On the weak equivalence of Elgot's flow-chart schemata

by Z. ÉSIK

Notions and notations

Algebraic theories were originally introduced in [10]. An equational presentation of algebraic theories can be found in [1]. Following this latter work, by an algebraic theory we shall mean a many-sorted algebra $T=(T(n,p);\ \cdot, \langle\,\rangle, \pi_p^i)$ where n, p are non-negative integers; composition, denoted by $\cdot$ or juxtaposition, maps $T(n,p)\times T(p,q)$ into $T(n,q)$; source-tupling associates a unique element $\langle f_1, \ldots, f_n\rangle \in T(n,p)$ with each family of scalar elements $f_1, \ldots, f_n \in T(1,p)$; finally, there is an injection $\pi_p^i \in T(1,p)$ for each i and p such that $i\in[p]$ $([p]=\{1, \ldots, p\})$. Furthermore, the following identities have to be satisfied by T:

(A$_1$) $f(gh)=(fg)h$ if $f\in T(m,n),\ g\in T(n,p),\ h\in T(p,q)$,

(A$_2$) $f\langle\pi_p^1, \ldots, \pi_p^p\rangle=f$ if $f\in T(n,p)$,

(A$_3$) $\pi_n^i\langle f_1, \ldots, f_n\rangle=f_i$ if $f_1, \ldots, f_n\in T(1,p)$,

(A$_4$) $\langle\pi_n^1 f, \ldots, \pi_n^n f\rangle=f$ if $f\in T(n,p)$.

Although identities (A$_1$), ..., (A$_4$) above are sufficient to characterize algebraic theories, in order to have identity $\langle f\rangle=f$ if $f\in T(1,p)$ we require identity

(A$_5$) $\langle\pi_1^1\rangle=\pi_1^1$ [1].

In case of $n=0$, (A$_4$) means that $T(0,p)$ is a one-element set, its unique element will be denoted by 0_p. It follows from the axioms that elements $1_n=\langle\pi_n^1, \ldots, \pi_n^n\rangle$ are identities with respect to composition. Therefore, an algebraic theory can be viewed as a small category. According to this analogy, we shall often write $f\colon n\to p\in T$ instead of $f\in T(n,p)$.

Pairing, denoted also by $\langle\,\rangle$, and separated sum, which will be denoted by $+$, are frequently used derived operations in algebraic theories. As regards the defini-

[1] In spite of the fact that identity $\langle\pi_1^1\rangle=\pi_1^1$ is used many times by several authors, it is usually not explicited stately. This is the case in [6] and [7], too.

tion of these derived operations cf. [3]. Given a mapping $\varrho\colon [n]\to[p]$, there is a corresponding base element $\varrho\colon n\to p\in T$. It is defined by $\varrho=\langle \pi_p^{\varrho(1)}, \dots, \pi_p^{\varrho(n)}\rangle$. If the mapping ϱ is surjective then the corresponding base element is also called surjective. Injective and bijective base elements are similarly defined. If $\varrho\colon [n]\to[n]$ is bijective then ϱ^{-1} denotes the inverse of ϱ.

Iteration theories were introduced in [2]. They were called generalized iterative theories in [6] and [7].

An iteration theory is an algebraic theory equipped with a new operation, called iteration and usually denoted by $\dagger$. In an iteration theory $I=(I(n,p);\ \cdot, \langle\ \rangle, \pi_p^i, \dagger)$ iteration maps $I(n, n+p)$ into $I(n,p)$. According to [6], iteration theories can be characterized by the following identities:

(B$_1$) $(0_n+f)^\dagger=f$ if $f\colon n\to p\in I$,

(B$_2$) $(f+0_q)^\dagger=f^\dagger+0_q$ if $f\colon n\to n+p\in I$,

(B$_3$) $\langle f,g\rangle^\dagger=\langle h^\dagger, (g\varrho)^\dagger\langle h^\dagger, 1_p\rangle\rangle$ where $f\colon n\to n+m+p\in I$,

$$g\colon m\to n+m+p\in I,\quad \varrho=\langle 0_m+1_n, 1_m+0_n\rangle+1_p,$$

$$h=f\langle 1_n+0_p, (g\varrho)^\dagger, 0_n+1_p\rangle,$$

(B$_4$) $\langle \pi_m^1\varrho g(\varrho_1+1_p), \dots, \pi_m^m\varrho g(\varrho_m+1_p)\rangle^\dagger=\varrho(g(\varrho+1_p))^\dagger$ if

$g\colon n\to m+p\in I$, and $\varrho\colon m\to n\in I$, $\varrho_1, \dots, \varrho_m\colon m\to m\in I$

are base with $\varrho_1\varrho=\dots=\varrho_m\varrho=\varrho$ and ϱ is surjective,

(B$_5$) $f\langle f^\dagger, 1_p\rangle=f^\dagger$ if $f\colon n\to n+p\in I$.

