On identities preserved by general products of algebras

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Equational classes of automata (i.e. unoids) obtained by general product were characterized in [1]. Here we present similar results for tree automata, i.e., arbitrary algebras. We show that the main result $K^{**}=HSP_g(K)=HSP_{\alpha_0}(K)=HSPP_{f\alpha_0}(K)$ in [1] remains valid in this generality, too.

First we briefly introduce the basic notions to be used. For all unexplained notions coming from universal algebra and tree-automata theory the reader is referred to [3] and [2].

By a rank-type we mean an arbitrary subset R of the set of nonnegative integers. A type corresponding to a rank-type R is a collection of operational symbols $F=\amalg(F_k|k\geqq 0)$ such that $F_k\neq\emptyset$ if and only if $k\in R$. In the sequel we fix a ranktype R and by a type always mean a type corresponding to R.

Algebras of type F constitute a similarity class $\mathcal{K}_F$. An algebra $\mathfrak{A}\in\mathcal{K}_F$ is a pair $(A, \{f_{\mathfrak{A}}|f\in F\})$ — (A, F) for short —, where $f_{\mathfrak{A}}$ is a k-ary operation on the nonvoid set A for any $f\in F_k$. By a class of algebras we shall mean an arbitrary nonvoid class of algebras.

We are going to deal with certain products of algebras. Let I be a nonvoid set linearly ordered by $\leqq$. Given a system $\mathfrak{A}_i=(A_i, F_i)$ $(i\in I)$ of algebras, by a general product we mean an algebra $\mathfrak{A}=(A, F)=\Pi(\mathfrak{A}_i, \varphi|i\in I)$, where $A=\Pi(A_i|i\in I)$, φ is a family of mappings of $(\Pi(A_i|i\in I))^k\times F_k$ into $\Pi((F_i)_k|i\in I)$, and finally, the operations in $\mathfrak{A}$ are defined in accordence with φ as follows. Let $a_1, \ldots, a_k, a\in A, f\in F_k$. Then, $f_{\mathfrak{A}}(a_1, \ldots, a_k)=a$ if and only if $a_i=(f_i)_{\mathfrak{A}_i}(a_{1i}, \ldots, a_{ki})$ holds for every $i\in I$ with $f_i=(\varphi(a_1, \ldots, a_k, f))_i=\varphi_i\ (a_1, \ldots, a_k, f)$. If for every nonnegative integer k, $\varphi_i(a_1, \ldots, a_k, f)$ depends on f and $a_{1j}, \ldots, a_{kj}$ with $j<i$ only, then $\mathfrak{A}$ is a so called α_0-product of the $\mathfrak{A}_i$-s. We shall denote by P_g and P_{α_0} the operators corresponding to the formations of general and α_0-products, resp. $P_{f\alpha_0}$ will denote the formation of finite α_0-products. Finite α_0-products will be written as $\Pi(\mathfrak{A}_1, \ldots, \mathfrak{A}_n, \varphi)$ where $I=\{1, \ldots, n\}$ with the usual ordering. The operators H, S and P have their usual meaning.

Also we fix a countable set $X=\{x_1, x_2, \ldots\}$ of variables and treat polynomial symbols of type F as trees built on X and F. T_F will denote the set of all trees of type F. If $\mathfrak{A}\in\mathcal{K}_F$ and $p\in T_F$ then $p_{\mathfrak{A}}\colon A^\omega\to A$ is the polynomial induced by p in $\mathfrak{A}$. If $a_1, a_2, \ldots$ is an ω-sequence of elements of A then $p_{\mathfrak{A}}(a_1, a_2, \ldots)$ denotes the value of $p_{\mathfrak{A}}$ on $a_1, a_2, \ldots$. If $\mathfrak{A}$ is the general product described previously then we can view φ as a mapping of $(\Pi(A_i|i\in I))^\omega\times T_F$ into $\Pi(T_{F_i}|i\in I)$ in a natural way. For each index $i\in I$ we shall denote by φ_i the i-th component-map of φ, as well.

