On the real-time recognition of formal languages in cellular automata

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1. Previous approaches to use cellular automata to recognize formal languages: appreciation and critic

To recognize formal languages by cellular automata (ca) already some approaches have been developed. A ca is defined by a d-dimensional Euclidean space $\mathbf{Z}^d$ ($d \geqq 0$) ($\mathbf{Z}$ denoting the set of integers), where each lattice point is occupied by a finite deterministic automaton, and all automata are identical and work synchronously. Each automaton is connected with a fixed finite number ($\geqq 2$) of neighbours, where all automata use the same interconnection scheme T, called *template* or *neighbourhood.* The best known templates are the (d-dimensional) *von Neumann templates* $T = H^d = \{\mathbf{0}, \pm u_1, \pm u_2, \ldots, \pm u_d\}$, where $\mathbf{0} = (0, 0, \ldots, 0)$ is the (d-dimensional) origin and u_i is the i^{th} d-dimensional unit vector and the (d-dimensional) *Moore templates* $T = M^d = \{-1, 0, 1\}^d$. In each transition step the behaviour of the automaton at point x depends only on the states of the automata at points $x+t$, where t ranges over T. In that sense, we will consider homogeneously occupied, homogeneously interconnected deterministic, single transition function ca. For details see, e.g., [13], [1], [3] or [14].

Using ca to recognize formal languages [9], one has to decide how to input the words, or chains of symbols. As in other abstract recognition devices, there are two main possibilities to do this. First, we have the "on-line" ca as defined by Cole [2]. In this case, the automaton at the origin is equipped with an additional input line from which it reads the input word, one letter at each time step. Let us take the state of the (distinguished) automaton at origin, immediately after having received the last symbol of the input word, in order to decide whether or not the word belongs to the language considered. The class of languages which can be recognized by such a d-dimensional ca is called "the class of the d-dimensional 'on-line' real-time recog-

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nizable languages" and is denoted by $\mathscr{L}_d^{\text{on}}$ ($d \geqq 0$). In [2], the following is obtained:

$$\mathscr{L}_0^{\text{on}} = \mathscr{L}_3,$$
$$\forall d \geqq 0: \mathscr{L}_d^{\text{on}} \subsetneqq \mathscr{L}_{d+1}^{\text{on}},$$
$$\exists L \in \mathscr{L}_2: \forall d \geqq 0: L \notin \mathscr{L}_d^{\text{on}},$$

where $\mathscr{L}_3$ denotes the class of regular languages and $\mathscr{L}_2$ the class of context-free languages.

As in Cole's approach the word to be analyzed has to be input letter by letter, it is in general impossible to get a (maximal) recognition time less than $1 \cdot n$, where n is the length of the actual input word. Moreover, as the information disperses only with finite speed from the origin into the space (in a pyramidal manner), most of the automata are activated "too late" and only few of the capabilities of parallelism are exploited. Therefore this approach of real-time recognition causes an exploitation factor of (approximately) only $1/d!$

These disadvantages can be removed, if the word to be analyzed is not read sequentially in n steps, but in a parallel manner, using only one step. In other words: The information is supposed to be written into the ca (i.e.: distributed over the single automata) at the beginning of the recognition process. The way to embed the words must be simple and as independent as possible from the actual word (in some sense). Moreover, no two symbols of the same word are allowed to occupy the same automaton. Smith [10] considers this procedure for the one-dimensional case. He presupposes that the input word is inscribed from left to right, beginning at the origin 0 and with no gaps allowed. Automata not occupied by the input are assumed to remain in a "boundary state" which does not alter during the whole evaluation process. After n steps (n as above) the state of the automaton at origin gives the decision whether the word belongs to the language or not.

If one considers only the von Neumann template, H^1, the languages recognizable in such a manner are called "one-dimensional 'off-line' real-time recognizable" and their class is denoted by $\mathscr{L}_1^{\text{off}}$. Smith proved that

$$\mathscr{L}_1^{\text{on}} \subsetneqq \mathscr{L}_1^{\text{off}}$$

and

$$\exists L \in \mathscr{L}_1^{\text{off}}: \forall d \geqq 0: L \notin \mathscr{L}_d^{\text{on}},$$

concluding that "off-line" ce are inherently faster than "on-line" ones. He explains this phenomenon with the higher degree of parallelism now available from the beginning of the analyzing process. In this approach, however, remains the fact that recognition times less than real-time are generally not achievable, too, because the most distant symbol cannot influence the cell at origin before the n^{th} step.

Generalizing the results of Smith, Seiferas [12] achieves recognition times of the form $d\left\lceil \sqrt[d]{n} \right\rceil$ in d-dimensional off-line ca. To do this, the word to be analyzed is inscribed into the cube $\left\{0, 1, \ldots, \left\lceil \sqrt[d]{n} \right\rceil - 1\right\}^d$ row by row and then surrounded by a special boundary symbol (sc. Fig. 1). Seiferas uses the templates $N_+^d := \{0, u_1, 2u_1, u_2, 2u_2, \ldots, u_d, 2u_d\}$. He proves that all regular languages can be recognized in such a way within the cited time. But it is easy to verify that, using this type of inscription

