On the index of concavity of neighbourhood templates

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Abstract

In automatic image analysis with parallel algorithms or parallel processors successive Minkowski-operations (like erosions and dilatations) with a given neighbourhood template (also referred to as structuring element), T, play an important role. It can be shown that, after a certain number of such steps, the neighbourhood template E, which contains only the extreme points of T, can be used instead of T. This number of steps is called the index of concavity of T. In bit-plane oriented parallel processors this fact can be used to speed-up pattern recognition algorithms. The speed-up is only assymptotical and its practical performance depends upon whether the index of concavity is low or high. In this paper it is shown that for the practical cases of convex or small templates the index is very small, namely at most 2 or 3 resp. which ensures speed-up for this type of templates. As against to this result it is, however, also shown that, theoretically, arbitrary high indices of concavity can be achieved for appropriately chosen (exotic) templates.

1. Introduction

Minkowski-operations play an important role in automatic image analysis, particulary in optical material control. Herein, after thresholding the video image (from camera) appropriatelly a binary image, b, (usually 256×256 or 512×512 pixels) is produced. b is also called a bit-plane. The bit-plane b is eroded repeatedly and after each step of erosion a measurement of area, boundary length and/or number of particles is done. Assembling these numbers in one (or 3) feature vector(s), convenient statistical classification procedures can be applied to get final decision of certain material properties. Depending on the material properties to be jugded upon various neighbourhood templates must be chosen (1, 2).

In bit-plane oriented parallel array processors (so called: bitplane processors) (3, 4, 5, 6) a straight forward implementation of this operations needs Ct elementary parallel bitwise logical operations where t is the number of elements in T (1, 2). There it is also shown that in case of convex, symmetric templates $Cu/2$ operations are sufficient where u is the number of boundary points of T. In (7) this result was improved be showing that, for any template T, assymptotically already Ce operations are also sufficient where e is the number of extreme points of T. This result relies on the fact that, after a certain number of steps, the Minkowski-operation with T can be replaced by the same operation using only the template E which contains just the extreme points of T. This number depends on T and is called the *index of concavity* of T. It is denoted $\mu(T)$.It is clear that the assymptotic speed gain is only achieved if $\mu(T)$ is low. It is the intent of this paper to show that for the practically important cases of small (i.e. 3×3) templates or (possibly big) convex templates the index of concavity does not exceed 3 or 2 (resp.). On the other hand, exotic templates (with some few and, however, wide spread points) can yield arbitrarily high indices of concavity.

After presenting some necessary mathematical definitions and facts in chapter 2 we derive our claim as cited above in chapter 3.

2. Basic definitions and facts

Definition 1. Let $\mathbf{Z}$ denote the set of integers. Any finite subset T of $\mathbf{Z}^2$ is called a *neighbourhood template*. Between two templates T and U the sum $T\oplus U$ is defined as $\{t+u/t\in T \text{ and } u\in U\}$ ($+$ is here the usual componentwise vector sum). For any template T the sequence $(kT)_{k\in \mathbf{N}}$ ($\mathbf{N}=\{0, 1, 2, \dots\}$) is recursively defined by

$$0T = \{\mathbf{0}\}, \tag{1}$$

$$(k+1)\,T = kT\oplus T \quad (k \geqq 0). \tag{2}$$

Here, $\mathbf{0}=(0, 0)$ is the 2-dimensional origin in $\mathbf{Z}^2$. $x\in T$ is called an *extreme point* of T, if any representation $x=\sum_{t\in T} a_t t$ with $a_t\geqq 0$, $a_t\in\mathbf{R}$ and $\sum_{t\in T} a_t=1$ implies $a_x=1$, and $a_t=0$ for $t\neq x$. The set of extreme points of T is denoted E or $E(T)$.

Proposition 1. (7) For any template T there is a $k_0\in\mathbf{N}$ such that

$$kT\oplus T = kT\oplus E \quad (k \geqq k_0). \tag{3}$$

Definition 2. For any template T let $\mu(T)$ denote the minimal k_0 such that Proposition 1 holds. $\mu(T)$ is called the *index of concavity* of T.

Definition 3. For any template T let $\bar{T}$ denote the *convex hull* of T (in $\mathbf{R}^2$), formally:

$$\bar{T} := \Big\{\sum_{t\in T} a_t t/a_t \geqq 0,\ a_t\in\mathbf{R},\ \sum_{t\in T} a_t = 1\Big\} \tag{4}$$

and $\dot{T}:=\bar{T}\cap\mathbf{Z}^2$. A template T is called *convex* if $T=\dot{T}$. The norm $\|T\|$ of T is the maximal absolute value of all occuring coordinates of all elements of T. T is called *small*, if $\|T\|\leqq 1$.