The notion of subtrees of a tree p as well as the height $h(p)$ of a tree will be used in an unexplained but obvious way. A subtree q of a tree p is called proper if $q\neq p$. sub (p) denotes the set of all proper subtrees of p. Also we shall in a natural way speak about an occurence of a subtree in a tree, and about the substitution of a tree for occurences of a subtree in a tree. If p is a tree then rt (p) denotes the root of p.

By a relabeling we mean any mapping $\varphi\colon T_F\to T_{F'}$ with the following properties:

(i) if $p\in F_0$ then $\varphi(p)\in F_0'$,

(ii) if $p\in X$ then $\varphi(p)=p$,

(iii) if $p=f(p_1, ..., p_k)$ with $f\in F_k$, $k>0$, $p_1, ..., p_k\in T_F$ then there exist an $f'\in F_k'$ such that $\varphi(p)=f'(\varphi(p_1), ..., \varphi(p_k))$.

Now we are in the position to give the most fundamental definitions. Let K be an arbitrary class of algebras. Then $K^*=\{K_F^*|F$ is a type$\}$, where K_F^* is the set of all identities $p=q$ $(p, q\in T_F)$ such that $\varphi(p)=\varphi(q)$ is in the usual sence a valid identity in $K\cap\mathcal{K}_{F'}$ for any relabeling $\varphi\colon T_F\to T_{F'}$. An algebra $\mathfrak{A}\in\mathcal{K}_F$ is in K^{**} if and only if all identities belonging to K_F^* are valid in $\mathfrak{A}$. Thus, $K^{**}\cap\mathcal{K}_F$ is an equational class of algebras. If $p, q\in T_F$, we write $K^*\models p=q$ to mean that $K_F^*\models p=q$.

If we consider unoids, i.e. we take $R=\{1\}$, then for any type F and $p, q\in T_F$ we have $p=q\in K_F^*$ if and only if $p=q$ is valid in the equational class $HSP_{\alpha_0}(K)\cap\mathcal{K}_F$. Consequently, $K^{**}=HSP_{\alpha_0}(K)$, or even, $K^{**}=HSP_g(K)=HSP_{\alpha_0}(K)=HSPP_{f\alpha_0}(K)$ (cf. [1]).

In general, the first statement fails to hold. Indeed, take $R=\{1, 2\}$ and for every type F let $K\cap\mathcal{K}_F$ be the equational class determined by the identities $g(x_1)=h(x_1)$ $(g, h\in F_1)$. Supposing $f\in F_2$, identity $f(g(x_1), g(x_1))=f(h(x_1), h(x_1))$ is obviously valid in $HSP_{\alpha_0}(K)\cap\mathcal{K}_F$, but this identity is not in K_F^*. However, we still have a somewhat weaker result:

Theorem 1. Let $p, q\in T_F$ be arbitrary trees of type F. Then $p=q$ is a valid identity in an equational class $HSP_{\alpha_0}(K)\cap\mathcal{K}_F$ if and only if $K^*\models p=q$.

Proof. Sufficiency follows by observing that general product preserves K^*, that is, $P_g(K)\subseteq K^{**}$. Therefore, also $HSP_{\alpha_0}(K)\subseteq K^{**}$. In order to prove the necessity of our Theorem, let Σ contain those valid identities $p=q$ of the equational class $HSP_{\alpha_0}(K)\cap\mathcal{K}_F$ for which our statement does not hold. Supposing $\Sigma\neq\emptyset$, choose $p=q\in\Sigma$ in such a way that $|\text{sub}(p)\cup\text{sub}(q)|$ is minimal.

