On minimal autonomous partitions of directed graphs and some applications to automata theory

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We are here concerned with a class of partitions which are similar to the well known cyclic partitions of Markov chains. Let $G=(V, E)$ be a directed graph with a non-empty (possibly infinite) vertex set V and a set of directed edges E. Consider partitions $\pi=\{B_\alpha\}$ of a graph G, where $\{B_\alpha\}$ is a family of disjoint non-empty subsets (or blocks) $B_\alpha \subseteqq V$ and $\bigcup_\alpha B_\alpha = V$. A partition π is called autonomous if for every block B_α either $\delta(B_\alpha)$ is empty or $\delta(B_\alpha)\subseteqq B_\beta$ for some block B_β. Here $\delta(B)$ denotes the set of all vertices which are reached in one step from $B\subseteqq V$. By the minimal autonomous partition (m.a.p.) of a directed graph G we mean such autonomous partition which is a refinement of any autonomous partition of this graph. Denote the m.a.p. of G by $\pi_{\min}(G)$, or simply $\pi_{\min}$ when non confusion is possible. The intersection of all autonomous partitions of G is an autonomous partition which is equal to the m.a.p. of G. Thus, the m.a.p. is uniquely determined for every directed graph.

These partitions turned out to be a very useful tool for studying some properties of automata and much of the motivation for the work discussed here derives from attempts to describe a structure of automata which are stable to the input-induced errors. My attention to examining the m.a.p. was also called by the paper [1] of A. Ádám, who introduced the autonomous partitions under the name P-partitions and considered these partitions from the graphtheoretical point of view. The main result of A. Ádám lies in the following. Let ϱ be the relation on a graph G such that for each pair of vertices $v, u\in V$ we have $(v, u)\in\varrho$ if and only if v and u are reached in equal number of steps from some vertex $w\in V$, i.e. there exist two paths of equal length from w to v and from w to u. Then $\varrho^T=\pi_{\min}$ for every sink-free directed graph, where ϱ^T denotes the transitive closure of ϱ. (Here and elsewhere we do not distinguish between partitions and the corresponding equivalence relations.) The above statement we shall call A. Ádám's theorem on minimal autonomous partitions.

The purposes of our paper are:

1) to describe the structure of m.a.p. for various types of directed graphs; and

2) to demonstrate the possibility of applications of A. Ádám's theorem to automata theory.

Minimal autonomous partitions

In this section we describe the structure of m.a.p. for arbitrary directed graphs, for graphs with finitely many sinks, for sink-free and source-free graphs and for strongly connected graphs.

By o we denote the trivial partition such that every block of o is a singleton. If τ is any relation on a graph, then we denote by τ^a the following relation: for each two vertices $v, u \in V$ we have $(v, u) \in \tau^a$ if and only if either $(v, u) \in \tau$ or there exists a finite sequence of pairs $v_i, u_i \in V$, $i=1, \dots, n$, such that $(v_1, u_1) \in \tau$, $v_i \in \delta(v_{i-1})$ and $u_i \in \delta(u_{i-1})$ for $i=2, \dots, n$ and $v_n = v$, $u_n = u$. τ^a will be called the autonomous closure of τ. It will be observed that o^a is exactly the relation ϱ defined above. By τ^T we denote the transitive closure of τ, i.e. $(v, u) \in \tau^T$ iff there exists a finite sequence of vertices $v_1, \dots, v_n$ such that $(v_{i-1}, v_i) \in \tau$ for $i=2, \dots, n$ and $v_1 = v$, $v_n = u$.

We begin with A. Ádám's theorem ([1], Propositions 5,6):

Theorem 1. For an arbitrary graph $o^{aT} \subseteq \pi_{\min}$. For an arbitrary sink-free graph $o^{aT} = \pi_{\min}$.

Remark 1. Although A. Ádám [1] dealt only with finite connected graphs, his proof of this theorem is valid for arbitrary graphs.

We are going to generalize A. Ádám's theorem in the following way. Let sink (G) be the number of sinks of G and $o^{n \times (aT)}$ means $o^{aT \dots aT}$, where aT is repeated n times. We shall first give some properties of the relations $o^{n \times (aT)}$. Put $o^{\infty (aT)} = \bigcup_{n=1}^{\infty} o^{n \times a(T)}$, i.e. $(v, u) \in o^{\infty (aT)}$ iff $(v, u) \in o^{n \times (aT)}$ for some $n \geqq 1$. Note that $o^{n \times (aT)} \subseteq o^{(n+1) \times (aT)}$, for each $n \geqq 1$, hence $o^{\infty (aT)}$ is a partition. Furthermore, one easily verifies that the following pairs of factor-graphs are isomorphic:

$$G/\pi_{\min}(G) \sim [G/o^{aT}]/[\pi_{\min}(G/o^{aT})] \tag{$*$}$$

$$G/o^{(n+1) \times (aT)} \sim [G/o^{aT}]/[o^{n \times (aT)}(G/o^{aT})] \tag{$**$}$$

or (by induction) for each $n, i \geqq 1$

$$G/\pi_{\min}(G) \sim [G/o^{i \times (aT)}]/[\pi_{\min}(G/o^{i \times aT})] \tag{$*$}$$

$$G/o^{(n+i) \times (aT)} \sim [G/o^{i \times (aT)}]/[o^{n \times (aT)}(G/o^{i \times (aT)})] \tag{$**$}$$

We are now in a position to prove the generalization of A. Ádám's theorem:

Theorem 2. For an arbitrary graph, $o^{\infty (aT)} = \pi_{\min}$. If sink $(G) \leqq n$, then $o^{(n+1) \times (aT)} = \pi_{\min}$.

Proof. 1) It follows from A. Ádám's theorem that $o^{n \times (aT)} \subseteq \pi_{\min}$ for each $n \geqq 1$. Thus $o^{\infty (aT)} \subseteq \pi_{\min}$. To prove $o^{\infty (aT)} = \pi_{\min}$, fix two vertices $(v, u) \in o^{\infty (aT)}$. We then have $(v, u) \in o^{k \times (aT)}$, for some $k \geqq 1$. Consequently, $(v', u') \in o^{(k+1) \times (aT)}$ if $v' \in \delta(v)$ and $u' \in \delta(u)$. From this it follows that $o^{\infty (aT)}$ is an autonomous partition. Suppose $o^{\infty (aT)} \neq \pi_{\min}$. Then $o^{\infty (aT)}$ is a proper refinement of $\pi_{\min}$ and the minimality of $\pi_{\min}$ gives the contradiction.