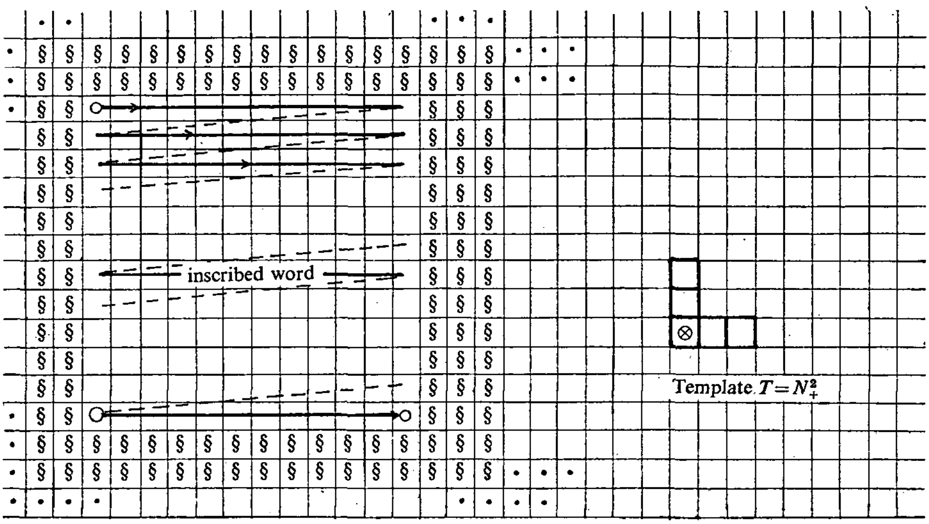

Fig. 1

Off-line recognition of regular languages in 2-dimensional cellular automata according to SEIFERAS [12]

and this template N_+^d, all symbols of the inscribed word can influence the origin already within time $\left\lceil \frac{d}{2} \left\lceil \sqrt[d]{n} \right\rceil \right\rceil$. This implies that Seiferas does not meet the lowest possible recognition time which perhaps could be reached in these structures. The aim of the following is to investigate and generalize this aspect in a more detailed way.

2. Introduction to a systematic approach: T-recognition of T-languages

If we consider, for example, the template N_+^d, then after the k^{th} step, the automaton at origin can be influenced by (approximately) $(2k)^d/d!$ other automata. This means that Seiferas uses only

$$\frac{(\lceil \sqrt[d]{n} \rceil)^d}{(2d\lceil \sqrt[d]{n} \rceil)^d/d!} \quad \left(= \frac{d!}{2^d d^d}\right)$$

of the supplied space. Similarly, we state that n points can be "reached" from the origin within

$$\left\lceil \frac{1}{2} \lceil \sqrt[d]{d!} \rceil \lceil \sqrt[d]{n} \rceil \right\rceil$$

steps. This fact implies that a speedup factor of (approximately)

$$\frac{2d}{\sqrt[d]{d!}} \quad \left(\geq \frac{4d}{d+1}\right)$$

can possibly be achieved, without changing the template N^d_+, if we replace the cubic representation of the word by one which is more adapted to the shape of the region eontaining all the points reachable by the oirigin within k steps. In this case it turns out to be a simplex. But let us consider these problems in an even more general way:

Given any template T, the region which can influence the origin within k steps is the set kT recursively defined by

$$0T := \{0\}$$

and

$$(k+1)T := kT+T \quad (k \geqq 0)\ ^1.$$

This means that, after the k^{th} step, the state at the automaton at origin $\mathbf{0}$ can be used to decide some property of that part of the input pattern which is contained in region kT. Vice versa, we can (ab-) use any ca with template T to classify patterns of the shape kT $(k \geqq 0)$. This is done in the following way: let us assume that the patterns to be classified contain only symbols of some subset A of the state set Z of the ca. Then, given any such pattern with shape kT, extend it to an (infinite) pattern in an arbitrary way and make work the ca exactly k steps. Afterwards the state of the automaton at origin is taken to classify the original finite pattern.

To formalize these ideas, let us call any finite, nonempty set A an *alphabet* and any mapping $w: kT \to A$ a *$T-A$-word* (*T-word* or, simply, *word*) *with shape kT* or *with T-diameter k* $(k \geqq 0)$. Then, formally, A^{kT} denotes the set of all such words with shape kT [2]. Furthermore, let $(T, A)^*$, defined by

$$(T, A)^* := \bigcup_{k \geqq 0} A^{kT}$$

denote the *set of all $T-A$-words (of any T-diameter)*. It is true that, depending on the underlying template T, the words of $(T, A)^*$ may have somewhat strange shapes (sc. Fig. 2). Any subset L of $(T, A)^*$ is called a *$T-A$-language* (or, simply, *T-language*). We say that a certain ca (with the same template T) *T-recognizes L* if its state set, Z, contains A and if there is a subset F of Z, the *set of accepting states*, such that for any word w of $(T, A)^*$ with shape kT it holds: w is an element of L iff, extending w as cited above, and starting running the ca exactly k steps[3], the automaton at origin enters a state of F. L is called *T-recognizable*, if there exists a ca (with template T) which T-recognizes L. Obviously, this notion of recognizability is the strongest real-time recognizability definable in off-line ca, because a pattern must be classified as soon as the whole information to be classified can have influenced the deciding cell.

Furthermore, if we want to apply this approach to the recognition of formal languages, we have to define how to represent the (conventional) words (e.g. of A^*) as $T-A$-words. Therefore we introduce the following notation: Any sequence $h=(h^k)_{k \geqq 0}$, where each member h^k represents some bijection[4]

$$h^k\colon kT \to \{1, 2, 3, \ldots, \operatorname{card}(kT)\},$$

[1] For two subsets M and N of $\mathbf{Z}^d$ and any element $x \in \mathbf{Z}^d$ let $M+N := \{y+z / y \in M$ and $z \in N\}$ and $x+M(:=M+x) := \{x\}+M$. $+(-)$ is the componentwise sum (difference).

[2] For two sets M and N let M^N denote the set of all mappings f from N into M (i.e.: $M^N := \{f / f\colon N \to M\}$).