Now take an algebra $\mathfrak{A}=(A, F)$ freely generated by the sequence $a_1, a_2, ...$ in the equational class $HSP_{\alpha_0}(K)\cap\mathcal{K}_F$. First we show that if we have $r_{\mathfrak{A}}(a_1, a_2, ...)=s_{\mathfrak{A}}(a_1, a_2, ...)$ for some trees $r, s\in\text{sub}(p)\cup\text{sub}(q)$, then $r=s$, i.e., the trees r and s coincide. Assume to the contrary that there exist different trees $r, s\in\text{sub}(p)\cup\text{sub}(q)$ with $r_{\mathfrak{A}}(a_1, a_2, ...)=s_{\mathfrak{A}}(a_1, a_2, ...)$. Let us fix a tree $r\in\text{sub}(p)\cup\text{sub}(q)$ with the property that if $\bar{r}\in\text{sub}(p)\cup\text{sub}(q)$ and $\bar{r}_{\mathfrak{A}}(a_1, a_2, ...)=r_{\mathfrak{A}}(a_1, a_2, ...)$ then $h(r)\leq h(\bar{r})$, and there is a distinct tree $s\in\text{sub}(p)\cup\text{sub}(q)$ with $r_{\mathfrak{A}}(a_1, a_2, ...)=s_{\mathfrak{A}}(a_1, a_2, ...)$. Given r, choose a different tree $s\in\text{sub}(p)\cup\text{sub}(q)$ such that $r_{\mathfrak{A}}(a_1, a_2, ...)=s_{\mathfrak{A}}(a_1, a_2, ...)$, and $h(\bar{s})\leq h(s)$ whenever $\bar{s}\in\text{sub}(p)\cup\text{sub}(q)$ and

$s_{\mathfrak{A}}(a_1, a_2, \ldots)=\bar{s}_{\mathfrak{A}}(a_1, a_2, \ldots)$. Obviously, we have $h(r)\leqq h(s)$. If $s\in \text{sub}(p)$ then let us substitute r for any occurrence of s in p, and denote the resulting tree by $\bar{p}$. If $s\notin \text{sub}(p)$ then put $\bar{p}=p$. Similar procedure when applied to q will produce the tree $\bar{q}$. Of course we have $\text{sub}(\bar{p})\cup\text{sub}(\bar{q})\subseteq\text{sub}(p)\cup\text{sub}(q)$, or even, the choise of r and s garantees that $s\notin\text{sub}(\bar{p})\cup\text{sub}(\bar{q})$. Thus, $|\text{sub}(\bar{p})\cup\text{sub}(\bar{q})|<|\text{sub}(p)\cup\text{sub}(q)|$. Similarly, $|\text{sub}(r)\cup\text{sub}(s)|<|\text{sub}(p)\cup\text{sub}(q)|$.

As $r_{\mathfrak{A}}(a_1, a_2, \ldots)=s_{\mathfrak{A}}(a_1, a_2, \ldots)$, it follows that $r=s$ is a valid identity in $HSP_{\alpha_0}(K)\cap\mathcal{K}_F$. As $|\text{sub}(r)\cup\text{sub}(s)|<|\text{sub}(p)\cup\text{sub}(q)|$ also $K^*\models r=s$. As $r=s$ is a valid identity in $HSP_{\alpha_0}(K)\cap\mathcal{K}_F$, also the equalities $p_{\mathfrak{A}}(a_1, a_2, \ldots)=\bar{p}_{\mathfrak{A}}(a_1, a_2, \ldots)$ and $q_{\mathfrak{A}}(a_1, a_2, \ldots)=\bar{q}_{\mathfrak{A}}(a_1, a_2, \ldots)$ are satisfied. Since $p=q$ was a valid identity in $HSP_{\alpha_0}(K)\cap\mathcal{K}_F$ and $\mathfrak{A}$ is freely generated by $a_1, a_2, \ldots$, also $\bar{p}=\bar{q}$ is a valid identity in $HSP_{\alpha_0}(K)\cap\mathcal{K}_F$. As $|\text{sub}(\bar{p})\cup\text{sub}(\bar{q})|<|\text{sub}(p)\cup\text{sub}(q)|$, by the choise of the identity $p=q$, we obtain that $K^*\models\bar{p}=\bar{q}$. The construction of the trees $\bar{p}$ and $\bar{q}$ shows that $\{r=s, \bar{p}=\bar{q}\}\models p=q$. We have already seen that $K^*\models r=s$, thus, $K^*\models p=q$. This is a contradiction.