2) Now prove the second assertion.
Induction base: If sink $(G)=0$, then $o^{aT}=\pi_{\min}$ by A. Ádám's theorem.
Induction step: If sink $(G)=n+1$ and $o^{aT}\neq\pi_{\min}$, then sink $(G/o^{aT})\leqq n$. Indeed, if $o^{aT}\neq\pi_{\min}$, then there exist blocks A, B, C of o^{aT} with $\delta(A)\cap B\neq\emptyset$, $\delta(A)\cap C\neq\emptyset$ and $B\cap C=\emptyset$. (Here $\emptyset$ denotes the empty set.) This means that there is a sink in A, but A, considered as a vertex of the factor-graph G/o^{aT}, is not a sink. Thus sink $(G/o^{aT})\leqq n$.

Suppose $o^{(n+1)\times(aT)}=\pi_{\min}$ holds for each graph G' with sink $(G')\leqq n$. Then

$$G'/\pi_{\min}(G')\sim G'/o^{(n+1)\times(aT)}(G').$$

where $G'=G/o^{aT}$ and $\sim$ means graph isomorphism. On the other hand, properties $(*)$ and $(**)$ give

$$G'/\pi_{\min}(G')\sim G/\pi_{\min}(G)$$

$$G'/o^{(n+1)\times(aT)}(G')\sim G/o^{(n+2)\times(aT)}(G).$$

Thus

$$G/\pi_{\min}(G)\sim G/o^{(n+2)\times(aT)}(G)$$

and consequently, $\pi_{\min}=o^{(n+2)\times(aT)}$ if sink $(G)=n+1$. Q.E.D.

Examples. Fig. 1 shows a graph G with sink $(G)=1$, $o^{aTa}\neq\pi_{\min}$, $o^{aTaT}=\pi_{\min}$. Fig. 2 shows a graph G with sink $(G)=2$, $o^{aTaTa}\neq\pi_{\min}$, $o^{aTaTaT}=\pi_{\min}$. For the graph in Fig. 3 we have sink $(G)=3$, $o^{aTaTaTa}\neq\pi_{\min}$, $o^{aTaTaTaT}=\pi_{\min}$.

These examples give rise to the following

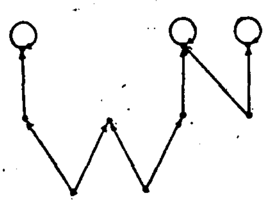

Fig. 1.

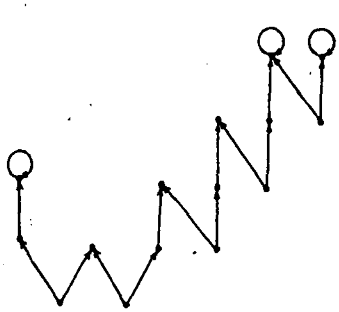

Fig. 2.

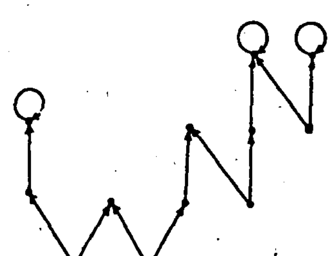

Fig. 3.

Proposition 1. For each integer n there exists a graph G such that $\operatorname{sink}(G)=n$ and $o^{n\times(aT)a}\neq\pi_{\min}$.

Question 1. Does Proposition 1 remain valid when we restrict ourselves to finite graphs without sources?

Question 2. For any n, characterize the graphs having exactly n sinks such that $o^{n\times(aT)a}=\pi_{\min}$ holds in Theorem 2.

Corollary 1. If G is finite, then $o^{n\times(aT)}=\pi_{\min}$ for some integer $n\leqq|V|$.

Proof. If every vertex of G is a sink, then $o=\pi_{\min}$. Elsewhise, $\operatorname{sink}(G)\leqq|V|-1$ and we can apply Theorem 2.

Question 3. What is the smallest number $f(k)$ such that $o^{f(k)\times(aT)}=\pi_{\min}$ for every finite graph with $|V|=k$?

Now let us consider sink-free graphs. The relation o^a is not transitive, in general, even if a graph has no sink and no source (see Fig. 4). (This example also provides a particular answer to Problem 3 in [1].) But for strongly connected graphs the relation o^a is always transitive.

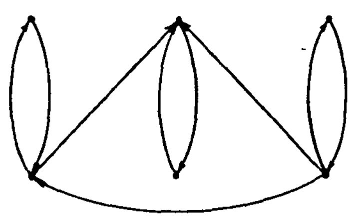

Fig. 4.

Proposition 2. For an arbitrary strongly connected graph, $o^a=\pi_{\min}$.

Proof. First, let G be a finite strongly connected graph. Let $B\in\pi_{\min}$ be an arbitrary block of its m.a.p. and let $c\in B$ be a vertex in this block. Consider the factor graph $G/\pi_{\min}$. Obviously, $G/\pi_{\min}$ is a cycle. Denote its length by p. Consider a sequence of sets $S_k=\delta^{kp}(c)$, $k=0,1,2,\ldots$, where $\delta^n(c)=\delta(\delta^{n-1}(c))$, $\delta^0(c)=c$, $\delta^1(c)=\delta(c)$. Note that $S_k\subseteqq B\in\pi_{\min}$ and for every pair of vertices $v,u\in S_k$ we have $(v,u)\in o^a$, for each $k=0,1,2,\ldots$. Since G is finite, the sequence S_k becomes stationary, i.e. there exist integers $l\leqq m$ such that $\delta^p(S_l)=S_{l+1}$, $\delta^p(S_{l+1})=S_{l+2},\ldots$ $\ldots,\delta^p(S_m)=S_l$. Since G is strongly connected, $\bigcup_{i=1}^{m}S_i=B$. If $l=m$, then the proposition is already proved (in this case $S_l=B$ and $(v,u)\in o^a$ for each pair $v,u\in B$). If not, suppose without the loss of generality that all sets in the system $S=\{S_l,\ldots$ $\ldots,S_m\}$ are different. A family of sets $\{S_\alpha\}\subseteqq S$, $\alpha\in A\subseteqq\{l,\ldots,m\}$, will be called a maximal system if and only if $\bigcap_{\alpha\in A}S_\alpha=\tilde S\neq\emptyset$ and for each n $(l\leqq n\leqq m)$ such that $n\notin A$ we have $\tilde S\cap S_n=\emptyset$. Let $\tilde{S}=\{\tilde S_1,\ldots,\tilde S_q\}$ be the family of intersections of the maximal systems. It is not difficult to see that $\delta^p(\tilde S_1)\subseteqq\tilde S_2,\ldots,\delta^p(\tilde S_q)\subseteqq\tilde S_1$ for a suitable numeration of the sets $\tilde{S}$. Furthermore, if $i\neq j$, then $\tilde{S}_i\cap\tilde{S}_j=\emptyset$. Since G is strongly connected, one has $\bigcup_{i=1}^{q} S_1=B$. Therefore we can consider the following partition:

$$\pi = \{\tilde{S}_1, \delta^1(\tilde{S}_1), \ldots, \delta^{p-1}(\tilde{S}_1), \tilde{S}_2, \ldots, \delta^{p-1}(\tilde{S}_q)\}.$$

Obviously, π is autonomous partition and π is a proper refinement of $\pi_{\min}$. This contradicts the minimality of $\pi_{\min}$. Hence $l=m$ and $o^a=\pi_{\min}$.

The general case, when G is an arbitrary strongly connected graph, is reducible to the previous one. Indeed, let B be an arbitrary block of the m.a.p. We are going to show that $(v, u)\in o^a$ for each pair of vertices $v, u\in B$. Since a strongly connected graph has no sink, therefore it fulfils the suppositions of A. Ádám's theorem. Hence $(v, u)\in o^{aT}$. This means that there exists a sequence of vetrices $v_1, \ldots, v_n\in B$ such that $v_1=v$, $v_n=u$ and $(v_i, v_{i+1})\in o^a$ for $i=1, \ldots, n-1$. Select two paths of equal length from v_i to v_i and from v_i to v_{i+1} for each $i=1, \ldots, n-1$ and take an arbitrary path from v_n to v_1. Consider the subgraph G' of G consisting all vertices and edges of selected paths. It is clear that G' is a finite strongly connected graph. Moreover, the vertices v and u belong to the same block of $\pi_{\min}(G')$. Consequently, $(v, u)\in o^a$ by the previous part of the proof (note that $o^a(G')$ is the refinement of $o^a(G)$). Q.E.D.

Remark 2. In addition, strongly connected graphs have another advantageous property: it is a well known fact in the theory of Markov chains that for such graphs the equality $p=p^*$ holds (see below).

Now we are going to generalize Proposition 2. When does $o^a=\pi_{\min}$ hold for sink-free and source-free graphs? This problem is closely related to Problem 2 in [1]: when is the length p of the cycle of the functional graph $G/\pi_{\min}$ equal to the greatest common divisor p^* of all cycle lengths of G? Let $\tilde{p}$ be the greatest common divisor of all cycle lengths of the induced subgraph spanned by all generators of G, i.e. $\tilde{p}=$g.c.d. {length (C): every vertex of cycle C is a generator of G}. (A vertex v is called generator if for each vertex u there exists a path from v to u.)

One has the following

Theorem 3. *If a finite connected graph G has no source, then $o^a=\pi_{\min}$ iff there exists at least one generator of G and $p=\tilde{p}$.*

Proof. Assume that $o^a=\pi_{\min}$ and G has no source. Then o^a is a transitive relation. It is not difficult to see that there exists a generator v of G. Let u be a vertex such that there is a path from u to v of length p. If $v=u$, then $p=\tilde{p}$. Otherwise, since the vertices v and u belong to the same block of $\pi_{\min}$ (therefore $(v, u)\in o^a$) and since v is a generator, there exist two paths from v to v and from v to u of equal length kp (for some integer $k\geqq 1$). It is clear that one can find two cycles, both containing v, with lengths kp and $kp+p$. Hence $p=\tilde{p}$.

Now let $p=\tilde{p}$ and suppose that there exists a generator v of G. Then there are two cycles C_1 and C_2 such that $v\in C_1$, $v\in C_2$ and the greatest common divisor of $l_1=$length (C_1) and $l_2=$length (C_2) equals p. Indeed, the subgraph $\tilde{G}$ of G spanned by all generators is strongly connected, hence we can apply Proposition 2, our assumption $p=\tilde{p}$, then Remark 2 and the construction of the previous part of this proof. It is clear that if $(v_1, v_2)\in o^a$, then $(v_1, v_2)\in\pi_{\min}$. We are going to show the converse implication. Let v_1 and v_2 be arbitrary vertices which belong to the same block of

$\pi_{\min}$. Since v is a generator of G, there exist two paths from v to v_1 (of the length m_1) and from v to v_2 (of the length m_2). It should be observed that $l_1=k_1p$, $l_2=k_2p$, $|m_1-m_2|=k_3p$, for some $k_1\geqq 1$, $k_2\geqq 1$, $k_3\geqq 0$. From the fact that the equation

$$l_1x+l_2y=|m_1-m_2|$$

is solvable in integers, it follows that there exist two paths of equal length from v to v_1 and from v to v_2, respectively. Q.E.D.

Corollary 1. If a finite connected graph G has no source, then $o^a=\pi_{\min}$ implies $p=p^*$.

Proof. The divisibility relations $p|p^*$ and $p^*|\tilde{p}$ are clear. If $p=\tilde{p}$, then $p=p^*$.

The converse implication in Corollary 1 is not valid, in general (see Fig. 5).

We finish this section with the remark that the above results were not intended as an overview of the m.a.p. and Problems 1—2 proposed by A. Ádám [1] are still open.

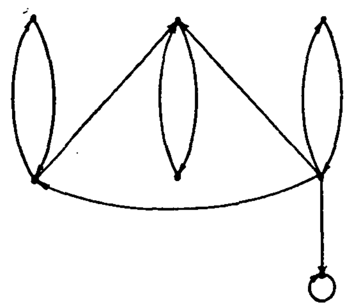

Fig. 5.