[3] Note that $k \neq k' \Rightarrow kT \neq k'T$ $(k, k' = 0, 1, 2, \ldots)$ (cp. [6, 7]).

[4] A mapping is said to be a bijection if it is onto and one-to-one.

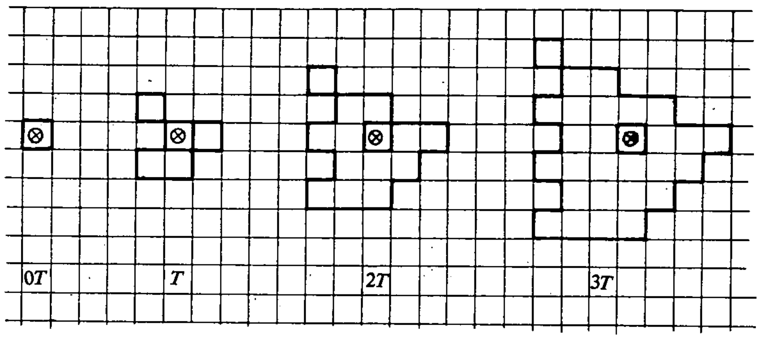

Fig.2

The shapes $0T=\{0\}$, $1T=T$, $2T$ and $3T$ for some 2-dimensional template $T(=\{(-1,-1), (0,-1), (0,0), (1,0), (-1,1)\})$

is called a *T-wrap*. Such a T-wrap $h=(h^k)_{k\geq 0}$ permits us to represent any word $p=a_1a_2a_3\ldots a_{\mathrm{card}\,(kT)}$ (i.e. of the length card (kT)) as the $T-A$-word w $(\in A^{kT}!)$, defined by

$$w(x) := a_{(h^k(x))}\quad (x\in kT)\ ^5.$$

Let us denote this (uniquely defined) word w as $\hat{h}(p)$. Thus, $\hat{h}$ can be considered as a partial mapping from A^* to $(T, A)^*$, which only maps words of the length 1, card (T), card $(2T)$, card $(3T)$, ... (This is no real striction because any nonfitting word can be filled up to the next fitting length.) For any formal language $S(\subseteq A)^*$ let $\hat{h}(S)$ be the $T-A$-language, defined by

$$\hat{h}(S) := \{\hat{h}(p)/p\in S \text{ and } \hat{h}(p) \text{ is defined}\}.$$

Now, with these notions, the "real-time recognition of formal languages by off-line ca" reduces to the problem:

Let S be a formal language, T a template and h a T-wrap. Is the $T-A$-language $\hat{h}(S)$ T-recognizable or not?

In this paper we give a partial answer to this question concerning the T-recognizability of regular and context-free formal languages.

First we restrict our considerations to the family of Moore templates, M^d and their capabilities to recognize regular languages. Smith [11] has shown that, in case $d=1$, for any regular language, R, there is a ca using template M^1 which M^1-recognizes $\hat{h}(R)$, where h is the straightforward inscription from left to right. Theorem 10 states that this inscription technique can not be generalized for $d>1$. There it is shown that no T-wrap which divides the admitted inscription areas (i.e.: kM^d into parallel rows and fills these individual rows strictly from left to right or right to left each can generally be used to recognize regular languages. In Theorem 11, however, it is shown that, for any d-dimensional template T, there exists a nontrivial T-wrap which makes possible the T-recognition of any regular language.

In case of context-free languages we obtained only negative answers which are the sharper the more extreme points the template contains. Ruling out the trivial

[5] Note that $k\neq k'\Rightarrow\mathrm{card}\,(kT)\neq\mathrm{card}\,(k'T)$ $(k, k'=0, 1, 2, 3, \ldots)$ (cp. [6, 7]).

case where T contains exactly one extreme point (and, consequently, kT is a singleton) we proved the following:

If T contains exactly 2 extreme points (which implies that each kT is a (possible sparsed) line), then the simple T-wrap along this line does not fit for all context-free languages (Theorem 12). It is, however, an open problem whether some other T-wraps will do it.

If T contains exactly 3 extreme points (and therefore any kT with $k \geqq 1$ contains also exactly 3 extreme points), then kT can be considered to consist of a series of lines which are parallel to one of the 3 extreme egdes of kT. In Theorem 13 we prove that there is no inscription rule which fills first the starting extreme egde and then the remaining lines in a strictly removing manner and which fits for all context-free languages. This is true even if it would be allowed to vary the internal order within each particular row arbitraryly and depending on the particular language.

Finally, if T contains 4 extreme points or more, then there is a context-free language, C, for which there is no T-wrap h at all such that $\hat{h}(C)$ is T-recognizable (Theorem 14).

3. The proofs of the results

First, we give some appropriate notations and some basic statements concerning them which have been developed previously ([6], [7]). Essentially, the notions mentioned in section 2 are treated in a more systematical way and some of them will be redefined or generalized.

Let $\mathbf{N}=\{0, 1, 2, \ldots\}$ denote the set of natural numbers. Let d $(d \geqq 1)$ be a dimension, M a finite subset of $\mathbf{Z}^d$ and A an alphabet. Then any function $w: M \to A$ is called a *(d-dimensional) word (over the alphabet A)* and M is called the *support* or *domain* of w, denoted as dom (w). The set of all d-dimensional words over alphabet A is denoted as $(d, A)^*$ and equals

$$\bigcup_{M \subseteq \mathbf{Z}^d :\, \mathrm{card}\,(M) < \infty} A^M.$$

d-dimensional words may be *displaced* and *restricted:* for any word $w \in (d, A)^*$ and any vector $x \in \mathbf{Z}^d$ let the word $w \oplus x$, the *x-displacement* of w, be defined by

$$\mathrm{dom}\,(w \oplus x) := \mathrm{dom}\,(w) + x$$

and

$$(w \oplus x) \quad (y) := w(y-x) \quad (y \in \mathrm{dom}\,(w \oplus x)).$$

Instead of $w \oplus (-x)$ we write $w \ominus x$, too. (Clearly, $w \oplus (x+y) = (w \oplus x) \oplus y$.) For any word w with $\mathrm{dom}\,(w) = M$ and any subset N of M let $w|_N$ be the wellknown *restriction* of w to N. Note that $\mathrm{dom}\,(w|_N) = N$.