So far we have shown that the equality $r_{\mathfrak{A}}(a_1, a_2, \ldots)=s_{\mathfrak{A}}(a_1, a_2, \ldots)$ is satisfied by trees $r, s\in\text{sub}(p)\cup\text{sub}(q)$ if and only if $r=s$. Next we are going to prove that $p=q\in K_F^*$. As $K^*\models p=q$ holds in this case evidently, this would again be a contradiction.

Assume that $p=q\notin K_F^*$. Then there is a type F' and a relabeling $\varphi\colon T_F\to T_{F'}$ such that $\varphi(p)=\varphi(q)$ is not a valid indentity in the class $K\cap\mathcal{K}_{F'}$. Therefore, there is an algebra $\mathfrak{B}=(B, F')\in K$ and elements $b_1, b_2, \ldots\in B$ with

$$\varphi(p)_{\mathfrak{B}}(b_1, b_2, \ldots)\neq\varphi(q)_{\mathfrak{B}}(b_1, b_2, \ldots).$$

Let $\mathfrak{C}=(C, F)$ be any α_0-product $\Pi(\mathfrak{A}, \mathfrak{B}, \psi)$ with ψ satisfying the following conditions for every $f\in F_k$ $(k\geqq 0)$:

(i) $\psi_1(f)=f$,

(ii) $\psi_2((p_1)_{\mathfrak{A}}(a_1, a_2, \ldots), \ldots(p_k)_{\mathfrak{A}}(a_1, a_2, \ldots), f)=\text{rt}(\varphi(f(p_1, \ldots, p_k)))$

if $f(p_1, \ldots, p_k)$ is a subtree of p or q.

In order to show that such an α_0-product exists, it is enough to see that whenever both $f(p_1, \ldots, p_k)$ and $f(q_1, \ldots, q_k)$ are subtrees of p or q and $(p_i)_{\mathfrak{A}}(a_1, a_2, \ldots)=(q_i)_{\mathfrak{A}}(a_1, a_2, \ldots)$ $(i=1, \ldots, k)$ then $\text{rt}(\varphi(f(p_1, \ldots, p_k)))=\text{rt}(\varphi(f(q_1, \ldots q_k)))$. But this can be seen immediately as φ is a mapping and $(p_i)_{\mathfrak{A}}(a_1, a_2, \ldots)=(q_i)_{\mathfrak{A}}(a_1, a_2, \ldots)$ implies that $p_i=q_i$.

As $HSP_{\alpha_0}(K)$ is closed under α_0-products, we get $\mathfrak{C}\in HSP_{\alpha_0}(K)\cap\mathcal{K}_F$. On the other hand, $\varphi(p)_{\mathfrak{B}}(b_1, b_2, \ldots)\neq\varphi(q)_{\mathfrak{B}}(b_1, b_2, \ldots)$ implies that $p_{\mathfrak{C}}((a_1, b_1), (a_2, b_2), \ldots)\neq q_{\mathfrak{C}}((a_1, b_1), (a_2, b_2), \ldots)$, contrary to our assumption that $p=q$ is valid in $HSP_{\alpha_0}(K)\cap\mathcal{K}_F$.

A set of identities $\Delta\subseteq T_F^2$ is called closed if whenever $\Delta\models p=q$ is valid for trees $p, q\in T_F$ then $p=q\in\Delta$. It is known from universal algebra that Δ is closed if and only if the following five conditions are satisfied by Δ:

(i) $x_i=x_i\in\Delta$ $(i=1, 2, \ldots)$,
(ii) $p=q\in\Delta$ implies that $q=p\in\Delta$,
(iii) $p=q$, $q=r\in\Delta$ implies that $p=r\in\Delta$,

(iv) if $p_i=q_i\in\Delta$ for all $i=1, \ldots, k$ $(k\geqq 0)$ and $f\in F_k$ then $f(p_1, \ldots, p_k)=f(q_1, \ldots, q_k)\in\Delta$,

(v) if $p=q\in\Delta$ and we get p' and q' from p and q by substituting all occurences of a variable x_i by an arbitrary tree $r\in T_F$ then $p'=q'\in\Delta$.

