absolute) minimality of some concrete configurations. An interesting phenomenon arises when dealing with the problem of the existence of calibrations for such mass minimizing currents.

Part 1

Differentiability of Lipschitz functions with respect to measures

CHAPTER 1

Differential forms and currents

Introduction to part I

The celebrated Rademacher theorem asserts that if $f : \mathbb{R}^d \to \mathbb{R}^m$ is a Lipschitz function, then it is differentiable almost everywhere with respect to the Lebesgue measure $\mathscr{L}^d$. If we consider a measure μ on $\mathbb{R}^d$ which is absolutely continuous with respect to $\mathscr{L}^d$, of course we can also say that every Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}^m$ is differentiable almost everywhere with respect to μ. Now take a $\mathscr{C}^1$-curve C in $\mathbb{R}^2$ and consider the measure $\mu = \mathscr{H}^1 \llcorner C$, which is the restriction of the 1-dimensional Hausdorff measure to the curve C. In general one cannot say that a Lipschitz function $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable μ-a.e.,

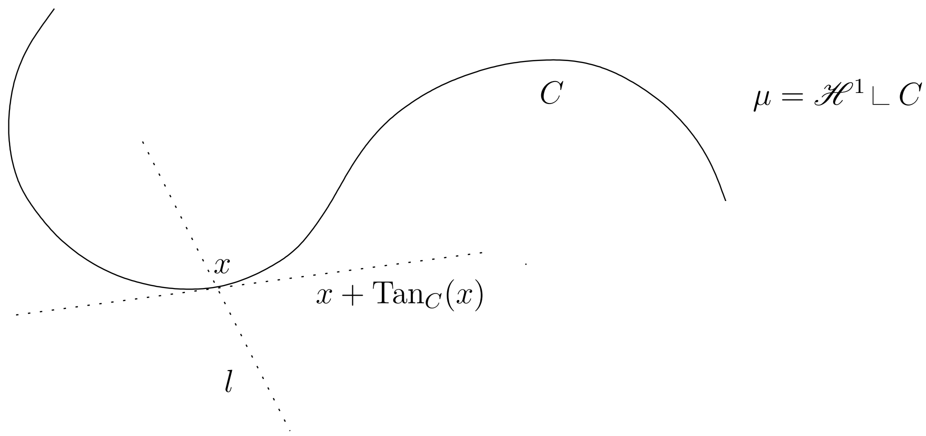

FIGURE 1.0.1

in fact for example the 1-Lipschitz function $g(x) = \mathrm{dist}(x, C)$ is non-differentiable at any point of C. Nevertheless it is easy to see that every Lipschitz function $f : \mathbb{R}^2 \to \mathbb{R}$ is μ-almost everywhere differentiable along the tangent bundle of the curve Tan_C, which means that the restriction of f to the line through x with the direction of $\mathrm{Tan}_C(x)$ is differentiable at x for μ-a.e. $x \in C$ (see Figure 1.0.1). Moreover the 1-Lipschitz function g defined above has this property: the restriction of g to any line l through x, which is not the tangent line, is non-differentiable at x, for every point $x \in C$. Therefore it is clear that the best possible version of the Rademacher theorem valid for the measure μ is the following: every Lipschitz function $f : \mathbb{R}^2 \to \mathbb{R}$ is μ-almost everywhere differentiable along $S := \mathrm{Tan}_C$.

Our aim is to prove an analogous result for every Radon measure μ on $\mathbb{R}^d$. Namely we want to define a map S mapping a point $x \in \mathbb{R}^d$ into a linear subspace $S(x)$ of $\mathbb{R}^d$ with the following property: every Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$ is differentiable along the vector space $S(x)$ for μ-almost every $x \in \mathbb{R}^d$. Then we want to find the analogous of the function g defined above, i.e. a Lipschitz function on $\mathbb{R}^d$ that is non-differentiable at μ-almost every $x \in \mathbb{R}^d$ along any line that is not a vector subspace of $S(x)$.

In Chapter 1 we introduce the main notation and we recall some basic facts about Geometric Measure Theory and in particular about the Theory of Currents. We give detailed proof of two results about 1-currents: in Proposition 1.3.13, we prove that it is possible to write every normal 1-current as an integral of integral 1-currents, without loss of mass; in Proposition 1.3.16, we describe the structure of integral 1-currents: they are sum of countably many closed oriented curves plus a finite number of open ones.

In Chapter 2 we collect some results on the differentiablity of Lipschitz maps. In the first section there are results concerning the existence of Lipschitz maps which are non-differentiable at all the points of a prescribed Lebesgue null set or, with a different point of view, which are non-differentiable almost everywhere with respect to a prescribed measure which is singular with respect to Lebesgue. In particular, in Theorem 2.1.2 we revisit an old theorem by Zahorski: we prove that in the class of 1-Lipschitz functions on the line, those which are non-differentiable at all the points of a prescribed compact null set form a residual set. In the second section we recall an important class of Lipschitz functions with a "large" non-differentiability set, namely distance functions of σ-porous sets. In Proposition 2.2.4 we show that not necessarily a singular measure is supported on a σ-porous set, therefore distance functions of σ-porous sets are not sufficient to prove, in any dimension, the existence of a Lipschitz function which is non-differentiable almost everywhere with respect to a prescribed singular measure.

In Chapter 3 we prove the first part of our main result, Theorem 4.2.11. In Theorem 3.1.1 we prove that given a normal 1-current on $\mathbb{R}^d$ (associated with a Radon measure μ and a vectorfield τ) then every Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$ is μ-a.e. differentiable along the vectorfield τ. This is essentially a consequence of Proposition 1.3.13 and the Disintegration Theorem 3.1.2. In the second section, given a Radon measure μ, we construct the map S mentioned above; we call it the decomposability bundle of μ and we prove the result of differentiability of Lipschitz functions μ-a.e. along S. The decomposability bundle is defined in terms of the possibility to write parts of the measure μ as an integral of 1-rectifiable measures. In this way, the measure μ can be associated with a sequence of normal 1-currents, to which we apply the previous result, together with a boundary formula (Proposition 3.2.6), essential to get the linearity of the directional derivatives.

Lastly, in Chapter 4 we prove the second part of the main result. In the first section we give a covering result for a special class of null sets in $\mathbb{R}^d$ that we call sets invisible along a cone. Such a set can be covered by a family of slabs determined by graphs of Lipschitz functions $f_i : \mathbb{R}^{d-1} \to \mathbb{R}$, in such a way that the sum of the thickness of the slabs is arbitrarily small. In the second section, we use this covering result to prove the existence of a Lipschitz function on $\mathbb{R}^d$ that is non-differentiable at μ-almost every $x \in \mathbb{R}^d$ along any line that is not a vector subspace of $S(x)$. This completes the main result of the first part, in fact this means that in general one cannot expect any differentiability of Lipschitz functions outside of the decomposability bundle. In the last section we give a simplified proof of this last result, inspired by the proof of Theorem 2.1.2. Namely we prove that the class of functions satisfying the property described above on an arbitrarly large set of points is residual on a suitable space of Lipschitz functions.

In the next sections of this chapter we review some notions of multilinear algebra and the Theory of Currents. Our aim is to fix the notation and to give the main theorems, together with some additional results that will be essential in the sequel. This presentation does not aim to be exhaustive.

1.1. Notation and preliminaries

Here we recall some basic definitions in Geometric Measure Theory and some results that we will use (often tacitly) through this thesis. The reader is referred to [**KP**] for a more detailed exposition.

We will call linear k-plane a k-dimensional linear subspace of $\mathbb{R}^d$ and, when V is a linear k-plane and x is a point in $\mathbb{R}^d$, the set $x + V$ will be called an affine k-plane. We will often use simply the word "k-plane", when there is no ambiguity.

The letter μ will always denote a positive Borel measure on $\mathbb{R}^d$. If no measure is mentioned in expressions like "almost everywhere", "negligible", "null set" and so on, we are assuming that the measure involved is the Lebesgue measure $\mathscr{L}^d$. Given a Borel set E, we will denote by $\mu \llcorner E$ the restriction of the measure μ to E, i.e. the measure defined by

$$\mu \llcorner E(A) = \mu(A \cap E),$$

for every Borel set A. If f is a μ-integrable function, then we denote by $f\mu$ the Borel measure defined by

$$f\mu(A) = \int_A f \, \mathrm{d}\mu,$$

for every Borel set A. A Borel measure μ is called *Borel regular* if, for every μ-measurable set A, there exists a Borel set $B \supset A$ such that $\mu(B \setminus A) = 0$. The measure μ is *locally finite* if every point has a neighborhood of finite measure, or equivalently if every compact set has finite measure. A locally finite, Borel regular measure is called a *Radon measure*. Radon measures enjoy the following regularity property.

PROPOSITION 1.1.1. Let μ be a Radon measure on $\mathbb{R}^d$. If $\mu(E) < \infty$, then for every $\varepsilon > 0$, there exist a compact set K and an open set A such that $K \subset E \subset A$ and $\mu(A \setminus K) \leq \varepsilon$.

We will often use the fact that Borel measurable functions are nearly continuous, as shown by the following result:

THEOREM 1.1.2 (Lusin Theorem). *Let μ be a finite Radon measure on $\mathbb{R}^d$ and let (X, d) be a separable, locally compact, topological vector space. Let $f : \mathbb{R}^d \to X$ be a Borel measurable function. Then for every $\varepsilon > 0$ there exists a continuous function $f_\varepsilon : \mathbb{R}^d \to X$ such that*

$$\mu(\{x \in \mathbb{R}^d : f_\varepsilon(x) \neq f(x)\}) < \varepsilon.$$

We endow the space $\mathscr{C}_c^0(\mathbb{R}^d)$ of continuous compactly supported functions on $\mathbb{R}^d$, with the usual topology of uniform convergence on compact sets. A functional L on $\mathscr{C}_c^0(\mathbb{R}^d)$ is called *positive* if $L(\phi) \geq 0$ for every $\phi \geq 0$. If μ is a locally finite, positive measure on $\mathbb{R}^d$, then the map

$$\phi \to \int \phi \, \mathrm{d}\mu$$

is a continuous, positive linear functional on $\mathscr{C}_c^0(\mathbb{R}^d)$. Actually every continuous, positive linear functional on $\mathscr{C}_c^0(\mathbb{R}^d)$ has such a representation, in fact we have the following:

THEOREM 1.1.3 (Riesz Theorem). *Let L be a continuous, positive linear functional on $\mathscr{C}_c^0(\mathbb{R}^d)$. Then there exists a locally finite, positive Borel measure μ on $\mathbb{R}^d$ such that*

$$L(\phi) = \int \phi \, \mathrm{d}\mu, \quad \text{for every } \phi \in \mathscr{C}_c^0(\mathbb{R}^d)$$

Therefore it is natural to endow the space $\mathscr{M}(\mathbb{R}^d)$ of locally finite, positive Borel measures with the weak* topology. In particular, we say that a sequence of locally finite positive measures $(\mu_n)_{n\in\mathbb{N}}$ on $\mathbb{R}^d$ converges weakly* to μ, and we write $\mu_n \overset{*}{\rightharpoonup} \mu$, if

$$\lim_n \int \phi \, d\mu_n = \int \phi \, \mathrm{d}\mu,$$

for every $\phi \in \mathscr{C}_c^0(\mathbb{R}^d)$. As usual on a space which is a dual of a separable space, the weak* topology enjoys a sequential compactness property. We say that a family $\{\mu_j\}_{j\in J}$ of measures is *uniformly locally bounded* if for every compact set K there exists a constant C_K such that $\mu_j(K) \leq C_K$ for every j.

THEOREM 1.1.4 (Compactness for measures). *Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of uniformly locally bounded positive measures on $\mathbb{R}^d$. Then there exists a subsequence converging to a locally finite measure μ.*

In the following proposition we collect some useful facts about weak* convergence of measures.

PROPOSITION 1.1.5. Let $(\mu_n)_{n\in\mathbb{N}}$ and μ be positive Radon measures on $\mathbb{R}^d$.

(i) If $\mathscr{A}$ is an algebra of sets generating the topology of $\mathbb{R}^d$ and if $\mu_n(A) \to \mu(A)$ for every $A \in \mathscr{A}$, then $\mu_n \overset{*}{\rightharpoonup} \mu$.

(ii) If $\mu_n \overset{*}{\rightharpoonup} \mu$, then

$$\mu(A) \leq \liminf_{n\to\infty} \mu_n(A), \quad \text{for every open set } A,$$

$$\mu(K) \geq \limsup_{n\to\infty} \mu_n(K), \quad \text{for every compact set } K.$$

In particular $\mu_n(E) \to \mu(E)$ for every set E such that $\mu(\partial E) = 0$.

Given a positive measure λ on $\mathscr{M}(\mathbb{R}^d)$, which is Borel (with respect to the weak* topology) and satisfies, for every compact set $K \subset \mathbb{R}^d$,

$$\int_{\mathscr{M}(\mathbb{R}^d)} \mu(K)\, \mathrm{d}\lambda(\mu) < +\infty,$$

we denote by

$$\int_{\mathscr{M}(\mathbb{R}^d)} \mu\, \mathrm{d}\lambda(\mu) \tag{1.1.1}$$

the measure ν satisfying

$$\nu(B) = \int_{\mathscr{M}(\mathbb{R}^d)} \mu(B)\, \mathrm{d}\lambda(\mu),$$

for every Borel set $B \subset \mathbb{R}^d$. In particular we have

$$\int_{\mathbb{R}^d} \phi\, \mathrm{d}\nu = \int_{\mathscr{M}(\mathbb{R}^d)} \left(\int_{\mathbb{R}^d} \phi\, \mathrm{d}\mu \right) \mathrm{d}\lambda(\mu),$$

for every $\phi \in \mathscr{C}^0_c(\mathbb{R}^d)$.

Let k be an integer with $1 \leq k \leq d$. With the symbol $\mathscr{H}^k$ we denote the k-dimensional Hausdorff measure on $\mathbb{R}^d$. A set $E \subset \mathbb{R}^d$ is called *$\mathscr{H}^k$-countably k-rectifiable* (or simply *k-rectifiable*) if $E \subset \bigcup_{i=0}^{\infty} E_i$, where

(i) $\mathscr{H}^k(E_0) = 0$,

(ii) $E_i = F_i(\mathbb{R}^k)$, for $i \geq 1$, where $F_i : \mathbb{R}^k \to \mathbb{R}^d$ is a Lipschitz function.

A set $U \subset \mathbb{R}^d$ is called *k-purely unrectifiable* if

$$\mathscr{H}^k(U \cap E) = 0,$$

for every k-rectifiable set E.

A *k-rectifiable measure* μ on $\mathbb{R}^d$ is a measure written as

$$\mu = \theta \mathscr{H}^k \llcorner E,$$

where E is a k-rectifiable set in $\mathbb{R}^d$ and θ is a Borel positive function defined on E, integrable with respect to $\mathscr{H}^k$.

1.2. Differential forms and Stokes theorem

Consider $(e_1, \ldots, e_d)$ the standard basis of $\mathbb{R}^d$. For every positive integer $k \le d$, denote by $I(d,k)$ the set of multi-indices $I = (i_1, \ldots, i_k)$, with $1 \le i_1 < \ldots < i_k \le d$. Associate with every index $I \in I(d,k)$ the formal expression

$$e_I = e_{i_1} \wedge \ldots \wedge e_{i_k}.$$

A generic linear combination

$$v = \sum_{I \in I(d,k)} \alpha_I e_I,$$

with $\alpha_I \in \mathbb{R}$, is called *k-vector* in $\mathbb{R}^d$. The space of k-vectors in $\mathbb{R}^d$ is denoted by $\bigwedge_k(\mathbb{R}^d)$, so we have $\bigwedge_1(\mathbb{R}^d) = \mathbb{R}^d$ and for convenience we set $\bigwedge_0(\mathbb{R}^d) = \mathbb{R}$ and $\bigwedge_k(\mathbb{R}^d) = 0$ if $k > d$.

For every $v \in \bigwedge_k(\mathbb{R}^d)$ and $w \in \bigwedge_h(\mathbb{R}^d)$, it is possible to define an operation, called *exterior product*, denoted by $v \wedge w$. The result is a $(k+h)$-vector in $\mathbb{R}^d$. The exterior product is characterized by the following properties: it is associative, linear in both arguments and alternating (i.e. $e_i \wedge e_j = -e_j \wedge e_i$).

A k-vector v is called *simple* if it can be written as the exterior product of certain 1-vectors, i.e.

$$v = v_1 \wedge \ldots \wedge v_k.$$

Remark 1.2.1. Notice that there are k-vectors which are not simple, for example the 2-vector

$$v = e_1 \wedge e_2 + e_3 \wedge e_4$$

in $\mathbb{R}^4$ is not simple. If it were simple, then it should be $v = v_1 \wedge v_2$, for some v_1 and v_2, hence $v \wedge v = (v_1 \wedge v_2) \wedge (v_1 \wedge v_2) = 0$, while an easy computation shows that $v \wedge v = 2e_1 \wedge e_2 \wedge e_3 \wedge e_4 \neq 0$.

Remark 1.2.2. Simple unitary vectors are the correct tool to represent k-dimensional oriented planes (through the origin). In fact it turns out that the simple vector $v = v_1 \wedge \ldots \wedge v_k$ is null if and only if the v_i's are linearly dependent. Moreover if $v'_1, \ldots, v'_k$ generate the same vector space generated by $v_1, \ldots, v_k$, then $v'_1 \wedge \ldots \wedge v'_k$ is a multiple of v.

Let $\{\mathrm{d}x_1, \ldots, \mathrm{d}x_d\}$ denote the standard orthonormal basis of $\mathbb{R}^{d^*}$, dual to $\{e_1, \ldots, e_d\}$. The dual space of $\bigwedge_k(\mathbb{R}^d)$ is called the space of *k-covectors* and it is denoted by $\bigwedge^k(\mathbb{R}^d)$. The union, over $I \in I(d,k)$, of the k-covectors

$$\mathrm{d}x_I = \mathrm{d}x_{i_1} \wedge \ldots \wedge \mathrm{d}x_{i_k}$$

is a basis for $\bigwedge^k(\mathbb{R}^d)$, dual to the basis $\{e_I\}$. The duality pairing $\langle\cdot;\cdot\rangle$ is as usual defined by the relation $\langle dx_I; e_J\rangle = \delta_{I,J}$. The exterior product for k-covectors is defined as that for k-vectors.

A *differential k-form* ω on $\mathbb{R}^d$ is a k-covector field, that is a map

$$\omega : \mathbb{R}^d \to \bigwedge^k(\mathbb{R}^d).$$

We can write ω using the standard basis of $\bigwedge^k(\mathbb{R}^d)$, as

$$\omega(x) = \sum_{I\in I(d,k)} \omega_I(x)\mathrm{d}x_I,$$

where the coordinates ω_I are real valued functions on $\mathbb{R}^d$. We say that a differential k-form has a certain regularity, when the coordinate functions have that regularity.

As usual, the *support* of a differential k-form ω is defined as the set $\mathrm{supp}(\omega)$ which is the closure of the set $\{x \in \mathbb{R}^d : \omega(x) \neq 0\}$.

The *exterior derivative* of a differential k-form ω of class $\mathscr{C}^1$ is the differential $(k+1)$-form:

$$\mathrm{d}\omega(x) = \sum_{I\in I(d,k)} \mathrm{d}\omega_I \wedge \mathrm{d}x_I,$$

where

$$\mathrm{d}\omega_I(x) = \sum_{i=1}^{d} \frac{\partial w_I}{\partial x_i}(x)\mathrm{d}x_i.$$

In addition to the euclidean norm $|\cdot|$ on $\bigwedge_k(\mathbb{R}^d)$ and $\bigwedge^k(\mathbb{R}^d)$, we consider the mass norm $\|\cdot\|$ on k-vectors and the comass norm $\|\cdot\|^*$ on k-covectors, defined as follows:

$$\|\phi\|^* = \sup\{|\langle\phi; v\rangle| : v \text{ is a simple } k\text{−vector, with } |v| = 1\},$$
$$\|v\| = \sup\{|\langle\phi; v\rangle| : \|\phi\|^* = 1\}.$$

Remark 1.2.3. Remark 1.2.2 establishes a one-to-one correspondence between simple k-vectors with unit euclidean norm and oriented k-dimensional vector subspaces of $\mathbb{R}^d$. This fact motivates the following definition: an *orientation* of a k-dimensional surface S of class $\mathscr{C}^1$ is a continuous map $\tau_S : S \to \bigwedge_k(\mathbb{R}^d)$ such that $\tau_S(x)$ is a simple unit k-vector spanning $\mathrm{Tan}_S(x)$ for every x.

If there exists an orientation of S, then there is a canonical orientation for the boundary of S, namely the one satisfying

$$\tau_S(x) = \nu(x) \wedge \tau_{\partial S}(x) \text{ for every } x \in \partial S,$$

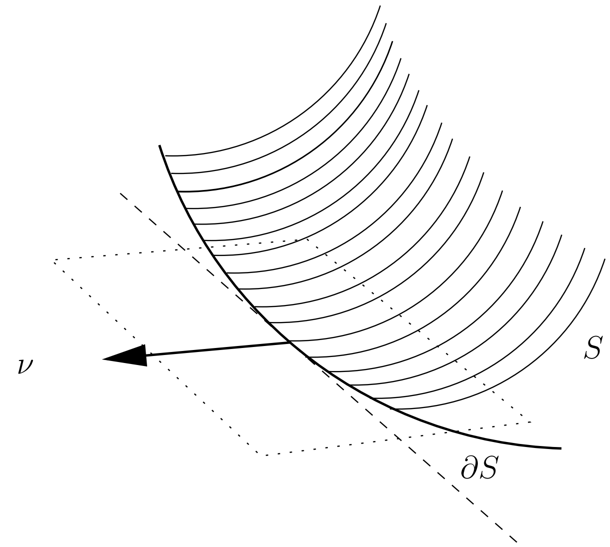

FIGURE 1.2.1

where ν is the outer normal to ∂S (see Figure 1.2.1).

The integral of differential k-form ω on an oriented k-surface S can be defined as follows

$$\int_S \omega = \int_S \langle \omega(x); \tau_S(x) \rangle \, \mathrm{d}\mathscr{H}^k(x).$$

Stokes theorem establishes that for every $(k-1)$-form of class $\mathscr{C}^1$ the following relation holds:

$$\int_{\partial S} \omega = \int_S \mathrm{d}\omega, \tag{1.2.1}$$

where the orientation of ∂S is the one defined above.

Next we want to define the pull-back, under a smooth map $f : \mathbb{R}^d \to \mathbb{R}^{d'}$ of a differential k-form on $\mathbb{R}^{d'}$. First, for any simple k-vector $v = v_1 \wedge \ldots \wedge v_k \in \bigwedge_k(\mathbb{R}^d)$ and a point $x \in \mathbb{R}^d$, define the push-forward of v as the simple k-vector

$$\mathrm{d}f_\sharp(v) = Df(x)v_1 \wedge \ldots \wedge Df(x)v_k.$$

This map is extended to all k-vectors by linearity. Then, for any differential k-form ω on $\mathbb{R}^{d'}$ define it is *pull-back* $f^\sharp\omega$ on $\mathbb{R}^d$ by

$$\langle f^\sharp\omega(x); v \rangle = \langle \omega(f(x)); f_\sharp(v) \rangle, \quad \text{for all } x \in \mathbb{R}^d. \tag{1.2.2}$$

1.3. Currents

Let $\mathscr{D}^k(\mathbb{R}^d)$ be the vector space of smooth differential k-forms on $\mathbb{R}^d$ with compact support, endowed with the locally convex topology τ constructed as the topology on the space $\mathscr{D}(\mathbb{R}^d)$ (of smooth compactly supported functions on $\mathbb{R}^d$), with respect to which distributions are dual. The dual of $\mathscr{D}^k(\mathbb{R}^d)$ is denoted by $\mathscr{D}_k(\mathbb{R}^d)$ and it is called the space of *k-dimensional currents* (or simply k-currents). As usual $\mathscr{D}_k(\mathbb{R}^d)$ is endowed with its weak* topology. In particular we will say that a sequence of k-currents $(T_n)_{n\in\mathbb{N}}$ converges to a k-current T and we write $T_n \stackrel{*}{\rightharpoonup} T$ if it converges in the weak* topology, that is:

$$\langle T_n; \omega\rangle \to \langle T; \omega\rangle,$$

for every $\omega \in \mathscr{D}^k(\mathbb{R}^d)$.

Remark 1.3.1. A simple example of a k-current on $\mathbb{R}^d$ is the integration over an oriented k-dimensional surface S of class $\mathscr{C}^1$. We will denote such a current with $[S]$. This motivates some authors to use the terminology "generalized surfaces" when they introduce currents.

Actually many geometric operations for surfaces have their analogue for currents, defined by duality with forms. We begin with the *boundary* ∂T of a k-current T, which is the $(k-1)$-current defined by

$$\langle \partial T; \phi\rangle = \langle T; \mathrm{d}\phi\rangle,$$

for every $\phi \in \mathscr{D}^{k-1}(\mathbb{R}^d)$. We can immediately see that $\partial^2 T = 0$, because $\mathrm{d}^2\phi = 0$. By Stokes theorem, this agrees with the usual definition of boundary if $T = [S]$ and S is an oriented surface of class $\mathscr{C}^1$, the orientation of ∂S being defined in Remark 1.2.3.

Secondly, if $f : \mathbb{R}^d \to \mathbb{R}^{d'}$ is a proper smooth map, then it is possible to define the *push-forward* of a k-current T on $\mathbb{R}^d$ as the k-current $f_\sharp T$ on $\mathbb{R}^{d'}$ defined by

$$\langle f_\sharp T; \omega\rangle = \langle T; f^\sharp \omega\rangle,$$

for every $\omega \in \mathscr{D}^k(\mathbb{R}^{d'})$. As expected, the boundary of the push forward is the push forward of the boundary.

Lastly, for a normal k-current T (that will be defined later in this section), it is possible to define the intersection with the generic level set $f^{-1}(y)$ of a smooth map $f : \mathbb{R}^d \to \mathbb{R}^{d'}$ (with $k \le d' \le d$). It turns out that, for almost every y, the resulting current is normal, with the expected dimension $d' - k$. This operation is called *slicing*, but we will not enter in the details here.

The *support* of a k-current in $\mathbb{R}^d$ is the set

$$\mathrm{supp}(T) = \mathbb{R}^d \setminus \bigcup\{U : U \text{ is open }, T(\omega) = 0 \text{ whenever } \omega \in \mathscr{D}^k(\mathbb{R}^d) \text{and } \mathrm{supp}(\omega) \subset U\}.$$

The *mass* of a current T is the quantity

$$\mathbb{M}(T) = \sup\{\langle T; \omega\rangle : \|\omega(x)\|^* \leq 1 \text{ for every } x\}.$$

It is easy to show that the mass is lower semicontinuous with respect to the weak* topology. Moreover for the current $[S]$ associated with an oriented k-dimensional surface S, we have $\mathbb{M}([S]) = \mathscr{H}^k(S)$, therefore the mass is a natural extension to k-currents of the notion of k-volume. Note that the norm

$$\sup\{\|\omega(x)\|^* : x \in \mathbb{R}^d\}$$

induces on $\mathscr{D}^k(\mathbb{R}^d)$ a weaker topology with respect to the one to which currents are dual, therefore a current may have (even locally) infinite mass. As an example, consider the 0-current T on $\mathbb{R}$ such that

$$T(\phi) = \phi'(0), \text{ for every } \phi \in \mathscr{D}(\mathbb{R}).$$

Another useful notion is the *flat norm* of a current:

$$\mathbb{F}(T) := \inf\{\mathbb{M}(R) + \mathbb{M}(S) : T = R + \partial S\}.$$

Remark 1.3.2. In a certain sense, the flat norm gives a better notion of distance between surfaces then the mass norm. For example consider the 1-current $T = [I_1] - [I_2]$ in $\mathbb{R}^2$, where I_1 and I_2 are two parallel segments with same orientation, same length l and ε is the (Hausdorff) distance between them. Then the flat norm of T does not exceed $(l+2)\varepsilon$, confirming the intuition that the two segments are close together, while the mass norm of T is $2l$. The importance of the flat norm is due the fact that (at least in the space of normal currents with a bound on the mass of the current and on the mass of the boundary) it metrizes the weak* topology.

By Riesz theorem, a k-current with finite mass can be represented as a bounded measure with values in $\bigwedge_k(\mathbb{R}^d)$, i.e. there exists a positive finite measure μ on $\mathbb{R}^d$ and a Borel measurable map $\tau : \mathbb{R}^d \to \bigwedge_k(\mathbb{R}^d)$ with $|\tau| = 1$ μ-a.e. such that

$$\langle T; \omega\rangle = \int_{\mathbb{R}^d} \langle \omega(x); \tau(x)\rangle \, \mathrm{d}\mu(x),$$

for every $\omega \in \mathscr{D}^k(\mathbb{R}^d)$. The mass of T equals the mass of the measure μ. We will often denote such a current with $T = \tau\mu$.

A k-current T is called *normal* if both T and ∂T have finite mass. The fact that $\mathscr{D}_k(\mathbb{R}^d)$ is dual to a separable space, implies the following result, which is an immediate consequence of the compactness theorem for vector valued measures.

Theorem 1.3.3 (Compactness theorem for normal currents). *Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of normal k-currents on $\mathbb{R}^d$ such that $\mathbb{M}(T_n) + \mathbb{M}(\partial T_n)$ is uniformly bounded. Then there exists a subsequence $(T_{n_i})_{i\in\mathbb{N}}$ converging to a normal k-current.*

Before giving the next definition, we need to recall a fundamental fact about k-rectifiable sets. Let $G(d,k)$ be the linear space of k-dimensional vector subspaces of $\mathbb{R}^d$. Given a Borel set E in $\mathbb{R}^d$, we call a *weak tangent field* to E a Borel map $\mathrm{Tan}_E : E \to G(d,k)$, such that for every k-dimensional $\mathscr{C}^1$-surface S on $\mathbb{R}^d$ there holds

$$\mathrm{Tan}_S(x) = \mathrm{Tan}_E(x) \text{ for } \mathscr{H}^k\text{−a.e } x \in S \cap E. \tag{1.3.1}$$

PROPOSITION 1.3.4. Every k-rectifiable set E in $\mathbb{R}^d$ admits a weak tangent field.

PROOF. Cover $\mathscr{H}^k$-a.e. point of E with a sequence of k-dimensional $\mathscr{C}^1$-surfaces $\{S_i\}_{i\in\mathbb{N}}$ and set $\mathrm{Tan}_E(x) = \mathrm{Tan}_{S_i}(x)$ if i is the smallest integer such that $x \in S_i$, $\mathrm{Tan}_E(x) = 0$ otherwise. The proof that Tan_E is a weak tangent field to E is a consequence of the following well known fact: if S and S' are k-dimensional $\mathscr{C}^1$-surfaces, then $\mathrm{Tan}_S(x) = \mathrm{Tan}_{S'}(x)$ for $\mathscr{H}^k$-a.e. $x \in S \cap S'$. □

In particular, given a k-rectifiable set E, one can define an orientation of E as a choice, for every point $x \in E$ of a simple unit k-vector τ_E spanning $\mathrm{Tan}_E(x)$.