(B$_5$) is called Elgot's fixed-point equation. It was shown in [8] that (B$_5$) is not independent from the other defining identities of iteration theories. Iteration theories are natural generalizations of iterative theories (cf. [3]) and rational algebraic theories (cf. [13]).

Given a ranked alphabet Σ — i.e. $\Sigma=\bigcup(\Sigma_n|n=0,1,\dots)$ with $\Sigma_n\cap\Sigma_m=\emptyset$ if $n\neq m$, and a fixed countable set of variables $X=\{x_1, x_2, \dots\}$, the iteration theory of all (partial infinite) Σ-trees on X play an important role in the fixed-point theory of program schemes. Denote by N the set of natural numbers $\{1,2,\dots\}$ and by X_n the set of the first n variables $\{x_1, x_2, \dots, x_n\}$ for each $n\in N$. Furthermore, denote by A^* the set of all strings over a set A. Then, according to [9], the set of n-ary Σ-trees is the set $T_\Sigma^\infty(X_n)$ consisting of all partial functions $f\colon N^*\Sigma \multimap \bigcup X_n$ satisfying the following condition:

if $f(wi)$ is defined where $w\in N^*$ and $i\in N$ then also $f(w)$ is defined, and there is an integer $m(\geqq i)$ with $f(w)\in\Sigma_m$.

The n-ary Σ-trees give rise to an iteration theory $T_\Sigma^\infty=(T_\Sigma^\infty(n,p);\ \cdot, \langle\ \rangle, \pi_p^i, \dagger)$, where $T_\Sigma^\infty(n,p)=T_\Sigma^\infty(X_p)^n$ $(n,p\geqq 0)$, composition is defined by tree substitution, source-tupling is the tupling of trees, injection π_p^i is the variable x_i considered to be a p-ary tree, and iteration is defined in the following way: let $f=\langle f_1, \dots, f_n\rangle\colon n\to n+p\in T_\Sigma^\infty$ and $g=\langle g_1, \dots, g_n\rangle\colon n\to p\in T_\Sigma^\infty$. Then $f^\dagger=g$ holds provided that

[2] $T_\Sigma^\infty(X_n)$ is denoted by $CT_\Sigma(X_n)$ in [9].

for any $i\in[n]$ and $w\in N^*$, $w\in \operatorname{dom} g_i$ if and only if there exist $r(\geqq 0)$, $i_0(=i)$, $i_1, \ldots, i_r\in[n]$ and $w_j\in \operatorname{dom} f_{i_j}$ $(j=0, \ldots, r)$ such that $f_{i_0}(w_0)=x_{i_1}, \ldots, f_{i_{r-1}}(w_{i_{r-1}})=x_{i_r}$, $f_{i_r}(w_r)\notin X_n$ and $w=w_0\ldots w_r$. Furthermore, in this case

$$g_i(w)=\begin{cases} f_{i_r}(w_r) & \text{if } f_{i_r}(w_r)\in\Sigma \\ x_j & \text{if } f_{i_r}(w_r)=x_{n+j}. \end{cases}$$

Everywhere in the paper $\perp_{np}$ denotes $(1_n+0_p)^\dagger$. In T_Σ^∞, $\perp_{np}=\langle\perp, \ldots, \perp\rangle$ (n-times), where $\perp$ is the totally undefined nullary tree.

By viewing an n-ary operational symbol $\sigma\in\Sigma_n$ as an n-ary tree, Σ can be embedded into T_Σ^∞ in a natural way. Denote by $R_\Sigma=(R_\Sigma(n, p);\ \cdot, \langle\rangle, \pi_p^i, {}^\dagger)$ the subalgebra generated by Σ in T_Σ^∞. R_Σ is freely generated by Σ in the class of all iteration theories. In more detail, any map $\varphi\colon \Sigma\to I$ into an iteration theory I can be uniquely extended to a homomorphism $\bar\varphi\colon R_\Sigma\to I$ provided that φ is a ranked alphabet map, i.e., $\varphi(\Sigma_n)\subseteq I(1, n)$ $(n\geqq 0)$.