By virtue of the previous Theorem, if K_F^* is closed for every type F, then whenever $p=q$ is a valid identity in an equational class $HSP_{\alpha_0}(K)\cap\mathcal{K}_F$ then $p=q\in K_F^*$. Conversely, if $p=q\in K_F^*$ then $p=q$ is a valid identity in $HSP_{\alpha_0}(K)\cap\mathcal{K}_F$. As K_F^* always satisfies conditions from (i) to (v) above except (iv), a necessary and sufficient condition for K_F^* to be closed is to satisfy condition (iv). In this way we get the following

Corollary. Assume that K_F^* satisfies condition (iv) for every type F. Then an identity $p=q$ is valid in an equational class $HSP_{\alpha_0}(K)\cap\mathcal{K}_F$ if and only if $p=q\in K_F^*$. Conversely, if we have the equivalence $p=q$ is valid in an equational class $HSP_{\alpha_0}(K)\cap\mathcal{K}_F$ if and only if $p=q\in K_F^*$ then K_F^* satisfies condition (iv).

Further on we shall need the following

Lemma. Let $\mathfrak{A}=(A, F)=\Pi(\mathfrak{A}_i, \varphi)|i\in I)$ be an arbitrary infinite α_0-product of algebras $\mathfrak{A}_i=(A_i, F_i)$ and let $J\subseteq I$ and $T\subseteq T_F$ be finite sets. For every sequence $a_1, a_2, \ldots\in A$ there is a finite α_0-product $\mathfrak{B}=(B, F)=\Pi(\mathfrak{A}_i, \psi|i\in J_1)$ with $J\subseteq J_1$ and such that $\psi_i(a_{1J_1}, a_{2J_1}, \ldots, p)=\varphi_i(a_1, a_2, \ldots, p)$ for any $p\in T$ and $i\in J$.[1]

Proof. Put $h=\max\{h(p)|p\in T\}$. If $h=0$ then our statement is obviously valid. We proceed by induction on h. Let $h>0$ and assume that the proof is done for $h-1$. For every $k>0$ and $f\in F_k$ set

$$U_f = \{p|p\in T,\quad h(p) = h,\quad \mathrm{rt}\,(p) = f\},$$

$$V_f = \{p|p\in \bigcup(\mathrm{sub}\,(q)|q\in T)\cup T,\quad \mathrm{rt}\,(p) = f\}.$$

Let $(p, q, i)\in U_f\times V_f\times J$ — say $p=f(p_1, \ldots, p_k)$, $q=f(q_1, \ldots, q_k)$ — be arbitrary. If $\varphi_i(p_{1\mathfrak{A}}(a_1, a_2, \ldots), \ldots, p_{k\mathfrak{A}}(a_1, a_2, \ldots), f)\neq\varphi_i(q_{1\mathfrak{A}}(a_1, a_2, \ldots), \ldots, q_{k\mathfrak{A}}(a_1, a_2, \ldots), f)$ then choose an index $i_0<i$ with $(p_{t\mathfrak{A}}(a_1, a_2, \ldots))_{i_0}\neq(q_{t\mathfrak{A}}(a_1, a_2, \ldots))_{i_0}$ for some $t\in\{1, \ldots, k\}$. Denote by I_0 the set of indices obtained in this way, and put $J'=J\cup I_0$, $T'=\bigcup(\mathrm{sub}\,(p)|\,p\in T)\cup\{p\in T|h(p)<h\}$. By the induction hypothesis, there exist a finite set J_1' and an α_0-product $\mathfrak{B}'=(B', F)=\Pi(\mathfrak{A}_i, \psi'|i\in J_1')$ with $J'\subseteq J_1'$ and satisfying $\psi_i'(a_{1_{J_1'}}, a_{2_{J_1'}}, \ldots, p)=\varphi_i(a_1, a_2, \ldots, p)$ for each $p\in T'$ and $i\in J_1'$.