A k-current T is called *rectifiable* if T admits the following representation

$$\langle T; \omega\rangle = \int_E \langle \omega(x); \tau_E(x)\rangle \theta(x) \, \mathrm{d}\mathscr{H}^k(x),$$

where E is a k-rectifiable set, τ_E is an orientation of E, and θ is a *multiplicity*, i.e. a real-valued function such that $\int_E \theta(x)\ d\mathscr{H}^k(x)$ is finite. We often use the notation $T = T(E, \tau_E, \theta)$. In particular we have

$$\mathbb{M}(T) = \int_E |\theta(x)| \, \mathrm{d}\mathscr{H}^k(x).$$

A rectifiable current whose multiplicity takes only integral values is called an *integer multiplicity rectifiable current*. If both T and ∂T are integer multiplicity rectifiable currents, than T is called an *integral current*.

REMARK 1.3.5. An integer multiplicity rectifiable 0-current in $\mathbb{R}^d$, T, admits the following representation:

$$T = \sum_{i=1}^{k} m_i \delta_{x_i},$$

where x_i are points in $\mathbb{R}^d$, $m_i \in \mathbb{Z}$ and δ_{x_i} represents the rectifiable 0-current supported on x_i with multiplicity 1. This means that the action of T on a smooth compactly supported function $f : \mathbb{R}^d \to \mathbb{R}$ is

$$\langle T; f\rangle = \sum_{i=1}^{k} m_i f(x_i).$$

Actually an integer multiplicity rectifiable current turns out to be an integral current, unless its boundary has infinite mass, in fact we have the following result (see Theorem 7.9.3 of [**KP**]).

THEOREM 1.3.6 (Boundary rectifiability theorem). *Let T be an integer multiplicity rectifiable current with $\mathbb{M}(\partial T) < \infty$. Then ∂T is an integer multiplicity rectifiable current.*

A fundamental theorem for integral currents is the closure theorem. Indeed it is stated as a compactness result: the reason for the name "closure theorem" is that the point is not the existence of a converging subsequence (already established by Theorem 1.3.3), but the fact that the limit is an integral current (see Theorem 7.5.2 of [**KP**]).

THEOREM 1.3.7 (Compactness theorem for integral currents). *Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of integral k-currents on $\mathbb{R}^d$ such that $\mathbb{M}(T_n) + \mathbb{M}(\partial T_n)$ is uniformly bounded. Then there exists a subsequence $(T_{n_i})_{i\in\mathbb{N}}$ converging to an integral k-current.*

The main historical motivation for the introduction of currents was to develop the correct framework to prove the existence of k-dimensional surfaces of minimal area, spanning a prescribed boundary. This is known as Plateau problem and the previous closure theorem provides the main tool for the solution.

THEOREM 1.3.8. *Let Γ be the boundary of an integral k-current in $\mathbb{R}^d(1 \le k \le d)$. Then there exists a current minimizing the mass among all integral currents T satisfying $\partial T = \Gamma$.*

PROOF. Let m be the infimum of $\mathbb{M}(T)$ among integral k-currents with $\partial T = \Gamma$. Let $(T_n)_{n\in\mathbb{N}}$ be a minimizing sequence. Since $\mathbb{M}(T_n)$ is bounded and $\mathbb{M}(\partial T_n)$ is constant, we can apply Theorem 1.3.7 to the sequence (T_n) and find a subsequence converging to an integral current T. By the continuity of the boundary operator we still have $\partial T = \Gamma$ and by lower semicontinuity of the mass we have $\mathbb{M}(T) \le m$. □

We define now a class of currents which contains the regular objects often used to approximate currents. A *polyhedral* k-current, is a rectifiable k-current of the form

$$T = \sum_{i=1}^{n} T(S_i, \tau_i, \theta_i),$$

where S_i is a k-dimensional simplex in $\mathbb{R}^d$, τ_i is a constant orientation of S_i and θ_i is a constant multiplicity.

The following approximation theorem is crucial for our purposes (see Theorem 4.2.24 of [**Fe1**]).

THEOREM 1.3.9 (Polyhedral approximation theorem). *Let T be a normal k-current in $\mathbb{R}^d$ and $\varepsilon > 0$. Then there exists a polyhedral k-current P such that $\mathbb{F}(T - P) \le \varepsilon$ and $\mathbb{M}(P) + \mathbb{M}(\partial P) \le \mathbb{M}(T) + \mathbb{M}(\partial T) + \varepsilon$. Moreover if ∂T is polyhedral it is possible to take $\partial P = \partial T$ and if T is integral it is possible to take P integral.*

We conclude this review with two additional results about 1-currents. They are proved here, even if their role in the Theory of Currents is less relevant with respect to the previous ones. The motivation is that the literature about them is not so wide, and we are going to make a substantial use of them in the sequel. Proposition 1.3.13 provides a decomposition of normal 1-currents as an average of integral currents without loss of mass. This result firstly appeared in [**S**]. To prove it, we need the following two lemmas. Lemma 1.3.10 describes a decomposition of every integral polyhedral 1-current as a sum of integral polyhedral 1-currents with mass of the boundary not exceeding 2. Via Lemma 1.3.11, we can put also a bound on the mass of the 1-currents appearing in the decomposition.

LEMMA 1.3.10. Let P be a polyhedral integral 1-current in $\mathbb{R}^d$. Then there exist finitely many polyhedral integral 1-currents P_i, with $\mathbb{M}(\partial P_i) \le 2$, such that

$$P = \sum_i P_i; \quad \mathbb{M}(P) = \sum_i \mathbb{M}(P_i); \quad \mathbb{M}(\partial P) = \sum_i \mathbb{M}(\partial P_i).$$

PROOF. We can write

$$P = \sum_{i=1}^{k} m_i S_i,$$

where $S_i = [[a_i, b_i]]$ is the integral 1-current associated with the segment $[a_i, b_i]$ oriented from a_i to b_i, with unit multiplicity. We can assume that the S_i's can intersect only at the extreme points and moreover

$$|b_i - a_i| \le 1, \tag{1.3.2}$$

for every i. Following the notation of Remark 1.3.5, we have

$$\partial P = \sum_{i=1}^{k} m_i(\delta_{b_i} - \delta_{a_i}) = \sum_{j=1}^{h} \alpha_j \delta_{x_j},$$

where α_j are non-zero integers and $x_j \in \bigcup_{i=1}^{k} \{a_i, b_i\}$ for every j.

Construct the polyhedral 1-current P_1 as follows. If there exists a point x_j such that $\alpha_j < 0$, then take a segment S_i such that $a_i = x_j$. If there is no such x_j, then start from any segment S_i. Consider the 1-current $P - S_i$. Take a segment, having positive multiplicity in $P - S_i$, whose first extreme point coincide with the second extreme point of S_i. Collect segments in the same way, until it is possible. When it is no longer possible to add a new segment, let P_1 be the sum of the segments chosen. The current P_1 satisfies:

$$\mathbb{M}(P) = \mathbb{M}(P - P_1) + \mathbb{M}(P_1); \quad \mathbb{M}(\partial P) = \mathbb{M}(\partial(P - P_1)) + \mathbb{M}(\partial P_1).$$

Repeat the same procedure for $P - P_1$ and so on. The procedure will stop after a finite number of steps. The collection $\{P_i\}$ gives the desired decomposition. □

LEMMA 1.3.11. Let P be a polyhedral integral 1-current in $\mathbb{R}^d$. Then there exist finitely many polyhedral integral 1-currents P_i, with $\mathbb{M}(P_i) \le 2$ and $\mathbb{M}(\partial P_i) \le 2$, such that

$$P = \sum_i P_i; \quad \mathbb{M}(P) = \sum_i \mathbb{M}(P_i); \quad \sum_i \mathbb{M}(\partial P_i) \le 2\mathbb{M}(\partial P) + 2\mathbb{M}(P).$$

Proof. Repeat the procedure described in the previous proof, with an additional rule. While building the current P_i, stop whenever the sum of the lengths of the segments is greater then or equal to 1. By (1.3.2) we have that

$$\mathbb{M}(P_i) \leq 2.$$

Moreover

$$\mathbb{M}(P) = \mathbb{M}(P_i) + \mathbb{M}(P - P_i),$$

and whenever

$$\mathbb{M}(\partial P_i) + \mathbb{M}(\partial(P - P_i)) = \mathbb{M}(\partial P) + 2$$

we have

$$\mathbb{M}(P_i) \geq 1.$$

As a consequence, it turns out that the number of P_i satisfying

$$\mathbb{M}(\partial P_i) + \mathbb{M}(\partial(P - P_i)) = \mathbb{M}(\partial P) + 2$$

is at most $\mathbb{M}(P)$. Obviously the number of P_i satisfying

$$\mathbb{M}(\partial P_i) + \mathbb{M}(\partial(P - P_i)) = \mathbb{M}(\partial P) + 1$$

is at most $\mathbb{M}(\partial P)$. In fact if P_i satisfy this, then there is a segment $S = [a, b]$ in P_i such that either a or b is a point in the support of ∂P and the number of such segments (counted with multiplicity) is bounded by $\mathbb{M}(\partial P)$. Hence the inequality in the decomposition of ∂P. □

Remark 1.3.12. In the previous decomposition, one could even require

$$\mathbb{M}(P_i) + \mathbb{M}(\partial P_i) \geq 1$$

for every index except at most one. In fact one can collect the currents P_i without boundary, in groups whose total mass is between 1 and 2 and define a new current as the sum of the currents in the same group. It is possible that in this procedure one group remains, whose total mass is less than one. This group determines the exceptional index. In conclusion it is possible to perform the previous decomposition with at most $3\mathbb{M}(P) + 2\mathbb{M}(\partial P) + 1$ currents P_i.

Let λ be a positive Borel measure on $\mathscr{D}_k(\mathbb{R}^d)$, supported on the set X of normal currents. If λ satisfies

$$\int_{T \in X} \mathbb{M}(T)\, \mathrm{d}\lambda(T) < \infty,$$

we denote by

$$N = \int_{T \in X} T\, \mathrm{d}\lambda(T) \tag{1.3.3}$$

the normal k-current defined by:

$$\langle N; \omega \rangle = \int_{T \in X} \langle T; \omega \rangle\, \mathrm{d}\lambda(T),$$

for every $\omega \in \mathscr{D}^k(\mathbb{R}^d)$.
Moreover, if

$$\int_{T \in X} \mathbb{M}(\partial T)\, \mathrm{d}\lambda(T) < \infty,$$

we also have, for every $\phi \in \mathscr{D}^{k-1}(\mathbb{R}^d)$

$$\langle \partial N; \phi \rangle = \langle N; \mathrm{d}\phi \rangle = \int_{T \in X} \langle T; \mathrm{d}\phi \rangle\, \mathrm{d}\lambda(T) = \int_{T \in X} \langle \partial T; \phi \rangle\, \mathrm{d}\lambda(T),$$

hence

$$\partial \left(\int_{T \in X} T\, \mathrm{d}\lambda(T) \right) = \int_{T \in X} \partial T\, \mathrm{d}\lambda(T). \tag{1.3.4}$$

Lastly, since the flat norm metrizes the topology on X, we can consider the standard notion of weak convergence of Borel measures, i.e. we say that λ_n weakly converge to λ (and we write $\lambda_n \overset{*}{\rightharpoonup} \lambda$) if

$$\int_X f\, \mathrm{d}\lambda_n \to \int_X f\, \mathrm{d}\lambda,$$

for every continuous bounded function f on X. Choosing as $f(T)$ the action of T on a generical element of $\mathscr{D}^k(\mathbb{R}^d)$, we immediately get the implication

$$\lambda_n \overset{*}{\rightharpoonup} \lambda \Rightarrow \int_{T \in X} T\, \mathrm{d}\lambda_n(T) \overset{*}{\rightharpoonup} \int_{T \in X} T\, \mathrm{d}\lambda(T). \tag{1.3.5}$$

Proposition 1.3.13. Let $\mathscr{I}$ be the set of integral 1-currents T in $\mathbb{R}^d$ with $\mathbb{M}(T) \leq 2$ and $\mathbb{M}(\partial T) \leq 2$. Every normal 1-current N in $\mathbb{R}^d$ can be written as

$$N = \int_{T \in \mathscr{I}} T\, \mathrm{d}\lambda(T),$$

where λ is a finite Borel measure on $\mathscr{I}$. Moreover

$$\mathbb{M}(N) = \int_{T \in \mathscr{I}} \mathbb{M}(T)\, \mathrm{d}\lambda(T)$$

and

$$\int_{T \in \mathscr{I}} \mathbb{M}(\partial T)\, \mathrm{d}\lambda(T) \leq 2\mathbb{M}(N) + 2\mathbb{M}(\partial N).$$

Proof. For every $n \in \mathbb{N}$ consider a polyhedral 1-current P_n satisfying

$$\mathbb{F}(N - P_n) \leq \frac{1}{n}; \quad \mathbb{M}(P_n) + \mathbb{M}(\partial P_n) \leq \mathbb{M}(T) + \mathbb{M}(\partial T) + \frac{1}{n}.$$

The existence of such a current is guaranteed by Theorem 1.3.9. Our first aim is to replace P_n with a multiple of an integral polyhedral 1-current. First write

$$P_n = \sum_{j=1}^{k_n} m_{n,j} [S_{n,j}],$$

where $S_{n,j} = [a_{n,j}, b_{n,j}]$ are segments oriented from $a_{n,j}$ to $b_{n,j}$, which can intersect each other only at the extreme points and $m_{n,j} > 0$ is their multiplicity. Define

$$d_n = \max\{1, \sum_{j=1}^{k_n} |b_{n,j} - a_{n,j}|\}.$$

Now, take

$$\alpha_n = \frac{1}{nk_n d_n}.$$

For $j = 1$ to k_n consider $l_{n,j} = \lfloor \frac{m_{n,j}}{\alpha_n} \rfloor$, where $\lfloor x \rfloor$ denotes the biggest integer less than x. The current

$$Q_n = \sum_{j=1}^{k_n} \alpha_n l_{n,j} S_{n,j}$$

have the following properties:

(i) $\alpha_n^{-1} Q_n$ is an integral polyhedral 1-current
(ii) $\mathbb{M}(Q_n - P_n) \le \alpha_n d_n = \frac{1}{nk_n} \le \frac{1}{n}$,
(iii) $\mathbb{M}(\partial Q_n - \partial P_n) \le 2\alpha_n k_n = \frac{1}{nd_n} \le \frac{1}{n}$.

By Lemma 1.3.11 and Remark 1.3.12, it is possible to write the polyhedral integral current $B_n := \alpha_n^{-1} Q_n$ as a sum of integral currents $\{B_{n,j}\}_{j=1}^{h_n}$ in such a way that:

$$\mathbb{M}(B_{n,j}) \le 2 \text{ and } \mathbb{M}(\partial B_{n,j}) \le 2, \text{ for } j = 1, \dots, h_n, \tag{1.3.6}$$

$$h_n \le 3\mathbb{M}(B_n) + 2\mathbb{M}(\partial B_n) + 1. \tag{1.3.7}$$

$$\mathbb{M}(B_n) = \sum_j \mathbb{M}(B_{n,j}) \tag{1.3.8}$$

$$\sum_j \mathbb{M}(\partial B_{n,j}) \le 2\mathbb{M}(\partial B_n) + 2\mathbb{M}(B_n) \tag{1.3.9}$$

In other words, defining

$$\lambda_n = \alpha_n \sum_{j=1}^{h_n} \delta_{B_{n,j}},$$

we can write

$$Q_n = \int_{T \in \mathscr{I}} T \, \mathrm{d}\lambda_n(T),$$

in such a way that

$$\mathbb{M}(Q_n) = \int_{T \in \mathscr{I}} \mathbb{M}(T) \, \mathrm{d}\lambda_n(T),$$

and

$$\int_{T\in\mathscr{I}} \mathbb{M}(\partial T)\,\mathrm{d}\lambda_n(T) \leq 2\mathbb{M}(Q_n) + 2\mathbb{M}(\partial Q_n).$$

By (1.3.7), $\|\lambda_n\|$ is controlled by $3\mathbb{M}(Q_n)+2\mathbb{M}(\partial Q_n)+1$, which is bounded by $3\mathbb{M}(N)+2\mathbb{M}(\partial N)+2$. So, up to subsequences, λ_n weakly converges to some positive measure λ. Since $Q_n \xrightarrow{*} N$, $\mathbb{M}(Q_n)\to\mathbb{M}(N)$ and $\mathbb{M}(\partial Q_n)\to\mathbb{M}(\partial N)$ we have:

$$N = \int_{T\in\mathscr{I}} T\,\mathrm{d}\lambda(T),$$
$$\mathbb{M}(N) = \int_{T\in\mathscr{I}} \mathbb{M}(T)\,\mathrm{d}\lambda(T)$$

and

$$\int_{T\in\mathscr{I}} \mathbb{M}(\partial T)\,\mathrm{d}\lambda(T) \leq 2\mathbb{M}(N) + 2\mathbb{M}(\partial N).$$

□

REMARK 1.3.14. In some cases it is more convenient to write the normal current N as an integral of a parametrized family of integral currents, where the parameter is in the unit interval $[0,1]$ and the measure on the set of parameters is the Lebesgue measure, i.e.

$$N = \int_0^1 T_t\,\mathrm{d}t.$$

This is always possible thanks to the following

THEOREM 1.3.15. *Let X be a polish space (homeomorphic to a complete separable metric space) and λ be a probability measure on X. Then there exists a Borel map*

$$m:[0,1]\to X$$

such that $m_\sharp(\mathscr{L}^1)=\lambda$, i.e. $\lambda(E)=\mathscr{L}^1(m^{-1}(E))$, for every Borel set $E\subset X$.

The previous result is very easy to prove when X is [0,1], being

$$m(x) = \inf\{t : \lambda([0,t]) \geq x\}.$$

The proof for the generical X easily follows from Theorem 2.12 of [**Pa**]. A warm thank to G. Letta for helping in finding this reference.
The next proposition is a characterization of integral 1-currents as a finite sum of open oriented curves plus a countable sum of closed ones. Given an interval I on the line, we denote with $[I]$ the integral 1-current in $\mathbb{R}$ associated with the interval I, the positive orientation and multiplicity 1.

PROPOSITION 1.3.16. (see section 4.2.25 of [**Fe1**]) Given an integral 1-current T on $\mathbb{R}^d$, there exists a sequence of Lipschitz maps $f_i : I=[0,1]\to\mathbb{R}^d$ such that $T=\sum_i T_i$, where $T_i = f_{i\sharp}[I]$, moreover

$$\mathbb{M}(T) = \sum_i \mathbb{M}(T_i)$$

and

$$\mathbb{M}(\partial T) = \sum_i \mathbb{M}(\partial T_i).$$

PROOF. For every $i \in \mathbb{N}$, let P_i be a polyhedral integral 1-current satisfying

$$\mathbb{F}(T - P_i) \le \frac{1}{i}, \tag{1.3.10}$$

$$\partial P_i = \partial T, \quad \mathbb{M}(P_i) \le \mathbb{M}(T) + \frac{1}{i}.$$

The existence of such a current for every i is guaranteed by Theorem 1.3.9. By Lemma 1.3.10 we may write $P_i = \sum_j Q_{i,j}$, where $Q_{i,j}$ are polyhedral integral 1-currents of the form

$$Q_{i,j} = {g_{i,j}}_\sharp[I], \tag{1.3.11}$$

for some sequence of Lipschitz maps $g_{i,j} : I \to \mathbb{R}^d$. The decomposition can be done in such a way that

$$\mathbb{M}(P_i) = \sum_j \mathbb{M}(Q_{i,j}) \text{ and } \mathbb{M}(\partial P_i) = \sum_j \mathbb{M}(\partial Q_{i,j}).$$

Denote

$$A_{i,0} = \{Q_{i,j} : \partial Q_{i,j} \ne 0\},$$
$$A_{i,1} = \{Q_{i,j} : \partial Q_{i,j} = 0 \text{ and } 1 \le \mathbb{M}(Q_{i,j}) < \mathbb{M}(T) + 1\}$$

and for $k \ge 2$,

$$A_{i,k} = \{Q_{i,j} : \partial Q_{i,j} = 0 \text{ and } 2^{-k+1} \le \mathbb{M}(Q_{i,j}) < 2^{-k+2}\}.$$

Notice that the families $A_{i,k}$ are disjoint and

$$P_i = \sum_k \left(\sum_{Q \in A_{i,k}} Q \right);$$

$$\sharp(A_{i,0}) \le \mathbb{M}(\partial T);$$

$$\sharp(A_{i,k}) \le 2^{k-1}(\mathbb{M}(T) + 1), \quad \text{for } k \ge 1.$$

Moreover, there is a positive constant C such that,

$$\mathbb{F}\left(\sum_{Q \in A_{i,k}} Q \right) \le C 2^{-k} \quad \text{for every } i. \tag{1.3.12}$$

For every i and for every k there there exists a constant C_k (independent on i) such that every $Q \in A_{i,k}$ admits a C_k-Lipschitz map $f : I \to \mathbb{R}^d$ such that

$$Q(\omega) = \int_{[0,1]} \langle \omega \circ f(t); f'(t) \rangle \, dt, \quad \text{for every } \omega \in \mathscr{D}^1(\mathbb{R}^d).$$

Hence, the compactness in these families of equi-Lipschitz functions, gives that when i goes to infinity along a sequence of indices $(i_{0,1}, i_{0,2}, \ldots)$ we have

$$\sum_{Q \in A_{i,0}} Q \overset{*}{\rightharpoonup} A_0 = \sum_{j=1}^{j_0} T_{0,j},$$

where $T_{0,j}$ are still integral 1-currents satisfying (1.3.11). Similarly when i goes to infinity along a subsequence $(i_{1,1}, i_{1,2}, \ldots)$ of $(i_{0,1}, i_{0,2}, \ldots)$ we have

$$\sum_{Q \in A_{i,1}} Q \overset{*}{\rightharpoonup} A_1 = \sum_{j=1}^{j_1} T_{1,j}$$

and so on. Properties (1.3.10) and (1.3.12) guarantee that

$$T = \sum_{i=0}^{\infty} A_i;$$

lower semicontinuity of the mass and continuity of the boundary operator, give the desired properties of the decomposition of T. $\square$

CHAPTER 2

Old and recent results on the differentiability of Lipschitz maps

This chapter is devoted to the description of the structure of the non-differentiability set of a Lipschitz function, namely the set of those points where the function is non-differentiable. Rademacher theorem states that a Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}^n$ is differentiable almost everywhere with respect to the Lebesgue measure $\mathscr{L}^n$. Thus, a set of positive measure cannot be contained in the non-differentiability set of a Lipschitz function. In dimension $d = 1$, by Zahorski theorem (see [**Zah**]), it turns out that every null set is contained in the non-differentiability set of some Lipschitz function. Actually the theorem gives a complete characterization of the non-differentiability set of a Lipschitz function $f : \mathbb{R} \to \mathbb{R}$: indeed $E \subset \mathbb{R}$ is the set of non-differentiability points of some Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ if and only if E is a $G_{\delta\sigma}$ set (a union of countably many sets, called G_δ, which are intersection of coutably many open set) with Lebesgue measure zero. A surprising theorem due to D. Preiss show that Zahorski result cannot be extended to dimension $d = 2$, where, however, a suitable counterpart (see Theorem 2.1.4) is true.

However it is possible to consider a different point of view: instead of fixing a null set E and looking for a Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}^n$ which is non-differentiable at any point of E, one can fix a measure μ on $\mathbb{R}^d$, singular with respect to the Lebesgue measure and look for a Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$ which is non-differentiable μ-a.e. In this framework, the dimension of the target space is irrelevant, as Lemma 2.1.6 points out. It's worth to mention the fascinating progress made in [**CJ**] for both the pointwise problem and the "almost everywhere" one. In this chapter, we will denote by X the complete metric space of real valued 1-Lipschitz functions on the line, endowed with the supremum distance.

2.1. Zahorski theorem

Here we prove a weaker version of Zahorski theorem, namely that every null set in the line is contained in the non-differentiability set of some Lipschitz function.

THEOREM 2.1.1. [**Zah**] *Let E be a set in $\mathbb{R}$ such that $\mathscr{L}^1(E) = 0$. Then there exists a Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ that is non-differentiable at any point of E.*

PROOF. Let $(E_n)_{n\in\mathbb{N}}$ be a decreasing sequence of open sets, of finite measure, containing E, satisfying the property:

$$\mathscr{L}^1(E_{n+1} \cap I) \leq 2^{-n}\mathscr{L}^1(I),$$

for every n and for every connected component I of E_n. Note that, in particular,

$$\mathscr{L}^1(E_{n+1}) \leq 2^{-n}\mathscr{L}^1(E_n).$$

Define

$$g_n(x) = \int_{-\infty}^{x} \chi_{E_n}(t)\,\mathrm{d}t$$

and

$$f_n = \sum_{k=1}^{n}(-1)^{k+1}g_k.$$

Since f_n is a Cauchy sequence in X, it converges to a 1-Lipschitz function f. Moreover, note that $|f_n - f| \leq |f_n - f_{n+1}|$ for every n.

Fix a point $x \in E$ and an odd integer n. Let I be the connected component of E_n containing x. For every $y \in I$ we have:

$$\frac{f(y)-f(x)}{y-x} = \frac{f(y)-f_n(y)+f_n(y)-f_n(x)+f_n(x)-f(x)}{y-x} \geq$$
$$\frac{f_n(y)-f_n(x)}{y-x} - \frac{|f_n(y)-f(y)|}{|y-x|} - \frac{|f_n(x)-f(x)|}{|y-x|} \geq$$
$$1 - \frac{|f_n(y)-f_{n+1}(y)|}{|y-x|} - \frac{|f_n(x)-f_{n+1}(x)|}{|y-x|} \geq 1 - 2\frac{\mathscr{L}^1(E_{n+1}\cap I)}{|y-x|}.$$

Choosing $y_0 \in I$ such that $|y-x| \geq \frac{\mathscr{L}^1(I)}{4}$ we have

$$\frac{f(y_0)-f(x)}{y_0-x} \geq 1 - 8\frac{\mathscr{L}^1(E_{n+1}\cap I)}{\mathscr{L}^1(I)} \geq 1 - 2^{-n+3}.$$

Since, for sufficiently large n, the length of I can be choosen arbitrarly small, then the upper derivative of f at x is 1. Analogously it can be proved that the lower derivative is 0 at every $x \in E$. $\square$

The following unpublished version underlines that the "size" of the family of 1-Lipschitz functions $f : \mathbb{R} \to \mathbb{R}$ which are not differentiable at any point of a fixed compact null set $E \subset \mathbb{R}$ is large, in the sense of category.

THEOREM 2.1.2. *Let E be a compact set in $\mathbb{R}$ such that $\mathscr{L}^1(E) = 0$. Then the family of Lipschitz functions $f : \mathbb{R} \to \mathbb{R}$ such that f is not differentiable at the points of E is a residual set in X.*

PROOF. Define inductively an infinitesimal sequence of positive numbers (ε_i) and a sequence of open sets (E_i), whith the following properties

- $E \subset E_{i+1} \subset E_i$;
- E_i is a finite union of disjoint open intervals;
- $\mathscr{L}^1(E_i) \leq \varepsilon_i$;
- Denoting $\alpha_i = \min_j\{\mathscr{L}^1(I^i_j)\}$, we have $\varepsilon_{i+1} \leq \alpha_i\varepsilon_i$.

Define the following subsets of X

$$U_i = \{g \in X : g(b) - g(a) > (b-a) - \varepsilon_{i+1}, \text{ for every } (a,b) \text{ connected component of } E_i\},$$
$$V_i = \{g \in X : g(b) - g(a) < \varepsilon_{i+1} - (b-a), \text{ for every } (a,b) \text{ connected component of } E_i\},$$
$$A_j = \bigcup_{i \geq j} U_i, \quad B_j = \bigcup_{i \geq j} V_i.$$

Obviously U_i and V_i are open sets for every i, and therefore A_j and B_j are also open, for every j. Moreover, U_i and V_i are $2\varepsilon_i$-nets, by which we mean that for every element $\phi \in X$ there is an element $\phi_i \in U_i$ (respectively V_i) such that $\operatorname{dist}(\phi, \phi_i) \leq 2\varepsilon_i$. To show this, for every function $\phi \in X$, consider the function

$$\phi_i(x) = \phi\left(x - \int_{-\infty}^{x} \chi_{E_i}(t)\, \mathrm{d}t\right) + \int_{-\infty}^{x} \chi_{E_i}(t)\, \mathrm{d}t,$$

which has the following property: $\phi_i'(x)\phi'(x)$ if $x \notin E_i$ and $\phi_i'(x) = 1$ if $x \in E_i$. This is clearly an element of U_i and $\|\phi - \phi_i\|_\infty \leq 2\varepsilon_i$. The proof that V_i is a $2\varepsilon_i$-net is analogous.

As a consequence, A_j and B_j are dense for every j. Finally,

$$A = \left(\bigcap_{j=1}^{\infty} A_j\right) \cap \left(\bigcap_{j=1}^{\infty} B_j\right)$$

is a residual set in X (in particular it is non empty).

Next we prove that every function $f \in A$ is not differentiable at any point of E. More precisely, we claim that

$$f_+'(x) = \limsup_{|h_n| \searrow 0} \frac{f(x+h_n) - f(x)}{h_n} = 1$$

and

$$f_-'(x) = \liminf_{|h_n| \searrow 0} \frac{f(x+h_n) - f(x)}{h_n} = -1$$

for every $x \in E$. Fix $\varepsilon > 0$ and take $i \in \mathbb{N}$ such that $3\varepsilon_i < \varepsilon$, and $f \in U_i$. Let $I = (a,b)$ be the connected component of E_i containing x. Take a point $y \in I$ such that

$$\operatorname{dist}(x,y) \geq \frac{\mathscr{L}^1(I)}{3}.$$

Let I' be the open interval with end points x and y. Since on (a,b) we have $f' \leq 1$ a.e. and $f(b) - f(a) \geq b - a - \varepsilon_{i+1}$, then we also have $\int_{I'} f'(t)\, \mathrm{d}t \geq |x-y| - \varepsilon_{i+1}$. Therefore we have:

$$\frac{f(y) - f(x)}{y - x} \geq \frac{|y-x| - \varepsilon_{i+1}}{|y-x|} \geq 1 - \frac{3\varepsilon_{i+1}}{\mathscr{L}^1(I)} \geq 1 - \frac{3\varepsilon_{i+1}}{\alpha_i} \geq 1 - 3\varepsilon_i \geq 1 - \varepsilon.$$

Analogously we can prove that $f_-'(x) = -1$ for every $x \in E$. $\square$

As we have already mentioned, in general it is not possible to extend Theorem 2.1.1 to higher dimension, as shown by the following theorem, due to D. Preiss.

THEOREM 2.1.3. [**Pr**] *There exist a Lebesgue null set E in the plane such that every Lipschitz function $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at least at one point of E.*

Actually Preiss' null set E is quite "large", in fact it is dense (one can choose E any G_δ set of measure zero containing countably many lines having a dense set of directions). In the recent paper [**DoMa1**] M. Doré and O. Maleva constructed a compact null set with the same property. They also proved in [**DoMa2**] that in every Banach space X with separable dual there exists a closed bounded set of Hausdorff dimension 1 containing at least one point of Fréchet differentiability for every Lipschitz function $f : X \to \mathbb{R}$.

These results point out that in order to find a possible converse of Rademacher theorem, one should change the setting. The following theorem shows that, in dimension 2, it is sufficient to enlarge the target space to obtain a counterpart of Theorem 2.1.1.