Restricting ourselves to finite Σ-trees we obtain the algebraic theory $T_\Sigma=(T_\Sigma(n, p);\ \cdot, \langle\rangle, \pi_p^i)$. In this theory $T_\Sigma(n, p)=T_\Sigma(X_p)^n$ and $T_\Sigma(X_p)=\{f\in T_\Sigma^\infty(X_p)\mid \operatorname{dom} f \text{ is finite}\}$. Note that T_Σ is a subtheory of R_Σ. Let

$$\bar T_\Sigma(X_n)=\{f\in T_\Sigma(X_n)\mid \forall w\in N^*,\ r>0,\ i\in[r],\ f(w)\in\Sigma_r\Rightarrow wi\in \operatorname{dom} f\}.$$

Put $\bar T_\Sigma=(\bar T_\Sigma(n, p);\ \cdot, \langle\rangle, \pi_p^i)$ where $\bar T_\Sigma(n, p)=\bar T_\Sigma(X_p)^n$ $(n, p\geqq 0)$. It is well-known that $\bar T_\Sigma$ is a subtheory of T_Σ and in fact it is freely generated by Σ in the class of all algebraic theories.

The trees in $\bar T_\Sigma(1, p)$ can also be represented as finite strings over the alphabet $\Sigma\cup X_p$. Namely, $\bar T_\Sigma(1, p)$ can be viewed as the smallest set satisfying

(i) $X_p, \Sigma_0\subseteq \bar T_\Sigma(1, p)$,

(ii) if $\sigma\in\Sigma_r$, $r>0$, $f_1, \ldots, f_r\in\bar T_\Sigma(1, p)$ then $\sigma f_1\ldots f_r\in\bar T_\Sigma(1, p)$.

Another interesting iteration theory is the theory $[A]=([A](n, p);\ \cdot, \langle\rangle, \pi_p^i, {}^\dagger)$ on a set A. Here $[A](n, p)$ stands for the set of all partial functions $f\colon A\times[n]\multimap A\times[p]$, $\cdot$ is the composition of partial functions, source-tupling is the source-tupling of partial functions, injection π_p^i is the mapping $a\mapsto(a, i)$ with $A\times[1]$ and A being indentified, finally, if $f\colon A\times[n]\multimap A\times[n+p]$ is a partial function then $f^\dagger$ is the least fixed-point of the mapping $g\mapsto f\langle g, 1_p\rangle$ $(g\colon A\times[n]\to A\times[p])$. Here least means least with respect to the natural ordering of partial functions.

Concerning flow-chart schemata we accept Elgot's definition of flow-chart schemata in [4], with the exception that in order to make iteration to be a totally defined operation rather than a partial one, we allow nodes to be unlabelled in a flow-chart scheme. In this manner, cf. [4], R_Σ becomes the iteration theory of the strong behaviours of finite flow-chart schemata on a ranked alphabet Σ. Therefore, we may treat flow-chart schemata on Σ as elements of R_Σ.

From now on we fix a ranked alphabet Σ with $\Sigma_n=\emptyset$ if $n\neq 1$ and $n\neq 2$, and denote Σ_1 and Σ_2 by Ω and Π, resp. Ω is called the set of action symbols and Π the set of predicate symbols. Furthermore, we shall assume that Π is finite, say $\Pi=\{\pi_1, \ldots, \pi_r\}$. Given a set A, by an interpretation $\mathscr{I}$ of Σ in A we mean any ranked alphabet map $\mathscr{I}\colon \Sigma\to[A]$ such that $\mathscr{I}(\pi)$ is a total predicate for each $\pi\in\Pi$. That is, if $\mathscr{I}(\pi)(a)=(b, i)$ $(a, b\in A,\ i\in[2])$ then $a=b$, and $\mathscr{I}(\pi)$ is totally defined.

Denote by $\mathscr{I}$ the unique homomorphic extension of $\mathscr{I}$ from R_Σ into $[A]$, as well. We say that $f, g \in R_\Sigma(n, p)$ $(n, p \geqq 0)$ are equivalent under $\mathscr{I}$ provided that $\mathscr{I}(f) = \mathscr{I}(g)$ holds. Moreover, f and g are called weakly equivalent, written $f \equiv g$, if $\mathscr{I}(f) = \mathscr{I}(g)$ holds for every interpretation $\mathscr{I}$ (cf. [4], [11], [12]).

Relation $\equiv$ is a congruence relation of the iteration theory R_Σ. The problem we are going to solve is the presentation of a generating system of this relation. If such a system is found then this system together with the defining identities of iteration theories can be viewed as an axiom system for the weak equivalence of finite flow-chart schemata on Σ.