Applications to automata theory

In this section we are going to describe a class of automata which are stable to the input-induced errors. First introduce some notations used below. By automaton we mean a system $A=(X, S, \delta)$, where X and S are arbitrary finite non-empty sets, called the input alphabet and the state set, respectively, and $\delta\colon S\times X\to S$ is called the transition function. By δ we also denote the natural extension of the transition function to a mapping $2^S\times X^*\to 2^S$, where 2^S is a family of subsets of S and X^* is a free monoid generated by X. By the m.a.p. of an automaton we mean the m.a.p. of its transition diagram. A block B of the m.a.p. of an automaton is called cyclic if $\delta(B, J)\subseteqq B$ for some non-empty $J\in X^*$. The set of states which belong to the cyclic blocks is denoted $C(S)$. By the period of automaton A we mean the least common multiple of all cycle lengths of the transition diagram of the factor automaton $A/\pi_{\min}$. We denote the period of A by $p(A)$. If $A_1=(X, S_1, \delta_1)$ and $A_2=(X, S_2, \delta_2)$ are automata with $S_1\cap S_2=\emptyset$ and the same input alphabet, then $A_1\times A_2= =(X, S_1\times S_2, \delta)$, where $\delta((s_1, s_2), x)=(\delta_1(s_1, x), \delta_2(s_2, x))$ for each $s_1\in S_1$, $s_2\in S_2$

and $x \in X$, is called the product of the automata and $A_1 + A_2 = (X, S_1 \cup S_2, \delta)$, where

$$\delta(s, x) = \begin{cases} \delta_1(s, x), & \text{if } s \in S_1 \\ \delta_2(s, x), & \text{if } s \in S_2, \end{cases}$$

is called the sum of the automata. An automaton $\tilde{A} = (X, \tilde{S}, \tilde{\delta})$ is called a subautomaton of $A = (X, S, \delta)$ if $\tilde{S} \subseteq S$ and $\tilde{\delta}(\tilde{s}, x) = \delta(\tilde{s}, x)$ for every choice of $\tilde{s} \in \tilde{S}$, $x \in X$. An automaton A is said to be strongly connected if for every pair of states $s, t \in S$ there are such words $J_1, J_2 \in X^*$ that $\delta(s, J_1) = t$ and $\delta(t, J_2) = s$. In other words, an automaton A is strongly connected iff the transition diagram of A is strongly connected. An automaton is said to be connected if its transition diagram, considered as a non-oriented graph, is connected. Note that every automaton A is a sum of connected automata $A = \sum_{\alpha} A_\alpha$. We say that an automaton A can be represented by a parallel composition of automata B and C if there exists a subautomaton D of $B \times C$ such that A is a homomorphic image of D. The onto mapping $h\colon S' \to S$ is called a homomorphism from $D = (X, S', \delta')$ to $A = (X, S, \delta)$ if $\delta(h(s), x) = h(\delta'(s, x))$ for every choice of $s \in S'$, $x \in X$.

An automaton A is called autonomous if $\delta(s, x) = \delta(s, y)$ for each $s \in S$ and $x, y \in X$. It should be observed that an automaton A is autonomous iff it is isomorphic to the factor automaton $A/\pi_{\min}$.

An automaton A is called to be directable (or cofinal) if there exists a word $J \in X^*$ such that $|\delta(S, J)| = 1$, where $|\cdot|$ denotes the cardinality. Such words J are called directing.

A directable automaton is called definite if there exists an integer n such that every word, whose length is greater than or equal to n, is directing.

The automata we will be concerned with belong to a class defined by the following properties.

Definition 1. An automaton $A = (X; S, \delta)$ is called correctable if there exists $J \in X^*$ such that $\delta(s, J_1 J) = \delta(s, J_2 J)$ for every state $s \in S$ and every two words $J_1, J_2 \in X^*$ of equal length. Such words J are called correcting.

Note that it is just the case of S. Winograd's automata which are synchronized with probability 1 with respect to the input-induced errors [6].

The automata of this type are capable of "forgetting" all previously occurred errors after accepting a specially selected correcting sequence of inputs. This provides the advantages of their use in technique.

The next assertion follows immediately from A. Ádám's theorem.

Correctability Criterion. An automaton A is correctable iff there exists a (correcting) word $J \in X^*$ such that $|\delta(B, J)| = 1$ for each block $B \in \pi_{\min}(A)$.

This criterion allows us to describe the structure of correctable automata more precisely.

First Decomposition Theorem. An automaton A is correctable iff it can be represented by a parallel composition of autonomous and directable automata.

Sketch of the proof. The following five lemmas imply the sufficiency of the theorem.

The next assertion is obvious:

Lemma 1. Every autonomous automaton is correctable.

Lemma 2. An automaton A is directable iff the following three conditions are fulfilled:

A is correctable,

A is connected,

$p(A)=1$.

Proof. Suppose that $p(A)=1$ holds for a connected correctable automaton A. Then $A/\pi_{\min}$ has only one cycle and this cycle is a loop. Denote by B the set of states $s\in S$ of A such that the natural homomorphism γ of A onto $A/\pi_{\min}$ carries s to the unique cyclic vertex of $A/\pi_{\min}$. It is easy to see that there exists a natural number n such that $\delta(S, I)\subseteq B$ whenever the length of the word I is at least n. The Correctability Criterion implies the existence of a word $J_1\in X^*$ such that $|\delta(B, J)|=1$. Let $J_2\in X^*$ be an arbitrary word whose length is at least n and let J be defined by $J=J_2J_1$. Then J is a directing word.

Conversely, assume that A is a directable automaton. Obviously, A is connected and correctable. Our last aim is to verify $p(A)=1$. We shall show that $p(A)>1$ leads to a contradiction. Indeed, $p(A)>1$ implies the existence of two states $s, t\in S$ such that s and t belong to different cyclic blocks of $\pi_{\min}$. Thus $\delta(s, J)\neq\delta(t, J)$ for every word $J\in X^*$ and the lemma follows.

Lemma 3. ([4]). A product of finitely many correctable automata is correctable.

Lemma 4. ([4]). Every subautomaton of a correctable one is correctable.

Lemma 5. Every homomorphic image (consequently every factor automaton) of a correctable automaton is correctable.

Proof. Consider a homomorphism h: $A\to B$, where A is correctable. Let us start with three states s_1, s_2, s_3 of B such that $\delta_B(s_1, J_1)=s_2$, $\delta_B(s_1, J_2)=s_3$ with some words J_1, J_2 which are of equal length. (Here δ_B means the transition function of B and δ_A denotes the transition function of A). Then obviously

$$\delta_A(s_1', J_1)\in h^{-1}(s_2), \quad \delta_A(s_1', J_2)\in h^{-1}(s_3)$$

for an arbitrary element s_1' of $h^{-1}(s_1)$; thus the correcting word J of A fulfils

$$\delta_A(s_1', J_1J) = \delta_A(s_1', J_2J).$$

Hence

$$\delta_B(s_2, J) = \delta_B(s_3, J)$$

and the lemma follows.