THEOREM 2.1.4. [**ACP**] *For every null set E in the plane, there exists a Lipschitz map $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ which is non-differentiable at every point $x \in E$.*

REMARK 2.1.5. Here, the non-differentiability at the points of E is intended in a sense (stronger than the usual one) that for every point of E, there exist a direction $e(x)$ such that at least one of the two components of f does not admit the directional derivative

$$f_i'(x, e(x)) = \lim_{t \to 0} \frac{f_i(x + te(x)) - f(x)}{t}.$$

As we said, changing the dimension of the target space, is not helpful for the "almost everywhere" problem. Indeed, given a singular measure μ on $\mathbb{R}^d$, if we have a Lipschitz map $f : \mathbb{R}^d \to \mathbb{R}^{d'}$ which is μ-a.e. non-differentiable, then we can find also a (real valued) Lipschitz function on $\mathbb{R}^d$ with the same property. An immediate implication of Theorem 2.1.1 is that given a singular measure μ on the line, there is a Lipschitz function which is μ-a.e. non-differentiable. The following lemma allow us to exploit Theorem 2.1.4 to obtain the same result in the plane.

LEMMA 2.1.6. Let μ be a finite measure on $\mathbb{R}^d$, let $e(x)$ be a vectorfield and let $f_1, f_2 : \mathbb{R}^d \to \mathbb{R}$ be two Lipschitz functions such that for μ-a.e. $x \in \mathbb{R}^d$ at least one of the f_i is non-differentiable along the direction $e(x)$. Then there exists a Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$ which is non-differentiable along the direction $e(x)$ for μ-a.e. x.

PROOF. Let

$$\delta(f_i, x) = \limsup_{t \to 0} \frac{f_i(x + te(x)) - f_i(x)}{t} - \liminf_{t \to 0} \frac{f_i(x + te(x)) - f_i(x)}{t} \quad \text{for } i = 1, 2.$$

We know that for every $x \in E$ at least one between $\delta(f_1, x)$ and $\delta(f_2, x)$ is non zero. For every $\lambda \in (0, 1]$ we have

$$\delta(f_1 + \lambda f_2, x) \geq |\delta(f_1, x) - \lambda \delta(f_2, x)|.$$

Note that for $\lambda \in (0,1]$ the sets E_λ given by

$$E_\lambda = \{x \in \mathbb{R}^d : |\delta(f_1, x) - \lambda\delta(f_2, x)| = 0\}_{\lambda\in(0,1]}$$

are pairwise disjoint. Therefore $\mu(E_\lambda) > 0$ for at most countably many λ. Thus for all remaining λ we have $\mu(E_\lambda) = 0$, i.e. the Lipschitz function $f_1 + \lambda f_2 : \mathbb{R}^2 \to \mathbb{R}$ is non-differentiable along $e(x)$ for μ-a.e. x. $\square$

COROLLARY 2.1.7. Given a measure μ on $\mathbb{R}^2$ which is singular with respect to Lebesgue, there exists a Lipschitz function $f : \mathbb{R}^2 \to \mathbb{R}$ which is non-differentiable μ-a.e.

REMARK 2.1.8. There exists a characterization of those sets that are contained in the non-differentiability set of some Lipschitz function in $\mathbb{R}^d$: up to the recent work [**CJ**] it was in a certain sense incomplete. Indeed, it was not known whether Lebesgue null sets belong to this family or not. The work of M. Csornyei and P. Jones gives a positive answer to this question.

2.2. σ-Porous sets and differentiability

For a positive real number $\delta < 1$, we say that a set $E \subset \mathbb{R}^d$ is *δ-porous at a point* $x \in E$ if there is sequence of points $y_n \to 0$ such that

$$B(x + y_n, \delta|y_n|) \cap E = \emptyset$$

for every $n \in \mathbb{N}$. In other words, at arbitrarily small scales centered at x, the complement of E contains a ball of fixed radius. A set E is *porous* if there is some positive δ such that E is δ-porous at all of its points and is *σ-porous* if it is a countable union of porous sets. The Lebesgue density theorem implies that porous sets (and therefore also σ-porous ones) are Lebesgue null. Moreover, a porous set is nowhere dense (i.e., its closure has empty interior), so a σ-porous set is a set of first category (countable union of nowhere dense sets). Zajicek Theorem 2.2.5 shows that the family of σ-porous sets does not contain all Lebesgue-null, first category sets.

The following remark shows that σ-porous sets seem to be good candidates to characterize those subsets of $\mathbb{R}^d$ for which most of the points are non-differentiability points of some Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$. It turns out that the condition is sufficient, but not necessary.

REMARK 2.2.1. It is not difficult to see that a set E is δ-porous at x, for some $\delta > 0$ if and only if the function

$$d_E(x) = \mathrm{dist}(x, E)$$

is non-differentiable at x. Let μ be a measure on $\mathbb{R}^d$ and assume that E_i is a sequence of porous sets whose union contains μ-a.e point in the support of μ. It is possible to show that there is a linear combination of the functions d_{E_i} which is non-differentiable μ-a.e. Unfortunately this is not enough to prove Corollary 2.1.7. Indeed for every $d \geq 1$ there exists a measure μ, singular with respect to the Lebesgue measure on $\mathbb{R}^d$, such that every porous set is μ-negligible (see Theorem 2.2.4). The rest of this section is devoted to the proof of this result.

Our proof of Theorem 2.2.4 is based on a blowup argument. Given a locally finite Borel measure μ on $\mathbb{R}^d$ and a point x we define the set $\mathrm{Tan}(\mu, x)$ of the blowups of μ at x, as the limits

$$\lim_{r_n \searrow 0} \kappa_n = \lim_{r_n \searrow 0} \frac{\mu_{x,r_n} \llcorner B(0,1)}{\mu(B(x, r_n))},$$

where, for every x and for every $r > 0$

$$\mu_{x,r}(A) = \mu(x + rA), \text{ for every Borel set } A.$$

The following lemma shows that if μ gives positive measure to some porous set, then there exists a μ-positive set of points A such that for every $x \in A$, $\mathrm{Tan}(\mu, x)$ contains a measure ν satisfying $\mathscr{L}^1 \not\ll \nu$ (i.e. the Lebesgue measure is not absolutely continuous with respect to ν). We just mention that the converse is also true.

Lemma 2.2.2. Let μ be a locally finite measure on the line, such that for μ-a.e.x and for every $\nu \in \mathrm{Tan}(\mu, x)$, $\mathscr{L}^1 \ll \nu$. Then $\mu(P) = 0$ for every porous set $P \subset \mathbb{R}$.

Proof. By contradiction, consider $\delta > 0$ and a δ-porous set P with $\mu(P) > 0$. It is a general fact that if E is a Borel set, then $\mathrm{Tan}(\mu \llcorner E, x) = \mathrm{Tan}(\mu, x)$ for μ-a.e $x \in E$. Then for μ-a.e. $x \in P$ every blowup ν of $\mu \llcorner P$ at x is an element of $\mathrm{Tan}(\mu, x)$, in particular ν gives positive measure to every non trivial interval $J \subset (-1, 1)$. We show how to find, for every $x \in P$, a blowup ν of $\mu \llcorner P$ at the point x such that $\nu(1 - \delta, 1) = 0$ or $\nu(-1, -1 + \delta) = 0$. Fix $x \in P$ and consider a sequence $y_n \to 0$ such that

$$B(x + y_n, \delta|y_n|) \cap P = \emptyset.$$

Possibly passing to a subsequence, we may assume that y_n has constant sign, let us say positive. It turns out that, if we take $r_n = y_n$, for every limit ν of some subsequence of κ_n, we have $\nu(1 - \delta, 1) = 0$. $\square$

On $[-1, 1]$ we call n-th generation of dyadic intervals all the intervals of the form

$$I = [a2^{-n}, (a + 1)2^{-n}], \text{ for } a = -2^n, \dots, 2^n - 1.$$

Theorem 2.2.3 (Martingale theorem). *Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of probability measures on $[-1, 1]$. Assume that $\mu_n = f_n \mathscr{L}^1$, where f_n is constant on the dyadic intervals of the n-th generation. Assume moreover that $\mu_m(I) = \mu_n(I)$ for every dyadic interval of the n-th generation, for every $m > n$. Then μ_n weakly converges to a probability measure μ, and the Radon Nikodym derivative f of the absolutely continuous part of μ satisfyes*

$$f = \lim_{n\to\infty} f_n, \ \mathscr{L}^1 - \text{a.e.}$$

Proof. By the compactness theorem for measures, there is a subsequence μ_{n_h} weakly converging to a measure μ. The proof that actually the whole sequence μ_n converges to μ is a straightforward application of property (i) of Proposition 1.1.5 to the algebra of sets generated by the dyadic intervals. To prove the second part of the theorem, take a point x which is a Lebesgue point for f. Assume moreover that x is a continuity point for every f_n. Let I_n be the dyadic intervals of the n-th generation containing x. This is a family of sets of bounded eccentricity. Therefore the Lebesgue theorem gives:

$$f_n(x) = \frac{\mu_n(I_n)}{\mathscr{L}^1(I_n)} = \frac{\mu(I_n)}{\mathscr{L}^1(I_n)} = \frac{\int_{I_n} f \, \mathrm{d}\mathscr{L}^1}{\mathscr{L}^1(I_n)} \to f(x), \text{ as } n \to \infty.$$

$\square$

We are now ready to prove the following theorem. This construction was suggested by B. Kirchheim.

THEOREM 2.2.4. *There exists a singular measure μ on the line such that $\mu(P) = 0$ for every porous set $P \subset \mathbb{R}$.*

PROOF. Take the 1-periodic function $\varphi : \mathbb{R} \to \mathbb{R}$ which agrees with $2\chi_{[0,1/2]} - 1$ on $[0, 1]$ and choose a decreasing sequence of positive numbers a_n such that $a_n \to 0$ and $\sum_n a_n^2 = +\infty$. Define on $[0, 1]$ the functions

$$\varphi_n(x) = a_n \varphi(2^n x), \quad \Phi_N = \sum_{n=1}^{N} \varphi_n, \quad \psi_n = 1 + \varphi_n, \ \Psi_N = \prod_{n=1}^{N} \psi_n.$$

Consider now the measures $\mu_N = \Psi_N \mathscr{L}^1$. By Theorem 2.2.3 there exists a measure μ such that $\mu_n \overset{*}{\rightharpoonup} \mu$ and moreover $\Psi_N \to \frac{d\mu_{abs}}{dx}$ (the Radon-Nikodym derivative of the absolutely continuous part of μ). Then it is sufficient to prove that $\liminf_N \Psi_N = 0 \mathscr{L}^1$-a.e. to guarantee that μ is singular with respect to Lebesgue. Notice now that for $|x| < 1$ there holds

$$\log(1 + x) \leq x - \frac{x^2}{8},$$

hence we have

$$\log(\Psi_N) = \sum_{n=1}^{N} \log(1 + \varphi_n) \leq \sum_{n=1}^{N} (\varphi_n - \frac{\varphi_n^2}{8}) = \Phi_N - \sum_{n=1}^{N} \frac{a_n^2}{8}.$$

Since the random variable Φ_N has expected value $E(\Phi_N) = 0$ and variance $\sigma^2(\Phi_N) = \sum_{n=1}^{N} a_n^2$, then Chebyshev inequality gives

$$\mathscr{L}^1 \left(\left\{ x \in [0, 1] : \Phi_N(x) > \sum_{n=1}^{N} \frac{a_n^2}{16} \right\} \right) \leq \frac{16^2}{\sum_{n=1}^{N} a_n^2} \to 0 \text{ as } N \to \infty,$$

because $\sum a_n^2 = +\infty$. Therefore we have

$$\liminf_N \Psi_N = \exp \left(\liminf_N \left(\Phi_N - \frac{\sum_{n=1}^{N} a_n^2}{8} \right) \right) = 0, \ \mathscr{L}^1 - \text{a.e.}$$

Now fix $a_n = n^{-1/2}$; we want to show that for μ-a.e. point $x \in (0,1)$, every blowup of μ at x gives positive measure to every non trivial interval $J \subset (-1,1)$. By Lemma 2.2.2, this guarantees that every porous set is μ-negligible.

Consider a point $x \in (0,1)$, a measure $\nu \in \mathrm{Tan}(\mu, x)$ and a sequence r_m with $r_m \leq \mathrm{dist}(x, B(0,1)^c)$ and $r_m \searrow 0$ such that $\nu = \lim_m \kappa_m$. For every $m \in \mathbb{N}$, there exist $n \in \mathbb{N}$ and a dyadic interval $I_n(x)$, of the n-th generation, containing x, such that it also contains $x + r_n$ or $x - r_n$, but no interval in the next generation has the same property. In particular we have $r_n \leq |I_n(x)| \leq 2r_n$. Denote by $I'_n(x)$ the neighbour dyadic interval of the same generation as $I_n(x)$, that together with $I_n(x)$ covers $(x - r_n, x + r_n)$. We want to show that, eventually in n, the ratio

$$c_n(x) = \frac{\mu(I_n(x))}{\mu(I'_n(x))}$$

satisfies $e^{-2} \leq c_n(x) \leq e^2$ for μ-a.e. $x \in (0,1)$: this is sufficient to prove that $\nu(J) > 0$ for every non trivial interval $J \subset (-1,1)$, for every $\nu \in \mathrm{Tan}(\mu, x)$.

For every $x \in (0,1)$ let $(\sigma_i(x))_{i \in \mathbb{N}}$ be the unique sequence made of 0's and 1's such that

$$\min\{I_n(x)\} = \sum_{i=0}^{n} 2^{-i} \sigma_i(x),$$

(see Figure 2.2.1) and analogously define $(\sigma'_i(x))_{i=1}^n$.

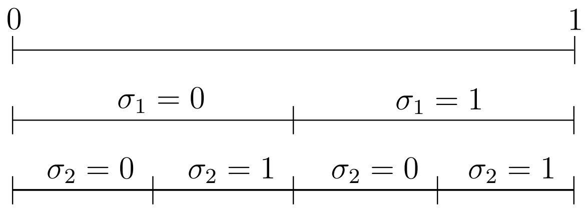

FIGURE 2.2.1

Obviously we have

$$\max\{c_n(x), c_n(x)^{-1}\} \leq \prod_{i=j_0+1}^{n} 1 + a_i,$$

where j_0 is the last index less than n such that $\sigma_{j_0}(x) = \sigma'_{j_0}(x)$. Notice that if $I'_n(x)$ is the left neighborhood of $I_n(x)$, we have $\sigma_{j_0+1}(x) = 1$ and $\sigma_i(x) = 0$ for every $i = j_0 + 2, \ldots, n$; viceversa if $I'_n(x)$ is the right neighborhood of $I_n(x)$, we have $\sigma_{j_0+1}(x) = 0$ and $\sigma_i(x) = 1$ for every $i = j_0 + 2, \ldots, n$.
For $j = 0, 1$, and for $n \geq 2$ denote

$$E_n^j = \{x \in (0,1) : \sigma_i(x) = j, \quad \text{for every } i \in [n - n^{1/2} + 2, n]\}.$$

It is easy to see that, for n sufficiently large, the set of points x such that $c_n(x) \notin [e^{-2}, e^2]$ is contained in $E_n^0 \cup E_n^1$. In fact if, for example, we had $c_n(x) > e^2$, then $\prod_{i=j_0+1}^n 1+i^{-1/2} > e^2$ and this means that $\sum_{i=j_0+1}^n 1 + i^{-1/2} > 2$. But $\sum_{i=j_0+1}^n 1 + i^{-1/2} < 2$ if $j_0 > n - n^{1/2}$. We have

$$\mu(E_n^j) \leq \prod_{i=n-n^{1/2}+2}^{n} \frac{1+i^{-1/2}}{2} \leq 2^{-n^{1/2}-2} \prod_{i=n-n^{1/2}}^{n} 1 + i^{-1/2} \leq 2^{-n^{1/2}+2}.$$

Therefore

$$\mu\left(\bigcap_{k=2}^{\infty} \bigcup_{n=k}^{\infty} \bigcup_{j=0,1} E_n^j\right) = 0$$

and since this set contains the set of points x such that $c(x) \notin [e^{-2}, e^2]$ frequently, we are done. Actually with slightly better extimates it is possible to prove that $c_n(x)$ goes to 1 μ-a.e. and this implies that the blowups are (a multiple of) the Lebesgue measure on $(-1, 1)$ at μ-a.e. point. $\square$

THEOREM 2.2.5. [**Zaj**] *In $\mathbb{R}^d$ there is a compact, first category, Lebesgue null set, which is not σ-porous.*

PROOF. It is sufficient to prove the result for $d = 1$. Let μ be the measure constructed in Theorem 2.2.4 and consider a Lebesgue null set N supporting μ. Take a compact subset K of N such that $\mu(K) \geq \frac{1}{2}$. Obviously K is Lebesgue null and first category (actually it is nowhere dense), moreover K is not σ-porous, because every porous subset of K has measure μ equal zero by Lemma 2.2.2. $\square$

Given a Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$, Rademacher theorem on the line and Fubini theorem are sufficient to guarantee the existence of directional derivatives $f'(x, v)$ for every direction v and for a.e. x. Of course the existence of many partial derivatives is not sufficient to have differentiability. The following result, points out that the set of points for which the two notions differ is, in a certain sense, small.

THEOREM 2.2.6. [**PZ**] *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a Lipschitz function. Then the set of those points at which there exist directional derivatives in d linearly independent directions, but f is not differentiable, is σ-porous.*

REMARK 2.2.7. As we said, σ-porous sets are negligible, therefore Theorem 2.2.6, together with the previous discussion, is sufficient to deduce Rademacher theorem in $\mathbb{R}^d$. Nowadays there are much simpler proofs of Rademacher theorem in $\mathbb{R}^d$: in this remark we just want to emphatize the gap between the existence of many directional derivatives and the differentiability. We will return on this in the next chapter.

CHAPTER 3

Differentiability of Lipschitz functions with respect to measures

The results of this chapter are original and are contained in [**AM**].

Definition 3.0.8. Consider a map

$$S : \mathbb{R}^d \to Y = \bigcup_{k=0}^{d} Gr_k(\mathbb{R}^d)$$

from $\mathbb{R}^d$ to the vector space Y which is the union over k of the Grasmannians of k-planes in $\mathbb{R}^d$. We say that a function $f : \mathbb{R}^d \to \mathbb{R}$ is *differentiable along* S at the point $x \in \mathbb{R}^d$ if the restriction of f to the plane $x + S(x)$ is differentiable at x.

We consider on the target space of S the topology inherited by distance which is given by the Hausdorff distance of the intersection of sets with the unit ball

$$d(V, W) = \mathrm{dist}_{\mathscr{H}}(V \cap B_1(0), W \cap B_1(0)).$$

When we refer to the measurability of S we intend it with respect to the Borel σ-algebra generated by this topology.

3.1. Differentiability w.r.t. normal 1- currents

The aim of this section is to prove the following theorem.

Theorem 3.1.1. [**AM**] *Let $T_0 = \tau_0 \mu_0$ be a 1-dimensional normal current in $\mathbb{R}^d$. Then every Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$ is differentiable along $\tau_0(x)$, at μ_0-a.e. $x \in \mathbb{R}^d$.*

Clearly, the theorem is valid for those 1-currents T for which there exists a Lipschitz function $\gamma : I \to \mathbb{R}^d$, satisfying

$$T = \gamma_\sharp([I]), \tag{3.1.1}$$

where I is the interval $[0, 1]$ in $\mathbb{R}$ and $[I]$ is the integral 1-current defined in Remark 1.3.1. Indeed, given a Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$, the set of points $\gamma(t) \in \mathbb{R}^d$ such that f is non-differentiable at $\gamma(t)$ along the direction $\gamma'(t)$ is contained in the set

$$A = \gamma(M) \cup \gamma(N) \cup \gamma(S),$$

where M and N are respectively the set of points in $(0, 1)$ such that γ and $\gamma \circ f$ are not differentiable and S is the set of points such that $\gamma' = 0$. Since $\gamma \circ f$ is a Lipschitz function, then, by Rademacher theorem on the line, N is $\mathscr{H}^1$-null. Since γ is Lipschitz, the image of M and N through γ are also $\mathscr{H}^1$-null. Lastly, $\gamma(S)$ is $\mathscr{H}^1$-null by Sard theorem. Since f is Lipschitz, the fact that $\gamma \circ f$ is differentiable on $(0,1) \setminus (M \cup N \cup S)$, with non zero derivative, implies that f is differentiable along γ' on $\gamma((0,1)) \setminus A$.

By Proposition 1.3.16, this fact has an easy extension to integral currents. Proposition 1.3.13 provides the correct tool to extend the result to normal 1-current. To complete the proof, we recall a basic result in Measure Theory, called disintegration theorem (see [**DeMe**]).

THEOREM 3.1.2. *Let Y and X be locally compact, separable metric spaces, μ a measure on X, $\pi : X \to Y$ a Borel map, and ν a measure on Y such that $\pi_\sharp \mu \ll \nu$. Then there exists a family $\{\mu_y\}_{y \in Y}$ of measures on X such that*

(i) *the function $y \mapsto \mu_y$ is Borel measurable, in the sense that $y \mapsto \mu_y(B)$ is Borel measurable for every Borel set B;*
(ii) *$\mu_y(X \setminus \pi^{-1}(y)) = 0$, for every $y \in Y$;*
(iii) *μ can be decomposed as $\mu = \int_Y \mu_y \, \mathrm{d}\nu(y)$, which means that*

$$\mu(B) = \int_Y \mu_y(B) \, \mathrm{d}\nu(y),$$

for every Borel set B contained in X.

Any family $\{\mu_y\}$ satisfying (i),(ii) and (iii) is called a disintegration of μ with respect to π and ν. The disintegration is ν-a.e. uniquely determined, i.e. for any other disintegration $\widetilde{\mu}_y$ there holds $\mu_y = \widetilde{\mu}_y$ for ν-a.e. y.

PROOF OF THEOREM 3.1.1. Apply Theorem 3.1.2, with

$$X = \mathbb{R}^d \times S^{d-1}; \quad Y = \mathbb{R}^d,$$

and

$$\pi : X \to Y$$

the natural projection. By Remark 1.3.14 there is a family T_t of integral 1-currents, with $\mathbb{M}(T_t) \leq 2$ and $\mathbb{M}(\partial T_t) \leq 2$ satisfying

$$T_0 = \int_0^1 T_t \, \mathrm{d}t; \quad \mathbb{M}(T_0) = \int_0^1 \mathbb{M}(T_t) \, \mathrm{d}t.$$

For every $t \in [0,1]$ define on X a positive measure μ_t such that

$$\pi_\sharp(\mu_t) = \|T_t\|,$$

($\|T_t\|$ being the measure associated with T_t) and such that

$$\mu_t(\{(x,v) : v \neq \tau_t(x)\}) = 0$$

(τ_t being the vector field associated with T_t).

Define on X the measure

$$\mu = \int_0^1 \mu_t \, \mathrm{d}t$$

and take $\nu = \pi_\sharp \mu$.

Now, given a Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$, define a function $g : X \to [0,1]$ such that $g(y,v) = 1$ if f is not differentiable at y in the direction v, $g(y,v) = 0$ otherwise.

The disintegration theorem gives:

$$0 = \int_0^1 \left(\int_X g \, \mathrm{d}\mu_t \right) \, \mathrm{d}t = \int_X g \, \mathrm{d}\mu = \int_Y \int_{\pi^{-1}(y)} g \, \mathrm{d}\mu_y \, \mathrm{d}\nu(y).$$

The second integral being 0 means that f is differentiable at ν-a.e y in the direction $\tau_t(y)$ for a.e. $t \in [0,1]$. In other words we have differentiability ν-a.e. along certain directions, but we need to show that ν is actually the measure associated with T_0 and that the directions coincide with the right one (the direction associated with T_0).

Define

$$\tau(y) = \int_{\pi^{-1}(y)} v \, \mathrm{d}\mu_y(v).$$

Firstly, we prove that the normal current T_0 satisfy $T_0 = \tau\nu$ (notice that, at this stage, we are not saying yet that $|\tau| = 1$, ν-a.e.). In fact, for every 1-covector $\psi \in \mathbb{R}^d$ and for every smooth compactly supported function $\varphi \in \mathscr{C}_c^\infty(\mathbb{R}^d)$ we have

$$\begin{aligned}
\langle T_0; \psi\varphi \rangle = \int_0^1 \langle T_t; \psi\varphi \rangle \, \mathrm{d}t = \\
\int_0^1 \int_{\mathbb{R}^d} \langle \psi; \tau_t \rangle \varphi \, \mathrm{d}\|T_t\| \, \mathrm{d}t = \\
\int_0^1 \int_{\mathbb{R}^d \times S^{d-1}} \langle \psi; v \rangle \varphi \, \mathrm{d}\mu_t \, \mathrm{d}t = \\
\int_{\mathbb{R}^d \times S^{d-1}} \langle \psi; v \rangle \varphi \, \mathrm{d}\mu = \\
\int_{y \in \mathbb{R}^d} \int_{v \in \pi^{-1}(y)} \langle \psi; v \rangle \varphi(y) \, \mathrm{d}\mu_y(v) \, \mathrm{d}\nu(y) = \\
\int_{y \in \mathbb{R}^d} \left\langle \psi; \int_{\pi^{-1}(y)} v \, \mathrm{d}\mu_y(v) \right\rangle \varphi(y) \, \mathrm{d}\nu(y) = \\
\int_{\mathbb{R}^d} \langle \psi; \tau \rangle \varphi \, \mathrm{d}\nu.
\end{aligned}$$

Secondly, we prove that μ_y coincides with the Dirac measure $\delta_{\tau(y)}$ for ν-a.e. y, hence ν coincides with μ_0 and f is differentiable ν-a.e along the right direction τ_0. We have

$$\int_Y |\tau| \, \mathrm{d}\nu = \mathbb{M}(T_0) = \int_0^1 \mathbb{M}(T_t) \, \mathrm{d}t = \int_0^1 \|\mu_t\| \, \mathrm{d}t = \|\mu\| = \|\nu\|,$$

hence $|\tau(y)| = 1$ for ν-a.e. y. Being $\tau(y)$ baricenter of a measure μ_y, living on S^{d-1}, we must have $\mu_y = \delta_{\tau(y)}$ for μ-a.e.y. $\square$

Remark 3.1.3. In the previous proof we skipped the check of the measurability of g. We prove here that g is Borel measurable. This is a consequence of the fact that the functions

$$\partial^+ f : (x,v) \to \frac{\partial^+ f}{\partial v}(x) = \limsup_h \frac{f(x+hv)-f(x)}{h}$$

and

$$\partial^- f : (x,v) \to \frac{\partial^- f}{\partial v}(x) = \liminf_h \frac{f(x+hv)-f(x)}{h}$$

are Borel measurable. In fact if we call

$$f_h(x,v) = \frac{f(x+hv)-f(x)}{h},$$

we have that $f_h(x,v)$ is measurable for every h, moreover

$$\partial^+ f = \inf_{n\in\mathbb{N}} \sup_{h\in\mathbb{Q}, |h|\le \frac{1}{n}} f_h$$

and similarly

$$\partial^- f = \sup_{n\in\mathbb{N}} \inf_{h\in\mathbb{Q}, |h|\le \frac{1}{n}} f_h.$$

3.2. Differentiability along the decomposability bundle

In this section we prove the main differentiability result. Given a Radon measure μ on $\mathbb{R}^d$ we define a map S that associates to every point x a vector subspace $S(x)$ of $\mathbb{R}^d$. Then we prove that every Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$ is differentiable at μ-a.e. point along S.

We say that a Radon measure μ on $\mathbb{R}^d$ is 1-*decomposable* provided μ admits a decomposition

$$\mu = \int_0^1 \mu_t \, \mathrm{d}t, \tag{3.2.1}$$

where μ_t are 1-rectifiable measures. We call (3.2.1) a 1-decomposition of μ. Notice that in (3.2.1) every measure μ_t is endowed with a weak tangent field τ_t, defined in (1.3.1), relative to the rectifiable set E_t supporting the measure.

Definition 3.2.1. Let μ be a Radon measure on $\mathbb{R}^d$. Let $\mathscr{F}$ be the class of all Borel maps

$$S : \mathbb{R}^d \to \bigcup_{k=0}^{d} Gr_k(\mathbb{R}^d),$$

such that:

(i) for every $\lambda \ll \mu$, such that $\lambda = \int_0^1 \lambda_t \, \mathrm{d}t$ is 1-decomposable and λ_t are 1-rectifiable measures supported on E_t, endowed with weak tangent fields τ_t, then

$$\langle \tau_t(x) \rangle \subset S(x) \quad \text{for } \lambda_t - \text{a.e. } x \in \mathbb{R}^d, \quad \text{for a.e. } t \in [0,1];$$

Among these there exists one which is minimal (see Remark 3.2.2) in the following sense:

(ii) for every S' satisfying (i), $S(x) \subset S'(x)$ for μ-a.e. x. This is called the *decomposability bundle* of μ.

REMARK 3.2.2. The existence of a minimal element in $\mathscr{F}$ can be proved as follows. One can take a sequence of Borel maps $(S_n)_{n\in\mathbb{N}}$, satisfying (i) and minimize the quantity

$$\int_{\mathbb{R}^d} \dim(S_i)\, d\mu,$$

then the decomposability bundle of μ is the Borel map $S(x) = \bigcap_{n\in\mathbb{N}} S_n(x)$.

DEFINITION 3.2.3. Let μ be a Radon measure in $\mathbb{R}^d$. Let $\mathscr{L}$ be the set of pairs (λ, T), where λ is a Radon measures, with $\lambda \ll \mu$ and $T = \tau\nu$ is a normal 1-current in $\mathbb{R}^d$, such that $\lambda \ll \nu$. Given a sequence of elements of $\mathscr{L}$, $(\lambda_n, T_n)_{n\in\mathbb{N}}$, we call *bundle generated by* (λ_n, T_n) a Borel map

$$G : \mathbb{R}^d \to \bigcup_{k=0}^{d} Gr_k(\mathbb{R}^d),$$

such that:

(i) for every $n \in \mathbb{N}$

$$\langle \tau_n(x) \rangle \subset G(x) \text{ for } \lambda_n - \text{a.e. } x \in \mathbb{R}^d;$$

(ii) for every G' satisfying (i), $G(x) \subset G'(x)$ for μ-a.e. x.

A *cone* with *axis* v and *angle* α is the set

$$C(v, \alpha) = \{x \in \mathbb{R}^d : |\langle x; v\rangle| > |x| \cos(\alpha)\}.$$

LEMMA 3.2.4. [**AM**] Let $\mu = \int_0^1 \mu_t \, dt$ be a 1-decomposable measure on $\mathbb{R}^d$, such that for every t, the tangent filed τ_t to the rectifiable set E_t supporting μ_t satisfies $\tau_t \in C(v, \alpha)$, for some $v \in \mathbb{R}^d$, and $\alpha \in (0, \pi/2)$. Then there exists a normal current $T = \sigma\nu$ such that $\mu \ll \nu$ and $\sigma \in C(v, \alpha)$, ν-a.e.