A generating system of the relation $\equiv$

In the sequel we shall frequently use some consequences of the defining identities of iteration theories. Among these identities there are identities of poor algebraic theories, which will be used without any reference. In the other part of these identities we have identities involving the † operation, and they are listed here:

(B$_6$) $(\varrho f(\varrho^{-1}+1_p))^\dagger = \varrho f^\dagger$ if $f\colon n \to n+p$ and $\varrho\colon n \to n$ is bijective,

(B$_7$) $\langle f\langle 1_{n+m}+0_{k+p}, h, 0_{n+m+k}+1_p\rangle, g, h\rangle^\dagger = \langle f, g, h\rangle^\dagger$ if $f\colon n \to n+m+k+p$, $g\colon m \to n+m+k+p$, $h\colon k \to n+m+k+p$,

(B$_8$) $(f(1_n+g))^\dagger = f^\dagger g$ where $f\colon n \to n+p$, $g\colon p \to q$,

(B$_9$) $(1_n+0_1)\langle \bar a_1, \dots, \bar a_n, \pi_n^j+0_{1+p}\rangle^\dagger = \langle a_1, \dots, a_n\rangle^\dagger$ where $a_1, \dots, a_n\colon 1 \to n+p$, $\bar a_k = a_k(1_n+0_1+1_p)$ if $k \neq i$, $\bar a_i = a_i(\langle 1_{j-1}+0_{n+2-j}, \pi_{n+1}^{n+1}, 0_j+1_{n-j}+0_1\rangle + 1_p)$, $i, j \in [n]$,

(B$_{10}$) $(1_n+0_m)\langle f(1_n+0_m+1_p), g\rangle^\dagger = f^\dagger$ if $f\colon n \to n+p$, $g\colon m \to n+m+p$,

(B$_{11}$) $\pi_{n+1}^1\langle \pi_{n+1+p}^2, 0_1+f\rangle^\dagger = \pi_n^1 f^\dagger$ if $f\colon n \to n+p$.

Now we present the system (C) and prove that this system constitutes a generating system of the weak equivalence relation. (C) consists of the following pairs, written as equalities:

(C$_1$) $\pi\langle f, f\rangle = f$ if $\pi \in \Pi$, $f\colon 1 \to p \in R_\Sigma$,

(C$_2$) $\pi\langle \pi'\langle f_1, f_2\rangle, \pi'\langle f_3, f_4\rangle\rangle = \pi'\langle \pi\langle f_1, f_3\rangle, \pi\langle f_2, f_4\rangle\rangle$ where $\pi, \pi' \in \Pi$, $f_1, \dots, f_4\colon 1 \to p \in R_\Sigma$,

(C$_3$) $\pi\langle \pi\langle f_1, f_2\rangle, \pi\langle f_3, f_4\rangle\rangle = \pi\langle f_1, f_4\rangle$ where $\pi \in \Pi$, $f_1, \dots, f_4\colon 1 \to p \in R_\Sigma$,

(C$_4$) $f = \perp$ if $f\colon 1 \to 0 \in R_\Sigma$,

(C$_5$) $f^\dagger = f\langle \perp_{1p}, 1_p\rangle$ if $f\colon 1 \to 1+p \in T_\Pi$.

Denote by θ the congruence relation induced by (C) in R_Σ. The following statement is immediate by $(C)\subseteqq\equiv$.

Lemma 1. $\theta\subseteqq\equiv$.

Later on the following statement will be frequently used.

Lemma 2. Let $f\colon n\to n+m+p+q$, $g\colon m\to n+m+p+q$, $h\colon p\to n+m+p+q$ be arbitrary elements in R_Σ. Assume that $(g(\langle 0_m+1_n, 1_m+0_n\rangle+1_{p+q}))^\dagger\theta\bar{g}$ holds where $\bar{g}\colon m\to n+p+q\in R_\Sigma$. Then also $\langle f, g, h\rangle^\dagger\theta\langle f, \bar{g}(1_n+0_m+1_{p+q}), h\rangle^\dagger$.

Proof. First suppose that $p=0$ and let $\varrho=\langle 0_m+1_n, 1_m+0_n\rangle+1_q$. Then $\langle f, g\rangle^\dagger=\langle a^\dagger, (g\varrho)^\dagger\langle a^\dagger, 1_q\rangle\rangle$ follows by (B_3), where $a=f\langle 1_n+0_q, (g\varrho)^\dagger, 0_n+1_q\rangle$. Put $\bar{a}=f\langle 1_n+0_q, \bar{g}, 0_n+1_q\rangle$. As $(g\varrho)^\dagger\theta\bar{g}$, also $a\theta\bar{a}$ and $\langle f, g\rangle^\dagger\theta\langle\bar{a}^\dagger, \bar{g}\langle\bar{a}^\dagger, 1_q\rangle\rangle$. However, $\langle\bar{a}^\dagger, \bar{g}\langle\bar{a}^\dagger, 1_q\rangle\rangle=\langle f, \bar{g}(1_n+0_m+1_q)\rangle^\dagger$ follows by (B_1) and (B_3).