Now we are going to prove the necessity. Let A be a correctable automaton. Consider the following three cases.

1. Let A be strongly connected. Then the partition classes mod $\pi_{\min}(A)$ can be denoted by $B_1, B_2, \ldots, B_q$ in such a manner that $\delta(B_i, x)\subseteq B_{i+1}$ if $1\leqq i\leqq q-1$ and $\delta(B_q, x)\subseteq B_1$ (for each $x\in X$). Let us choose a set $C=\{s_1, s_2, \ldots, s_q\}$ such that $s_i\in B_i$ for each i $(1\leqq i\leqq q)$.

Consider the family of all sets $\delta(C, J)$ where (C is fixed and) J runs through all the elements of X^*. Since A is a finite automaton, this family consists of a finite number of different members. Denote the members of the family by $C_1, C_2, \ldots, C_n$ (the ordering is arbitrary). $C_1, C_2, \ldots, C_n$ are pairwise different (but not necessarily disjoint) state sets and their union $C_1 \cup \ldots \cup C_n$ equals S (otherwise we could get a contradiction to the strongly connectedness of A). It is easy to see that $|C_i \cap B| = 1$ for every choice of C_i and B, where $1 \leqq i \leqq n$ and B is a block mod $\pi_{\min}(A)$. For every choice of C_i and $x \in X$ there exists a unique C_j such that $\delta(C_i, x) = C_j$ $(1 \leqq i \leqq n, 1 \leqq j \leqq n)$.

Consider the automaton $A_d = (X, \{C_i\}, \delta_d)$, where $\delta_d(C_i, x) = C_j$ iff $\delta(C_i, x) = C_j$. Denote $A_a = A/\pi_{\min}$. It is not difficult to see that A_d is a directable automaton. This assertion follows from the Correctability Criterion. Indeed, since A is correctable, then there exists a correcting word $J \in X^*$ such that $|\delta(B, J)| = 1$ for each block B mod $\pi_{\min}(A)$. Consider two arbitrary states C_i, C_j of A_d. Let $C_i = \{s_1, \ldots, s_q\}$ and $C_j = \{s'_1, \ldots, s'_q\}$, where $s_\alpha \in B_\alpha$, $s'_\alpha \in B_\alpha$. Since s_α and s'_α belong to the same block mod $\pi_{\min}(A)$, we have $\delta(s_\alpha, J) = \delta(s'_\alpha, J)$ for each α $(1 \leqq \alpha \leqq q)$. Put $s''_\alpha = \delta(s_\alpha, J)$. Obviously, $\{s''_1, \ldots, s''_q\}$ is a member of the family $C = \{C_1, \ldots, C_n\}$. Let $\{s''_1, \ldots \ldots, s''_q\} = C_k$ where $1 \leqq k \leqq q$. Then

$$\delta_d(C_i, J) = \delta_d(C_j, J) = C_k,$$

hence A_d is directable. Obviously, A_a is autonomous. The mapping from $A_a \times A_d$ to A taking a pair of states $B \in \pi_{\min}(A)$ (this is a state of A_a) and $C_i \in C$ (this is a state if A_d) into the state $s = B \cap C_i$ of A is a required homomorphism.

2. Let A be connected.

Lemma 6. ([4]). Every connected correctable automaton contains a unique strongly connected subautomaton (which is evidently correctable).

Remark 3. It will be noted that Lemma 6 is not valid in general for infinite automata.

Denote by $\tilde{A} = (X, \tilde{S}, \tilde{\delta})$ the strongly connected subautomaton of A. Let $\tilde{A}_a = \tilde{A}/\pi_{\min}(\tilde{A})$ and $\tilde{A}_d = (X, \{\tilde{C}_i\}, \tilde{\delta}_d)$, $\tilde{C}_i \subseteq \tilde{S}$, be autonomous and directable components of $\tilde{A}$, constructed analogously to the previous part of this proof, i.e. $\tilde{A}$ can be represented by a parallel composition of $\tilde{A}_a$ and $\tilde{A}_d$. Consider the automaton $A_d = (X, (S \setminus \tilde{S}) \cup \{C_i\}, \delta_d)$, where

$$\delta_d(b, x) = \begin{cases} \delta(b, x), & \text{if } b \in S \setminus \tilde{S} \text{ and } \delta(b, x) \in S \setminus \tilde{S}; \\ \text{arbitrary } \tilde{C}_i \in \{\tilde{C}_i\} \text{ such that } \delta(b, x) \in \tilde{C}_i, & \text{if } b \in S \setminus \tilde{S} \text{ and } \delta(b, x) \in \tilde{S}; \\ \tilde{\delta}_d(b, x), & \text{if } b \in \{\tilde{C}_i\}. \end{cases}$$

The automaton A_d is directable. Indeed, it is easy to see that there exists a word $J_1 \in X^*$ such that $\delta_d((S \setminus \tilde{S}) \cup \{\tilde{C}_i\}, J_1) \subseteq \{\tilde{C}_i\}$. Let $J_2 \in X^*$ be a correcting word of $\tilde{A}$, hence J_2 is a directing word of $\tilde{A}_d$. Let J be defined by $J = J_1 J_2$. Then J is a directing word of A_d. Denote $A_a = A/\pi_{\min}(A)$. Our last aim is to find a subauomtaton of $A_a \times A_d$ which can be mapped homomorphically onto A. Consider the subautomaton $\bar{A}$ whose states are all the pairs (B, b), where $B \in \pi_{\min}(A)$, $b \in (S \setminus \tilde{S}) \cup \{\tilde{C}_i\}$, such that $B \cap b \neq \emptyset$. Note that $\bar{A}$ really is a subautomaton. Then the mapping $(B, b) \to \to B \cap b$ is a required homomorphism.

3. Let A be an arbitrary correctable automaton. Represent A in the form $A = \sum_\alpha A_\alpha$, where A_α is connected for each α.