PROOF. For every $t \in A$, cover $\mathscr{H}^1$-a.e. point in F_t with a sequence of $\mathscr{C}^1$-curves $(\gamma_t^n)_{n\in\mathbb{N}}$ such that γ_t^n goes in the direction of $C(v, \alpha)$. Possibly extending the curves, we may assume that their length is at least 1. For every $t \in A$, we denote by N_t the 1-current having the following property: $N_t = \sum_{n\in N} R_t^n$, where R_t^n is the rectifiable 1-current supported on γ_t^n with orientation given by the positive part of the cone $C(v, \alpha)$ and with constant multiplicity θ_t^n satisfying the property

$$\int_{\gamma_t^n} \theta_t^n = \lambda_t(\gamma_t^n) - \lambda_t\left(\gamma_t^n \cap \bigcup_{m<n} \gamma_t^m\right).$$

It is easy to see that $\mathbb{M}(\partial R_t^n) \leq \mathbb{M}(R_t^n)$, therefore R_t is a normal 1-current. $\square$

PROPOSITION 3.2.5. [**AM**] Let μ be a finite Radon measure on $\mathbb{R}^d$. Then there exists a sequence of elements of $\mathscr{L}$, $(\lambda_n, T_n)_{n\in\mathbb{N}}$, such that the decomposability bundle of μ coincides with the bundle generated by (λ_n, T_n).

PROOF. Among all sequences of elements of $\mathscr{L}$, choose one, $(\lambda_n, T_n)_{n\in\mathbb{N}}$, which maximizes the quantity

$$\int_{\mathbb{R}^d} \dim(G(x))\, \mathrm{d}\mu(x), \tag{3.2.2}$$

where G is the bundle generated by the sequence. Notice that this quantity is bounded by $d\|\mu\|$. Now we prove that the decomposability bundle S of μ coincides with the bundle G generated by this sequence.
By Proposition 1.3.13 and by Remark 1.3.14, we have $G(x) \subset S(x)$ μ-a.e., then it is sufficient to prove that G satisfies condition (i) in Definition 3.2.1.
Assume by contradiction that there exists a Radon measure $\lambda \ll \mu$ with the following property: there exists a 1-decomposition $\lambda = \int_0^1 \lambda_t\ \mathrm{d}t$, (with λ_t 1-rectifiable measure supported on a 1-rectifiable set E_t endowed with tangent field τ_t) and a set $A \subset [0,1]$, with positive Leesgue measure, such that for every $t \in A$ there exists a $\mathscr{H}^1$-positive set $F_t \subset E_t$ satisfying

$$\langle \tau_t(x) \rangle \not\subset G(x) \quad \text{for every } t \in A, \quad \text{for every } x \in F_t.$$

Possibly considering subsets of A and F_t, we may assume that there exists $v \in S^{d-1}$, $\alpha \in (0, \pi/2)$ such that $\tau_t(x)$ belongs to the cone $C(v,\alpha)$ and $S(x) \cap C(v,\alpha) = \emptyset$ for every $t \in A$ and for every $x \in F_t$. From Lemma 3.2.4 we know that there exists a normal current $N = \sigma\nu$ such that the positive measure

$$\widetilde{\mu} = \int_A \lambda_t \llcorner F_t\, \mathrm{d}t$$

(which is absolutely continuous with respect to μ) satisfies $\widetilde{\mu} \ll \nu$ and moreover $\sigma \in C(v,\alpha)$, ν-a.e., therefore the line with direction $\sigma(x)$ is not a vector subspace of $S(x)$ for $\widetilde{\mu}$-a.e. x. Adding $(\widetilde{\mu}, N)$ to the sequence (λ_n, T_n), the quantity 3.2.2 (evaluated on the new sequence) strictly increases, which is a contradiction. $\square$

In particular, as a consequence of Theorem 3.1.1, we have differentiability of every Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$ along "many" vectorfields (namely the vectorfields associated with the currents T_n). Now we want to look for differentiability along higher dimensional planes. Therefore we need a tool to ensure, at least, the linearity of the directional derivatives.

PROPOSITION 3.2.6. [**AM**] Let $T = \tau\nu$ be a nomal 1-current with compact support and $f : \mathbb{R}^d \to \mathbb{R}$ be a Lipschitz function. Then $fT = \tau f \nu$ is a normal current, moreover there exists ν-a.e. the directional derivative $\frac{\partial f}{\partial \tau}$ and satisfies

$$\partial(fT) = f\partial T + \frac{\partial f}{\partial \tau}\nu.$$

PROOF. If T is an integral 1-current satisfying (1.3.11), then the theorem is just a consequence of the Fundamental Theorem of Calculus. By Proposition 1.3.16 the theorem extends to every integral 1-current. To prove the result for a normal 1-current T, we use again Proposition 1.3.13. We write as usual

$$T = \int_0^1 T_t \, \mathrm{d}t.$$

Denote by $T(\Sigma_t, \tau_t, \theta_t)$ the integral 1-current T_t and by $\nu_t = \theta_t \mathscr{H}^1 \llcorner \Sigma_t$. Recall that in the proof of Theorem 3.1.1 we showed in particular that $\tau(x) = \tau_t(x)$ for ν_t-a.e. x and a.e. t. Hence we have, for a.e. t,

$$\frac{\partial f}{\partial \tau}(x) = \frac{\partial f}{\partial \tau_t}(x),$$

for ν_t-a.e. $x \in \Sigma_t$. Therefore for a.e. t we can write

$$\partial(fT_t) = f\partial T_t + \frac{\partial f}{\partial \tau}\nu_t. \tag{3.2.3}$$

Since we have also

$$\mathbb{M}(T) = \int_0^1 \mathbb{M}(T_t) \, \mathrm{d}t,$$

then we deduce

$$\nu = \int_0^1 \nu_t \, \mathrm{d}t.$$

Therefore, integrating on t in (3.2.3), we obtain the thesis. $\square$

Now, we are ready to prove the first part of our main theorem.

THEOREM 3.2.7. [**AM**] *Let μ be a Radon measure on $\mathbb{R}^d$ and let S be the decomposability bundle of μ. Then every Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$ is differentiable along $S(x)$ at μ-a.e. $x \in \mathbb{R}^d$.*

PROOF. Let $(\lambda_n, T_n)_{n\in\mathbb{N}}$ be the sequence given in Proposition 3.2.5. Write $T_n = \tau_n \nu_n$ for every $n \in \mathbb{N}$. Consider $(e_i)_{i\in\mathbb{N}}$ the standard basis of ℓ_1, and the countable set $A = \bigcup_{m\in\mathbb{N}} A_m$, dense in ℓ_1, where A_m is the set of elements $\sum_{i=1}^m a_i e_i$ with a_i integer multiple of 2^{-m}. For μ-a.e. $x \in \mathbb{R}^d$ and for every $\tau \in S(x)$ there exist $n_1, \ldots, n_d$ such that $\tau \in \langle \tau_{n_1}(x), \ldots, \tau_{n_d}(x) \rangle$, and the measure λ_{n_i} satisfies

$$\frac{\mathrm{d}\lambda_{n_i}}{\mathrm{d}\mu}(x) > 0, \forall i = 1, \ldots, d.$$

Hence for every $\varepsilon > 0$ there exists $a = (a_i)_{i\in\mathbb{N}} \in A$ such that

$$|\tau - \tau_a(x)| \leq \varepsilon,$$

where τ_a is the vectorfield associated with the normal 1-current $a_{i_1} T_{i_1} + \ldots + a_{i_d} T_{i_d}$

Now fix a Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$. Consider a set N such that for every $n \in \mathbb{N}$, $\nu_n(N) = 0$ and there exists the directional derivative $\frac{\partial f}{\partial \tau_n}$ on $\mathbb{R}^n \setminus N$. For every $a \in A$, call

$T_a = \sum_{i=1}^{\infty} a_i T_i = \tau_a \nu_a$. Denote with Γ the collection of the vectors τ_a. By Proposition 3.2.6 we can assume that on $\mathbb{R}^n \setminus N$ there exists the directional derivative $\frac{\partial f}{\partial \tau_a}$, for every $a \in A$ and we may also assume that it is linear with respect to the direction τ_a. We shall now show that f is differentiable along $S(x)$ for every $x \in \mathbb{R}^d \setminus N$. For every $x \in \mathbb{R}^d \setminus N$, consider the linear operator $L(x)$ on $S(x)$ defined by the values of $\frac{\partial f}{\partial v}$ in the directions $v = \tau_a$, and a vector $e \in S(x)$. For every $h > 0$, take $v_h \in \Gamma$ such that $|e - v_h| \leq h$. Then compute:

$$\frac{|f(x+he) - f(x) - hL(e)|}{h} = \frac{|f(x+he) - f(x+hv_h) - hL(e-v_h)|}{h} +$$
$$+ \frac{|f(x+hv_h) - f(x) - hL(v_h)|}{h} \leq h\mathrm{Lip}(f) + h|L| + o(1).$$

Therefore f is differentiable along $S(x)$ on $\mathbb{R}^d \setminus N$.

□

CHAPTER 4

Non-differentiability results

In this chapter, we describe a technique to construct a Lipschitz function which is non-differentiable at the points of a given "small" set in $\mathbb{R}^d$. Given a Radon measure μ on $\mathbb{R}^d$, we use this technique to obtain a Lipschitz function which is μ-almost everywhere non-differentiable along the directions which are not vector subspaces of the decomposability bundle of μ. This is a simplified version of the construction given in [**ACP**]. In the last section we give a new proof of the existence of such a function, inspired to the proof of Theorem 2.1.2.

4.1. Structure of invisible sets

In the sequel E is a set in $\mathbb{R}^d$, $v \in S^{d-1}$ is a direction, $\alpha \in (0, \frac{\pi}{2})$ is an angle and $\gamma : [0,1] \to \mathbb{R}^d$ is a curve, whose image in $\mathbb{R}^d$ is Γ.
We say that γ *goes in the direction of* the cone $C(v, \alpha)$, if

$$\gamma(s) - \gamma(t) \in C^+(v, \alpha), \quad \text{for every } s, t \in [0,1],\ s \geq t,$$

where

$$C^+(v, \alpha) = \{x \in \mathbb{R}^d : \langle x; v \rangle \geq |x| \cos\alpha\}.$$

We say that E is *invisible along the cone* $C(v, \alpha)$, if

$$\mathscr{H}^1(E \cap \Gamma) = 0,$$

for every curve γ going in the direction of $C(v, \alpha)$.
We say that a set $E \subset \mathbb{R}^d$ is *invisible along the direction* v, if

$$\mathscr{H}^1(E \cap \Gamma) = 0,$$

for every $\alpha \in (0, \frac{\pi}{2})$ and for every curve γ going in the direction of $C(v, \alpha)$.
Given $\varepsilon > 0$, $A \subset v^\perp$ open in $v^\perp$ and $f : A \to \mathbb{R}$ a Lipschitz function, we call v-*slab of thickness* $w(I) = \varepsilon$ around f the following set

$$I = \left\{x + tv : x \in A, t \in \left(f(x) - \frac{\varepsilon}{2}, f(x) + \frac{\varepsilon}{2}\right)\right\}.$$

If L is a Lipschitz constant for f, we say that I is an L-Lipschitz slab. If f is of class $\mathscr{C}^1$, we say that I is a slab of class $\mathscr{C}^1$.
In a partially ordered set, an *antichain* is a set of elements no two of which are comparable to each other. A *chain* is a totally ordered subset. The *length* of a chain (or an antichain)

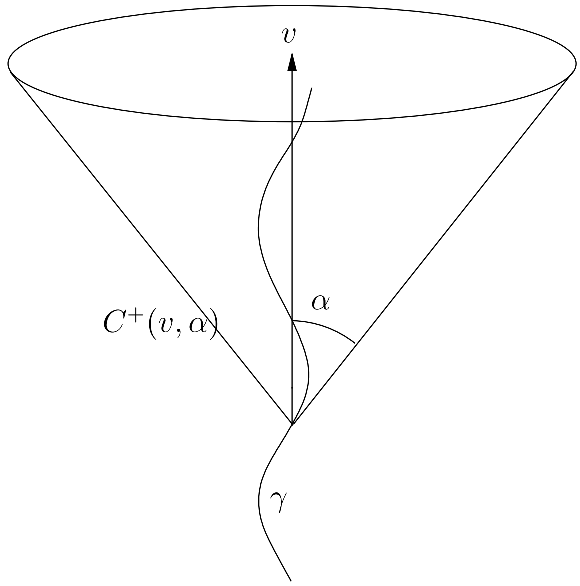

Figure 4.1.1

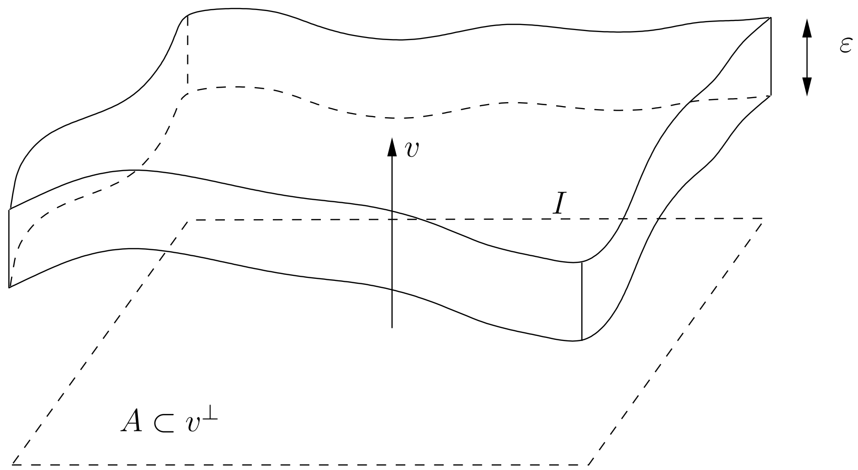

Figure 4.1.2

is just the number of its points. The following theorem is a dual version of the classical Dilworth theorem ([**D**]).

Theorem 4.1.1. [**Mi**] *In a finite partially ordered set $(X, \leq)$, the size of the largest chain equals the smallest number of antichains into which the set can be partitioned.*

Proof. For every $x \in X$, let $l(x)$ be the length of the largest chain having x as maximal element. Let

$$L = \max_{x \in X} l(x).$$

For every $n = 1, \dots, L$ the set

$$A_n = \{x \in X : l(x) = n\}$$

is an antichain and $\{A_n\}_{n=1}^{L}$ is a partition of X. Obviously it is not possible to find a partition with a smaller number of antichains, since every two elements of the largest chain must belong to different antichains. □

The next theorem is derived from a brilliant geometric interpretation of the previous result. It is possible to find several applications of this idea in [**ACP**].

Theorem 4.1.2. *Let E be a compact set in $\mathbb{R}^d$, invisible along the cone $C(v, \alpha)$. Then E can be covered by (finitely many)* $\cot(\alpha)$*-Lipschitz v-slabs in such a way that the sum of the thickness of the slabs is arbitrarily small.*

Proof. Without loss of generality, we may assume

$$E \subset G = [0, \tan(\alpha)]^{d-1} \times [0, 1]$$

and $v = e_d$. Let G_k be the grid obtained dividing each edge of G into k equal parts. Let E_k be the set of the centers of the cells of G_k intersecting E. Define a partial order on E_k by setting, for every $y_1, y_2 \in G_k$:

$$y_1 \leq y_2 \quad \text{if } y_2 - y_1 \in C^+(v, \alpha)$$

. We want to show that the length of the largest chain in E_k has lower order with respect

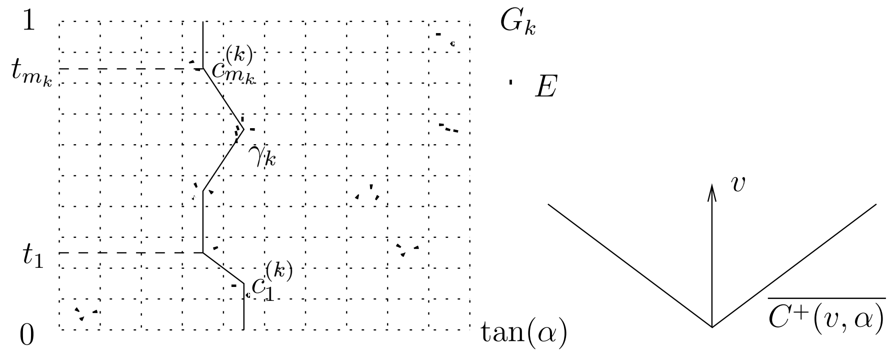

Figure 4.1.3

to k. Assume by contradiction that there exist $l > 0$ such that for infinitely many k there is a chain $C_k = (c_1^{(k)}, \dots, c_{m_k}^{(k)})$ of length at least lk. Define

$$\gamma_k : [0, 1] \to G, \quad \text{such that } \gamma_k(t_1) = c_1^{(k)}, \dots, \gamma_k(t_{m_k}) = c_{m_k}^{(k)},$$

where $t_i = \langle c_i^{(k)}; e_n\rangle$. Define

$$\gamma_k(0) = \gamma_k(t_1) - t_1 e_n, \quad \gamma_k(1) = \gamma_k(t_{m_k}) + (1 - t_{m_k})e_n$$

and γ_k affine on $[0, t_1]$, on $[t_{m_k}, 1]$ and on $[t_i, t_{i+1}]$ for every $i = 1, \ldots, m_{k-1}$ (see Figure 4.1.3). Up to subsequences, γ_k converges to a curve γ going in the direction of $C(v, \alpha)$. We want to show that $\mathscr{H}^1(\gamma \cap E) > 0$, which is a contradiction. For every k define

$$g_k : [0,1] \to \mathbb{R}, \quad \text{such that } g_k(t) = \operatorname{dist}(\gamma_k(t), E).$$

Since γ_k uniformly converges (up to subsequences) to γ, then g_k uniformly converges to the continuous function $g(t) = \operatorname{dist}(\gamma(t), E)$. By construction we have

$$g_k \leq k^{-1}(\sqrt{d}\tan(\alpha) + 1), \quad \text{on a set of length } l, \text{ for every } k.$$

For $\varepsilon > 0$, take k such that $|g_k - g| \leq \varepsilon$ and $k^{-1}(\sqrt{d}\tan(\alpha) + 1) \leq \varepsilon$. Then $g \leq 2\varepsilon$ on a set of length l. This proves that $g = 0$ on a set of length l, then the contradiction that $\mathscr{H}^1(\gamma \cap E) \geq l$.

By Theorem 4.1.1, E_k can be covered by $o(k)$ antichains. Every antichain A is the graph of a $\cot(\alpha)$-Lipschitz function f_A from a discrete set contained in $G \cap \{e_n = 0\}$ with values in $[0,1]$. Take a $\cot(\alpha)$-Lipschitz extension g_A of f_A to $\{e_n = 0\}$ in such a way that the image of g_A is contained in $[0,1]$. A slab of thickness $k^{-1}(\sqrt{d} + 1)$ around g_A contains every cell intersected by the graph of f_A. Therefore E can be covered by $o(k)$ $\cot(\alpha)$-Lipschitz v-slabs of thickness $k^{-1}(\sqrt{d} + 1)$. ☐

For some reasons, it could be convenient to have disjoint $\mathscr{C}^1$ slabs in the covering. The next corollary shows that this could be done, as long as one is willing to lose a small set. In the sequel, the word box indicates an n-dimensional rectangle.

Corollary 4.1.3. Let E be a compact set in $\mathbb{R}^d$, invisible along the cone $C(v, \alpha)$, with E contained in some closed box Q with one axis parallel to v. Let μ be a finite Radon measure supported on E. Then it is possible to cover μ-a.e. point of E by (finitely many) disjoint $\cot(\alpha)$-Lipschitz v-slabs of class $\mathscr{C}^1$, contained in an arbitrarily small neighborhood of Q and such that the sum of the thickness of the slabs is arbitrarily small.

Proof. Assume $E \subset Q = [0, \tan(\alpha)]^{d-1} \times [0,1]$ and $v = e_d$ and consider the covering of E given by Theorem 4.1.2. Fix $\varepsilon > 0$ and consider the open box

$$Q_\varepsilon = (-\varepsilon, \tan(\alpha) + \varepsilon)^{d-1} \times (-\varepsilon, 1 + \varepsilon).$$

For every index k sufficiently large, the slabs constructed in the previous proof, intersected with

$$(-\varepsilon, \tan(\alpha) + \varepsilon)^{d-1} \times \mathbb{R}$$

are contained in

$$(-\varepsilon, \tan(\alpha) + \varepsilon)^{d-1} \times \left(-\frac{\varepsilon}{4}, 1 + \frac{\varepsilon}{4}\right),$$

and the sum of their thickness is less than $\frac{\varepsilon}{4}$. We want to replace them with disjoint slabs of class $\mathscr{C}^1$ with the same Lipschitz constant and "almost" the same thickness. Let $A_1, \dots, A_m$ be the antichains associated with these slabs, and $g_{A_1}, \dots, g_{A_m}$ as in the previous proof.

Without loss of generality we may assume that $g_{A_i} \leq g_{A_j}$ if $i < j$. In fact if this is not the case one can define

$$i_1(x) = \min_{i=1,\dots,m} \{i : g_{A_i}(x) \leq g_{A_j}(x) \text{ for every } j = 1, \dots, m\}$$

and take $\widetilde{g}_{A_1}(x) = g_{A_{i_1(x)}}(x)$. Then for every $n = 2, \dots, m$ define recursively

$$I_n(x) = \{i_j(x) : j < n\}$$

and

$$i_n(x) = \min_{i \notin I_n(x)} \{i : g_{A_i}(x) \leq g_{A_j}(x) \text{ for every } j \notin I_n(x)\}$$

and take $\widetilde{g}_{A_n}(x) = g_{A_{i_n(x)}}(x)$. The new fucntions $\widetilde{g}_{A_i}$ satisfy the above property.

Now let

$$h = (2k)^{-1}(\sqrt{d} + 1)$$

be half of the thickness of the slabs and define

$$g_1 = g_{A_1} - h$$

and let f_1 be a $\mathscr{C}^1$ function with the same Lipschitz constant of g_1 and such that

$$0 \leq g_1 - f_1 \leq h$$

Let $w_1 \in [3h, 4h]$ be such that

$$\mu(\text{graph}(f_1 + w_1)) = 0.$$

It is possible to choose such an w_1 because the family

$$\{\text{graph}(f_1 + t)\}_{t \in [3h,4h]}$$

is uncountable and disjoint. Let I_1 be the slab of thickness w_1 around $f_1 + \frac{w_1}{2}$. Define

$$g_2(x) = \max\left(f_1(x) + w_1; g_{A_2}(x) - h\right).$$

Let f_2 be a $\mathscr{C}^1$ function with the same Lipschitz constant of g_2 and such that

$$f_2 \geq f_1 + w_1; \quad 0 \leq g_2 - f_2 \leq h.$$

Let $w_2 \in [3h, 4h]$ be such that

$$\mu(\text{graph}(f_2 + w_2)) = 0.$$

Let I_2 be the slab of thickness w_2 around $f_2 + \frac{w_2}{2}$ (See Figure 4.1.4).

After at most m steps, the union of the slabs $I_1, \dots, I_m$ covers μ-a.e. point in the union of $A_1, \dots, A_m$, in fact, for every $x \in Q \cap \{e_d = 0\}$ and for every $i = 1, \dots, m$ the set $\bigcup_{j=1}^{i} \overline{I_j}$ contains the intervals $(g_{A_j}(x) - h, g_{A_j}(x) + h)$ for every $j \leq i$. The choice

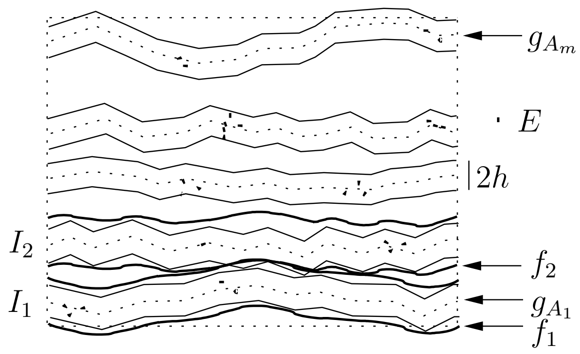

FIGURE 4.1.4

of w_i guarantees that the measure of $\bigcup_{i=1}^m \partial I_i$ is zero. Moreover $I_1, \ldots, I_m$ are disjoint $\cot(\alpha)$-Lipschitz v-slabs of class $\mathscr{C}^1$. Their intersection with

$$(-\varepsilon, \tan(\alpha) + \varepsilon)^{d-1} \times \mathbb{R}$$

is contained in Q_ε. □

REMARK 4.1.4. For a set which is invisible along a direction, the previous covering can be done with slabs of arbitrarily small Lipschitz constant. In particular for 1-purely unrectifiable sets, both the direction v of the slabs and the Lipschitz constant can be choosen arbitrarily.

4.2. Non-differentiability outside of the decomposability bundle

LEMMA 4.2.1. Let $E \subset \mathbb{R}^d$, $v(x)$ a vectorfield. Let $f : \mathbb{R}^d \to \mathbb{R}$ and $f_n \to f$ uniformly. For every $x \in E$ let $\{i_n\}$ and $\{j_n\}$ be two increasing sequences of indices and let y_n and z_n be corresponding sequences of points, both converging to x (but they are never equal to x). Assume there exist two real numbers $\alpha > \beta$ and an infinitesimal sequence ε_n such that, for every $n \in \mathbb{N}$, the following properties are satisfied:

$$\frac{f_{i_n}(y_n) - f_{i_n}(x)}{|y_n - x|_x} \geq \alpha, \tag{4.2.1}$$

(where $|y|_x = |y|$ if $\langle y\ x\rangle \geq 0$, $|y|_x = -|y|$ otherwise);

$$\frac{f_{j_n}(z_n) - f_{j_n}(x)}{|z_n - x|_x} \leq \beta; \tag{4.2.2}$$

$$y_n - x \text{ and } z_n - x \text{ are parallel to } v(x); \tag{4.2.3}$$

$$\|f - f_{i_n}\|_\infty \leq \varepsilon_n |y_n - x|; \tag{4.2.4}$$

$$\|f - f_{j_n}\|_\infty \leq \varepsilon_n |z_n - x|. \tag{4.2.5}$$

Then f is non-differentiable along $v(x)$ for every $x \in E$. In particular $f'_+(x,v) \geq \alpha$ and $f'_-(x,v) \leq \beta$

PROOF. We compute the difference quotient along y_n:

$$\frac{f(y_n)-f(x)}{|y_n-x|_x} = \frac{f(y_n)-f(x)+f_{i_n}(y_n)-f_{i_n}(x)-f_{i_n}(y_n)+f_{i_n}(x)}{|y_n-x|_x} \geq$$
$$\frac{f_{i_n}(y_n)-f_{i_n}(x)}{|y_n-x|_x} - 2\frac{\|f_{i_n}-f\|_\infty}{|y_n-x|} \geq \alpha - 2\frac{\|f_{i_n}-f\|_\infty}{|y_n-x|} \geq \alpha - 2\varepsilon_n.$$

Analogously, along z_n:

$$\frac{f(z_n)-f(x)}{|z_n-x|_x} = \frac{f(z_n)-f(x)+f_{j_n}(z_n)-f_{j_n}(x)-f_{j_n}(z_n)+f_{j_n}(x)}{|z_n-x|_x} \leq$$
$$\frac{f_{j_n}(z_n)-f_{j_n}(x)}{|z_n-x|_x} + 2\frac{\|f_{j_n}-f\|_\infty}{|z_n-x|} \leq \beta + 2\frac{\|f_{j_n}-f\|_\infty}{|z_n-x|} \leq \beta + 2\varepsilon_n.$$

□

We describe now a construction that will be useful in the sequel. Consider a vector v, a closed box Q with one axis parallel to v and a Radon measure μ supported on a compact set $E \subset Q$ such that E is invisible along the cone $C(v, \frac{\pi}{2} - \alpha)$. We want to construct a 2-Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$ having roughly speaking the following properties:

- f is supported on a small neighborhood of Q and $\|f\|_\infty$ is small;
- The Lipschitz constant of f along $v^\perp$ is small;
- f is $\mathscr{C}^1$ on an open set A with "large" measure μ and ∇f is small on A;
- for a large set of points $x \in A$, the slope of f in the direction v at x is almost 1 at a certain scale and it is 0 at some smaller scale.

For simplicity we describe the costruction for $Q = [0,1]^n$ and $v = e_n$. Fix $\varepsilon > 0, \lambda > 0$, $M \in \mathbb{N}$. Now consider the functions g, h, f and the set A defined as follows.

STEP 1: By Corollary 4.1.3, we can consider a covering

$$\mathscr{A}_1 = \{I_1, \ldots, I_k\}$$

of μ-a.e. point of E with a finite number of disjoint $\tan(\alpha)$-Lipschitz v-slabs of class $\mathscr{C}^1$, such that the sum of the thickness of the slabs in $\mathscr{A}_1$ is less than ε and the slabs are contained in

$$Q_\varepsilon = (-\varepsilon, 1+\varepsilon)^d.$$

Denote

$$A^1 = \bigcup_{I \in \mathscr{A}_1} I.$$

Define the function $\widetilde{g} : Q_\varepsilon \to \mathbb{R}$:

$$\widetilde{g}(x) = \int_{-\infty}^{0} \chi_{A^1}(x + tv)\, \mathrm{d}t.$$

Notice that $\widetilde{g}$ is of class $\mathscr{C}^1$ everywhere on Q_ε, except on the boundary of the slabs and it is $\tan(\alpha)$-Lipschitz along $v^\perp$ on Q_ε. Moreover $\|\widetilde{g}\|_\infty \leq \varepsilon$. Extend $\widetilde{g}$ to a Lipschitz function g defined on $\mathbb{R}^d$, which is null on the complement of the set $(Q)_{2\varepsilon}$. The extension can be done in such a way that $\|g\|_\infty \leq 2\varepsilon$, g is 2-Lipschitz on $\mathbb{R}^d$ and of class $\mathscr{C}^1$ everywhere except on the boundary of the slabs.

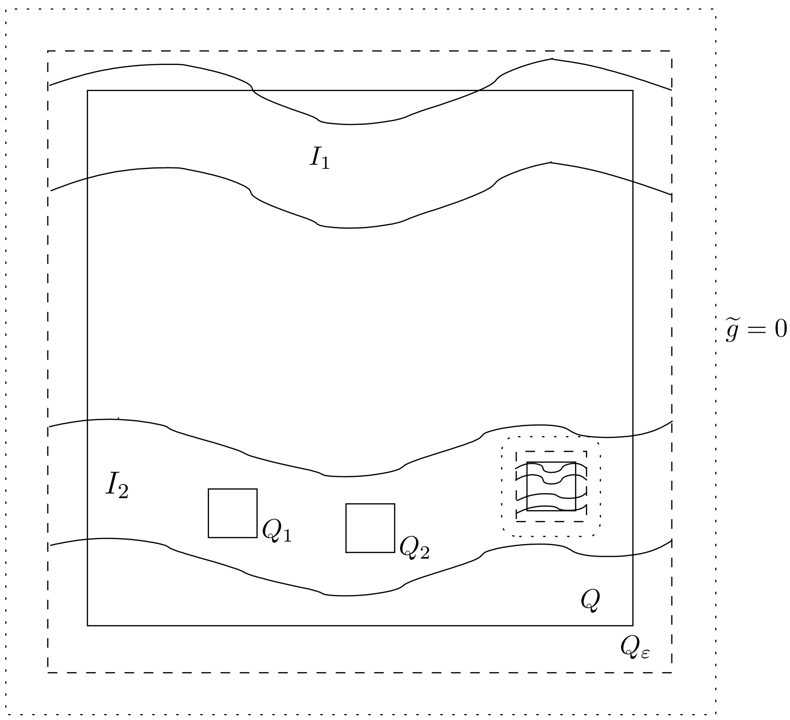

FIGURE 4.2.1

STEP 2 Take a finite number of disjoint closed boxes $Q_1, \ldots, Q_m$ with one axis parallel to v, such that $Q_i \subset A^1$ and

$$\mu\left(A^1 \setminus \bigcup_{i=1}^{m} Q_i\right) \leq \lambda.$$

Let ω be the smallest among the thickness of the slabs in $\mathscr{A}_1$ and $\widetilde{d}$ the minimal distance between two boxes Q_i. Define

$$\underline{d} = \min\left(\widetilde{d}, \operatorname{dist}\left(\bigcup_i Q_i, \mathbb{R}^d \setminus A^1\right)\right); \quad \overline{d} := \min\left(M^{-1}w, \frac{\underline{d}}{4\sqrt{d}}\right).$$

Consider for every $i = 1, \ldots, m$ a covering

$$\mathscr{A}_2^i = \{I_1^i, \ldots, I_{k(i)}^i\}$$

of μ-a.e. point of Q_i with a finite number of disjoint $\tan(\alpha)$-Lipschitz v-slabs of class $\mathscr{C}^1$, such that the sum of the thickness of the slabs in $\mathscr{A}_2^i$ is less than $\overline{d}$. This can be done in such a way that the slabs in $\mathscr{A}_2^i$ are contained in $(Q_i)_{\overline{d}}$, (the definition of $(Q_i)_{\overline{d}}$ is analogous to that of Q_ε).