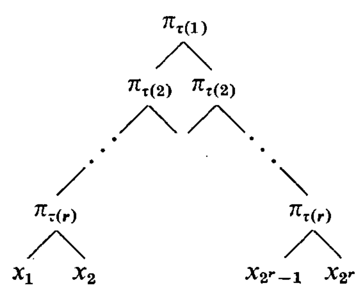
If $p>0$ then define $f_1=f(\alpha+1_q)$, $g_1=g(\alpha+1_q)$ and $h_1=h(\alpha+1_q)$ where $\alpha=1_n+\langle 0_p+1_m, 1_p+0_m\rangle$. Then $(g_1(\langle 0_m+1_{n+p}, 1_m+0_{n+p}\rangle+1_q))^\dagger\theta\bar{g}$ holds by $(\alpha+1_q)\cdot(\langle 0_m+1_{n+p}, 1_m+0_{n+p}\rangle+1_q)=\langle 0_m+1_n, 1_m+0_n\rangle+1_{p+q}$. Thus, $\langle f_1, h_1, g_1\rangle^\dagger\theta$ $\theta\langle f_1, h_1, \bar{g}(1_{n+p}+0_m+1_q)\rangle^\dagger$ by the previous case. From this the result follows by

$$(B_6)\colon\ \langle f, g, h\rangle^\dagger=\alpha\alpha^{-1}\langle f, g, h\rangle^\dagger=$$

$$=\alpha(\alpha^{-1}\langle f, g, h\rangle(\alpha+1_q))^\dagger=\alpha\langle f_1, h_1, g_1\rangle^\dagger\theta\alpha\langle f_1, h_1, \bar{g}(1_{n+p}+0_m+1_q)\rangle^\dagger=$$

$$=(\alpha\langle f_1, h_1, \bar{g}(1_{n+p}+0_m+1_q)\rangle(\alpha^{-1}+1_q))^\dagger=\langle f, \bar{g}(1_n+0_m+1_{p+q}), h\rangle^\dagger.$$

Let $\tau\colon[r]\to[r]$ be any bijection. We shall denote by $\bar{\pi}_\tau\colon 1\to 2^r\in T_\Pi$ the balanced tree visualized in the following figure:



In the case that τ is the identity mapping, the index τ will be omitted in $\bar{\pi}_\tau$.

Lemma 3. For any $f\colon 1\to p\in\bar{T}_\Pi$ there exists a (unique) base element $\varrho\colon 2^r\to p$ with $f\theta\bar{\pi}\varrho$.

Proof. This statement is well-known. In spite of this, for the sake of completeness, a proof will be outlined here. We shall show a little bit more than it is stated by our lemma. Namely, we show that for any $f\colon 1\to p\in\bar{T}_\Pi$ and bijective $\tau\colon[r]\to[r]$ there is a base element $\varrho\colon 2^r\to p$ with $f\theta\bar{\pi}_\tau\varrho$.

If $f=x_i$ ($i\in[p]$) then put $\varrho=\langle\overbrace{\pi_p^i, \dots, \pi_p^i}^{2^r\text{-times}}\rangle$. Then $f\theta\bar{\pi}_\tau\varrho$ follows by applications of (C_1). We proceed by structural induction of f. Suppose that $f=\pi_i f_1 f_2$, $f_1\theta\bar{\pi}_\alpha\varrho_1$

and $f_2\theta\bar{\pi}_\alpha\varrho_2$ where $i\in[r]$ and α: $[r]\to[r]$ is any bijection with $\alpha(1)=i$. Then, by (C_3), we obtain $f\theta\bar{\pi}_\alpha\varrho'$ for a suitable ϱ': $2^r\to p$. However, $\bar{\pi}_\alpha\varrho'\theta\bar{\pi}_\tau\varrho$ holds by (C_2) for a satisfactory choise of ϱ.

Lemma 4. $(f+0_p)^\dagger\theta\perp_{np}$ holds for every f: $n\to n\in R_\Sigma$.

Proof. By (B_2) it is enough to deal with the case $p=0$. If $n=0$ then the statement is obviously valid by $R_\Sigma(0,p)=\{0_p\}$. Assuming $n>0$ we have $f=\langle f_1,f_2\rangle$ where $f_1=(1_1+0_{n-1})f$, $f_2=(0_1+1_{n-1})f$. Thus, by (B_3), $f^\dagger=\langle h^\dagger,(f_2\varrho)^\dagger h^\dagger\rangle$, where $\varrho=\langle 0_{n-1}+1_1, 1_{n-1}+0_1\rangle$, $h=f_1\langle 1_1,(f_2\varrho)^\dagger\rangle$. As $h\in R_\Sigma(1,1)$, $h^\dagger\theta\perp$ holds by (C_4). Therefore, $\pi_n^1 f^\dagger\theta\perp$. From this the result follows by (B_6).