Given any template, T $(2 \leqq \mathrm{card}\,(T) < \infty)$, then we get, for $k, k' \in \mathbf{N}$ with $k \neq k'$ $kT \neq k'T$. This is proved in [7] using the strictly increasing (Euclidean) diameter of the sets kT. Thus, $(T, A)^*$, as defined in section 2, is the disjoint union of its constituting subsets A^{kT} $(k \in \mathbf{N})$.

Therefore, for any $w \in (T, A)^*$, the natural number k with $w \in A^{kT}$ is uniquely determined and is denoted by $D(w)$ or $D_T(w)$ and named: the *(T-)diameter* of w. Note that, because of

$$(k+m)T = \bigcup_{x \in kT} x + mT \quad (k, m \in \mathbf{N});$$

for any word $w\in(T, A)^*$, any i with $0\leqq i\leqq D(w)$ and any $x\in iT$, the word $(w\ominus x)|_{(D(w)-i)T}(=w|_{x+(D(w)-i)T}\ominus x)$ is also a word in $(T, A)^*$ (with diameter $D(w)-i$). Furthermore, let $(T, A)^+$ be defined as the set $\{w/w\in(T, A)^*$ and $D_T(w)\geqq 1\}$.

Now let us formalize the notion of T-recognition. A T-recognizing cellular automaton (Trca) **A** is a quintuple $\mathbf{A}=(T, A, Z, f, F)$ where T is a template, Z is an alphabet, called the *state alphabet*, $A(\subseteq Z)$ is another alphabet, called the *input alphabet*, f is some function $f\colon Z^T\to Z$ and called the *local transition function* and $F(\subseteq Z)$ is called the *set of accepting states* or, shortly, *accepting set*. Now, identifying the set $Z^{\{0\}}$ with state alphabet, Z, we may extend domain and range of f, widening f to be a function $f\colon (T, Z)^+\to(T, Z)^*$ such that, for any $w\in(T, Z)^+$

$$\operatorname{dom}(f(w)) := (D(w)-1)T$$

with

$$f(w)(x) := f((w\ominus x)|_T) \quad (x\in(D(w)-1)T).$$

(This is possible because of $(k+1)T=\bigcup_{x\in kT} x+T$ $(k\in\mathbf{N})$.) Then, clearly, the function $f^*\colon (T, A)^*\to Z^{\{0\}}(=Z)$, defined by

$$f^*(w) := f^{D(w)}(w) \quad (w\in(T, A)^*),$$

is well defined. Now, given the Trca $\mathbf{A}=(T, A, Z, f, F)$ and the $T-A$-language $L(\subseteq(T, A)^*)$, we say that **A** T-recognizes L iff

$$\forall w\in(T, A)^*\colon w\in L \Leftrightarrow f^*(w)\in F.$$

The $T-A$-language $L(\subseteq(T, A)^*)$ is said to be T-recognizable iff there exist Z, f, G such that the Trca $\mathbf{A}=(T, A, Z, f, G)$ T-recognizes L.

Now we give some basic notions and results concerning T-recognizability. Because of their importance within this section, we will cite them as explicit definitions and theorems. They are presented here as in [7].

Definition 1. Let A and Z be two (arbitrary) alphabets, T a template and g some function $g\colon (T, A)^*\to Z$. Then let the function $\bar{\bar{g}}\colon (T, A)^+\to Z^T$ be defined by

$$\bar{\bar{g}}(w)(x) := g((w\ominus x)|_{(D(w)-1)T}) \quad (x\in T,\ w\in(T, A)^+).$$

Using this notion we get

Theorem 2. The T-language $L(\subseteq(T, A)^*)$ is T-recognizable iff there is an alphabet Z, a function $g\colon (T, A)^*\to Z$, a function $f\colon Z^T\to Z$ and a subset F of Z such that

$$\forall w\in(T, A)^+\colon f(\bar{\bar{g}}(w)) = g(w)$$

and

$$\forall w\in(T, A)^*\colon (w\in L \Leftrightarrow g(w)\in F)$$

hold.

Now we introduce a new notion of equivalence relation.

Definition 3. Let A be an alphabet, T a template and L a $T-A$-language. Then, for any $k\in\mathbf{N}$, any two words, $w, w'\in(T, A)^*$ with $D(w)\geqq k$ and $D(w')\geqq k$ are said to be $k-L$-*equivalent* iff

$$\forall i(0 \leqq i \leqq k): \forall x \in iT: \left((w \ominus x)|_{(D(w)-i)T} \in L \Leftrightarrow (w' \ominus x)|_{(D(w')-i)T} \in L\right).$$

Let E_{k-L} denote the number of equivalence classes the set $\{w/w \in (T, A)^*$ and $D(w) \geqq k\}$ is divided into by this relation.