For $i = 1, \ldots, m$ define the set

$$A_i^2 = \bigcup_{j=1}^{k(i)} I_j^i$$

and on $(Q_i)_{\overline{d}}$ define the function

$$h^i(x) = \int_{-\infty}^0 \chi_{A_i^2}(x + tv)\, \mathrm{d}t.$$

Note that, for every i, h^i is of class $\mathscr{C}^1$ everywhere on $(Q_i)_{\overline{d}}$, except on the boundary of the slabs and it is $\tan(\alpha)$-Lipschitz along $v^\perp$ on $(Q_i)_{\overline{d}}$. Moreover $\|h^i\|_\infty \leq \overline{d}$ and

$$\operatorname{dist}((Q_i)_{\overline{d}}, (Q_j)_{\overline{d}}) \geq 2\overline{d}, \quad \text{for } i \neq j.$$

Define

$$A = \bigcup_{i=1}^m A_i^2.$$

Define a function $\widetilde{h}$ on $\bigcup_{i=1}^m (Q_i)_{\overline{d}}$:

$$\widetilde{h} = \sum_{i=1}^m \chi_{(Q_i)_{\overline{d}}} h^i.$$

Extend $\widetilde{h}$ to a function h defined on $\mathbb{R}^d$, which is null on the complement of the set $\bigcup_{i=1}^{m_1} (Q_i)_{2\overline{d}}$.

The extension can be done in such a way that $\|h\|_\infty \leq 2\overline{d}$, h is 2-Lipschitz and of class $\mathscr{C}^1$ everywhere except on the boundary of the slabs. Notice that in particular h is null on $\mathbb{R}^d \setminus A^1$.

STEP 3 Consider the function

$$f = g - h,$$

defined on $\mathbb{R}^d$. The following properties hold:

- On A, f is constant along v and it is $(2\tan(\alpha))$-Lipschitz along $v^{\perp}$, therefore we have $|\nabla f| \leq 2\tan(\alpha)$ on A;
- $\|f\|_{\infty} \leq 2\varepsilon + 2\overline{d}$;
- $\|h\|_{\infty} \leq 2M^{-1}w(I)$ for all $I \in \mathscr{A}_1$ (remember that $\omega(I)$ is the thickness of the slab I);
- given $I \in \mathscr{A}_1$, we have $\frac{g(y)-g(x)}{y-x} = 1$, for every x, y in I such that $y - x$ is parallel to v;
- given $I \in \mathscr{A}_2^i$, we have $\frac{f(y)-f(x)}{y-x} = 0$, for every x, y in I such that $y - x$ is parallel to v, for every $i = 1, \dots, m$.

If a compact set is invisible along one direction, the previous construction can be iterated countably many times and it gives the following result.

THEOREM 4.2.2. *Let $E \subset \mathbb{R}^d$ be a compact set which is invisible along the direction v, let $\varepsilon > 0$ and let μ be a finite Radon measure supported on E. Then there exists a Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$ which is not differentiable along the direction v at any point in a set A with $\mu(\mathbb{R}^d \setminus A) \leq \varepsilon$.*

PROOF. Fix

$$\begin{aligned} \varepsilon_1 &= \frac{1}{2}, \\ \alpha_i &= 2^{-i}, \\ M_i &= 4^i, \\ \lambda_i &= \varepsilon 4^{-i}. \end{aligned}$$

Consider a box Q containing E with one axis parallel to v Depending on parameters $\varepsilon_1, \alpha_1, M_1$ and λ_1, construct the functions g, h, f and the set A described in the previous construction. Denote them by g_1, h_1, f_1 and A_1 respectively.

Take a finite number of disjoint closed boxes $Q_1, \dots, Q_m$, with one axis parallel to v, such that $Q_i \subset A_1$ and

$$\mu\left(A_1 \setminus \bigcup_i Q_i\right) \leq \lambda_2.$$

Let w_1 be the smallest thickness of the slabs whose union gives A_1 and $\widetilde{d_1}$ the minimal distance between two boxes Q_i. Define

$$\overline{d_1} = \min\left(\widetilde{d_1}, \operatorname{dist}\left(\bigcup_i Q_i, \mathbb{R}^d \setminus A_1\right)\right); \quad d_1 := \min\left((2M_1)^{-1} w_1, \frac{\overline{d_1}}{4\sqrt{d}}\right).$$

In every box Q_i build the functions g_2, h_2, f_2 depending on parameters

$$\begin{aligned} \alpha &= \alpha_2, \\ \varepsilon &= \varepsilon_2 = d_1, \\ M &= M_2, \\ \lambda &= \lambda_2. \end{aligned}$$

Construct in the same way $g_3, h_3, f_3 \dots$ and consider the function

$$f = \sum_{i=1}^{\infty} f_i.$$

This sum is absolutely convergent. Let $s_{2m} = \sum_{i=1}^{m} f_i$ and $s_{2m+1} = s_{2m} + g_{m+1}$. The function $f = \lim_{m\to\infty} s_m$ is Lipschitz, because, for every $i > 1$, the support of f_i is contained in the set A_{i-1}, where s_{i-1} is $\mathscr{C}^1$ and satisfies

$$|\nabla s_{i-1}| \le 2\sum_{j=1}^{i-1} \tan(\alpha_j).$$

So the Lipschitz constant of s_i does not exceed the quantity

$$\max\left(\mathrm{Lip}(s_{i-1}); \mathrm{Lip}(f_i) + 2\sum_{j=1}^{i-1}\tan(\alpha_j)\right) \le \max_i\{\mathrm{Lip}(f_i)\} + 2\sum_{j=1}^{\infty}\tan(\alpha_j).$$

For every point $x \in A = \bigcap_{m\in\mathbb{N}} A_m$ it is possible to apply Lemma 4.2.1 to the sequence s_n, with $\alpha = 1, \beta = 0$, $i_n = 2n+1$, $j_n = 2n$, $\varepsilon_n = 8M_n^{-1}$, z_n any point in A_n such that $x - z_n$ is parallel to v and $|x - z_n| \ge \frac{1}{4}w(I)$ (where $w(I)$ is the thickness of the slab of A_n containing x) and y_n is choosen analogously in the slab of the "next generation". Therefore it is possible to conclude that

$$1 = f^+(x, v) \ne f'_-(x, v) = 0 \qquad \text{for every } x \in A.$$

□

Remark 4.2.3. Notice that for every $m \in \mathbb{N}$ it is possible to write:

$$f = \sum_{i=1}^{m} f_i + \sum_{i=m+1}^{\infty} f_i.$$

Let $r_m = f - s_m$. Since s_m is of class $\mathscr{C}^1$ on the points of A_m, then r_m is such that the difference between upper and lower derivative along v is 1 on the points of A and it is $(2\sum_{i=m+1}^{\infty}\alpha_i)$-Lipschitz along $v^\perp$. This implies that r_m (and therefore f) is non-differentiable at the points of A along all the directions s such that the tangent of the angle between s and v is less then $(4\sum_{i=m+1}^{\infty}\alpha_i)^{-1}$.

For m sufficiently large, the angle can be chosen arbitrarily close to $\frac{\pi}{2}$, so the following improvement of Theorem 4.2.2 holds.

Theorem 4.2.4. *Let $E \subset \mathbb{R}^d$ be a compact set which is invisible along the direction v, let $\varepsilon > 0$ and let μ be a finite Radon measure supported on E. Then there exists a Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$ which is not differentiable along any direction, except for the directions orthogonal to v, at any point in a set A, with $\mu(\mathbb{R}^d \setminus A) \le \varepsilon$.*

Now, we are look for a further improvement. Precisely we wish to obtain a statement in which the expression "at any point in a set A, with $\mu(\mathbb{R}^d \setminus A) \leq \varepsilon$" is replaced by "at μ-a.e. point". We will use the following two general facts:

LEMMA 4.2.5. Let f be an L-Lipschitz function defined on $\mathbb{R}^d$ and let K be a compact set. Then there exists a function $\widetilde{f}$ defined on $\mathbb{R}^d$ such that $\widetilde{f} = f$ on K, $\widetilde{f}$ is smooth on $\mathbb{R}^d \setminus K$ and $5L$-Lipschitz on $\mathbb{R}^d$. Moreover $\|\widetilde{f}\|_\infty \leq 4\|f\|_\infty$.

PROOF. For Every $i \in \mathbb{N}$, $i \geq 2$ define

$$V_i := \left\{ x \in \mathbb{R}^d : \frac{1}{i+1} < \operatorname{dist}(x, K) < \frac{1}{i-1} \right\}.$$

Let λ_i be a partition of unity associated with V_i. It is possible to write

$$f = f\chi_K + \sum_i f\lambda_i.$$

Let ρ be a convolution kernel supported on $B(0,1)$ and for every $\varepsilon > 0$ let

$$\rho_\varepsilon(x) = \varepsilon^{-d} \rho\left(\frac{x}{\varepsilon}\right).$$

For a sequence $\varepsilon_i \searrow 0$ we define $\rho_i = \rho_{\varepsilon_i}$ and

$$\widetilde{f} = f\chi_K + \sum_i (f\lambda_i) * \rho_i.$$

If ε_i is chosen sufficiently small, then we can assume that $\lambda_i * \rho_i$ is supported on $V_{i-1} \cup V_i \cup V_{i+1}$, for every i. In particular for every point in $\mathbb{R}^d \setminus K$ $\widetilde{f}$ is a sum of up to four smooth functions, hence it is smooth and there holds $\|\widetilde{f}\|_\infty \leq 4\|f\|_\infty$. We are left with the proof that $\widetilde{f}$ is $5L$-Lipschitz. We can write

$$\widetilde{f} = f + \sum_i [(f\lambda_i) * \rho_i - (f\lambda_i)],$$

hence we have

$$D\widetilde{f} = Df + \sum_i [D(f\lambda_i) * \rho_i - D(f\lambda_i)] =$$

$$= Df + \sum_i [(Df\lambda_i) * \rho_i - Df\lambda_i] + \sum_i [(fD\lambda_i) * \rho_i - fD\lambda_i].$$

Denoting

$$g_i := fD\lambda_i; \quad h_i := (Df\lambda_i) * \rho_i,$$

we can write

$$D\widetilde{f} = Df\chi_K + \sum_i h_i + \sum_i (g_i * \rho_i - g_i)$$

Again, since given a point x we have $h_i = 0$ for all but at most four indices i, then we have

$$\|Df\chi_K + \sum_i h_i\|_\infty \leq 4L.$$

Moreover, every g_i is uniformly continuous, therefore for a choice of sufficiently small ε_i we can obtain

$$\|(g_i * \rho_i - g_i)\|_\infty \leq \frac{1}{4}L$$

for every i. Hence

$$\|\sum_i h_i + \sum_i (g_i * \rho_i - g_i)\|_\infty \leq L,$$

which completes the proof. □

LEMMA 4.2.6. For every $i \in \mathbb{N}$ let f_i be an L_i-Lipschitz function on $\mathbb{R}^d$, Assume that there is an open set E such that every f_i is differentiable on E and:

$$\sum_{i=1}^{\infty} L_i \leq +\infty; \tag{4.2.6}$$

$$\sum_{i=1}^{\infty} \|f_i\|_\infty \leq +\infty. \tag{4.2.7}$$

Then the sum of the f_i converges to a Lipschitz function f which is differentiable on E.

PROOF. Of course the sum converges to a Lipschitz function f because of (4.2.6) and (4.2.7). We want to prove that f is differentiable on E. Let

$$s_n = \sum_{i=1}^{n} f_i.$$

Fix a point $x \in E$. Let

$$v(x) = \sum_{i=1}^{\infty} \nabla f_i(x).$$

Fix $\varepsilon > 0$. There exists $m \in \mathbb{N}$ such that $\mathrm{Lip}(s_m - f) \leq \varepsilon$ and $\|(s_m - f)\|_\infty \leq \varepsilon$. Moreover there exists $r_0 > 0$ such that:

$$|s_m(x+h) - s_m(x) - \nabla s_m(x)h| \leq \varepsilon h; \quad \text{whenever } |h| \leq r_0.$$

For every $|h| \leq r_0$, we have:

$$|f(x+h) - f(x) - v(x)h| \leq$$
$$|(f-s_m)(x+h)-(f-s_m)(x)|+|\nabla s_m(x)h-v(x)h|+|s_m(x+h)-s_m(x)-\nabla s_m(x)h| \leq 3\varepsilon h.$$

□

Finally, we get the following result.

THEOREM 4.2.7. *Let $E \subset \mathbb{R}^d$ be a compact set which is invisible along the direction v and let μ be a finite Radon measure supported on E. Then there exists a Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$ which is not differentiable along any direction, except for the directions orthogonal to v, at μ a.e point.*

PROOF. Consider the function f and the set A given by Theorem 4.2.4 applied to the set E, consider a compact set K_1 "slightly smaller" than A, and apply Lemma 4.2.5 to the function f_1 and the set K_1, obtaining a function $\widetilde{f_1}$ which agrees with f_1 on K_1 and is smooth on $\mathbb{R}^d \setminus K_1$. In the next step apply Theorem 4.2.4 to some compact set K_2 disjoint from K_1 and perform the same construction. Repeat the procedure countably many times, choosing K_i disjoint in such a way that:

$$\mu\left(\mathbb{R}^d \setminus \bigcup_i K_i\right) = 0.$$

Apply Lemma 4.2.6 to the sequence $2^{-i}\widetilde{f_i}$. For every i, the function

$$\sum_{j\neq i} 2^{-j}\widetilde{f_j}$$

is differentiable on K_i and therefore the function

$$\sum_i 2^{-i}\widetilde{f_i}$$

is a Lipschitz function (because $\widetilde{f_i}$ are equi-Lipschitz) which is not differentiable on K_i along any direction, except for the directions orthogonal to v, because so is $\widetilde{f_i}$. $\square$

REMARK 4.2.8. Actually we do not need that E is invisible in one direction to perform this construction. In fact we only use that, at some small scale, the set E is locally invisible along a cone with axis v and an angle arbitrarily close to $\frac{\pi}{2}$. In the next theorem we will prove that the procedure works even if the axis v of the cones is allowed to vary in a continuous way.

In order to get the main non-differentiability result, we need the following lemma.

LEMMA 4.2.9 (Rainwater Lemma 9.4.3 of [**R**]). Let Γ be a compact set of Radon measures on $\mathbb{R}^d$. If

$$\lambda \perp \int_\Gamma \mu \,\mathrm{d}P$$

for every probability P on Γ, then there exists an F_σ set E (countable union of closed sets) such that λ is supported on E and $\mu(E) = 0$ for all $\mu \in \Gamma$.

THEOREM 4.2.10. *Let μ be a finite Radon measure on $\mathbb{R}^d$. Let S be the decomposability bundle of μ. Then there exists a Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$ such that, for μ-a.e. x, f is not differentiable at x along any direction which is not in $S(x)$.*

Proof. Using Lemma 4.2.5 and Lemma 4.2.6 as in the proof of Theorem 4.2.7, it is sufficient to prove the theorem when $K := \mathrm{supp}(\mu)$ is compact, $\dim(S^{\perp}) = m$ is constant on K and $v_1(x), \dots, v_m(x)$ is an orthonormal basis of $S^{\perp}(x)$ and we can assume that the v_i's are continuous on K for every $i = 1, \dots, m$. In fact, once we can find, for every compact set E_i of a disjoint sequence, a Lipschitz function which is non differentiable on the set E_i, then the two lemmas allows to construct a Lipschitz function which is non differentiable on the union $\bigcup_{i \in \mathbb{N}} E_i$.

Let $\mathscr{I}$ be the set

$$\mathscr{I} = \{(i, j) : i = 1, \dots, d; j \in \mathbb{N}\}.$$

Define a total order on $\mathscr{I}$, given by:

$$(i_1, j_1) \leq (i_2, j_2) \iff (j_1 < j_2) \text{ or } (j_1 = j_2 \text{ and } i_1 \leq i_2).$$

Fix an infinitesimal sequence $(\varepsilon_I)_{I \in \mathscr{I}}$.

Consider the family of closed boxes

$$\{Q_{x,r}^{(1,1)}\}_{x \in K}$$

with faces parallel to $v_1(x), \dots, v_m(x)$ such that

$$\langle v_1(x); t \rangle \leq \sin(\varepsilon_{(1,1)})|t| \text{ for every } t \in S(y), \text{ for every } y \in K \cap Q_{x,r}^{(1,1)}.$$

This is a fine covering of K (i.e. for every point of K there are arbitrarily small sets containig it). Consider a finite disjoint subfamily of (closed) boxes

$$\mathscr{Q}^{(1,1)} = \left\{Q_i^{(1,1)}\right\}_{i=1}^{m_{(1,1)}}$$

centered at some points $x_i^{(1,1)}$ such that

$$\mu\left(\mathbb{R}^d \setminus \bigcup_i \mathrm{int}(Q_i^{(1,1)})\right) \leq \varepsilon_{1,1}$$

We can apply Lemma 4.2.9 to the measure $\lambda = \mu \llcorner \mathrm{int}(Q_i^{(1,1)})$ and Γ is the compact set of 1-rectifiable measures, with unit multiplicity, supported on some curve going in the direction of the cone

$$C(v_1(x_i^{(1,1)}), \frac{\pi}{2} - \varepsilon_{(1,1)}).$$

This implies that $\mu \llcorner \mathrm{int}(Q_i^{(1,1)})$ is supported on a set $E_i^{(1,1)}$ which is invisible along the cone $C(v_1(x_i^{(1,1)}), \frac{\pi}{2} - \varepsilon_{(1,1)})$. With the same technique used in the proof of Theorem 4.2.2 we can construct for every $i = 1, \dots, m_{(1,1)}$ functions $g_i^{(1,1)}, h_i^{(1,1)}, f_i^{(1,1)}$ such that $f_i^{(1,1)}$ is null outside a small box containing $Q_i^{(1,1)}$. In particular we can take the support of $f_i^{(1,1)}$ disjoint from the support of $f_j^{(1,1)}$ whenever $i \neq j$. Let $A_i^{(1,1)} (i = 1, \dots, m_{(1,1)})$ be set described in the discussion after Lemma 4.2.1 arising from the construction relative to the box $Q_i^{(1,1)}$, and define

$$A^{(1,1)} = \bigcup_{i=1}^{m_{(1,1)}} A_i^{(1,1)}.$$

At the next step, indexed by $(2,1)$, consider the family of closed boxes

$$\{Q_{x,r}^{(2,1)}\}_{x\in A^{(1,1)}}$$

contained in $A^{(1,1)}$, with faces parallel to $v_1(x), \dots, v_m(x)$ such that

$$\langle v_2(x);t\rangle \leq \sin(\varepsilon_{(2,1)})|t| \text{ for every } t\in S(y), \text{ for every } y\in K\cap Q_{x,r}^{(2,1)}.$$

Proceeding as in the proof of Theorem 4.2.2, choose a finite family of boxes

$$\mathscr{Q}^{(2,1)} = \left\{Q_i^{(2,1)}\right\}_{i=1}^{m_{(2,1)}}$$

such that

$$\mu\left(A^{(1,1)}\setminus\bigcup_i \operatorname{int}(Q_i^{(2,1)})\right) \leq \varepsilon_{2,1}$$

and build analogously the functions $f_i^{(2,1)}$ with respect to these boxes (the functions $g_i^{(2,1)}$ and $h_i^{(2,1)}$ are obtained by integrating along the vector $v = v_2(x_i^{(2,1)})$).

Repeat this construction for every index $I \in \mathscr{I}$. Define the function

$$f = \sum_{I\in\mathscr{I}} f^I.$$

The proof that f is Lipschitz is analogous to the proof given in Theorem 4.2.2. Let $s^I = \sum_{J\leq I} f^J$. The function $f = \lim_I s^I$ exists (provided the sum of the ε^I's is small enough) and it is Lipschitz. The proof of this and of the fact that, for every point in the set $\bigcap_{I\in\mathscr{I}} A^I$, f is not differentiable along any direction which is not in $S(x)$ is analogous to the proof of Theorem 4.2.4. □

Summing up the two main results of this first part (Theorem 3.2.7 and Theorem 4.2.10), we have the following result:

THEOREM 4.2.11. *Given a Radon measure μ on $\mathbb{R}^d$, there exists a Borel map*

$$S:\mathbb{R}^d \to \bigcup_{k=0}^{d} Gr_k(\mathbb{R}^d)$$

such that every Lipschitz function $f:\mathbb{R}^d\to\mathbb{R}$ is μ-a.e. differentiable along S. Moreover there exists a Lipschitz function $g:\mathbb{R}^d\to\mathbb{R}$ such that, for μ-a.e. x, g is non-differentiable at x along any direction which is not in $S(x)$.

4.3. A Baire proof

In this section we give a new proof of Theorem 4.2.10, inspired by the proof of Theorem 2.1.2. Here, the existence of the function f is obtained by a Baire argument. Hence, Theorem 4.2.10 is in a certain sense improved: the family of functions which (on a "large" set) enjoy the same non-differentiability property as f is residual in a suitable space of Lipschitz functions. This result is contained in [**AM**].

Let μ be a finite Radon measure on $\mathbb{R}^d$. Let S be the decomposability bundle of μ. We may assume that K =supp(μ) is compact, hence without loss of generality, we assume $K \subset B_1(0)$. Given $\varepsilon > 0$, there exists a compact set K_ε with $\mu(\mathbb{R}^d \setminus K_\varepsilon) \leq \varepsilon$, such that we can find $v_1(x), \ldots, v_d(x) : B_1(0) \to S^{d-1}$ continuous, satisfying the following properties:

- for μ-a.e $x \in K_\varepsilon$, if $\dim(S(x)^\perp) = k$, then $S(x)^\perp$ is generated by
$$\{v_1(x), \ldots, v_k(x)\};$$
- for every x, $(v_1(x), \ldots, v_d(x))$ is an orthonormal basis of $\mathbb{R}^d$.

Define

(4.3.1)
$$X = \{u : B_1(0) \to \mathbb{R}\ \sqrt{d}\text{–Lipschitz, s.t. } |\langle \nabla u; v_i \rangle| \leq 1 \ \ \mathscr{L}^d\text{–a.e. for } i = 1, \ldots, d\}.$$

Notice that X, endowed with the supremum distance, is a complete metric space.
First, we want to prove that piecewise affine functions satisfying a strict inequality in (4.3.1) are dense in X.

Consider $(e_1, \ldots, e_d)$ the standard basis of $\mathbb{R}^d$. Let $\mathscr{G}_0 = \{T_n\}_{n \in \mathbb{N}}$ be a tiling of $\mathbb{R}^d$ made by uniformly bounded simplexes, i.e. the elements of $\mathscr{G}_0$ have the properties:

$$\bigcup_{n \in \mathbb{N}} T_i = \mathbb{R}^d \text{ and } \overset{\circ}{T}_n \cap \overset{\circ}{T}_m = \emptyset, \text{ for } n \neq m.$$

One can construct such a tiling by induction on d. For $d = 2$ one can firstly tile the unit square with the four triangles obtained as the convex envelop of one side of the square and the baricenter of the square. Then it is possible to extend this tiling on $\mathbb{R}^2$ using the fact that the space can be tiled by squares. For $d > 2$ one can tile the unit cube of $\mathbb{R}^d$ with the simplexes obtained as the convex envelop of the baricenter of the cube and one of the $(d-1)$-dimensional simplexes that by induction can be used to tile the faces of the d-dimensional cube.

For every $n \in \mathbb{N}$, let $\mathscr{G}_n = \{T_{n,m}\}_{m \in \mathbb{N}}$ be a tiling of $\mathbb{R}^d$ made by uniformly bounded simplexes such that the elements of $\mathscr{G}_n$ are contained in the elements of $\mathscr{G}_{n-1}$ and every element of $\mathscr{G}_n$ is contained in a cube of diameter 2^{-n}. Let G_n be the set

$$G_n = \bigcup_m \partial T_{n,m}$$

and let

$$G = \bigcup_{n \in \mathbb{N}} G_n.$$

It is easy to see that there exists $v \in \mathbb{R}^d$ such that $\mu(G+v) = 0$. In fact, assume by contraddiction that for every $v \in \mathbb{R}^d$ there holds $\mu(G+v) > 0$. Since G has only countably many faces, one can find an uncountable set $\{v_t\}_{t\in\mathbb{R}}$ with $v_t \in \mathbb{R}^d$ such that for every choice of distinct $v_1, \dots, v_{d+1} \in \{v_t\}_{t\in\mathbb{R}}$, there holds

$$\bigcap_{i=1}^{d+1} G + v_i = \emptyset.$$

This means that $\mu(G+v)$ is positive for at most only countably many $v \in \{v_t\}_{t\in\mathbb{R}}$. For simplicity, from now on we assume $v = 0$.

Lemma 4.3.1. [**AM**] For every $u \in X$ there is a sequence of functions $(u_n)_{n\in\mathbb{N}} \subset X$ uniformly converging to u, such that u_n is affine on every $T_{n,m}$. Moreover u_n satisfies

$$|\langle \nabla u_n; v_i \rangle| \leq 1 - \frac{3}{n} \quad \mathscr{L}^d\text{–a.e.} \quad \text{for every } i = 1, \dots, d. \tag{4.3.2}$$

Proof. First we prove that smooth functions satisfying (4.3.2) are dense in X. Since v_i are continuous on K, then for every $n \in \mathbb{N}$ we can find $\rho_n > 0$ such that

$$|v_i(x) - v_i(y)| \leq \frac{1}{n}, \text{ whenever } |y - x| \leq \rho_n, \text{ for every } i = 1, \dots, d.$$

Let ϕ be a convolution kernel supported on $B_1(0)$ and define

$$\phi_n(x) = \rho_n^{-d}\phi(\rho_n x).$$

Now take $u \in X$. The functions $u * \phi_n$ uniformly converge to u as $n \to \infty$ and satisfy

$$\langle \nabla(u * \phi_n); v_i \rangle \leq 1 + \frac{1}{n}, \text{ for every } i = 1, \dots, d.$$

Therefore the functions

$$\widetilde{u}_n = \left(1 - \frac{4}{n}\right) u * \phi_n$$

uniformly converge to u as $n \to \infty$ and satisfy

$$\langle \nabla \widetilde{u}_n; v_i \rangle \leq 1 - \frac{3}{n}. \text{ for every } i = 1, \dots, d.$$

Now, for every $n \in \mathbb{N}$ it is sufficient to consider the function u_n which is affine on each simplex $T_{n,m}$ and whose values on $T_{n,m}$ are determined by the values of $\widetilde{u}_n$ at the vertices of $T_{n,m}$. $\square$

THEOREM 4.3.2. [**AM**] *Let μ be a finite Radon measure on $\mathbb{R}^d$. Let S be the decomposability bundle of μ. Fix $\varepsilon > 0$. Then there exists a set A with $\mu(\mathbb{R}^d \setminus A) \leq \varepsilon$, such that the following condition holds. Let $\mathscr{F}$ be the set of all Lipschitz functions $f \in X$ which are non-differentiable at every $x \in A$, along the directions which are not in $S(x)$. Then $\mathscr{F}$ is residual in X.*

REMARK 4.3.3. Since we are saying that the family $\mathscr{F}$ is residual, one may wonder why we are not taking a countable intersection of families $\mathscr{F}_i$ with the corresponding ε_i going to zero to obtain a residual subset of X where the non-differentiablity property holds μ-a.e. The point is that the complete metric space X itself depend on the ε used in the theorem above. And the reason is that we need to fix a continuous base of the decomposability bundle S. To assume this continuity we are using Lusin theorem, and therefore we need to modify the bundle on a small set.

PROOF OF THEOREM 4.3.2. Thanks to the observation made at the beginning of this section, here we can assume that $d - k = \dim(S)$ is constant and that $v_1(x), \dots, v_k(x)$ is a continuous orthonormal basis of $S(x)^\perp$.
Take a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ such that

$$\sum_{n\in\mathbb{N}} k\varepsilon_n \leq \varepsilon.$$

For every $n \in \mathbb{N}$ and for every $i = 1, \dots, k$, let

$$Q_1, \dots, Q_{m_n}$$

be disjoint, closed cubes, centered at

$$x_1, \dots, x_{m_n}$$

with side length a_l ($l = 1, \dots, m_n$) and contained in $\mathbb{R}^d \setminus G_n$ such that:

- the edges of Q_l are parallel to $v_i(x_l)$ for $l = 1, \dots, m_n$;
- $|\langle v_i(y); v_i(x_l)\rangle| \geq \cos(1/n)$ for every $y \in 2Q_l$.
- $\mu(\mathbb{R}^d \setminus \bigcup_{l=1}^k Q_l) \leq \varepsilon_n/2$;

Denote

$$\widetilde{d} = \min_{l\neq p}\{\mathrm{dist}(Q_l, Q_p)\}.$$

and

$$\overline{d} = d^{-1/2}\min\{\min_l\{a_l\}, \widetilde{d}, \mathrm{dist}(G_n, \bigcup_l Q_l)\}.$$

For every l, use Corollary 4.1.3 and Lemma 4.2.9 to cover μ-a.e $x \in Q_l$ with a finite number of (open) $(\tan(1/n))$-Lipschitz $v_i(x_l)$-slabs of class $\mathscr{C}^1$ supported in $(Q_l)_{\overline{d}/4}$ and total thickness less than $\overline{d}/16n$. The choice of $\overline{d}/4$ is due to the fact that in the sequel we will have a function defined on the boxes $(Q_l)_{\overline{d}/4}$ and one defined outside of the boxes $(Q_l)_{\overline{d}/2}$ and we want to use the gap of $\overline{d}/4$ to have a continuous extension. Notice that the boxes $(Q_l)_{\overline{d}/2}$ are pairwise disjoint and also disjoint from the boundary of the simplexes, so

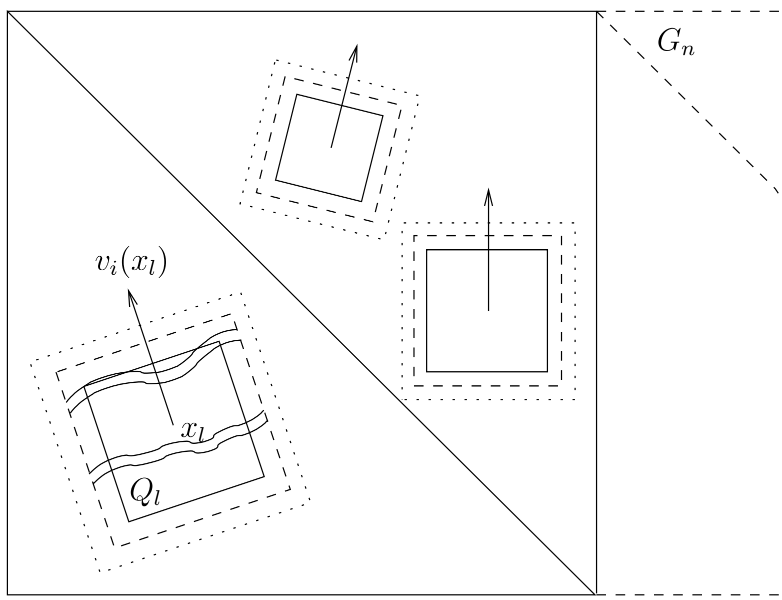

FIGURE 4.3.1

this operation can be done independently on each box. The choice of the total thickness $\overline{d}/16n$ will guarantee that the gradient of the extension remains sufficiently small. Let $\mathscr{A}_l$ be the family of the slabs I in $(Q_l)_{\overline{d}/4}$ and define

$$A_l = \bigcup_{I \in \mathscr{A}_l} I.$$

Lastly, consider a compact set $K_{n,i}$ such that $\mu(\mathbb{R}^d \setminus K_{n,i}) \leq \varepsilon_n$ and $K_{n,i}$ is a subset of

$$\left(\bigcup_{l=1}^{m_n} A_l \right) \cap \left(\bigcup_{l=1}^{m_n} Q_l \right).$$

Remember that the index i individuates one of the k components of $S^{\perp}$. Some ideas for the construction of the sets $K_{n,i}$ are illustrated in Figure 4.3.1.