Lemma 7. ([4]). Suppose $A = \sum_{i=1}^{n} A_i$; then A is correctable iff A_i is correctable for each $i = 1, \ldots, n$. If A is correctable and $A = \sum_{\alpha} A_\alpha$ (where there is an infinity of summands), then A_α is correctable for each α.

Let A_a^α and A_d^α be automous and directable components of A_α, constructed analogously to the first and second parts of our proof. Then A can be represented by a parallel composition of $\sum_{\alpha} A_a^\alpha$ and $\prod_{\alpha} A_d^\alpha$. It will be noted that a sum of autonomous automata is autonomous and a product of finitely many directable automata is directable. Since every correcting word of A is a directing word of $\prod_{\alpha} A_d^\alpha$, the automaton $\prod_{\alpha} A_d^\alpha$ is directable, even if there were an infinite number of multiplicands. Q.E.D.

Remark 4. It follows from part 1 of the previous proof that every strongly connected correctable automaton is a homomorphic image of a product of strongly connected autonomous and strongly connected directable automata.

A. Ádám's theorem can be applied to the description the other types of automata.

Definition 2. A correctable automaton A is called self-correctable if there exists an integer n such that every word, whose length is greater than or equal to n, is correcting.

The smallest n which satisfies the above condition is called the correction time and denoted by $n(A)$.

Self-correctability Criterion. An automaton A is self-correctable iff there exists an integer n such that $|\delta(B, J)| \doteq 1$ for each block $B \in \pi_{\min}(A)$ and each word $J \in X^n$. The smallest n which satisfies this condition equals $n(A)$.

Second Decomposition Theorem. An automaton A is self-correctable iff it can be represented by a parallel composition of autonomous and definite automata.

The proof might have been arranged analogously to the proof of First Decomposition Theorem, but we are here suggested a simpler way of proving this theorem.

First we establish some preliminary results on self-correctable automata.

The following result is obvious:

Lemma 8. An automaton A is self-correctable iff there exists an integer n such that the equality

$$\delta(s, I_1 J) = \delta(s, I_2 J)$$

holds for every choice of the state s and the words I_1, I_2, J where I_1, I_2 are of equal length and the length of J is n.

Suppose that A is self-correctable, let the smallest possible n $(=n(A))$ be considered (cf. Lemma 8). For every t $(\geqq n)$ we denote by $F_t(s, J)$ the state $\delta(s, IJ)$ where s is a state J is a word of length n and I is an abritrary word whose length is $t-n$.

Supplement to Lemma 8. The sequence of functions

$$F_n, F_{n+1}, F_{n+2}, \ldots \qquad (***)$$

is periodic.

Proof. The set of states of A and the set of words of length n are finite. Thus the sequence $(***)$ contains only a finite number of different members. Let q be the smallest number such that there is a number t_0 for which $n \leqq t_0 < q$ and $F_{t_0} = F_q$ are valid. We can show without difficulty that $F_t = F_{t'}$ implies $F_{t+1} = F_{t'+1}$. Consequently, the sequence $(***)$ is periodic; the length of its period and pre-period are $p = q - t_0$ and t_0, respectively. (In other words: $F_t = F_{t'}$ if and only if $t \geqq t_0$, $t' \geqq t_0$ and $t \equiv t' \pmod p$ are true.)

Remark 5. The period p of $(***)$ equals the period $p(A)$ of the automaton A.

Proof of the Second Decomposition Theorem. Since Lemmas 1—7 are valid for self-correctable automata after replacing the words correctable by self-correctable and directable by definite, then a composition of autonomous and definite automata is self-correctable. Now let A be a self-correctable automaton. Then one can consider the definite component of A as a connection of storage device on a shift register (for preservation of last $n(A)$ inputs) and a set of $p(A)$ devices for computing functions $F_{t_0}, \ldots, F_{t_0+p(A)-1}$. It is easy to see that the definite component is a definite in fact automaton. In this case the autonomous component $A/\pi_{\min}$ determines the function which value corresponds to the present state of A. Q.E.D.

Remark 6. One can show that every strongly connected self-correctable automaton is a homomorphic image of a product of strongly connected autonomous and strongly connected definite automata (cf. Remark 4). M. Ito and J. Duske proved in [3] that every strongly connected definite automaton is a homomorphic image of a shift register. (Recall that a shift register in [3] is an automaton (X, X^n, δ) where X is finite, $n \geqq 1$ and $\delta((x_1, \ldots, x_n), x) = (x_2, \ldots, x_n, x)$ for every choice of $x \in X$, $x_i \in X$, $i = 1, \ldots, n$.) Thus, every strongly connected self-correctable automaton is a homomorphic image of a product of strongly connected autonomous automaton and a shift register.

Now let us estimate the correction time $n(A)$ of self-correctable automata.

Theorem 4. Let A be a self-correctable automaton and let k $(\geqq 0)$ be the smallest number such that each $J \in X^{n(A)+k}$ satisfies:

1) $\delta(S, J) \subseteqq C(S)$; and
2) if $\delta(B_1, J) \subseteqq B$ and $\delta(B_2, J) \subseteqq B$ for some $B_1, B_2, B \in \pi_{\min}(A)$, then

$$\delta(B_1, J) = \delta(B_2, J).$$

Then

$$n(A) + k \leqq |S| - m,$$

where m is the maximum of all cycle lengths of the transition diagram of $A/\pi_{\min}$. If A is a strongly connected self-correctable automaton, then

$$p(A) \times |X|^{n(A)} \geqq |S|.$$

Remark 7. In particular, if A is a connected self-correctable automaton, then

$$n(A) + k \leqq |S| - p(A).$$

Remark 8. Since $J \in X^{n(A)+k}$ and $k \geqq 0$, then $|\delta(B_i, J)| = 1$ in Theorem 4 $(i = 1, 2)$. Therefore one can write $\delta(B_i, J) \in B$ instead of

$$\delta(B_i, J) \subseteqq B.$$

Proof of Theorem 4. Let us first suppose that A is a strongly connected self-correctable automaton. Since $C(S)=S$, then $k=0$ in this case. Given an integer n consider the relation

$$P_n = \{(s_1, s_2)\in S^2\colon\ \delta(s_1, J) = \delta(s_2, J) \text{ for each } J\in X^n\}.$$