With respect to E_{k-L}, two sequences of numbers turn out to be important:

Definition 4. Let T be a template; then the sequences $(d_{k,T})_{k \in \mathbf{N}}$ and $(e_{k,T})_{k \in \mathbf{N}}$ are defined by

$$d_{k,T} := \operatorname{card}(kT) \quad (k \in \mathbf{N})$$

and

$$e_{k,T} := \sum_{i=0}^{k} d_{i,T} \quad \left(= \sum_{i=0}^{k} \operatorname{card}(iT)\right) \quad (k \in \mathbf{N}).$$

Theorem 5. Let A be an alphabet, T a template and L a $T-A$-language. Then, generally, it holds

$$E_{k-L} \leqq 2^{(e_{k,T})} \quad (k \in \mathbf{N}).$$

If L is T-recognizable, then it holds

$$E_{k-L} \leqq C^{(d_{k,T})} \quad (k \in \mathbf{N})$$

for some appropriate positive constant C.

The following theorem serves as a widely applicable general information compression argument which is proved in full details in [7]. Essentially, it states the following:

Let M and N be two disjoint subsets of kT and i an integer with $0 \leqq i \leqq k$ such that $M-iT$ and $N-iT$ are disjoint on $(k-i)T$. Moreover, let $(w_{m,n})$ be some family of words of A^{kT} where m and n range over some index sets $\underset{\sim}{M}$ and $\underset{\sim}{N}$ respectively such that all $w_{m,n}$ are identical outside $M \cup N$, words with the same index m are identical on M and words with the same index n identical on N. Let L be a $T-A$-language such that, for any pair (n, n') with $n \neq n'$, there exists an m such that $w_{m,n}$ and $w_{m,n'}$ are separated by L, then any Trca which T-recognizes L contains at least C states where

$$C^{\operatorname{card}((k-i)T \cap (N-iT))} \geqq \operatorname{card}(\underset{\sim}{N}).$$

This is true because, after starting the Trca with a word $w_{m,n}$ as input and running it exactly i-times, the information about the index n must be preserved in the field $(k-i)T \cap (N-iT)$. For the area outside $N-iT$ can not be influenced from input information on N, the area inside $(N-iT) \cap (k-i)T$ can not be influenced from information on M (about m) and information outside $(k-i)T$ can not influence the deciding cell at origin within the remaining $k-i$ steps.

The theorem as presented below is a more applicable reformulation of this elementary fact, using the T-diameter k as running index and i as an additional free parameter which, in typical applications, is chosen as an appropriate function on k.

Theorem 6. If the $T-A$-language L is T-recognizable, then there is a (positive) constant C such that the following assertion holds:

Let $k\in\mathbf{N}$, M_k and N_k two sets with $N_k, M_k \subseteq kT$ and $M_k\cap N_k=\emptyset$, $\underset{\sim}{M}_k$ and $\underset{\sim}{N}_k$ two non empty finite sets of indices and $(w^k_{m,n})_{n\in \underset{\sim}{N}_k, m\in \underset{\sim}{M}_k}$ a family of words such that

$$\forall m\in\underset{\sim}{M}_k\colon \forall n\in\underset{\sim}{N}_k\colon w^k_{m,n}\in A^{kT}, \tag{1}$$

$$\forall m\in\underset{\sim}{M}_k\colon \forall n, n'\in\underset{\sim}{N}_k\colon w^k_{m,n|M_k} = w^k_{m,n'|M_k} \tag{2}$$

$$\forall n\in\underset{\sim}{N}_k\colon \forall m, m'\in\underset{\sim}{M}_k\colon w^k_{m,n|N_k} = w^k_{m',n|N_k} \tag{3}$$

$$\forall n, n'\in\underset{\sim}{N}_k\colon \forall m, m'\in\underset{\sim}{M}_k\colon w_{m,n|kT\setminus(M_k\cup N_k)} = w_{m',n'|kT\setminus(M_k\cup N_k)} \tag{4}$$

$$\forall n, n'\in\underset{\sim}{N}_k (n\neq n')\colon \exists m\in\underset{\sim}{M}_k\colon w^k_{m,n}\in L \quad\text{and}\quad w^k_{m,n'}\notin L \quad\text{or}$$
$$w^k_{m,n}\notin L \quad\text{and}\quad w^k_{m,n'}\in L. \tag{5}$$

Then we have, for any i $(0\leqq i\leqq k)$ for which additionally holds

$$(M_k-iT)\cap(N_k-iT)\cap(k-i)T=\emptyset, \tag{6}$$

the necessary inequality

$$\operatorname{card}(\underset{\sim}{N}_k)\leqq C^{\operatorname{card}((N_k-iT)\cap(k-i)T)}.$$

Because the topic of this paper is the treatment of T-languages which are the T-wraps of some (conventional) string languages, we have to provide for some tools to construct T-wraps or to compose complicated ones from simpler ones. To compose them it serves

Notation 7. Let G be any set with the (partially defined) associative binary operation $\square$ and identity element λ, let I be any finite set of indices with $\operatorname{card}(I)=n$ and let $\lessdot$ be a total linear ordering of I. Then, for any family $(g_i)_{i\in T}$ of elements of G, we define the abbreviation

$$\underset{i\in I}{\overset{\lessdot}{\square}} g_i := \lambda\,\square\, g_{i_1}\square\, g_{i_2}\square\ldots\square\, g_{i_n}$$

where $I=\{i_j|1\leqq j\leqq n\}$ and $i_1\lessdot i_2\lessdot i_3\ldots\lessdot i_n$.

Three such associative operations play some role, two of which are wellknown from automata theory and another one which is introduced in [6].