Let $U_{n,i}$ be the set of all $u \in X$ such that the following property holds.

- For every $x \in K_{n,i}$ there exists r_x with $|r_x| \leq \frac{1}{n}$ such that for every $z \in S^{d-1}$ with

$$\langle v_i(x); z \rangle \geq \sin\left(\frac{2}{n}\right)$$

  there holds

$$u\left(x + z r_x \langle z; v_i(x) \rangle^{-1}\right) - u(x) > \frac{1}{2} r_x.$$

In other words, for every element u of $U_{n,i}$, for every point $x \in K_{n,i}$ there is a "small" scale at which you can see at least slope $\frac{1}{2}$ along the direction $v_i(x)$ and an analogous inequality holds for all the directions z in a "large" cone with axis $v_i(x)$ (notice that the larger is the angle between z and v_i, the smaller is the slope).
Denote by $V_{n,i}$ the set defined analogously to $U_{n,i}$, except that the inequality

$$u\left(x + zr_x\langle z; v_i(x)\rangle^{-1}\right) - u(x) > \frac{1}{2}r_x$$

is replaced by

$$u\left(x + zr_x\langle z; v_i(x)\rangle^{-1}\right) - u(x) < -\frac{1}{2}r_x.$$

Denote

$$U = \bigcap_{i=1}^{d}\bigcap_{j\in\mathbb{N}}\bigcup_{n\geq j} U_{n,i}, \quad V = \bigcap_{i=1}^{d}\bigcap_{j\in\mathbb{N}}\bigcup_{n\geq j} V_{n,i}$$

and

$$A = \bigcap_{n\in\mathbb{N}}\bigcap_{i=1}^{d} K_{n,i}.$$

It is easy to see that every (Lipschitz) function in $U \cap V$ is non-differentiable at μ-a.e $x \in A$, along the directions which are not in $S(x)$. To prove that $U \cap V$ is residual, we need to show that, for every $i = 1, \ldots, n$ and for every $n \in \mathbb{N}$, $U_{n,i}$ and $V_{n,i}$ are open and that $\bigcup_{n\geq j} U_{n,i}$ and $\bigcup_{n\geq j} V_{n,i}$ are dense for every $i = 1, \ldots, d$ and for every j.

We prove that $U_{n,i}$ is open. The proof for $V_{n,i}$ is analogous. Denote

$$r_0 = \min\{r_x : x \in K_{n,i}\}.$$

We have $r_0 > 0$ because $K_{n,i}$ is compact. Denote also

$$\delta_0 = \min_{x\in K_{n,i}}\left\{r_x^{-1}\left|u\left(x + zr_x\langle z; v_i(x)\rangle^{-1}\right) - u(x)\right| : z \in S^{d-1} \text{ with } \langle v_i(x); z\rangle \geq \sin\left(\frac{2}{n}\right)\right\}.$$

We have $\delta_0 > \frac{1}{2}$. It is easy to see that every $f \in X$ satisfying

$$\|f - u\|_\infty < \frac{1}{2}r_0\left(\delta_0 - \frac{1}{2}\right)$$

belongs to $U_{n,i}$.

To prove that $\bigcup_{n\geq j} U_{n,i}$ is dense take $u \in X$ and fix $n \in \mathbb{N}$. Choose a function u_{n_0} given by Lemma 4.3.1 such that $n_0 \geq n$ and $\|u - u_{n_0}\|_\infty \leq 1/n$. Let $Q_1, \ldots, Q_m$ be the boxes containing $K_{n_0,i}$. For every l define on the box $(Q_l)_{\overline{d}/4}$ the function

$$\widehat{u}_l(x) = u_{n_0}(x - g_l(x)) + g_l(x),$$

where

$$g_l(x) = \left(1 - \frac{1}{2n_0}\right) \int_{-\infty}^{0} \chi_{A_l}(x + tu_i(x_l))$$

and A_l is the union of the slabs in $(Q_l)_{\overline{d}/4}$. It is easy to see that

$$\|\widehat{u}_l - u_{n_0} \llcorner (Q_l)_{\overline{d}/4}\| \leq 2\overline{d}/16n_0 = \overline{d}/8n_0.$$

Moreover $\widehat{u}_l$ satisfies

$$\langle \nabla \widehat{u}_l ; v_j \rangle \leq 1 - \frac{1}{n_0}, \quad \mathscr{L}^d\text{−a.e.}$$

for every $j = 1, \dots k$. It is possible to extend $\widehat{u}_l$ to a function $\widetilde{u}_l$ defined on $(Q_l)_{\overline{d}/2}$ in such a way that $\|\widetilde{u}_l - u\|_\infty \leq 2/n$, $\widetilde{u}_l = u_{n_0}$ on the boundary of $(Q_l)_{\overline{d}/2}$ and $\widetilde{u}_l$ satisfies

$$|\langle \nabla \widetilde{u}_l ; v_j \rangle| \leq 1 \;\; \mathscr{L}^d\text{−a.e.} \quad \text{for every } j = 1, \dots, k.$$

The function $\widetilde{u}$ obtained repeating the same procedure for every l, extended to the whole space in such a way that it agrees with u_{n_0} outside of all the enlarged boxes, belongs to $U_{n_0,i}$ and satisfies $\|\widetilde{u} - u\|_\infty \leq 2/n$. □

Part 2

Steiner tree problem revisited through rectifiable G-currents

CHAPTER 5

Rectifiable currents over a coefficient group

Introduction to part 2

The Steiner tree problem is a classical minimization problem in Calculus of Variations: given n distinct points $p_1, \ldots, p_n$ in $\mathbb{R}^d$, find the shortest connected set containing them. Some examples are given in Figure 5.0.1.

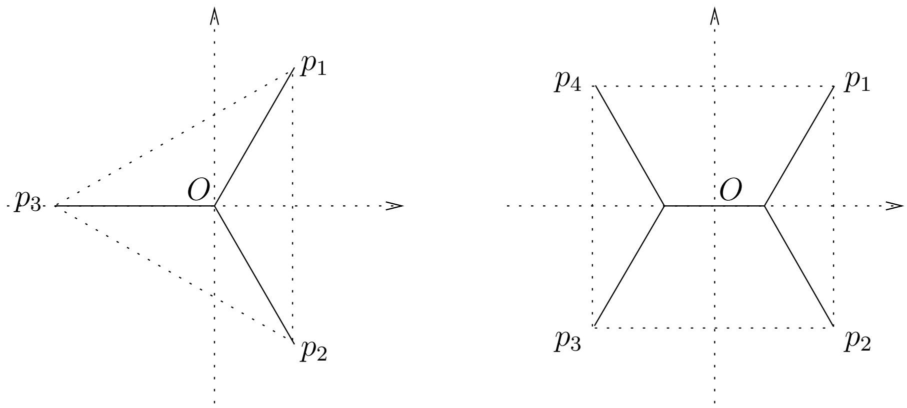

FIGURE 5.0.1. Solutions for the vertices of an equilateral triangle and a square

In $\mathbb{R}^2$ the problem is completely solved and there exists a wide literature on the subject, mainly devoted to improve the efficiency of algorythms for the construction of solutions: see, for instance, [**GP**] and [**IT**] for a survey of the problem. The recent papers [**PS**] and [**PU**] witness the current studies on the problem and its generalizations.

Our aim is to understand Steiner tree problem as a mass minimization problem, suitably replacing connected sets by integral 1-currents. Here the equivalence simply means that it is easily to pass from the solution of one problem to the other and viceversa. In the framework of currents, we are allowed to exploit techniques and tools arising from Calculus of Variations and Geometric Measure Theory. The results of this part of the thesis are contained in [**MM**].

The next examples show that classical polyhedral chains (and integral 1-currents, as well) are not the right environment.[1]

Firstly one should replace the initial data of the Steiner problem with the boundary assigned in the mass minimization problem: the points $p_1, \ldots, p_n$ must be substituted with the integral polyhedral 0-chain supported on $p_1, \ldots, p_n$, with some multiplicities $m_1, \ldots, m_n$. Notice that $m_1 + \ldots + m_n = 0$ is a necessary condition for the 0-chain to be the boundary of a compactly supported 1-chain.

In the example with the vertices of the triangle, see Figure 5.0.1, we have to break the symmetry at last, because $m_3 = -(m_1 + m_2)$, then we get the minimizer in Figure 5.0.2, not even close to the one in Figure 5.0.1.

In the second example, again from Figure 5.0.1, even though all multiplicities in the boundary have modulus 1, we get the "wrong" minimizer: its support is not connected, as we can see in Figure 5.0.2.

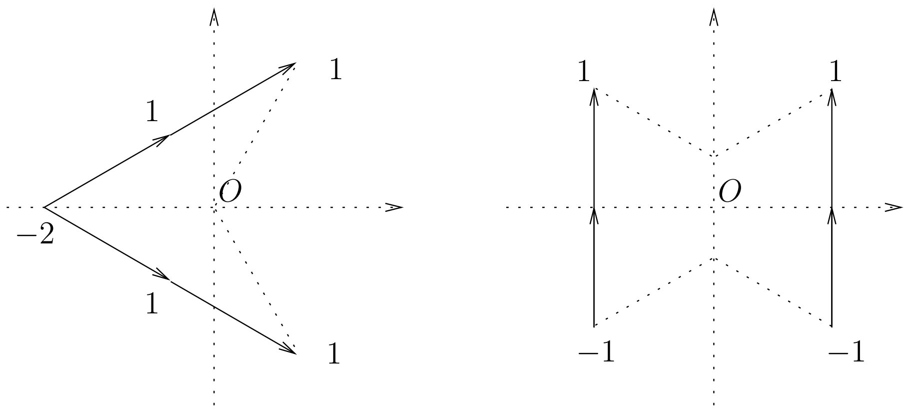

FIGURE 5.0.2. Solutions for the mass minimization problems among polyhedral chains with integer coefficients

These examples show that $\mathbb{Z}$ is not the right group of coefficients.

In Chapter 5 we introduce currents with coefficient in a normed abelian group G. Currents with coefficients in a group were introduced by W. Fleming: there is a very interesting literature starting from the seminal paper [**Fl**], passing through the work of B. White in [**W3**] and [**W2**] and proceeding, more recently, in [**DeHa**] and in [**A**].

In Chapter 6 we recast Steiner problem in terms of mass minimization over currents with coefficients in a discrete group G, chosen only on the basis of the number of points. This construction provides us a method to pass from a mass minimizer to a Steiner solution and viceversa.

[1] For the sake of simlicity, in this introduction we will talk about 1-dimensional chains (with coefficients in Z) or polyhedral integral currents, instead of general integral currents

Once we have established a way to deal with the Steiner tree problem through currents with coefficients in a group, we focus on calibrations as a sufficient condition for the minimality (see Chapter 7).
Classically a calibration ω associated with a given oriented k-submanifold $S \subset \mathbb{R}^d$ is a unit closed k-form taking value 1 on the tangent space of S. The existence of a calibration guarantees the minimality of S among oriented submanifolds with the same boundary ∂S. In fact

$$\mathrm{vol}(S) = \int_S \omega = \int_{S'} \omega \leq \mathrm{vol}(S')$$

for any submanifold S' sharing the same boundary of S, thanks to the assumptions on ω and Stokes Theorem.

In order to define calibrations in the G-currents framework it is convenient to view currents as linear functionals on forms, which is not possible in the usual setting of currents with coefficints in groups. This motivates the preliminary work in Chapter 5, where we embed the group G in a normed linear space E and we construct the currents with coefficients in E in the classical way.
In Definition 7.1.5 the notion of calibration is possibly weakened in order to include piecewise smooth forms, which appear in Examples 7.1.9 and 7.1.10, where we exhibit calibrations for the problem in Figure 5.0.1 and for the Steiner tree problem on the vertices of a regular hexagon plus the center. It is worth to underline here that even though we made explicit computations only on 2-dimensional configurations, the theory works for every dimension. Since the existence of a calibration is a sufficient condition for a manifold to be a minimizer, then it is natural to wonder whether this condition is also necessary or not.
Let us clarify, firstly, that a smooth (or piecewise smooth, like in Definition 7.1.7) calibration cannot always exist, nevertheless, we can still hope for a “weak” calibration, like a differential form with bounded measurable coefficients.
In Section 7.2 we discuss a strategy in order to get the existence of such a weak calibration. Thanks to a duality argument of H. Federer, [**Fe2**], a weak calibration exists for mass-minimizing normal currents and, in our setting, for mass-minimizing normal currents with coefficients in the normed vector space E.
Therefore an equivalence principle between minima among normal and integral 1-currents with coefficients in E and G, respectively, is sufficient to conclude. Theorem 7.2.4 guarantees the equivalence between minima in the case of integral 1-currents, hence the weak calibration always exists. The proof of this result is subject to the validity of the homogeneity property in Remark 7.2.5. Exemple 7.2.6 shows that for 1-dimensional G-currents an interesting new phenomenon appears: in fact, at least in a non-euclidean setting, the homogeneity property does not hold. It seems that in this case the problem of the equivalence of minima could depend on some property of the ambient space. The problem of the existence of a calibration in the Euclidean space is still open.

In the next sections of this chapter we provide definitions for currents over a coefficient group, with some basic examples.

5.1. E^*-valued differential forms

Fix an open set $U \subset \mathbb{R}^d$ and a normed vector space $(E, \|\cdot\|_E)$ with finite dimension $m \geq 1$. We will denote by $(E^*, \|\cdot\|_{E^*})$ its dual space endowed with the dual norm

$$\|f\|_{E^*} := \sup_{\|v\|_E \leq 1} \langle f; v\rangle \ .$$

Definition 5.1.1. We say that a map

$$\omega : \Lambda_k(\mathbb{R}^d) \times E \to \mathbb{R}$$

is a E^*-valued k-covector in $\mathbb{R}^d$ if

(i) $\forall\, \tau \in \Lambda_k(\mathbb{R}^d), \quad \omega(\tau, \cdot) \in E^*$, that is $\omega(\tau, \cdot) : E \to \mathbb{R}$ is a linear function.
(ii) $\forall\, v \in E, \quad \omega(\cdot, v) : \Lambda_k(\mathbb{R}^d) \to \mathbb{R}$ is a (classical) k-covector.

Sometimes we will use $\langle \omega; \tau, v\rangle$ instead of $\omega(\tau, v)$, in order to simplify the notation.
The space of E^*-valued k-covectors in $\mathbb{R}^d$ is denoted by $\Lambda^k_E(\mathbb{R}^d)$ and it is endowed with the comass norm

$$\|\omega\| := \sup\left\{\|\omega(\tau, \cdot)\|_{E^*} \ : \ |\tau| \leq 1, \tau \text{ simple}\right\} \ . \tag{5.1.1}$$

Remark 5.1.2. Fix an orthonormal system of coordinates in $\mathbb{R}^d$, $(\mathbf{e}_1, \ldots, \mathbf{e}_d)$; the corresponding dual base in $(\mathbb{R}^d)^*$ is $(dx_1, \ldots, dx_d)$. Consider a complete biorthonormal system, i.e. a pair

$$(v_1, \ldots, v_m) \in E^m; (w_1, \ldots, w_m) \subset (E^*)^m$$

such that $\|v_i\|_E = 1$, $\|w_i\|_{E^*} = 1$ and $\langle w_i; v_j\rangle = \delta_{ij}$. Given an E^*-valued k-covector ω, we denote

$$\omega^j := \omega(\cdot, v_j).$$

For each $j \in \{1, \ldots, m\}$, ω^j is a k-covector in the usual sense. Hence the biorthonormal system $(v_1, \ldots, v_m)$, $(w_1, \ldots, w_m)$ allows to write ω in "components" $\omega = (\omega^1, \ldots, \omega^m)$, in fact we have

$$\omega(\tau, v) = \sum_{j=1}^{m} \langle \omega^j; \tau\rangle \langle w_j; v\rangle \ .$$

In particular ω^j admits the usual representation

$$\omega^j = \sum_{1 \leq i_1 < \ldots < i_k \leq d} a^j_{i_1 \ldots i_k} dx_{i_1} \wedge \ldots \wedge dx_{i_k} \qquad j = 1, \ldots, m.$$

Definition 5.1.3. An E^*-valued differential k-form in $U \subset \mathbb{R}^d$, or just a k-form when it is clear which group we are referring to, is a map

$$\omega : U \to \Lambda^k_E(\mathbb{R}^d);$$

we say that ω is $\mathscr{C}^\infty$-regular if every component ω^j is so (see Remark 5.1.2). We denote by $\mathscr{C}_c^\infty(U, \Lambda_E^k(\mathbb{R}^d))$ the vector space of $\mathscr{C}^\infty$-regular E^*-valued k-forms with compact support in U.

We are mainly interested in E^*-valued 1-forms, nevertheless we analyze k-forms in wider generality, in order to ease other definitions, such as the differential of an E^*-valued form and the boundary of an E-current.

Definition 5.1.4. We define the differential $\mathrm{d}\omega$ of a $\mathscr{C}^\infty$ regular E^*-valued k-form ω by components:

$$\mathrm{d}\omega^j = \mathrm{d}(\omega^j) : U \times T_{k+1}(\mathbb{R}^d) \to \mathbb{R} \qquad j = 1, \dots, m \ ,$$

Moreover, $\mathscr{C}_c^\infty(U, \Lambda_E^1(\mathbb{R}^d))$ has a norm, denoted by $\|\cdot\|$, given by the supremum of the comass norm of the form defined in (5.1.1). Hence we mean

$$\|\omega\| := \sup_{x \in U} \|\omega(x)\| \ . \tag{5.1.2}$$

5.2. E-currents

Definition 5.2.1. A k-dimensional current T in $U \subset \mathbb{R}^d$, with coefficients in E, or just an E-current when there is no doubt on the dimension, is a linear and continuous function

$$T : \mathscr{C}_c^\infty(U, \Lambda_E^k(\mathbb{R}^d)) \longrightarrow \mathbb{R} \ ,$$

where the continuity is meant with respect to the locally convex topology on the space of E^*-valued k-forms with compact support in U, built on the framework of the topology on $\mathscr{C}_c^\infty(\mathbb{R}^n)$, with respect to which distributions are dual. This defines the weak* topology on the space of k-dimensional E-currents. Convergence in this topology is equivalent to the convergence of all the "components" in the space of classical k-currents, by which we mean the following. We define for every k-dimensional E-current T their components T^j, for $j = 1, \dots m$, and we will write

$$T = (T^1, \dots, T^m),$$

denoting

$$\langle T^j; \varphi \rangle := \langle T; \widetilde{\varphi}_j \rangle \ ,$$

for every (classical) compactly supported differential k-form φ on $\mathbb{R}^d$. Here $\widetilde{\varphi}_j$ denotes the E^*-valued differential k-form on $\mathbb{R}^d$ such that

$$\widetilde{\varphi}_j(\cdot, v_j) = \varphi, \tag{5.2.1}$$

$$\widetilde{\varphi}_j(\cdot, v_i) = 0 \quad \text{for } i \neq j \ . \tag{5.2.2}$$

It turns out that a sequence of k-dimensional E-currents T_h weakly* converges to an E-current T (in which case we write $T_h \overset{*}{\rightharpoonup} T$) if and only if the sequence of the components T_h^j converge to T^j in the space of classical k-currents, for $j = 1, \dots, m$.

Definition 5.2.2. For a k-current T over E we define the boundary operator

$$\langle \partial T; \varphi\rangle := \langle T; \mathrm{d}\varphi\rangle \quad \forall\, \varphi = (\varphi^1, \ldots, \varphi^m) \in \mathscr{C}_c^\infty(U, \Lambda_E^{k-1}(\mathbb{R}^d))$$

and the mass

$$\mathbb{M}(T) := \sup_{\|\omega\|\le 1} \langle T; \omega\rangle.$$

As one can expect, the boundary $\partial(T^j)$ of every component T^j is the relative component $(\partial T)^j$of the boundary ∂T.

Definition 5.2.3. A k-dimensional normal E-current in $U \subset \mathbb{R}^d$ is an E-current T with $\mathbb{M}(T) < +\infty$ and $\mathbb{M}(\partial T) < +\infty$. Thanks to the Riesz Theorem, T admits the following representation:

$$\langle T; \omega\rangle = \int_U \langle \omega(x); \tau(x), v(x)\rangle \,\mathrm{d}\mu_T \ , \quad \forall\, \omega \in \mathscr{C}_c^\infty(U, \Lambda_E^{k-1}(\mathbb{R}^d)) \ .$$

where μ_T is a Radon measure on U and $v : U \to E$ is summable with respect to μ_T and $|\tau| = 1$, μ_T-a.e. An analogous representation holds for the boundary ∂T.

Definition 5.2.4. A rectifiable k-current T in $U \subset \mathbb{R}^d$, over E, or a rectifiable E-current is an E-current admitting the following representation:

$$\langle T; \omega\rangle := \int_\Sigma \langle \omega(x); \tau(x), \theta(x)\rangle \,\mathrm{d}\mathscr{H}^k(x), \quad \forall\, \omega \in \mathscr{C}^\infty(\mathbb{R}^d, \Lambda_E^k(\mathbb{R}^d))$$

where Σ is an $\mathscr{H}^k$-rectifiable set contained in U, $\tau(x) \in T_x\Sigma$ with $|\tau(x)| = 1$ for $\mathscr{H}^k$-a.e. x and $\theta \in L^1(U; E)$. We will refer to such a current as $T = T(\Sigma, \tau, \theta)$.
If B is a Borel set and $T(\Sigma, \tau, \theta)$ is a rectifiable E-current, we denote by $T \llcorner B$ the current $T(\Sigma \cap B, \tau, \theta)$.

Consider now a discrete subgroup $G < E$, endowed with the restriction of the norm $\|\cdot\|_E$. If the multiplicity θ takes only values in G, and if the same representation holds for ∂T, we call T a rectifiable G-current.
Pay attention to the fact that, in the framework of currents over the coefficient group E, rectifiable E-currents play the role of (classical) rectifiable current, while rectifiable G-currents correspond to (classical) integral currents. Actually this correspondence is an equality, when E is the group $\mathbb{R}$ (with the euclidean norm) and G is $\mathbb{Z}$.

Example 5.2.5. Let $E = \mathbb{R}^d$ and let G be the additive subgroup generated by m elements $g_1, \ldots, g_m$.
Given $m+1$ points $p_1, \ldots, p_m, p_{m+1} \in \mathbb{R}^2$, consider the cone C over $(p_1, \ldots, p_m)$ with respect to p_{m+1}: if Σ_r is the oriented segment from p_{m+1} to p_r, $r = 1, \ldots, m$, then

$$C = \bigcup_{r=1}^m \Sigma_r \ .$$

We can define a rectifiable G-current supported on C as

$$\langle T; \omega \rangle := \sum_{r=1}^{m} \int_{\Sigma_r} \langle \omega(x); \tau_r(x), g_r \rangle \, \mathrm{d}\mathscr{H}^1(x),$$

where τ_r is the unit tangent vector to Σ_r, pointing towards p_r.
It is easy to see that, denoting $g_{m+1} = -(g_1 + \ldots + g_m)$ we can represent the 0-dimensional rectifiable G-current ∂T with the points $p_1, \ldots, p_{m+1}$ with multiplicities $g_1, \ldots, g_{m+1}$, respectively. From now on we will denote such a current as $g_1\delta_{p_1} + \ldots + g_{m+1}\delta_{p_{m+1}}$.

PROPOSITION 5.2.6. Let $T = T(\Sigma, \tau, \theta)$ be a rectifiable E-current, then

$$\mathbb{M}(T) = \int_{\Sigma} \|\theta(x)\|_G \, \mathrm{d}\mathscr{H}^1(x) \; .$$

Since the mass is lower semicontinuous, we can apply the direct method of Calculus of Variations for the existence of minimizers with given boundary, once we provide the following compactness result. Here we assume for simplicity that G is the subgroup of E generated by $v_1, \ldots, v_m$. A similar argument works for every discrete subgroup G.

THEOREM 5.2.7. [**MM**] *Let $(T_h)_{h\geq 1}$ be a sequence of rectifiable G-currents such that there exists a positive finite constant C satisfying*

$$\mathbb{M}(T_h) + \mathbb{M}(\partial T_h) \leq C \quad \text{for every } h \geq 1 \; .$$

Then there exists a subsequence $(T_{h_i})_{i\geq 1}$ and a rectifiable G-current T such that

$$T_{h_i} \stackrel{*}{\rightharpoonup} T.$$

PROOF. The statement of the theorem can be proved component by component. In fact, let $T_h^1, \ldots, T_h^m$ be the components of T_h. Since $(v_1, \ldots, v_m), (w_1, \ldots, w_m)$ is a biorthonormal sistem, we have

$$\mathbb{M}(T_h^j) + \mathbb{M}(\partial T_h^j) \leq \mathbb{M}(T_h) + \mathbb{M}(\partial T_h) \leq C \; ,$$

hence, (since we are dealing with only finitely many components) up to subsequences, $\big(T_h^j\big)_{h\geq 1}$ weakly* converges to some integral current T^j for every $j = 1, \ldots, m$. Then, denoting by T the rectifiable G-current, whose components are $T^1, \ldots, T^m$, there exists a subsequence $(T_{h_i})_{i\geq 1}$ such that

$$T_{h_i} \stackrel{*}{\rightharpoonup} T.$$

□

CHAPTER 6

Steiner tree Problem revisited

In this chapter we establish the equivalence between the Steiner tree problem and a mass minimization problem in a family of G-currents. We first need to choose the right group of coefficients G. Once we fix the n points in the Steiner problem, we look for a subgroup $(G, \|\cdot\|_G)$, of a normed vector space $(E, \|\cdot\|_E)$, (where $\|\cdot\|_G$ is the restriction to G of the norm $\|\cdot\|_E$) satisfying the following properties:

(P1) there exist $g_1, \dots, g_{n-1} \in G$ and $h_1, \dots, h_{n-1} \in E^*$ such that $(g_1, \dots, g_{n-1})$, $(h_1, \dots, h_{n-1})$ is a complete biorthonormal system for E, and G is additively generated by $g_1, \dots, g_{n-1}$;

(P2) $\|g_{i_1} + \ldots + g_{i_k}\|_G = 1$ whenever $1 \leq i_1 < \ldots < i_k \leq n-1$;

(P3) $\|g\|_G \geq 1$ for every $g \in G \setminus \{0\}$.

For the moment we will assume the existence of G and E. The proof of their existence and an explicit representation, useful for the computations, will be given later in this chapter.

The next lemma has a fundamental role: through it, we can give a nice structure of 1-dimensional rectifiable G-current to every suitable competitor for the Steiner tree problem. From now on we will denote $g_n = -(g_1 + \ldots + g_{n-1})$.

LEMMA 6.0.8. [**MM**] Let B be a connected 1-rectifiable set with finite length in $\mathbb{R}^d$, containing $p_1, \dots, p_n$. Then there exists a connected set $B' \subset B$ containing $p_1, \dots, p_n$ and a 1-dimensional rectifiable G-current $T_{B'} = T(B', \tau, \theta)$, such that

(i) $\|\theta(x)\|_E = 1$ for a.e. $x \in B'$,

(ii) $\partial T_{B'}$ is the 0-dimensional G-current $g_1\delta_{p_1} + \ldots + g_n\delta_{p_n}$

Proof. Since B is a connected set of finite length, B is connected by paths of finite length (see Lemma 3.12 of [**Fa**]).

Consider a path B_1 contained in B going from p_n to p_1. In analogy with Example 5.2.5, associate it with a current T_1 with constant multiplicity g_1 and orientation going from p_n to p_1.

Repeat this procedure keeping the starting point p_n and replacing at each step p_1 with $p_2, \dots, p_{n-1}$.

The set $B' = B_1 \cup \ldots \cup B_{n-1} \subset B$ is a connected set containing $p_1, \dots, p_n$ and the 1-dimensional rectifiable G-current $T = T_1 + \ldots + T_{n-1}$ satisfies the requirements of the lemma, in particular condition (i) comes from (P2). $\square$

Via the next lemma, we can say that mass minimizers for our problem have connected support.

LEMMA 6.0.9. [**MM**] Let T be a 1-dimensional rectifiable G-current, such that ∂T is the 0-current $g_1\delta_{p_1} + \ldots + g_n\delta_{p_n}$. Then there exists a rectifiable G-current $\widetilde{T} = T(\widetilde{\Sigma}, \widetilde{\tau}, \widetilde{\theta})$ such that

(i) $\partial\widetilde{T} = \partial T = g_1\delta_{p_1} + \ldots + g_n\delta_{p_n}$;
(ii) $\mathbb{M}(\widetilde{T}) \leq \mathbb{M}(T)$ and the equality holds only if $\widetilde{T} = T$;
(iii) The support of $\widetilde{T}$ is a connected 1-rectifiable set containing $\{p_1, \ldots, p_n\}$ and it is contained in the support of T;
(iv) $\mathscr{H}^1(\mathrm{supp}(\widetilde{T}) \setminus \widetilde{\Sigma}) = 0$.