Lemma 5. Given f: $n\to n+p\in T_\Pi$ there exists a g: $n\to p\in T_\Pi$ with $f^\dagger\theta g$.

Proof. The statement is obvious if $n=0$. Now assume that $n>0$ and proceed by induction on n. Define $f_1=(1_1+0_{n-1})f$, $f_2=(0_{n-1}+1_1)f$. By (C_5) and the induction hypothesis there exist $\bar{f}_1$: $1\to n-1+p$, $\bar{f}_2$: $n-1\to 1+p$ with $f_1^\dagger\theta\bar{f}_1$ and $(f_2(\langle 0_{n-1}+1_1, 1_{n-1}+0_1\rangle+1_p))^\dagger\theta\bar{f}_2$. Therefore, $f^\dagger\theta\langle 0_1+\bar{f}_1,\bar{f}_2(1_1+0_{n-1}+1_p)\rangle^\dagger$ holds by Lemma 2. By identity (B_7),

$$\langle 0_1+\bar{f}_1,\bar{f}_2(1_1+0_{n-1}+1_p)\rangle^\dagger = \langle \bar{f}_1\langle\bar{f}_2,0_1+1_p\rangle(1_1+0_{n-1}+1_p),\bar{f}_2(1_1+0_{n-1}+1_p)\rangle^\dagger.$$

Now let $\bar{h}=\bar{f}_1\langle\bar{f}_2,0_1+1_p\rangle$ and apply identity (B_8): $(\bar{h}(1_1+0_{n-1}+1_p))^\dagger=0_{n-1}+\bar{h}^\dagger$. As $\bar{h}\in T_\Pi(1,1+p)$ there is an element h: $1\to p\in T_\Pi$ with $\bar{h}^\dagger\theta h$. Thus,

$$f^\dagger\theta\langle 0_n+h,\bar{f}_2(1_1+0_{n-1}+1_p)\rangle^\dagger$$

is valid by Lemma 2. Put $g=\langle h,\bar{f}_2\langle h,1_p\rangle\rangle$. Then, by ($B_1$), ($B_3$) and ($B_8$), $\langle 0_n+h,\bar{f}_2(1_1+0_{n-1}+1_p)\rangle^\dagger=(0_n+g)^\dagger=g$. As $g\in T_\Pi(n,p)$, this proves Lemma 5.

Definition 1. Let $f=\langle f_1,\dots,f_n\rangle$: $n\to n+p\in T_\Sigma$ and let $i,j\in[n]$ be arbitrary. We say that f_i directly depends on f_j if there is an occurrence of variable x_j in f_i, i.e., $f_i(w)=x_j$ holds for some $w\in N^*$. The dependency relation is the transitive closure of direct dependency. A component f_i is called coaccessible provided that either there is an occurrence of a variable from $\{x_{n+1},\dots,x_{n+p}\}$ in f_i or there is an integer j with f_i depends on f_j and f_j is coaccessible.

Lemma 6. Suppose that f: $n\to n+p\in T_\Sigma$. Then there is an element g: $n\to n+p\in T_\Sigma$ which only contains coaccessible or undefined components, and such that $f^\dagger\theta g^\dagger$ holds.

Proof. Put $f=\langle f_1,\dots,f_n\rangle$ and let $f_{i_1},\dots,f_{i_m}$ $(1\leqq i_1<\dots<i_m\leqq n)$ be all those components of f which are not coaccessible. First suppose that $i_j=j$ holds for each $j\in[m]$. In this case there is an element a: $m\to m\in T_\Sigma$ with $(1_m+0_{n-m})f=$ $=a+0_{n-m+p}$. Thus, $f^\dagger\theta g^\dagger$ holds by Lemma 4 and Lemma 2, where

$$g = \langle \perp_{mn+p},(0_m+1_{n-m})f\rangle.$$

On the other hand, g only contains coaccessible or undefined components.

The general case, where $i_1,\dots,i_m$ are arbitrary, is reducible to the previous one by (B_6).

Definition 2. An element $a\colon n\to n+p\in T_\Sigma$ $(n\geq 1)$ is in normal form provided that each of its components $\pi_n^i a$ has one of the following four forms for every $i\in[n]$:

(i) $\pi_n^i a = \bar{\pi}\varrho+0_p$ where $\varrho\colon 2^r\to m$ is base,
(ii) $\pi_n^i a = \omega\varrho+0_p$ where $\omega\in\Omega$ and $\varrho\colon 1\to n$ is base,
(iii) $\pi_n^i a = \perp_{1n+p}$,
(iv) $\pi_n^i a = 0_n+\varrho$ where $\varrho\colon 1\to p$ is base.