Definition 8. For any alphabet A let $\circ$ denote the usual *concatenation* of words of A^*; the *empty word* ε serves as identity element. For any finite automaton (with input alphabet A) let σ be the set of all its transition functions σ_p where p ranges over A^*. Let $\hat{\circ}$ denote their product with $\sigma_p\hat{\circ}\sigma_{p'}=\sigma_{p\circ p'}$. For any state s of the finite automaton let $s\cdot\sigma_p$ denote the state assumed after p is input into the automaton starting with state s. Let $d\geqq 1$, M and N be two disjoint finite subsets of $\mathbf{Z}^d$ and let us assume that $h'\colon M\to\{1, 2, \ldots, \operatorname{card}(M)\}$ and $h''\colon N\to\{1, 2, \ldots, \operatorname{card}(N)\}$ both are bijections, then we denote by $h'\triangle h''$ the bijection $h\colon M\cup N\to\{1, 2, \ldots, \operatorname{card}(M)+\operatorname{card}(N)\}$ defined by

$$h(x) := \begin{cases} h'(x) & \text{if} \quad x\in M \\ \operatorname{card}(M)+h''(x) & \text{if} \quad x\in N. \end{cases}$$

Note that $h\colon \emptyset\to\emptyset$ serves as identity element and that $\triangle$ is not commutative.

The following theorem ([6], [8]) will deliver our T-wraps:

Theorem 9. Let T be a template. Then there is an alphabet Q, a function $t\colon Q\to 2^{T\times Q}$[6] and a family $(M_q^k)_{k\in\mathbf{N},q\in Q}$ of sets with

$$M_q^k \subseteqq kT \quad (k\in\mathbf{N},\ q\in Q), \tag{7}$$

$$kT = \dot{\bigcup_{q\in Q}} M_q^k \quad (k\in\mathbf{N}) \tag{8}$$

and

$$M_q^{k+1} = \dot{\bigcup_{(x,r)\in t(q)}} x+M_r^k \quad (k\in\mathbf{N},\ q\in Q) \tag{9}$$

where $\dot{\bigcup}$ means that all participating sets are disjoint.

Using these notions and results we turn to prove our claims of section 2.

Theorem 10. Let $T=M^d$. (Then kT represents a cube with side length $2k+1$ and the origin as centre.) Let $h_d=(h_d^k)_{k\geqq 0}$ be some T-wrap such that h_d^k fills the cube kT row by row, where all rows are parallel to each other and the order within each row is strictly from left to right or right to left, but the order of rows may be chosen arbitrarily. Then there exists a regular language R_1 which can be chosen independently of d, such that, for any $d\geqq 2$, $\hat{h}_d(R_1)$ is not M^d-recognizable.

Proof. Let $A:=\{a,b,c\}$ and consider the regular language $R_1=(ab^+a)^*\cdot(cb^+c)^*(ab^+a)^*$. Now let $d\geqq 2$ be any dimension and T the d-dimensional Moore template, M^d. Furthermore, let $h=(h^k)_{k\in\mathbf{N}}$ be any T-wrap where h^k maps any two row neighbours onto two successive natural numbers. We claim that the $T-A$-language $L:=\hat{h}(R_1)$ is not T-recognizable. To show this, we assume without loss of generality that any row of kT consists of the points $(n_1,n_2,\ldots,n_{d-1},j)$ where $-k\leqq j\leqq k$. Thus any row is entirely characterized by some row address $n=(n_1,n_2,\ldots,n_{d-1})$ $(\in kM^{d-1})$. Let us call this row the *n-row*.

Now let us assume that L is T-recognizable. Then we may apply Theorem 6. Let C be chosen such that the assertion cited in that theorem holds. Now take any $k\geqq 1$ and consider, for any two row-addresses n and m $(\in kM^{d-1})$, the word $w_{m,n}^k(\in A^{kT})$ defined by

$$w_{m,n}^k(x_1,x_2,\ldots,x_d) := \begin{cases} b & \text{iff} \quad -(k-1)\leqq x_d\leqq (k-1),\\ c & \begin{cases}\text{if} & (x_1,x_2,\ldots,x_{d-1})=m\wedge x_d=-k\\ \text{or} & (x_1,x_2,\ldots,x_{d-1})=n\wedge x_d=k,\end{cases}\\ a & \text{else.}\end{cases}$$

In short: the two sides of $w_{m,n}^k$ which mark row ends (or row beginnings) are entirely filled with a's except the leftmost element of the m-row and the rightmost element of the n-row which, in turn, exhibit two c's. The residue of $w_{m,n}^k$, i.e., the entire space between these two sides, is filled with b's. (For the case $d=3$, this is visualized in Fig. 3.)

Now, as one verifies easily, independently upon whether the rows are T-wrapped from left to right or from right to left and whether the wrapping direction alternates

[6] For any set M let 2^M denote the set of all subsets of M.