It is clear that $P_0=\{(s, s)\colon s\in S\}$. Since A is strongly connected, then $P_n\subseteqq\pi_{\min}(A)$ holds for each $n\geqq 0$. It follows from Self-correctability Criterion that $P_{n(A)}\supseteqq\pi_{\min}(A)$ where $n(A)$ is the correction time of A. Thus, $P_{n(A)}=\pi_{\min}(A)$. Moreover, $P_n\subseteqq P_{n+1}$ and $P_n\neq P_{n+1}$ iff $n<n(A)$. The strong connectedness of A implies that the transition diagram of $A/\pi_{\min}$ is a cycle. Therefore the number of blocks of $\pi_{\min}(A)$ is equal to the period $p(A)$ of A. Obviously, the number of blocks of $P_{n(A)-1}$ is greater than or equal to $p(A)+1$. Similarly, if $0\leqq i\leqq n(A)$ then the number of blocks of $P_{n(A)-i}\ \geqq p(A)+i$. In particular, with $i=n(A)$, we get

$$n(A) \leqq |S|-p(A).$$

Now consider an arbitrary self-correctable automaton A. Without the loss of generality we may restrict ourselves to the case where A is connected.

The proof of Theorem 4 will be continued after verifying a lemma.

Lemma 9. Let A be a connected self-correctable automaton. Then there exists a partition $\pi=\{B_i\}$, $i=1, \dots, n$, of A such that

1) for any i $(1\leqq i\leqq n)$, the elements of $B_1\cup B_2\cup\dots\cup B_i$ form a subautomaton of A; moreover, (X, B_1, δ) is strongly connected and self-correctable;

2) if $1\leqq i\leqq n$ and J is a word whose length is denoted by l, then $\delta(B_i, J)\subseteqq\subseteqq B_1\cup\dots\cup B_{i-l^*}$, where $l^*=\min(l, i-1)$.

Proof. One can choose (using Lemma 6 and the remark at the beginning of the proof of the Second Decomposition Theorem) the unique selfcorrectable strongly connected subautomaton (X, B_1, δ) of A.

Consider a sequence of sets:

$$B_2 = \{s\in S\setminus B_1\colon\ \delta(s, x)\in B_1 \text{ for each } x\in X\};$$

$$B_3 = \{s\in S\setminus(B_1\cup B_2);\ \delta(s, x)\in B_1\cup B_2 \text{ for each } x\in X\};$$

....

$$B_i = \{s\in S\setminus(B_1\cup\dots\cup B_{i-1})\colon\ \delta(s, x)\in B_1\cup\dots\cup B_{i-1} \text{ for each } x\in X\};$$

....

Clearly, one can find an integer m $(\geqq 1)$ such that $B_i=\emptyset$ iff $i>m$. Since the family of disjoint sets $\{B_i\}$, $i=1, \dots m$, satisfies the conditions $1-2$, we only have to prove that $\{B_i\}$ is really a partition, i.e. $\bigcup_{i=1}^{m} B_i=S$. Put $\tilde{S}=S\setminus(B_1\cup\dots\cup B_m)$. We are going to show that $\tilde{S}=\emptyset$.

Assume now that $\tilde{S}\neq\emptyset$. We derive a contradiction from this assumption. It is easy to see that there exists $r\geqq 1$ such that $\delta(S, J)\subseteqq C(S)$ for all words $J\in X^r$. Using the definition of $\tilde{S}$ one also easily obtains that for any $s\in\tilde{S}$ there exists a sequence of words $J_i^s\in X^i$ such that $\delta(s, J_i^s)\in\tilde{S}$ where $i=1, 2, \dots$. Let $\tilde{s}$ be an arbitrary state of $\tilde{S}$, then $s=\delta(\tilde{s}, J_r^s)$ belongs to a cyclic block of $\pi_{\min}(A)$. A moment's consideration shows that there exists a state t of B_1 which belongs to the same block of the m.a.p. Indeed, let the cyclic blocks of $\pi_{\min}(A)$ be denoted by $C_1, C_2, \ldots, C_q$. Since (X, B_1, δ) is a subautomaton of the connected automaton A, $B_1 \cap C_j \neq \emptyset$ for any $j=1, \ldots, q$. Now let $s \in C_j$, we choose an arbitrary state t of $B_1 \cap C_j$. Thus, the states $s \in \tilde{S}$ and $t \in B_1$ belong to the same (cyclic) block of $\pi_{\min}(A)$. Furthermore, $\delta(s, J_i^s) \in \tilde{S}$ for all $i=1, 2, \ldots$ and $\delta(t, J) \in B_1$ for each word $J \in X^*$. Obviously, $B_1 \cap \tilde{S} = \emptyset$. Therefore $\delta(s, J_i^s) \neq \delta(t, J_i^s)$ for every choice of $i=1, 2, \ldots$ (Note that the length of J_i^s equals i). This contradicts the Self-correctability Criterion and the lemma follows.

Proof of Theorem 4 (final part). Now (using Lemma 9) let us select the strongly connected subautomaton $\tilde{A}=(X, B_1, \delta)$ of A and let r $(\geqq 1)$ be the smallest number which satisfies $\delta(S, J) \subseteqq B_1$ for all $J \in X^r$. Clearly $\delta(S, J) \subseteqq C(S)$ when $J \in X^r$ and it follows from Lemma 9 that

$$r \leqq |S \setminus B_1|. \tag{1}$$

Since $\tilde{A}$ is strongly connected, therefore by the previous part of the proof one has

$$n(\tilde{A}) \leqq |B_1| - p(\tilde{A}). \tag{2}$$

It will be noted that

$$p(\tilde{A}) = p(A). \tag{3}$$

Also note that

$$n(A) + k \leqq n(\tilde{A}) + r. \tag{4}$$

Clearly (1), (2), (3) and (4) jointly imply

$$n(A)+k \leqq n(\tilde{A})+r \leqq |B_1| - p(\tilde{A}) + |S \setminus B_1| = |S| - p(A),$$

and the first assertion of Theorem 4 is proved.

But it follows immediately from Remark 6 that

$$p(A) \times |X|^{n(A)} \geqq |S|$$

holds for strongly connected self-correctable automata. Q.E.D.

Now let A be definite, then the smallest n which satisfies:

$$|\delta(S, J)| = 1 \quad \text{for all} \quad J \in X^n$$

is called the degree of A and is denoted by $d(A)$.