Furthermore, a is required to satisfy all conditions (v), (vi), (vii) and (viii) as well:

(v) if $\pi_n^i a$ is of type (i) then $\pi_n^j a$ has to have one of the forms (ii), (iii) or (iv) for each $j\in\varrho([2^r])$,
(vi) if $\pi_n^i a$ is of type (ii) then $\pi_n^{\varrho(i)} a$ is of type (i),
(vii) $\pi_n^1 a$ is of type (i),
(viii) every component $\pi_n^i a$ of type (ii) is coaccessible.

Lemma 7. For every $f\colon 1\to p\in R_\Sigma$ there is an element $y\colon k\to k+p$ in normal form such that $f\theta\pi_k^1 y^\dagger$.

Proof. By a simple modification of Theorem 2.5.1 in [5] we obtain that there is an element $a\colon n\to n+p\in T_\Sigma$ with $f=\pi_n^1 a^\dagger$ and such that each of its components $\pi_n^i a$ $(i\in[n])$ has one of the three forms (ii), (iii) or (iv), or $\pi_n^i a=a_i+0_p$ holds for some $a_i\in\bar{T}_\Pi(1,n)$. Furthermore, by identity (B_9), we may assume a to satisfy the following modified version of (vi): if $\pi_n^i a=\omega\varrho+0_p$ for some $\omega\in\Omega$ and $\varrho\colon 1\to n$ then $\pi_n^{\varrho(1)} a=\pi_{n+p}^{j}$ is valid for an integer $j\in[n]$. Finally, we may assume that $\pi_n^n a= =\perp_{1n+p}$ since otherwise a can be replaced by $\langle a(1_n+0_1+1_p),\ \perp_{1n+1+p}\rangle$ (cf. (B_{10})).
Let $i_1,\dots,i_m\in[n]$ $(i_1<\dots<i_m)$ be all those indices for which $\pi_n^{i_j} a$ is in $\bar{T}_\Pi(1,n+p)-\{\pi_{n+p}^{n+1},\dots,\pi_{n+p}^{n+p}\}$. First suppose that $i_j=j$ holds for each j. Put $b_i=(1_m+0_{n-m})a$, $c=(0_m+1_{n-m})a$. Then $a=\langle b_1, c\rangle$ holds obviously. Observe that $b_1=\bar{b}_1+0_p$ holds for some $\bar{b}_1\colon m\to n\in\bar{T}_\Pi$. Therefore, by Lemma 5 and (B_2), there exists $b_2\colon m\to n-m\in T_\Pi$ with $b_1^\dagger\theta b_2+0_p$. Thus, by Lemma 2, $\langle 0_m+b_2+0_p, c\rangle^\dagger\theta a$. There is an element $b_3\colon m\to n-m+1\in\bar{T}_\Pi$ with $b_2=b_3(1_{n-m}+\perp)$. Put $b_4=b_3\langle 1_{n-m}, \pi_{n-m}^{n-m}\rangle$. Then $\langle 0_m+b_2+0_p, c\rangle^\dagger=\langle 0_m+b_4+0_p, c\rangle^\dagger$ follows by (B_7) and $\pi_n^n a=\perp_{1n+p}$. On the other hand, by Lemma 3, we have

$$\langle 0_m+b_4+0_p, c\rangle^\dagger\theta\langle 0_m+b_5+0_p, c\rangle^\dagger$$

for some $b_5\colon m\to n-m$ whose each component is of type $\bar{\pi}\varrho$ for a suitable base element $\varrho\colon 2^r\to n-m$. Next, by an application of Lemma 6, we get an element $d\colon n\to n+p$ whose each component is either coaccessible or undefined, and $\langle 0_m+b_5+0_p, c\rangle^\dagger\theta d^\dagger$ holds. It follows from the proof of Lemma 6 that d satisfies all conditions in Definition 2 except possibly (vii). If d does not satisfy (vii) then let $g=\langle\bar{\pi}\varrho+0_p, 0_1+d\rangle$ where $\varrho\colon 2^r\to n+1$ is defined by $\varrho(i)=2$, $i\in[2^r]$. Otherwise put $g=d$. In both cases g is in normal form and $\pi_n^1 g^\dagger\theta f$ (cf. (B_{11}) and Lemma 3).
The general case, i.e. where $i_1,\dots,i_m$ are arbitrary, is reducible to the special one above (cf. (B_6)).

Lemma 8. Let $a\colon n\to n+p\in T_\Sigma$ and $b\colon m\to m+p\in T_\Sigma$ be in normal form. Then $\pi_1^n a^\dagger\equiv\pi_m^1 b^\dagger$ if and only if $\pi_n^1 a^\dagger=\pi_m^1 b^\dagger$.