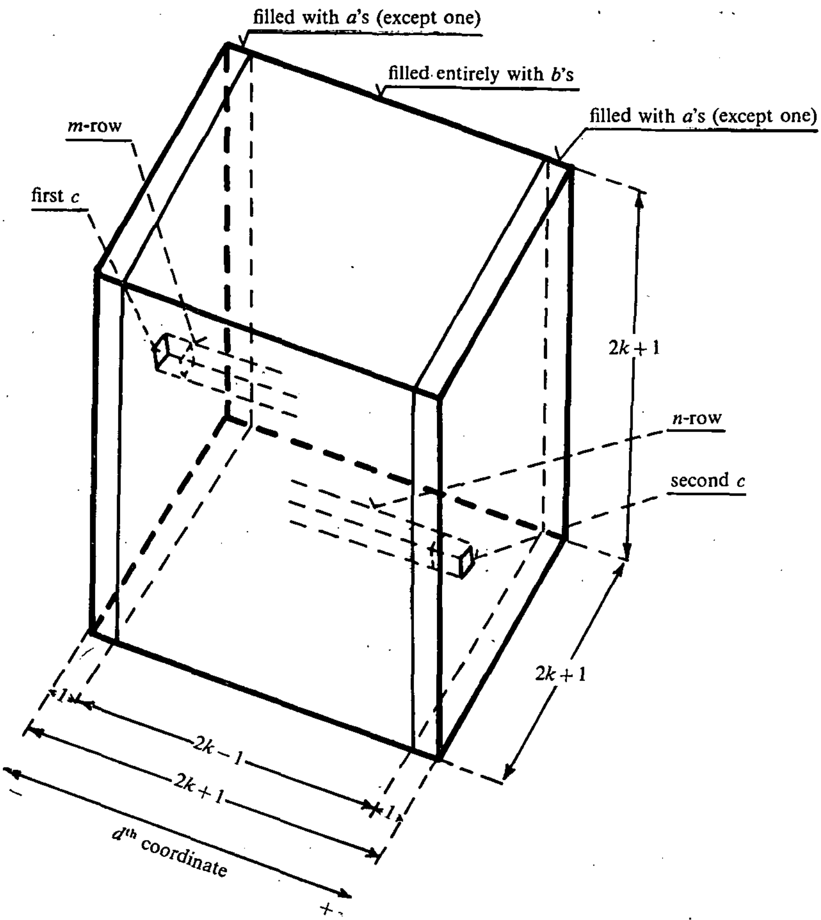

Fig.3

The words $w^k_{m,n}$ used in the proof of Theorem 10 (case $d=3$)

between some rows or not and independently upon which row is wrapped first, which second and so on, the following holds: $w^k_{m,n}$ is a word in L iff $n=m$, i.e.: iff the two exhibited c's are set vis-à-vis. Now, to continue applying Theorem 6, we set $M_k:=(kM^{d-1})\times\{-k\}$, $N_k:=(kM^{d-1})\times\{k\}$ (i.e.: the left and right side of kT resp.) and $\underset{\sim}{N}_k:=\underset{\sim}{M}_k:=kM^{d-1}$ (i.e.: $\underset{\sim}{N}_k$ and $\underset{\sim}{M}_k$ represent all possible row addresses in kT). We verify (1)—(5) step by step: Clearly, $w^k_{m,n}$ is an element of A^{kT} (1). $w^k_{m,n}$ and $w^k_{m,n'}$ differ not on M_k and $w^k_{m,n}$ and $w^k_{m',n}$ differ not upon N_k for arbitrary row addresses m, n, m' and n' ((2), (3)). On $kT\setminus(M_k\cup N_k)$, all admitted words $w^k_{m,n}$ exhibit only b's (4). For two distinct row addresses n and n', set $m=n$; then, clearly, $w^k_{m,n}$ is a member of L whereas $w^k_{m,n'}$ is not (5).

Now, let $i:=k-1$. Then M_k-iT contains only points whose d^{th} coordinates are less than 0 and N_k-iT contains only points whose d^{th} coordinates are greater than 0. Thus $(M_k-iT)\cap(N_k-iT)=\emptyset$ which, clearly, implies $(M_k-iT)\cap(N_k-iT)\cap\cap(k-i)T=\emptyset$. Thus (6) is verified, too, and according to Theorem 6, we get (with $k-i=1$)

$$(2k+1)^{d-1} = \mathrm{card}(kM^{d-1}) = \mathrm{card}\,(N_k) \leqq C^{\mathrm{card}((N_k-iT)\cap T)} \leqq C^{\mathrm{card}(T)} = C^{(d^3)}$$

independently of k ($k\geqq 1$). But for $d\geqq 2$ this is a contradiction. Thus, for such d's, $L=\hat{h}(R_1)$ is not T-recognizable, which proves Theorem 10. □

Theorem 11. For any dimension d and any template $T(\subseteq \mathbf{Z}^d)$ there is some T-wrap, h_T, such that, for any regular language R, $\hat{h}_T(R)$ is T-recognizable.

Proof. Let A be an alphabet, R a regular language $(\subseteq A^*)$ and $T(\subseteq \mathbf{Z}^d)$ a template. We will derive a T-wrap which does not depend on R such that $\hat{h}(R)$ is T-recognizable. To do this, we rely on Theorem 9 according to which there exists an alphabet Q, a function $t\colon Q\to 2^{T\times Q}$ and a family $(M_q^k)_{k\in\mathbf{N},q\in Q}$ of sets with (7), (8) and (9). Because all unions encountered in Theorem 9 are disjoint, we can define a T-wrap $h=(h^k)_{k\in\mathbf{N}}$ in the following recursive way:

First some notational simplification: for any finite subset U of $\mathbf{Z}^d$, a bijection $u\colon U\to\{1,2,\ldots,\mathrm{card}\,(U)\}$ and vector $x\in\mathbf{Z}^d$ let $(u\oplus x)\colon (U+x)\to\{1,2,\ldots,\mathrm{card}\,(U+x)\}$ $(=\{1,2,\ldots,\mathrm{card}\,(U)\})$ be defined by

$$(u\oplus x)\quad(y):=u(y-x)\quad(y\in U+x).$$

Now, let $h_q^k\colon M_q^k\to\{1,2,\ldots,\mathrm{card}\,(M_q^k)\}$ be recursively defined by

$$h_q^0\colon \emptyset\to\emptyset \quad\text{if}\quad M_q^0=\emptyset\quad(q\in Q)$$

$$h_q^0(\mathbf{0})=1\quad\text{if}\quad M_q^0=\{\mathbf{0}\}\quad(q\in Q)$$

(note that $\{\mathbf{0}\}=0T=\dot{\bigcup\limits_{q\in Q}} M_q^0$) and, for $k\geqq 0$,

$$h_q^{k+1}:=\widehat{\underset{(x,r)\in t(q)}{\lessdot}}\; h_r^k\oplus x\quad(k\in\mathbf{N},\,q\in Q)$$

where $\lessdot$ is any (fixed) total ordering of $T\times Q$ (which does not vary with k). Now, let $h^k\colon kT\to\{1,2,\ldots,d_{k,T}\}$ be defined by

$$h^k:=\widehat{\underset{q\in Q}{<}}\; h_q^k\quad(k\in\mathbf{N})$$

where $<$ is some fixed total ordering of Q (not depending on k). Clearly, $h=(h^k)_{k\in\mathbf{N}}$ is a T-wrap.