Proof. Let $T^j = T(\Sigma^j, \tau^j, \theta^j)$ be the components of T, for $j = 1, \ldots, n-1$ (with respect to the biorthonormal system $(g_1, \ldots, g_{n-1})$, $(h_1, \ldots, h_{n-1})$)

For every j, we can use Proposition 1.3.16 and write

$$T^j = \sum_{k=1}^{K_j} T_k^j + \sum_{\ell \geq 1} C_\ell^j \ ,$$

where T_k^j and C_ℓ^j are integral 1-currents associated with Lipschitz curves, with $\partial C_\ell^j = 0$ for every $\ell \geq 1$. Notice that, for every $j = 1, \ldots, n-1$, if θ_k^j denotes the multiplicity of T_k^j, then we have

$$\sum_{k=1}^{K_j} \left|\theta_k^j\right| \leq |\theta^j| \quad \mathscr{H}^1\text{−a.e. on } \mathrm{supp}(T^j). \tag{6.0.3}$$

This is because in the decomposition of Proposition 1.3.16 there is no loss of mass, (i.e.

$$\mathbb{M}(T^j) = \sum_{k=1}^{K_j} \mathbb{M}(T_k^j) + \sum_{\ell \geq 1} \mathbb{M}(C_\ell^j),$$

for every j).

We choose $\widetilde{T}$ the rectifiable G-current whose components are

$$\widetilde{T}^j := \sum_{k=1}^{K_j} T_k^j \ .$$

Again, because of the conservation of the mass in the decomposition of Proposition 1.3.16, we have $\mathrm{supp}(\widetilde{T}) \subset \mathrm{supp}(T)$ (the cyclic part of T^j never cancels the acyclic one). Property (i) is easy to check. Property (ii) is a consequence of (6.0.3) and of the following property of the norm $\|\cdot\|_G$. If $\theta = \sum_{j=1}^{n-1} \theta^j g_j$ and $\widetilde{\theta} = \sum_{j=1}^{n-1} \widetilde{\theta}^j g_j$ with $0 \leq \widetilde{\theta}^j \leq \theta^j$ if $\theta^j \geq 0$ and $0 \geq \widetilde{\theta}^j \geq \theta^j$ if $\theta^j \leq 0$, then $\|\widetilde{\theta}\|_G \leq \|\theta\|_G$ (this property follows from the fact that $(g_1, \ldots, g_{n-1})$, $(h_1, \ldots, h_{n-1})$ is a complete biorthonormal system for E). Property (iv) is

also easy to check, because the corresponding property holds for every T^j_k and therefore for every component $\widetilde{T}^j$. It remains to prove property (iii). By construction $\widetilde{T}$ is a finite sum of oriented curves with multiplicities; since we are considering curves with ending points (closed sets), $\mathrm{supp}(\widetilde{T})$ has a finite number of (closed) connected components far apart: consider S a connected component of $\mathrm{supp}(\widetilde{T})$ and the related restriction $\widetilde{T} \llcorner S$. We notice that S has positive distance from any other connected component of $\mathrm{supp}(\widetilde{T})$. Assume that S contains a non-empty subset of $\{p_1, \ldots, p_n\}$, let us relabel the points such that $S \supset \{p_1, \ldots, p_{\widetilde{n}}\}$, with $1 \leq \widetilde{n} \leq n$, and $p_j \notin S$ if $j > \widetilde{n}$. Thus $\partial(\widetilde{T} \llcorner S)$ is the 0-current associated with $p_1, \ldots, p_{\widetilde{n}}$ with multiplicities $g_1, \ldots, g_{\widetilde{n}}$.
Assume by contradiction that $\widetilde{n} < n$. Then we can choose an element $w \in E^*$ such that $w(g_j) = 1$ for $j = 1, \ldots, \widetilde{n}$ and take $\varphi \in \mathscr{C}^\infty_c(\mathbb{R}^d, \Lambda^1_E(\mathbb{R}^d))$ a smooth E^*-valued 1-form such that

$$\begin{aligned} \varphi &\equiv w && \text{on } S \\ \varphi &\equiv 0 && \text{on } \mathrm{supp}(\widetilde{T}) \setminus S \ . \end{aligned}$$

Then $0 = \widetilde{T} \llcorner S(d\varphi) = \partial(\widetilde{T} \llcorner S)(\varphi) = \widetilde{n}$, which is clearly a contradiction.
Therefore there is no boundary for the restriction of $\widetilde{T}$ to every connected component of its support, but one. Possibly replacing $\widetilde{T}$ by its restriction to this non-trivial connected component, we get the thesis. $\square$

Before stating the main theorem, let us point out that the existence of a solution to the mass minimization problem is a consequence of the direct method of Calculus of Variations.

THEOREM 6.0.10. [**MM**] *Assume that $T_0 = T(\Sigma_0, \tau_0, \theta_0)$ is a mass-minimizer among rectifiable 1-dimensional G-currents with boundary*

$$\delta_0 = g_1 \delta_{p_1} + \ldots + g_n \delta_{p_n} .$$

Then $S_0 = \mathrm{supp}(T_0)$ is a solution of the Steiner tree problem. Conversely, given a set C which is a solution of the Steiner problem for the points $p_1, \ldots, p_n$, there exists a canonical 1-dimensional G-current, supported on C, minimizing the mass among the currents with boundary δ_0.

Proof. The existence of T_0 is a direct consequence of Theorem 5.2.7. Moreover, since T_0 is a mass minimizer, then it must coincide with the current $\widetilde{T_0}$ given by Lemma 6.0.9. In particular, Lemma 6.0.9 guarantees that S_0 is a connected set. Let S be a competitor for the Steiner tree problem and let S' and $T_{S'}$ be the connected set and the rectifiable 1-current given by Lemma 6.0.8, respectively.
Hence we have

$$\mathscr{H}^1(S) \geq \mathscr{H}^1(S') \stackrel{(i)}{=} \mathbb{M}(T_{S'}) \stackrel{(ii)}{\geq} \mathbb{M}(T_0) \stackrel{(iii)}{\geq} \mathscr{H}^1(\Sigma_0) \stackrel{(iv)}{=} \mathscr{H}^1(S_0) \ ,$$

in fact

(i) thanks to the second property of Lemma 6.0.8 and Proposition 5.2.6, we obtain

$$\mathbb{M}(T_{S'}) = \int_{S'} \|\theta_{S'}(x)\|_G \,\mathrm{d}\mathscr{H}^1(x) = \mathscr{H}^1(S') \ ;$$

(ii) we assumed that T_0 is a mass-minimizer;

(iii) from property (P3), we get

$$\mathbb{M}(T_0) = \int_{\Sigma_0} \|\theta_0(x)\|_G \,\mathrm{d}\mathscr{H}^1(x) \geq \int_{\Sigma_0} 1 \,\mathrm{d}\mathscr{H}^1(x) = \mathscr{H}^1(\Sigma_0) \ .$$

(iv) is property (iv) in Lemma 6.0.9.

To prove the second part of the Theorem, apply Lemma 6.0.8 to the set C. Notice that with the procedure described in the lemma, the rectifiable G-current $T_{C'}$ is uniquely determined, because for every point p_i, C contains exactly one path from p_n to p_i, in fact it is well known that solutions of the Steiner tree problem cannot contain cycles. By Lemma 6.0.9 $T_{C'}$ is a solution of the mass minimization problem. □

Eventually, we give an explicit representation for G and E. Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the standard basis of $\mathbb{R}^n$; we consider on $\mathbb{R}^n$ the seminorm

$$\|u\|_\star := \max_{i=1,\ldots,n} u \cdot \mathbf{e}_i - \min_{i=1,\ldots,n} u \cdot \mathbf{e}_i \ .$$

We now take the quotient

$$E := \frac{\mathbb{R}^n}{\mathrm{Span}\{\mathbf{e}_1 + \ldots + \mathbf{e}_n\}} \ ,$$

denoting with π the standard projection from $\mathbb{R}^n$ to E. According to the relation in the quotient, we get $[(u_1, \ldots, u_n)] = [(u_1 + c, \ldots, u_n + c)]$, for every $c \in \mathbb{R}$ and for every $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$ (here $[u]$ denotes the element of the quotient associated with the vector $u \in \mathbb{R}^n$).
Since $\|u\|_* = \|u + v\|_*$ for every $u \in \mathbb{R}^n, v \in \mathrm{Span}\{\mathbf{e}_1 + \ldots + \mathbf{e}_n\}$, then it is well defined the corresponding seminorm $\|\cdot\|_E$ induced on E and it is actually a norm satisfying

$$\|v\|_E := \inf_{\pi(u)=v} \|u\|_\star = \|u\|_\star \quad \text{for any } u \in \pi^{-1}(v) \ .$$

For the sake of completeness, we remark that, with this notation, the dual space E^* can be represented as $E^* = \{(z_1, \ldots, z_n) \in \mathbb{R}^n \,:\, \sum_{i=1}^n z_i = 0\}$ and its dual norm $\|\cdot\|_{E^*}$ coincides with $\frac{1}{2}\|\cdot\|_1$. In fact, for every $[u] \in E$ with $\|[u]\|_E = 1$ we can choose a representative u, such that $|u_i| \leq \frac{1}{2}$, $i = 1, \ldots, n$ and then

$$\|z\|_{E^*} = \sup_{\|[u]\|_E = 1} \sum_{i=1}^n z_i u_i = \frac{1}{2} \sum_{i=1}^n |z_i| \ .$$

The choice of E as a quotient is motivated by the idea that the sum of the coefficients $\mathbf{e}_i$ must be zero, for boundary reasons. Anyway, we find that a slightly different representation of E, would ease computations later and we would rather introduce G with this new representation.
Consider

$$F := \{v \in \mathbb{R}^n \,:\, v \cdot \mathbf{e}_n = 0\} \subset \mathbb{R}^n$$

and the omomorphism $\phi : \mathbb{R}^n \to F$ such that

$$\phi(u_1, \dots, u_n) := (u_1 - u_n, \dots, u_{n-1} - u_n, 0) \;; \tag{6.0.4}$$

the seminorm $\|\cdot\|_\star$ is a norm on F.

The omomorphism ϕ in (6.0.4) induces an isometrical isomorphism $\widetilde{\phi} : E \to F$ defined by the relation $\widetilde{\phi} \circ \pi = \phi$: in fact, if $v \in E$ and $u \in \pi^{-1}(v)$, then $\|v\|_E = \|u\|_\star = \|\phi(u)\|_\star = \|\widetilde{\phi}(v)\|_\star$.
For every $i = 1, \dots, n-1$, define $g_i = \widetilde{\phi}^{-1}(\mathbf{e}_i)$ and define $g_n = -(g_1 + \ldots + g_{n-1})$. Let G be the subgroup of E generated by $g_1, \dots, g_{n-1}$.
For every $i = 1, \dots, n-1$ denote by h_i the element of E^* satisfying $h_i(g_j) = \delta_{ij}$: $(g_1, \dots, g_{n-1})$, $(h_1, \dots, h_{n-1})$ is a biorthonormal system.
With these coordinates, an element $v \in E$ has unit norm $\|v\|_E = 1$ if and only if

$$\|v\|_E = \|\widetilde{\phi}(v)\|_\star = \max_{i=1,\dots,n-1}(v_i \vee 0) - \min_{i=1,\dots,n-1}(v_i \wedge 0) = 1 \;. \tag{6.0.5}$$

The norm $\|\cdot\|_{E^*}$ of an element $w = w_1 h_1 + \ldots w_{n-1} h_{n-1} \in E^*$ can be characterized in the following way: let us abbreviate $w^P := \sum_{i=1}^{n-1}(w_i \vee 0)$ and $w^N := -\sum_{i=1}^{n-1}(w_i \wedge 0)$ and $\lambda(v) = \max_{i=1,\dots,n-1}(v_i \vee 0) \in [0,1]$, then

$$\begin{aligned} \|w\|_{E^*} &= \sup_{\|v\|_E = 1} \sum_{i=1}^{n-1} w_i v_i = \sup_{\|v\|_E=1} [\lambda(v) w^P + (1 - \lambda(v)) w^N] \\ &= \sup_{\lambda \in [0,1]} [(\lambda w^P + (1-\lambda) w^N] = w^P \vee w^N \;. \end{aligned} \tag{6.0.6}$$

Notice that, recalling the notation of Chapter 5, $m = n - 1$. Properties (P1), (P2) and (P3) are easy to check. In the sequel, we will fix both the normed space E and the group G, where n is the number of points in the corresponding Steiner tree problem that we want to solve.

Remark 6.0.11. We already know that the elements $g_1, \dots, g_n$ are the multiplicities of the n points in the boundary, for the Steiner tree problem. The definition we just gave does not seem to be "symmetric", in fact g_n has, in a certain sense, a privileged role, while the n points in the Steiner tree problem have of course all the same importance. To restore this lost symmetry, one may note that the group E is represented in $\mathbb{R}^n$ as the hyperplane $P := \{x_1 + \ldots + x_n = 0\}$ with a norm which is a multiple of the norm induced on P by the norm $\|\cdot\|_\star$ of $\mathbb{R}^n$. Here $g_1, \dots, g_n$ are the orthogonal projections on P of $\mathbf{e}_1, \dots, \mathbf{e}_{n-1}$ and $-(\mathbf{e}_1 + \ldots + \mathbf{e}_{n-1})$ respectively. It is easy to see that these points of π are the vertices of an $(n-1)$-dimensional regular tetrahedron. In particular the unit elements of G are the vertices of a convex $(n-1)$-dimensional polyhedron which is symmetric with respect to the origin. The vertices of the polyhedron are all the points of the form $g_{i_1} + \ldots + g_{i_k}$ with $1 \leq i_1 < \ldots < i_k \leq n-1$ and their inverses. It is clear that in this representation the role of the p_i's is perfectly symmetric.

CHAPTER 7

Calibrations

7.1. Definitions and examples

As we recalled in the Introduction, our interest in calibrations is the reason why we have chosen to provide an integral representation for E-currents, in fact the existence of a calibration guarantees the minimality of the associated current, as we will see in Proposition 7.1.2.

Definition 7.1.1. A smooth calibration associated with a k-dimensional rectifiable G-current $T(\Sigma, \tau, \theta)$ is a smooth compactly supported E^*-valued differential k-form ω, with the following properties:

(i) $\langle \omega(x); \tau(x), \theta(x)\rangle = \|\theta(x)\|_G$ for $\mathscr{H}^k$-a.e. $x \in \Sigma$;
(ii) $\mathrm{d}\omega = 0$;
(iii) $\|\omega\| \leq 1$, i.e. $\|\langle \omega; \tau\rangle\|_{E^*} \leq 1$, for every simple k-vector τ with $|\tau| = 1$.

Proposition 7.1.2. [**MM**] A rectifiable G-current T which admits a smooth calibration ω is a minimizer for the mass among the normal E-currents with boundary ∂T.

Proof. Fix a competitor T' which is a normal E-current associated with the vector-field τ', the multiplicity θ' and the measure $\mu_{T'}$, with $\partial T' = \partial T$. Since $\partial(T - T') = 0$, then $T - T'$ is a boundary of some current S in $\mathbb{R}^d$, and then

$$\mathbb{M}(T) = \int_\Sigma \|\theta\|_G \,\mathrm{d}\mathscr{H}^k \tag{7.1.1}$$

$$\overset{\text{(i)}}{=} \int_\Sigma \langle \omega(x); \tau(x), \theta(x)\rangle \,\mathrm{d}\mathscr{H}^k = \langle T; \omega\rangle \tag{7.1.2}$$

$$\overset{\text{(ii)}}{=} \langle T'; \omega\rangle = \int_{\mathbb{R}^d} \langle \omega(x); \tau'(x), \theta'(x)\rangle \,\mathrm{d}\mu_{T'} \tag{7.1.3}$$

$$\overset{\text{(iii)}}{\leq} \int_{\mathbb{R}^d} \|\theta'\|_G \,\mathrm{d}\mu T' = \mathbb{M}(T')\ , \tag{7.1.4}$$

where each equality (respectively inequality) holds because of the corresponding property of ω, as established in Definition 7.1.1. In particular, equality in (ii) follows from

$$\langle T - T'; \omega\rangle = \langle \partial S; \omega\rangle = \langle S; \mathrm{d}\omega\rangle = 0.$$

□

REMARK 7.1.3. If T is a rectifiable G-current calibrated by ω, then every mass minimizer with boundary ∂T is calibrated by the same form ω.
In fact, choose a mass minimizer $T' = T(\Sigma', \tau', \theta')$ with boundary $\partial T' = \partial T$: obviously we have $\mathbb{M}(T) = \mathbb{M}(T')$, then equality holds in (7.1.4), which means

$$\langle \omega(x); \tau'(x), \theta'(x) \rangle = \|\theta'(x)\|_G \quad \text{for } \mathscr{H}^k - \text{a.e. } x \in \Sigma' \ .$$

At this point we need a short digression on the representation of a E^*-valued 1-form ω; we will consider $d = 2$, all our examples being for the Steiner tree problem in $\mathbb{R}^2$. Remember that in Chapter 6 we fixed a basis $(h_1, \ldots, h_{n-1})$ for E^*, dual to the basis $(g_1, \ldots, g_{n-1})$ for E. We will represent

$$\omega = \begin{pmatrix} \omega_{1,1}\,\mathrm{d}x_1 + \omega_{1,2}\,\mathrm{d}x_2 \\ \vdots \\ \omega_{n-1,1}\,\mathrm{d}x_1 + \omega_{n-1,2}\,\mathrm{d}x_2 \end{pmatrix} ,$$

so that, if $\tau = \tau_1 \mathbf{e}_1 + \tau_2 \mathbf{e}_2 \in T_1(\mathbb{R}^2)$ and $v = v_1 g_1 + \ldots + v_{n-1} g_{n-1} \in E$, then

$$\langle \omega; \tau, v \rangle = \sum_{i=1}^{n-1} v_i (\omega_{i,1} \tau_1 + \omega_{i,2} \tau_2) \ .$$

EXAMPLE 7.1.4. Consider the vector space E and the group G defined in Chapter 6 with $n = 3$; let

$$p_0 = (0,0), p_1 = (1/2, \sqrt{3}/2), p_2 = (1/2, -\sqrt{3}/2), p_3 = (-1, 0)$$

(see Figure 5.0.1). Consider the rectifiable G-current T supported in the cone over (p_1, p_2, p_3), with respect to p_0, with piecewise constant weights $g_1, g_2, g_3 = -(g_1 + g_2)$ on $\Sigma_1, \Sigma_2, \Sigma_3$ respectively (recall Example 5.2.5 for notation and orientation). This current T is a minimizer for the mass. In fact, a constant G-calibration ω associated with T can be represented as

$$\omega := \begin{pmatrix} \frac{1}{2}\,\mathrm{d}x_1 + \frac{\sqrt{3}}{2}\,\mathrm{d}x_2 \\ \frac{1}{2}\,\mathrm{d}x_1 - \frac{\sqrt{3}}{2}\,\mathrm{d}x_2 \end{pmatrix} .$$

Condition (i) is easy to check and condition (ii) is trivially verified because ω is constant. To check condition (iii) we note that, for the generical vector $\tau = \cos\alpha\,\mathbf{e}_1 + \sin\alpha\,\mathbf{e}_2$, we have

$$\langle \omega; \tau, \cdot \rangle = \begin{pmatrix} \frac{1}{2}\cos\alpha + \frac{\sqrt{3}}{2}\sin\alpha \\ \frac{1}{2}\cos\alpha - \frac{\sqrt{3}}{2}\sin\alpha \end{pmatrix} .$$

In order to calculate the comass norm of ω, we could stick to the method explained in Chapter 6, but for $n = 3$ computations are simpler. Since the unit ball of E is convex, and its extreme points are the unit points of G, then it is sufficient to evaluate $\langle \omega; \tau, \cdot \rangle$ on

$\pm g_1, \pm g_2, \pm(g_1+g_2)$(remember that $\|g_1-g_2\|_E = 2$). We have

$$|\langle\omega;\tau,g_1\rangle| = |\langle\omega;\tau,-g_1\rangle| = \left|\sin\left(\alpha+\frac{\pi}{6}\right)\right| \leq 1 \ ,$$
$$|\langle\omega;\tau,g_2\rangle| = |\langle\omega;\tau,-g_2\rangle| = \left|\sin\left(\alpha+\frac{5}{6}\pi\right)\right| \leq 1 \ ,$$
$$|\langle\omega;\tau,g_1+g_2\rangle| = |\langle\omega;\tau,-(g_1+g_2)\rangle| = |\cos\alpha| \leq 1 \ .$$

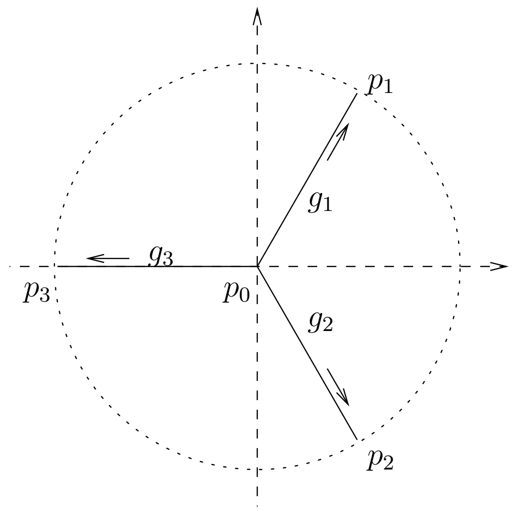

FIGURE 7.1.1. Solution for the problem with boundary on the vertex of an equilateral triangle

An interesting way to generalize this result will be recalled in Remark 7.1.14.

In Definition 7.1.1 we intentionally kept vague the regularity of the form ω. Indeed ω has to be a compactly supported[1] smooth form, a priori, in order to fit Definition 5.2.1. Nevertheless, in some situations it will be useful to consider calibrations with lower regularity, for instance piecewise constant forms. As long as (7.1.2)-(7.1.4) remain valid, it is meaningful to do so; for this reason we introduce the following very general definition.

DEFINITION 7.1.5. A *generalized calibration* associated with a k-dimensional normal E-current T is a linear and bounded functional ϕ on the space of normal E-currents satisfying the following conditions:

(i) $\phi(T) = \mathbb{M}(T)$;
(ii) $\phi(\partial R) = 0$ for any $(k+1)$-dimensional normal E-current R;
(iii) $\|\phi\| \leq 1$.

[1]Since we deal with currents that are compactly supported, we can easily drop the assumption that ω has compact support.

REMARK 7.1.6. The thesis in Proposition 7.1.2 is still true, since for every competitor T' with $\partial T = \partial T'$, there holds

$$\mathbb{M}(T) = \phi(T) = \phi(T') + \phi(\partial R) \leq \mathbb{M}(T') ,$$

where R is chosen such that $T - T' = \partial R$. Such R exists because T and T' are in the same homology class.

As examples, we present the calibrations for two well-known Steiner tree problems in $\mathbb{R}^2$. Both calibrations in Example 7.1.9 and in Example 7.1.10 are piecewise constant 1-forms (with values in normed vector spaces of dimension 2 and 6, respectively), so first of all we have to establish a compatibility condition which brings piecewise constant forms back to Definition 7.1.5.

DEFINITION 7.1.7. Fix a 1-dimensional rectifiable G-current T in $\mathbb{R}^2$, $T = T(\Sigma, \tau, \theta)$. Assume we have a collection $\{C_r\}_{r \geq 1}$ which is a locally finite, Lipschitz partition of $\mathbb{R}^2$, i.e. $\bigcup_{r \geq 1} C_r = \mathbb{R}^2$, the boundary of every set C_r is a Lipschitz curve and $C_r \cap C_s = \emptyset$ whenever $r \neq s$. Assume moreover that ∂C_r is a connected set for every r and that C_r contains the connected non-empty interior of its closure. Let us consider a compactly supported piecewise constant E^*-valued 1-form ω with

$$\omega \equiv \omega_r \quad \text{on } C_r$$

where $\omega_r \in \Lambda^1_E(\mathbb{R}^2)$ for every r. In particular $\omega \neq 0$ only on finitely many elements of the partition. Then we say that ω represents a *compatible calibration* for T if the following conditions hold:

(i) for almost every $x \in \Sigma$, $\langle \omega(x); \tau(x), \theta(x) \rangle = \|\theta(x)\|_G$;
(ii) for $\mathscr{H}^1$-almost every point $x \in \partial C_r \cap \partial C_s$ we have

$$\langle \omega_r - \omega_s; \tau(x), \cdot \rangle = 0,$$

where τ is the weak tangent field of ∂C_r;
(iii) $\|\omega_r\| \leq 1$ for every r.

We will refer to condition (ii) with the expression of *compatibility condition* for a piecewise constant form.

PROPOSITION 7.1.8. [**MM**] Let ω be a compatible calibration for the rectifiable G-current T. Then T minimizes the mass among the normal E-currents with boundary ∂T.

PROOF. Firstly we see that a suitable counterpart of Stokes Theorem holds. Namely, given a component ω^j of ω and a classical integral 1-current $T = T(\Sigma, \tau, 1)$ in $\mathbb{R}^2$, without boundary, then the quantity

$$\langle \omega^j; T \rangle := \int_\Sigma \langle \omega^j(x); \tau(x) \rangle \mathrm{d}\mathscr{H}^1(x)$$

is well defined, and we claim that it is equal to zero. The fact that it is well defined is a direct consequence of the compatibility condition (ii) in Definition 7.1.7. To prove that it is equal to zero, note that it is possible to find at most countably many unit multiplicity integral 1-currents $T_i = T(\Sigma_i, \tau_i, 1)$ in $\mathbb{R}^2$, without boundary, each one supported in a single tile C_r, such that $\sum_i T_i = T$. Since

$$\int_{\Sigma_i} \langle \omega^j(x); \tau_i(x) \rangle \mathrm{d}\mathscr{H}^1(x) = 0$$

for every i, then the claim follows from (ii). As a consequence we have that there exists a family of Lipschitz functions $\phi_j : \mathbb{R}^2 \to \mathbb{R}$ such that for every (classical) integral 1-current T with $\mathbb{M}(\partial T) \leq 2$ (in particular $\partial T = \delta_{x_T} - \delta_{y_T}$, with $x_T = y_T$ if and only if $\partial T = 0$) there holds:

$$\langle \omega^j; T \rangle = \phi_j(x_T) - \phi_j(y_T), \quad \text{for every } j.$$

In fact it is sufficient to choose

$$\phi_j(x) = |x| \int_0^1 \langle \omega^j(tx); \frac{x}{|x|} \rangle \ \mathrm{d}t.$$

Moreover it is easy to see that every ϕ_j is constant outisde of the support of ω^j, so we can assume, possibly subtracting a constant, that ϕ_j is compactly supported.

Now, take a 2-dimensional normal E-current T. Let $\{T^j\}_j$ be the components of T. For every j, use Proposition 1.3.13 to write $S^j := \partial T^j = \int_0^1 S_t^j \ \mathrm{d}t$. Then we have

$$\langle \omega; \partial T \rangle = \sum_j \int_0^1 \langle \omega^j; S_t^j \rangle \ \mathrm{d}t = \sum_j \int_0^1 \phi_j(x_{S_t^j}) - \phi_j(y_{S_t^j}) \ \mathrm{d}t.$$

Since for every j we have

$$0 = \partial(\partial T^j) = \int_0^1 \delta_{x_{S_t^j}} - \delta_{y_{S_t^j}} \ \mathrm{d}t,$$

then, for every j, we must have

$$\int_0^1 g(x_{S_t^j}) - g(y_{S_t^j}) \ \mathrm{d}t = 0,$$

for every compactly supported Lipschitz function g, in particular for every ϕ_j. Hence we have $\langle \omega; \partial T \rangle = 0$. □

Example 7.1.9. Consider the points

$$p_1 = (1, 1), p_2 = (1, -1), p_3 = (-1, -1), p_4 = (-1, 1) \in \mathbb{R}^2.$$

The length-minimizer graphs for the classical Steiner tree problem are those represented in Figure 5.0.1. We associate with each point p_j with $j = 1, \ldots, 4$ the coefficients $g_j \in G$, where G has "dimension" $m = 3$: let us call

$$B := g_1\delta_{p_1} + g_2\delta_{p_2} + g_3\delta_{p_3} + g_4\delta_{p_4} \ .$$

By this, we denote the 0-dimensional rectifiable G-current B such that

$$\langle B;\omega\rangle = \sum_{j=1}^{4}\langle \omega(p_j); g_j\rangle,$$

for every $\omega \in \mathscr{C}_c^\infty(\mathbb{R}^2, \Lambda_E^0(\mathbb{R}^2))$. This 0-dimensional current is our boundary.
Intuitively our mass-minimizing candidates among 1-dimensional rectifiable G-currents are those represented in Figure 7.1.2: these currents $T_{\rm hor}, T_{\rm ver}$ are supported, respectively, in the graphs of Figure 5.0.1 and have piecewise constant coefficients intended to satisfy the boundary condition $\partial T_{\rm hor} = B = \partial T_{\rm ver}$.

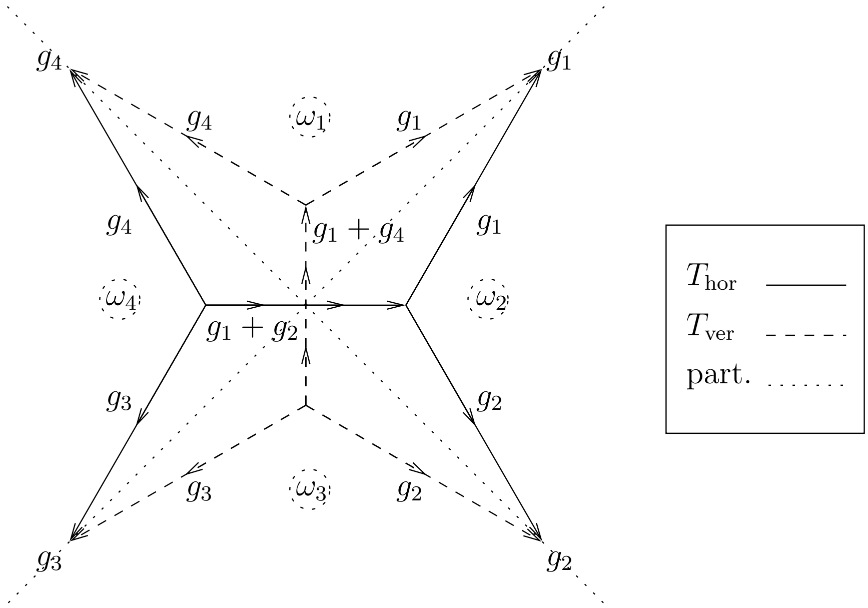

FIGURE 7.1.2. Solution for the mass minimization problem

In this case, a compatible calibration for both $T_{\rm hor}$ and $T_{\rm ver}$ is defined piecewise as follows (the notation is the same as in Example 7.1.4 and the partition is delimited by the dotted lines):

$$\omega_1 \equiv \begin{pmatrix} \frac{\sqrt{3}}{2}\mathrm{d}x_1 + \frac{1}{2}\mathrm{d}x_2 \\ \left(1-\frac{\sqrt{3}}{2}\right)\mathrm{d}x_1 - \frac{1}{2}\mathrm{d}x_2 \\ \left(-1+\frac{\sqrt{3}}{2}\right)\mathrm{d}x_1 - \frac{1}{2}\mathrm{d}x_2 \end{pmatrix} \qquad \omega_2 \equiv \begin{pmatrix} \frac{1}{2}\mathrm{d}x_1 + \frac{\sqrt{3}}{2}\mathrm{d}x_2 \\ \frac{1}{2}\mathrm{d}x_1 - \frac{\sqrt{3}}{2}\mathrm{d}x_2 \\ -\frac{1}{2}\mathrm{d}x_1 - \left(1-\frac{\sqrt{3}}{2}\right)\mathrm{d}x_2 \end{pmatrix}$$

$$\omega_3 \equiv \begin{pmatrix} \left(1-\frac{\sqrt{3}}{2}\right)\mathrm{d}x_1 + \frac{1}{2}\mathrm{d}x_2 \\ \frac{\sqrt{3}}{2}\mathrm{d}x_1 - \frac{1}{2}\mathrm{d}x_2 \\ -\frac{\sqrt{3}}{2}\mathrm{d}x_1 - \frac{1}{2}\mathrm{d}x_2 \end{pmatrix} \qquad \omega_4 \equiv \begin{pmatrix} \frac{1}{2}\mathrm{d}x_1 + \left(1-\frac{\sqrt{3}}{2}\right)\mathrm{d}x_2 \\ \frac{1}{2}\mathrm{d}x_1 - \left(1-\frac{\sqrt{3}}{2}\right)\mathrm{d}x_2 \\ -\frac{1}{2}\mathrm{d}x_1 - \frac{\sqrt{3}}{2}\mathrm{d}x_2 \end{pmatrix}$$

It is easy to check that ω satisfies both condition (i) and the compatibility condition of Definition 7.1.7. To check that condition (iii) is satisfied, we use formula (6.0.6).