Proof. Sufficiency is obvious. Conversely, let $f=\pi_n^1 a^\dagger$, $g=\pi_m^1 b^\dagger$ and suppose that $f\equiv g$. Define $\bar{f}\colon N^*\multimap\Sigma^*$ by $\bar{f}(\lambda)=f(\lambda)$ and $\bar{f}(wi)=\bar{f}(w)f(i)$ if $w\in N^*$ and

$i\in[n]$. As $f\equiv g$ and a, b are in normal form, $f^{-1}(x_i)=g^{-1}(x_i)$, i.e. $\{w|f(w)=x_i\}=$ $=\{w|g(w)=x_i\}$ holds for any $i\in[p]$. Furthermore, if $w\in\bigcup(f^{-1}(x_i)|i\in[p])$ then $\bar{f}(w)=\bar{g}(w)$ where $\bar{g}$ is similarly defined with respect to g as $\bar{f}$ was defined with respect to f. The above equalities are essentially known from [4] (cf. also [11], [12]).

Suppose that $f\neq g$. Then, as $f^{-1}(x_i)=g^{-1}(x_i)$ holds for each $i\in[p]$, there is a string $w\in N^*$ with $f(w)\neq g(w)$ and both $f(w)$ and $g(w)$ are in Ω or one of them is undefined. Thus two cases arise. However, similar order of ideas yields a contradiction in both cases. Therefore we assume that $f(w)\in\Omega$. By the last condition in the definition of normal forms, there is a string $v\in N^*$ with $wv\in\bigcup(f^{-1}(x_i)|i\in[p])$. As $f(w)\neq g(w)$ also $\bar{f}(wv)\neq\bar{g}(wv)$. This is a contradiction.

Now we are ready to state our

Theorem. $\theta=\equiv$.

Proof. $\theta\subseteq\equiv$ is valid by Lemma 1. Conversely, it is enough to show that $f\equiv g$ implies $f\theta g$ for arbitrary $f, g\in R_\Sigma(1,p)$. But this is immediate by Lemma 7 and Lemma 8.

An equational characterization of the strong equivalence of Elgot's flow-chart schemata was given in [6]. Here we present an equational characterization for the weak equivalence. An extended abstract of this paper has been already appeared in [14].

References

[1] Bloom, S. L. and C. C. Elgot, The existence and construction of free iterative theories, J. Comput System Sci. v. 12, 1976, pp. 305—318.
[2] Bloom, S. L., C. C. Elgot and J. B. Wright, Vector iteration in pointed iterative theories, SIAM J. Comput., v. 9. 1980, pp. 525—540.
[3] Elgot, C. C., Monadic computation and iterative algebraic theories, in Logic Colloquium'73, eds. Rose, H. E. and J. C. Shepherdson, v. 80, Studies in Logic, North Holland, Amsterdam, 1975, pp. 175—230.
[4] Elgot, C. C., Structured programming with and without GO TO statements, IEEE Trans. Soft. Engineering, v. SE—2, 1976, pp. 41—53.
[5] Elgot, C. C., S. L. Bloom and R. Tindell, On the algebraic structure of rooted trees, J. Comput. System Sci., v. 16, 1978, pp. 362—399.
[6] Ésik, Z., Identities in iterative and rational algebraic theories, Computational Linguistics and Computer Languages, v. XIV, 1980, pp. 183—207.
[7] Ésik, Z., On generalized iterative theories, Computational Linguistics and Computer Languages, v. XV, 1982, pp. 95—110.
[8] Ésik, Z., Algebras of iteration theories, J. Comput. Syst. Sci. 27 (1983), 291—303.
[9] Goguen, J. A., J. W. Thatcher, J. W. Wagner and J. B. Wright, Initial algebraic semantics and continuous algebras, J. Assoc. Comput. Mach., v. 24., 1977, pp. 68—95.
[10] Lawvere, F. W., Functorial semantics of algebraic theories, Proc. Nat. Acad. Sci. USA 50, 1963, pp. 869—872.
[11] Luckham A., A. Park and A. Paterson, On formalized computer programs, J. Comput. Syst. Sci., v. 4., 1970, pp. 220—249.
[12] Rutledge, J. D., On Ianov's program schemata, J. Ass. Comput. Mach., v. 11., 1964, pp. 1—9.
[13] Wagner, E. G., J. B. Wright, J. A. Goguen and J. W. Thatcher, Some fundamentals of order-algebraic semantics, in Proceedings, Mathematical Foundations of Computer Science, 1976, ed. Mazurkievicz, A., LNCS 45, 1976, pp. 151—168.
[14] Ésik, Z., On Elgot's flow-chart schemes, Syst. Theoretical Aspects in Comp. Sci., Salgótarján, 1982, pp. 99—102.

Received Oct. 12, 1982