Set $J_1=J_1'$ and define the α_0-product $\mathfrak{B}=(B, F)=\Pi(\mathfrak{A}_i, \psi|i\in J_1)$ so that the following two conditions are satisfied:

(i) $\psi(b_1, \ldots, b_k, f)=\psi'(b_1, \ldots, b_k, f)$ if $f\in F_k$ $(k\geqq 0)$ and there exist trees $p_1, \ldots, p_k\in T_F$ with $f(p_1, \ldots, p_k)\in T'$ and $b_t=p_{t\mathfrak{B}'}(a_{1_{J_1}}, a_{2_{J_1}}, \ldots)$ $(t=1, \ldots, k)$,

(ii) $\psi_i(b_1, \ldots, b_k, f)=\varphi_i(c_1, \ldots, c_k, f)$ if $i\in I$, $f\in F_k$ $(k>0)$, and there exist trees $p_1, \ldots, p_k\in T_F$ with $f(p_1, \ldots, p_k)\in U_f$ and $b_t=p_{t\mathfrak{B}'}(a_{1_{J_1}}, a_{2_{J_1}}, \ldots)$, $c_t=p_{t\mathfrak{A}}(a_1, a_2, \ldots)$ $(t=1, \ldots, k)$.

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[1] The ordering on J_1 is the restriction of the ordering on I to J_1. If $a\in\Pi(A_i|i\in I)$ then $a_{J_1}\in\Pi(A_i|i\in J_1)$ is determined by $(a_{J_1})_i=a_i$ for any $i\in J_1$.

Such an α_0-product exist, since otherwise we would have an index $i \in I$ together with distinct trees $p=f(p_1, \ldots, p_k) \in U_f$, $q=f(q_1, \ldots, q_k) \in V_f$ $(f \in F_k, k>0, p_t, q_t \in T_F)$ such that $(p_{t\mathfrak{A}}(a_1, a_2, \ldots))_j = (q_{t\mathfrak{A}}(a_1, a_2, \ldots))_j$ $(t=1, \ldots, k)$ for all $j<i$ but $\varphi_i(p_{1\mathfrak{A}}(a_1, a_2, \ldots), \ldots, p_{k\mathfrak{A}}(a_1, a_2, \ldots), f) \neq \varphi_i(q_{1\mathfrak{A}}(a_1, a_2, \ldots), \ldots, q_{k\mathfrak{A}}(a_1, a_2, \ldots), f)$. Also the equalities $\psi_i(a_{1_{J_1}}, a_{2_{J_1}}, \ldots, p) = \varphi_i(a_1, a_2, \ldots, p)$ $(i \in I, p \in T)$ follow in an easy way.

Theorem 2. $HSPP_{f\alpha_0}(K) = HSP_{\alpha_0}(K) = HSP_g(K) = K^{**}$ holds for any class K of algebras.

Proof. The last two equalities immediately follow by Theorem 1 and Birkhoff's Theorem. $HSPP_{f\alpha_0}(K) \subseteq HSP_{\alpha_0}(K)$ is obvious. We claim that also $HSP_{\alpha_0}(K) \subseteq$ $\subseteq HSPP_{f\alpha_0}(K)$. This can be seen by showing that if F is an arbitrary type and an identity $p=q$ $(p, q \in T_F)$ is not valid in $P_{\alpha_0}(K) \cap \mathcal{K}_F$ then the same holds for $P_{f\alpha_0}(K) \cap \mathcal{K}_F$. But this is a trivial consequence of our Lemma.

Theorem 2 is in a close connection with the characterization theorem of metrically complete systems of algebras in [2]. It turns out that a system K of algebras having finite types is metrically complete if and only if K^* contains only trivial identities. In other words this means that K is complete (that is, $HSP_g(K)$ is the class of all algebras) if and only if K is metrically complete.

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(Received Jan. 18, 1983)