Equipped with these preliminaries we proceed to prove our claims.

3. The index of concavity of certain classes of templates

Theorem 1. For any $k\in\mathbf{N}$, there is a neighbourhood template T with $\|T\|\leqq \leqq\sqrt{(k/3)}+5$ such that $\mu(T)\geqq k$.

Proof. Let $k\in\mathbf{N}$, and consider the template $T=\{x_1, x_2, x_3, x_4\}$ with $x_1=(n, 0)$, $x_2=(0, n-1)$, $x_3=(-(n-2), -(n-2))$, and $x_4=(0, 0)$ where n is the greatest odd natural number less than or equal to $\sqrt{(k/3)}+5$. Note that, in all cases, n is odd and not smaller than 5. Let $hT\oplus E=hT\oplus T$. We show that $h\geqq k$. Because

$$\mathbf{0} = (0, 0)\in hT\oplus T, \quad \mathbf{0}\in hT\oplus E.$$

Thus $\mathbf{0}=k_1x_1+k_2x_2+k_3x_3$, where $k_1, k_2, k_3\geqq 0$ and $h+1\geqq k_1+k_2+k_3>0$ ($k_1, k_2, k_3\in\mathbf{N}$). So at least one k_i is greater than 0 and $k_1n=k_3(n-2)=k_3(n-2)=k_2(n-1)$. Because $n, n-1$ and $n-2$ have no common divisor except unity we conclude that $k_1\geqq(n-1)(n-2)$, $k_2\geqq(n-2)n$, and $k_3\geqq n(n-1)$. Thus $h+1\geqq 3(n-2)^2\geqq 3(\sqrt{(k/3)}+ +5-2)^2 \geqq k+1$.

Q.E.D.

Theorem 2. For any small template we have $\mu(T)\leqq 3$.

Proof. The validity of this claim was checked by an appropriate computer program: For all small templates, T, their sets of extreme points, E, were computer and the first k were searched for which $kT\oplus E=kT\oplus T$. One proves easily that these k equal $\mu(T)$.

Q.E.D.

Theorem 3. For any convex template T we have $\mu(T)\leqq 2$.

Proof. A proof can be obtained by combining some partial results of (8) and (9). In (8) it is shown that $\dot{\overline{(d+1)T}}=\dot{\overline{dT}}\oplus E$ for any (d-dimensional) template T which yields, for our case $d=2$, the claim $\dot{\overline{3T}}=\dot{\overline{2T}}\oplus E$. In (9) it is shown that $\dot{\overline{kT}}=kT$ for all 2-dimensional convex templates and any $k\geqq 0$. Thus, we get

$$3T = \dot{\overline{3T}} = \dot{\overline{2T}}\oplus E = 2T\oplus E. \tag{5}$$

This proves our theorem.

Q.E.D.

In case of rectangular convex templates we get even lower indices:

Theorem 4. For the rectangular template $T=\{n, n+1, \ldots, n+i\}\times\{m, m+1, \ldots \ldots, m+j\}$, we have $\mu(T)=1$.

Proof. Let $x=(x_1, x_2)\in 2T$. Then $2n\leqq x_1\leqq 2n+2i$ and, consequently, $n\leqq \leqq(x_1-n)\leqq n+2i$. If $x_1-n>n+i$ then $n-i\leqq x_1-(n+i)\leqq n+i$ and $n\leqq x_1- -(n+i)\leqq n+i$. A similar argument shows that either $m\leqq x_2-(m+j)\leqq m+j$ or $m\leqq x_2-m\leqq m+j$. This proves our theorem because $E(T)=\{n, n+i\}\times\{m, m+j\}$.

Q.E.D.

4. Summary

In a former paper (7) the author had proved that, for any neighbourhood template T, there is a number, $\mu(T)$, such that $kT\oplus T=kT\oplus E$ $\big(k\geqq\mu(T)\big)$ where E is the template containing only the extreme points of T. $\mu(T)$ is called the index of concavity of T. In image analysis with bit-oriented parallel computers this fact can be used to speed-up pattern classification algorithms which make excessive use of

Minkowski-operations like erosion, dilation, opening and closing by appropriately chosen neighbourhood templates. This speed-up is only achieved if $\mu(T)$ is low. In this paper, it is shown that this is, in fact, true for all practically important templates, i.e., for (arbitrary) convex ones and small ones. Nevertheless, exotic templates can be derived having arbitrarily high indices of concavity.

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