EXAMPLE 7.1.10. Consider the vertices of a regular hexagon plus the center, namely

$$\begin{gathered} p_1 = (1/2, \sqrt{3}/2), \quad p_2 = (1,0), \quad p_3 = (1/2, -\sqrt{3}/2), \\ p_4 = (-1/2, -\sqrt{3}/2), \quad p_5 = (-1,0), \quad p_6 = (-1/2, \sqrt{3}/2), \quad p_7 = (0,0) \end{gathered}$$

and associate with each point p_j the corresponding multiplicity $g_j \in G$, where G is the group with dimension $m = 6$. A mass-minimizer for the problem with boundary

$$B = \sum_{j=1}^{7} g_j \delta_{p_j}$$

is illustrated in Figure 7.1.3, the other one can be obtained with a $\pi/3$-rotation of the picture.

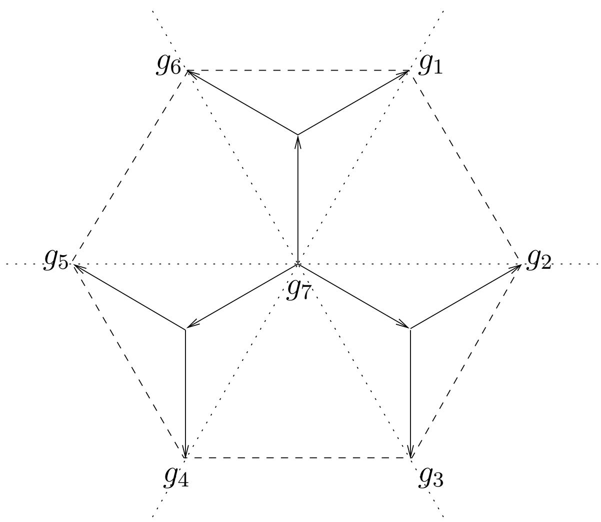

FIGURE 7.1.3. Solution for the mass minimization problem

Let us divide $\mathbb{R}^2$ in 6 cones of angle $\pi/3$, as in Figure 7.1.3; we will label each cone with a number from 1 to 6, starting from that containing $(0,1)$ and moving clockwise. A compatible calibration for the two minimizers is the following

$$
\omega_1 = \begin{pmatrix} -\frac{\sqrt{3}}{2}\mathrm{d}x_1+\frac{1}{2}\mathrm{d}x_2 \\ \frac{\sqrt{3}}{2}\mathrm{d}x_1+\frac{1}{2}\mathrm{d}x_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \omega_2 = \begin{pmatrix} 0 \\ \mathrm{d}x_2 \\ \frac{\sqrt{3}}{2}\mathrm{d}x_1-\frac{1}{2}\mathrm{d}x_2 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \omega_3 = \begin{pmatrix} 0 \\ 0 \\ \frac{\sqrt{3}}{2}\mathrm{d}x_1+\frac{1}{2}\mathrm{d}x_2 \\ -\mathrm{d}x_2 \\ 0 \\ 0 \end{pmatrix}
$$
$$
\omega_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{\sqrt{3}}{2}\mathrm{d}x_1-\frac{1}{2}\mathrm{d}x_2 \\ -\frac{\sqrt{3}}{2}\mathrm{d}x_1-\frac{1}{2}\mathrm{d}x_2 \\ 0 \end{pmatrix} \qquad \omega_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\mathrm{d}x_2 \\ -\frac{\sqrt{3}}{2}\mathrm{d}x_1+\frac{1}{2}\mathrm{d}x_2 \end{pmatrix} \qquad \omega_6 = \begin{pmatrix} \mathrm{d}x_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{\sqrt{3}}{2}\mathrm{d}x_1-\frac{1}{2}\mathrm{d}x_2 \end{pmatrix} \tag{7.1.5}
$$

Again, it is not difficult to check that ω satisfies both condition (i) and the compatibility condition of Definition 7.1.7. To check that condition (iii) is satisfied, we use formula (6.0.6).

Remark 7.1.11. We may wonder whether or not the calibration given in Example 7.1.10 can be adjusted so to work for the set of the vertices of the hexagon (without the seventh point in the center): it does not, in fact the support of the current in Figure 7.1.3 is not a solution for the Steiner tree problem on the six points, the perimeter of the hexagon minus one side being shorter.

Remark 7.1.12. In both Examples 7.1.9 and 7.1.10, once we fixed the group G and we decided to look for a piecewise constant calibration for our candidates, the construction of ω was forced by both conditions (i) of Definition 7.1.1 and the compatibility condition of Definition 7.1.7. Notice that the calibration for the Example 7.1.10 has evident analogies with the one exhibited in the Example 7.1.4. Actually we obtained the first one simply pasting suitably "rotated" copies of the second one.

In the following remarks we intend to underline the analogies and the connections with calibrations in similar contexts.

Remark 7.1.13. There is an interesting and deep analogy between calibrations and null-lagrangians, analogy that keeps unaltered in the group coefficients framework.
Consider some points $\{\eta_1, \dots, \eta_n\} \subset \mathbb{R}^m$, with

$$
|\eta_i - \eta_j| = 1 \quad \forall\, i \neq j \ ; \tag{7.1.6}
$$

for instance, the vertices of the regular n-tetrahedron with unit edge in $\mathbb{R}^{n-1}$ satisfy condition (7.1.6) (see Remark 6.0.11 to deepen the analogy with our group G in Chapter 6). We fix an open set with Lipschitz boundary $\Omega \subset \mathbb{R}^d$, for example $\Omega = B(0,1)$ and consider a bounded variation map $u : \Omega \to \{\eta_1, \dots, \eta_n\}$. Let us call $S[u] \subset \Omega$ the jump set associated with u: if ν is the unit normal according to some orientation of $S[u]$, let us say that u^+ and u^- are the traces of the BV function from above and from below the jump set (with respect to ν) respectively. We are interested in BV maps because

$$\int_\Omega |Du(x)|\,\mathrm{d}x = \int_{S[u]} |u^+(x) - u^-(x)|\,\mathrm{d}x = \mathscr{H}^{d-1}(S[u])\ ,$$

thanks to condition (7.1.6).
Therefore it is natural to study the variational problem

$$\min\left\{\int_\Omega |Du|\,:\, u \in BV\left(\Omega;\{\eta_1,\dots,\eta_p\}\right), u_{|\partial\Omega} \equiv u_0\right\}\ . \tag{7.1.7}$$

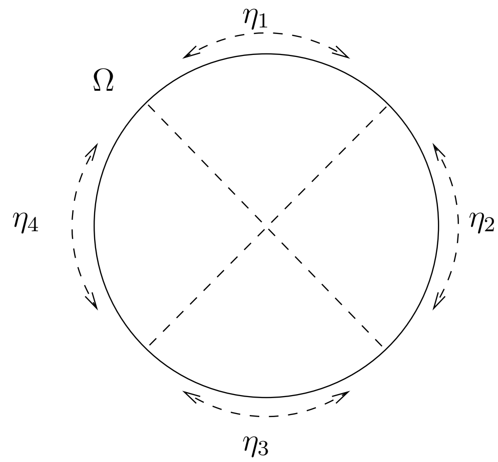

FIGURE 7.1.4. Boundary data

Assume there exists a vector field $V : \Omega \times \{\eta_1,\dots,\eta_n\} \to \mathbb{R}^d$ such that the following conditions hold:

(i) for every $x \in S[u]$,

$$[V(x,u^+(x)) - V(x,u^-(x))]\cdot\nu(x) = 1\ ;$$

(ii) marking $v_i(x) := V(x,\eta_i)$, $i = 1,\dots,n$,

$$\mathrm{div}_x V(x,\eta_i) = \mathrm{div}\, v_i(x) = 0\ ;$$

(iii) for every $i,j = 1,\dots,n$,

$$|v_i(x) - v_j(x)| \le 1\ .$$

In this case we can say that the functional $u \mapsto \int_\Omega \mathrm{div}(V(x,u(x)))\,\mathrm{d}x$ is a null-lagrangian, because it depends only on the boundary value u_0. As it happens in Proposition 7.1.2, if u admits a vector field V with the previous properties, then u is a minimizer for the variational problem (7.1.7) with $u_0 = u_{|\partial\Omega}$, because

$$\begin{aligned}
\int_\Omega |Du|\,\mathrm{d}x = \mathscr{H}^{d-1}(S[u]) &\overset{(i)}{=} \int_\Omega \operatorname{div}(V(x,u(x)))\,\mathrm{d}x = \int_\Omega \operatorname{div}(V(x,u'(x)))\,\mathrm{d}x \\
&\overset{(ii)}{=} \int_\Omega V_u(x,u'(x))\cdot\nabla u'(x)\,\mathrm{d}x \\
&= \int_{S[u']} \big(V(x,(u')^+(x)) - V(x,(u')^-(x))\big)\cdot\nu(x)\,\mathrm{d}\mathscr{H}^{d-1}(x) \\
&\overset{(iii)}{\leq} \int_{S[u']} |(u')^+ - (u')^-|\,\mathrm{d}\mathscr{H}^{d-1}(x) = \int_\Omega |Du'|\,\mathrm{d}x\ .
\end{aligned}$$

where u' is a competitor in $BV(\Omega;\{\eta_1,\dots,\eta_n\})$ with the same trace as u on $\partial\Omega$.
In order to clarify the similarity of the Null Lagrangian problem with the Steiner tree problem, consider the trace u_0 in Figure 7.1.4.
The minimizers of the problem (7.1.7) are showed in Figure 7.1.5. As a matter of fact, the minimizers $u_{\mathrm{hor}}, u_{\mathrm{ver}}$ admit a Null Lagrangian vector field, satisfying a compatibility condition and clearly related to the calibration ω defined above.

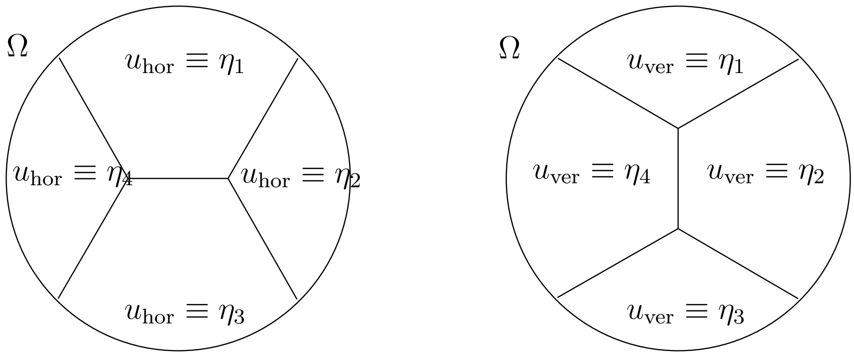

FIGURE 7.1.5. Minimizers

REMARK 7.1.14. In [**Mo**], F. Morgan applies flat chains with coefficients in a group G to soap bubble clusters and immiscible fluids, following the idea of B. White in [**W1**]. The model (in $\mathbb{R}^d$ for m immiscible fluids) associates to each fluid a coefficient $f_i \in G$, where $G \cong \mathbb{Z}^m \subset \mathbb{R}\otimes G \cong \mathbb{R}^m$ throughout the paper. Naturally, we are looking for least-energy interfaces, that is a mass-minimizing $(d-1)$-dimensional flat chain with coefficient in G. The mass norm is induced by the largest norm in $\mathbb{R}\otimes G$ such that

$$\|f_i\|_G = a_i \quad \forall\, i \in \{1,\dots,m\}$$

and

$$\|f_i - f_j\| = a_{ij} \quad \forall\, i,j\ .$$

Concerning soap bubble clusters, we choose $a_i = a_{ij} = 1$; hence, if $m = 2$, the unit ball is pictured below in Figure 7.1.6.

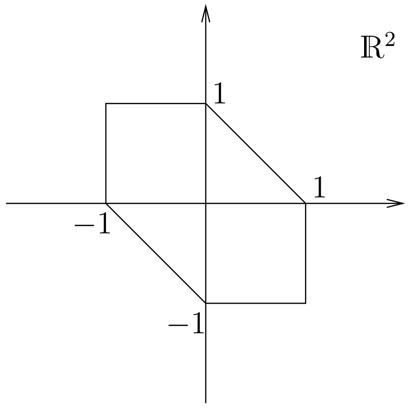

FIGURE 7.1.6. Unit ball in F. Morgan's model for soap bubble clusters

Following the idea in [**Mo**], a calibration for a rectifiable m-chain T in $\mathbb{R}^d$ is a homomorphism

$$\omega : G \to \Lambda^m(\mathbb{R}^d)$$

with the following properties:

(i) $\langle \vec{T}(x); \omega(g)(x) \rangle = \|g\|_G$ for a.e. $x \in \operatorname{supp}(T)$;
(ii) $\omega(g)$ is a closed differential m-form for every $g \in G$;
(iii) $\|\omega(g)\| \leq \|g\|_G$ for every $g \in G$, where $\Lambda^m(\mathbb{R}^d)$ is naturally endowed with the comass norm.

These properties guarantee that T is a mass-minimizer among flat chains with the same boundary; the proof is by all means analogous to the one given in Proposition 7.1.2.
Notice that this definition for the calibration works truly well in the case of a free abelian group, because we are considering homomorphisms with values in a vector space and every finite order subgroup is trivialized by such a homomorphism.
As F. Morgan shows in Proposition 4.5 of [**Mo**], in this framework it is easy to prove a generalization of Example 7.1.4: consider a cone $C = \sum_{i=1}^n g_i v_i$ in $\mathbb{R}^d$ of unit vectors v_i with coefficients in $G = \operatorname{span}\{g_i\}$ and assume that

$$\left| \sum_{i=1}^n \lambda_i \|g_i\|_G v_i \right| \leq \left\| \sum_{i=1}^n \lambda_i g_i \right\|_G \quad \forall \lambda_i \geq 0 \, ,$$

then C is a minimizer because it admits a calibration with constant coefficients.

7.2. Existence of the calibration and open problems

Once we established that the existence of a calibration is a sufficient condition for a rectifiable G-current to be a mass-minimizer, we may wonder if the converse is also true: does a (sort of) calibration exist for every mass-minimizing rectifiable G-current?

Let us step backward: does it occur for classical integral currents? The answer is quite articulate, but we can briefly summarize the state of the art we will rely upon.

REMARK 7.2.1. An actual calibration cannot exist for every minimizer. In fact there are currents which minimize the mass among integral currents with a fixed boundary, but not among normal currents (in some cases the two problems have different minima). This means that such currents cannot be calibrated, infact the existence of a calibration proves the minimality among normal currents.

REMARK 7.2.2. For every mass-minimizing classical normal k-current T, there exists a generalized calibration ϕ in the sense of Definition 7.1.5. Moreover, by means of the Riesz Representation Theorem, ϕ can be represented as a measurable map $U \to \Lambda^k(\mathbb{R}^d)$. This result is contained in [**Fe2**].

In particular, Remark 7.2.2 provides a positive answer to the existence of a generalized calibration for mass-minimizing integral currents of dimension $k = 1$, because minima among both normal and integral currents coincide, as we prove in Proposition 7.2.4.
It is possible to apply the same technique in the class of normal E-currents, therefore we have the following proposition.

PROPOSITION 7.2.3. For every mass minimizing normal E-current T, there exists a generalized calibration.

In order to guarantee the existence of a generalized calibration also for 1-dimensional mass-minimizing rectifiable G-currents, we need the analogous of Proposition 7.2.4 in the framework of G-currents. Namely, we need to prove that the minimum of the mass among 1-dimensional normal E currents with the same boundary coincides with the minimum calculated among rectifiable G-currents. Here the boundary is of course a 0-dimensional rectifiable G-current. This is a well known issue for classical k-dimensional currents: for $k \geq 2$ it is not even know whether the two minima are commensurable, i.e. whether or not there exist a constant C such that, for every fixed $(k-1)$-dimensional integral boundary B, the minimum of the mass among integral k-currents with boundary B is less then C times the minimum among normal k-currents with the same boundary.
From the argument used in the proof of Proposition 7.2.4 we realize that the equality of the two minima in the framework of 1-dimensional E-currents is equivalent to the homogeneity property in Remark 7.2.5. This property, which is trivially verified for classical integral currents, seems to be an interesting issue in the class of rectifiable G-currents. In Example 7.2.6 we exhibit a subset $M \subset \mathbb{R}^2$ such that, if our currents are forced to be supported on M, then the homogeneity property does not hold. In other words, we can say that equality of the two minima does not hold in the framework of 1-dimensional

E-currents on the metric space M. We can see the same phenomenon if we substitute the metric space M with the metric space $\mathbb{R}^2$ endowed with a density, which is unitary on the points of M and very high outside.

The following proposition is probably in the folklore, we give a proof here because we were not able to find any literature on it.

PROPOSITION 7.2.4. [**MM**] Consider the boundary of an integral 1-current in $\mathbb{R}^d$, represented as

$$\partial_0 = -\sum_{i=1}^{N_-} a_i \delta_{x_i} + \sum_{j=1}^{N_+} b_j \delta_{y_j}, \quad a_i\, b_j \in \mathbb{N} \ . \tag{7.2.1}$$

If we denote

$$\mathscr{M}_N(\partial_0) := \min\{\mathbb{M}(T) \ : \ T \text{ is a normal current }, \partial T = \partial_0\}$$

and

$$\mathscr{M}_I(\partial_0) := \min\{(T) \ : \ T \text{ is an integral current}\, \partial T = \partial_0\} \ ,$$

then the minima of the mass of 1-currents with boundary ∂_0 among normal 1-currents and among integral 1-currents coincide, that is

$$\mathscr{M}_N(\partial_0) = \mathscr{M}_I(\partial_0) \ .$$

Proof. Let us assume that the minimum among normal currents is attained at some current T_0, that is

$$\mathbb{M}(T_0) = \mathscr{M}_N(\partial_0) \ .$$

By definition

$$\mathscr{M}_N(\partial_0) \leq \mathscr{M}_I(\partial_0) \ .$$

Let $\{T_h\}_{h\in\mathbb{N}}$ be an approximation of T_0 made by polyhedral 1-currents, such that

- $\mathbb{M}(T_h) \to \mathbb{M}(T_0)$ as $h \to \infty$,
- $\partial T_h = \partial_0$ for all $h \in \mathbb{N}$,
- the multiplicities allowed in T_h are only integer multiples of $\frac{1}{h}$.

The existence of such a sequence is a consequence of the Polyhedral Approximation Theorem.
It is possible to decompose such a T_h as a sum of two addenda:

$$T_h = P_h + C_h \ , \tag{7.2.2}$$

with

$$\mathbb{M}(T_h) = \mathbb{M}(P_h) + \mathbb{M}(C_h) \quad \forall\, h \geq 1$$

and

- $\partial C_h = 0$, so C_h collects the cyclical part;
- P_h does not admit any decomposition $P_h = A + B$ satisfying $\partial A = 0$ and $\mathbb{M}(P_h) = \mathbb{M}(A) + \mathbb{M}(B)$

It is clear that P_h is the sum of a certain number of polyhedral currents $P_h^{i,j}$ each one having boundary a non-negative multiple of $-\frac{1}{h}\delta_{x_i} + \frac{1}{h}\delta_{y_j}$ and satisfying

$$\mathbb{M}(P_h) = \sum_{i,j} \mathbb{M}(P_h^{i,j})$$

We replace each $P_h^{i,j}$ with the oriented segment $Q^{i,j}$, from x_i to y_j having the same boundary as $P_h^{i,j}$ (therefore having multiplicity a non-negative multiple of $\frac{1}{h}$). This replacement is represented in Figure 7.2.1

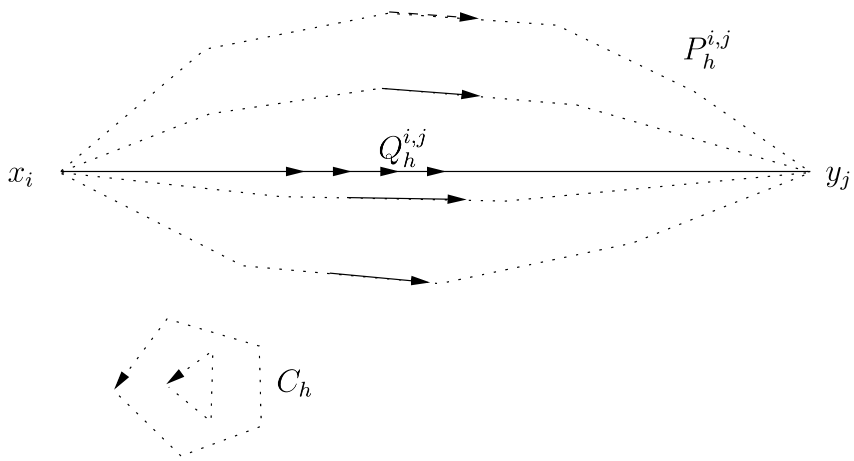

FIGURE 7.2.1

Since this replacement obviously does not increase the mass, there holds $\mathbb{M}(P_h) \geq \mathbb{M}(Q_h)$, where $Q_h = \sum_{i,j} Q_h^{i,j}$. In other words we can write $Q_h = \int_I T \,\mathrm{d}\lambda_h$, as an integral of currents, with respect to a discrete measure λ_h supported on the finite set I of unit multiplicity oriented segments with the first extreme among the points $x_1, \dots, x_{N_-}$ and second extreme among the points $y_1, \dots, y_{N_+}$. It is also easy to see that the total variation of λ_h has eventually the following bound from above

$$\|\lambda_h\| \leq \frac{\mathbb{M}(T_h)}{\min_{i\neq j} d(x_i, y_j)} \leq \frac{\mathbb{M}(T_0) + 1}{\min_{i\neq j} d(x_i, y_j)} \ .$$

Hence, up to subsequences, λ_h converges to some positive measure λ on I and so the normal 1-current

$$Q = \int_{T\in I} T \,\mathrm{d}\lambda$$

satisfies

$$\partial Q = \partial_0 \tag{7.2.3}$$

and

$$\mathbb{M}(Q) \leq \mathbb{M}(T_0) = \mathscr{M}_N(\partial_0) \ .$$

In order to conclude the proof of the theorem, we need to show that Q can be replaced by an integral current R with same boundary and mass $\mathbb{M}(R) = \mathbb{M}(Q) \leq \mathscr{M}_N(\partial_0)$.
Since I is the set of unit multiplicity oriented segments Σ^{ij} from x_i to y_j, we can obviously represent

$$Q = \sum_{i,j} k^{ij}\Sigma^{ij} \quad \text{with } k^{ij} \in \mathbb{R} \ ,$$

and, again, thanks to (7.2.3),

$$\sum_{i=1}^{N_-} k^{ij} = b_j \quad \text{and} \quad \sum_{j=1}^{N_+} k^{ij} = a_i \ .$$

If $k^{ij} \in \mathbb{Z}$ for any i, j, then Q itself is integral and then we are done; if not, let us consider the finite set of non-integer multiplicities

$$K_{\mathbb{R}\setminus\mathbb{Z}} := \left\{k^{ij} \ : \ i = 1, \dots, N_-, \ j = 1, \dots, N_+\right\} \setminus \mathbb{Z} \neq \emptyset \ .$$

We fix $k \in K_{\mathbb{R}\setminus\mathbb{Z}}$ and we choose an index (i_0, j_0), such that k is the multiplicity of the oriented segment $\Sigma^{i_0 j_0}$ in Q.
It is possible to track down a non-trivial cycle $\overline{Q}$ in Q with the following algorithm: after $\Sigma^{i_0 j_0}$, choose a segment from $x_{i_1} \neq x_{i_0}$ to y_{j_0} with non-integer multiplicity, it must exist because $\partial_0 = \partial Q$ is integral. Then choose a segment from x_{i_1} to $y_{j_1} \neq y_{j_0}$ with non-integer multiplicity and so on. Since $K_{\mathbb{R}\setminus\mathbb{Z}}$ is finite, at some moment we will get a cycle. Up to reordering the indices i and j we can write

$$\overline{Q} = \sum_{l=1}^{n} (\Sigma^{i_l j_l} - \Sigma^{i_{l+1} j_l}) \ .$$

We will denote by

$$\begin{aligned} \alpha &:= \min_l (k^{i_l j_l} - \lfloor k^{i_l j_l} \rfloor) > 0 \\ \beta &:= \min_l (k^{i_{l+1} j_l} - \lfloor k^{i_{l+1} j_l} \rfloor) > 0 \ . \end{aligned}$$

Finally notice that both $Q - \alpha\overline{Q}$ and $Q + \beta\overline{Q}$ have lost at least one non-integer coefficient; in addition, we claim that either

$$\mathbb{M}(Q - \alpha\overline{Q}) \leq \mathbb{M}(Q) \quad \text{or} \quad \mathbb{M}(Q + \beta\overline{Q}) \leq \mathbb{M}(Q) \ . \tag{7.2.4}$$

In fact we can define the linear auxiliary function

$$F(t) := \mathbb{M}(Q) - \mathbb{M}(Q - t\overline{Q}) = \sum_l (k^{i_l j_l} - t) d(x_{i_l}, y_{j_l}) + (k^{i_{l+1} j_l} + t) d(x_{i_{l+1}}, y_{j_l})$$

for which $F(0) = 0$, so either

$$F(\alpha) \geq 0 \quad \text{or} \quad F(-\beta) \geq 0 \ .$$

Iterating this procedure finitely many times, we obtain an integral current without increasing the mass. □

Now, we want to know whether the analogous of this result holds also in the framework of 1-dimensional E-currents. Fix a 0-dimensional rectifiable G-current R in $U \subset \mathbb{R}^d$. Do the minima for the mass among 1-dimensional normal E-currents and rectifiable G-currents with boundary R coincide?

REMARK 7.2.5. The answer to the previous question is positive if and only if the following is true: given $R = \sum_{i=1}^n g_i \delta_{x_i}$ with $\|g_i\|_G = 1$ and T a rectifiable G-current which is mass-minimizer with $\partial T = R$, then for every $k \in \mathbb{N}$ we have that

$$\min\{\mathbb{M}(S) \,:\, S \text{ rectifiable } G-\text{current}, \partial S = kR\} = k\mathbb{M}(T)\ . \tag{7.2.5}$$

Notice that, using the notation introduced in Theorem 7.2.4, (7.2.5) can be meaningfully written as

$$\mathscr{M}_I(kR) = k\mathscr{M}_I(R)\ . \tag{7.2.6}$$

The condition 7.2.6 is clearly necessary to have the equality of the two minima. It is also sufficient, in fact one can approximate a normal E-current with polyhedral currents with coefficients in $\mathbb{Q}G$.

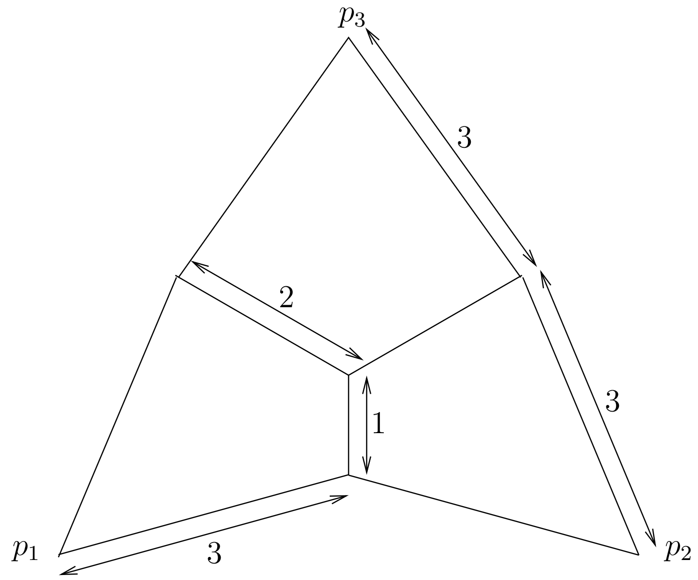

FIGURE 7.2.2. Metric space in the Example 7.2.6

EXAMPLE 7.2.6. Consider a very simple subset $M \subset \mathbb{R}^2$ with few paths[2] to move on, as in Figure 7.2.2.
Consider the group G, with $n = 3$, introduced in Chapter 6 and let $R = g_1\delta_{p_1} + g_2\delta_{p_2} + g_3\delta_{p_3}$.

[2]The length of each segment is explicitly declared in Figure 7.2.2, mind that the set is symmetric with respect to the vertical axis.

We will show that (7.2.6) does not hold even when $k = 2$. in fact it is trivial to prove that

$$\mathscr{M}_I(R) = 12 \ .$$

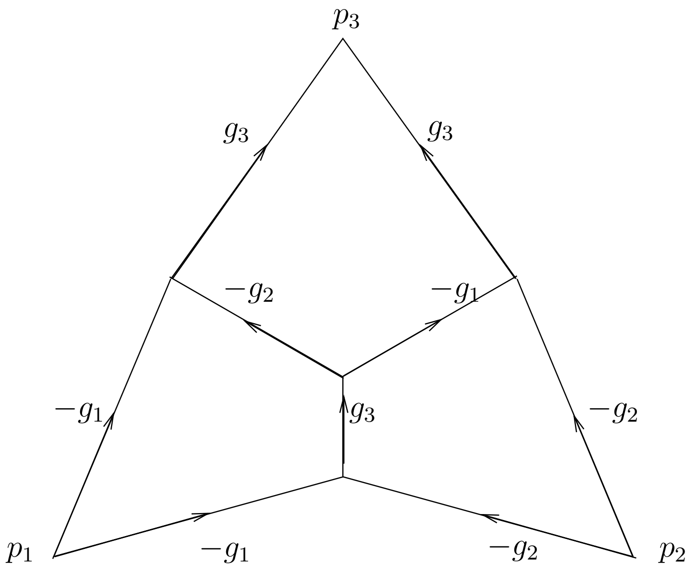

FIGURE 7.2.3. Counterexample to (7.2.6)

Nevertheless, concerning $\mathscr{M}_I(2R)$, it is proved in Figure 7.2.3 that

$$\mathscr{M}_I(2R) \leq 23 < 24 = 2\mathscr{M}_I(R) \ .$$

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