Corollary 1. Let A be a definite automaton, then

$$d(A) \leqq |S| - 1.$$

If A is a strongly connected definite automaton, then

$$|X|^{d(A)} \geqq |S|.$$

Proof. If A is definite, then it is self-correctable. By Lemma 2 and remark at the beginning of the proof of the Second Decomposition Theorem one has $p(A)=1$. Since A is definite, it is connected, therefore the maximum m of all cycle lengths of the transition diagram of $A/\pi_{\min}$ equals $p(A)=1$. Finally we show that $d(A)=n(A)+k$ (cf. Theorem 4). It is clear that $d(A) \geqq n(A)+k$. Now let $J \in X^*$ be defined by $J=J_1J_2$ where the length of J_1 equals k and the length of J_2 equals $n(A)$. Then all the states of $\delta(S, J_1)$ belong to the unique cyclic block of $\pi_{\min}(A)$. Therefore, by the Self-correctability Criterion, one has $|\delta(S, J)|=1$. Thus, $d(A)\leqq n(A)+k$ and the first assertion of Corollary 1 follows.

In order to prove the second assertion it will suffice to note that $d(A)=n(A)$ holds for strongly connected definite automata.

Remark 9. The first assertion of Corollary 1 is well-known (e.g. see V. I. Levenshtejn [5, Lemma 11]). In [3] M. Ito and J. Duske obtained the estimation: $|X|^{d(A)}\geqq|S|$.

Although we only dealt with finite automata in this section, some results are valid for arbitrary automata. First, one easily sees that the validity of the Correctability (Self-correctability) Criterion does not depend on the cardinality of the state set and of the input alphabet. One can also prove Decomposition Theorems for arbitrary automata.

A word should be said here about the structure of semigroups of correctable automata. Recall that the semigroup S_A of A is the factor semigroup $X^*/\equiv$ where $J_1\equiv J_2$ iff $\delta(s, J_1)=\delta(s, J_2)$ for all states $s\in S$. It is easy to see that the set of all correcting words forms an ideal of S_A. (Here and elsewhere we do not distinguish between semigroup's elements $J\in S_A$ and corresponding words $J\in X^*$.) If A is finite, then S_A is a finite semigroup, therefore there exists the kernel Ker (S_A) of S_A. One can show that $J\in$ Ker (S_A) iff 1) J is a correcting word; and 2) J satifies conditions 1—2 of Theorem 4. Note that conditions 1—2 of Theorem 4 actually means that the set $\{J, J^2, J^3, \ldots\}$ (where $J^2=JJ$, $J^3=JJJ, \ldots$) forms a subgroup of S_A. Recall that the kernel of an arbitrary compact (in particular, finite) semigroup can be written as a union of pairwise disjoint maximal isomorphic groups: $\mathrm{Ker}=\bigcup_{\alpha} G_\alpha$. The groups G_α are called the group-components of the kernel. If A is a correctable automaton, then each group-component G_α is cyclic and the period of G_α equals $p(A)$. Moreover, $S_A\cdot G_\alpha=G_\alpha$ for each α. One easily see that the semigroups of self-correctable automata possess the following additional property: the equality $S_A\cdot G=G$ holds for any maximal subgroup $G\subseteqq S_A$. In other words, the group-components of S_A (where A is a correctable automaton) are "generalized right zeros" of S_A. If A is self-correctable, then every maximal subgroup of S_A is a "generalized right zero".

Input-induced errors

Here we suggest an equivalent form of A. Ádám's theorem for automata. Let us consider the input-induced errors. Recall that an error (s, t) of storing state t instead of state s is said to be input-induced iff there exist a state v and two words J_1, J_2 of equal length such that $\delta(v, J_1)=s$ and $\delta(v, J_2)=t$. The partition (relation) π is said to be corresponding to the input-induced errors iff π is the smallest (i.e. most refined) partition such that for any input-induced error (s, t) we have $(s, t)\in\pi$. All these concepts were introduced by J. Hartmanis and R. E. Stearns in [2].

Thes next proposition actually was a base of our consideration in the previous section of the paper.

Proposition 3. Let A be an arbitrary (possibly infinite) automaton. Then the partition π corresponding to the input-induced errors is equal to the minimal autonomous partition $\pi_{\min}(A)$.

Note added in Proof. If a connected graph G has at least one semiwalk with positive net length, p of the (unique) cycle of $G/\pi_{\min}$ is equal to the greatest common divisor of all closed semiwalk net lengths of G (G. S. Bloom and S. A. Burr [7, Theorem 3.2]). Elsewise, $G/\pi_{\min}$ has no cycles and consequently $G/\pi_{\min}$ is a directed path (cf. [7, Theorem 3.3]).

Acknowledgement

I am greatly indebted to Professor A. Ádám for many helpful suggestions and comments.

Резюме

Рассматриваются ориентированные графы $G=(V,E)$, где V — множество вершин и E — множество дуг. Разбиение $\pi=\{B_\alpha\}$ множества вершин графа на непересекающиеся блоки $B_\alpha \subseteq V$ называется автономным, если для каждого блока B_α, содержащего хотя бы одну вершину с ненулевой полустепенью исхода, найдется такой блок B_β, что все вершины, достижимые из B_α за один шаг, лежат в B_β. Минимальное автономное разбиение (м.а.р.) графа — это такое его автономное разбиение, которое является собственным подразбиением любого другого автономного разбиения этого графа. Аналоги м.а.р. хорошо известны — это разбиения состояний марковских цепей и автоматов на циклические классы. В нашей работе изучается строение м.а.р. для различных типов ориентированных графов. Мы привели достаточно подробное описание структуры м.а.р. для графов с конечным числом стоков, графов, не содержащих истоков и стоков, а также для сильно связных графов. Можно показать, что м.а.р. графа с конечным числом стоков можно получить из тривиального разбиения этого графа (т.е. разбиения, каждый блок которого содержит в точности одну вершину) путем применения к нему конечного числа операций транзитивного и автономного замыканий, а именно, достаточно $2\times n+2$ таких операций, где n-число стоков графа. При этом для произвольных графов всегда достаточно счетного числа операций. Количество необходимых операций — важная характеристика м.а.р. и его оценкам собственно и посвящена первая часть настоящей статьи.

С помощью м.а.р. оказалось удобным описывать строение автоматов, устойчивых к индуцированным входными искажениями ошибкам. Этим вопросам посвящена вторая часть статьи, где, в частности, решается задача о декомпозиции таких автоматов.

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(Received Dec. 13, 1987)