For two disjoint finite subsets M and N of $\mathbf{Z}^d$, two bijections $m\colon M\to\{1,2,\ldots,\mathrm{card}\,(M)\}$, $n\colon N\to\{1,2,\ldots,\mathrm{card}\,(N)\}$ and two words $w\colon M\to A$, $v\colon N\to A$, let $\overline{m}(w)$ be the word $p=a_1a_2\ldots a_l$ $(\in A^*)$ with $l=\mathrm{card}\,(M)$ and $a_j:=w(m^{-1}(j))$ $(1\leqq j\leqq l)$ and $\overline{n}(v)$ defined similarly. Then we have for any word $u\colon M\cup N\to A$

$$\overline{m\vartriangle n}(u)=\overline{m}(u|_M)\circ\overline{n}(u|_N)$$

and for any word $w\colon M\to A$ and any point $x\in\mathbf{Z}^d$

$$\overline{m\oplus x}(w\oplus x)=\overline{m}(w).$$

Using these notations we get

$$\overline{h^k}(w) = \mathop{\underset{\smile}{<}}\limits_{q\in Q} \overline{h^k_q}(w|_{M^k_q}) \quad (k\in\mathbf{N},\ w\in A^{kT},\ q\in Q)$$

and

$$\overline{h^{k+1}_q}(w|_{M^{k+1}_q}) = \mathop{\overset{\frown}{\underset{\smile}{\nless}}}\limits_{(x,r)\in t(q)} \overline{h^k_r}((w\ominus x)|_{kT|M^k_r}) \quad (k\in\mathbf{N},\ w\in A^{(k+1)T},\ q\in Q).$$

The first formula is trivial, whereas the second one is derived in the following way:

$$\overline{h^{k+1}_q}(w|_{M^{k+1}_q}) = \overline{\mathop{\widehat{\nless}}\limits_{(x,r)\in t(q)} (h^k_r\oplus x)}(w|_{M^{k+1}_q}) = \mathop{\overset{\frown}{\underset{\smile}{\nless}}}\limits_{(x,r)\in t(q)} \overline{(h^k_r\oplus x)}(w|_{x+M^k_r}) =$$

$$= \mathop{\overset{\frown}{\underset{\smile}{\nless}}}\limits_{(x,r)\in t(q)} \overline{(h^k_r\oplus x)}(w|_{x+kT|x+M^k_r}) = \mathop{\overset{\frown}{\underset{\smile}{\nless}}}\limits_{(x,r)\in t(q)} \overline{(h^k_r\oplus x)}\big((w\ominus x)|_{kT|M^k_r}\oplus x\big) =$$

$$= \mathop{\overset{\frown}{\underset{\smile}{\nless}}}\limits_{(x,r)\in t(q)} \overline{h^k_r}((w\ominus x)|_{kT|M^k_r}).$$

Now, let σ be the finite semigroup of transition functions of some finite deterministic automaton which recognizes the language R. Let s_0 be its initial state and G its set of accepting states. As it is well known, we have $R=\{p/p\in A^* \text{ and } s_0\cdot\sigma_p\in G\}$. Thus we get

$$\hat{h}(R) = \{w/w\in(T,A)^* \quad\text{and}\quad s_0\cdot\sigma_{(\overline{h^k}(w))}\in G\}.$$

Now we apply Theorem 2 to show T-recognizability of $\hat{h}(R)$. To do this, we choose $Z:=\sigma^Q$ and function $g\colon (T,A)^*\to Z$ such that

$$g(w)(q) := \sigma_{(\overline{h^k_q}(w|_{M^k_q}))} \quad (k\in\mathbf{N},\ w\in A^{kT},\ q\in Q).$$

Then we have for any $k\in\mathbf{N}$ and $w\in A^{kT}$:

$$s_0\cdot\sigma_{(\overline{h^k}(w))} = s_0\cdot\sigma_{\left(\mathop{\underset{\smile}{<}}\limits_{q\in Q}\overline{h^k_q}(w|_{M^k_q})\right)} = s_0\cdot\mathop{\overset{\frown}{\underset{\smile}{<}}}\limits_{q\in Q}\sigma_{(\overline{h^k_q}(w|_{M^k_q}))} = s_0\cdot\mathop{\overset{\frown}{\underset{\smile}{<}}}\limits_{q\in Q} g(w)(q).$$

Thus, we get

$$\forall w\in(T,A)^*\colon\ w\in\hat{h}(R) \Leftrightarrow \big(s_0\cdot\mathop{\overset{\frown}{\underset{\smile}{<}}}\limits_{q\in Q} g(w)(q)\big)\in G$$

or, equivalently,

$$\forall w\in(T,A)^*\colon\ w\in\hat{h}(R) \Leftrightarrow g(w)\in F$$

where $F(\subseteq Z)$ is defined to be

$$F := \Big\{z/z\in Z \quad\text{and}\quad s_0\cdot\mathop{\overset{\frown}{\underset{\smile}{<}}}\limits_{q\in Q} z(q)\in G\Big\}.$$

Moreover, let $f\colon Z^T\to Z$ be defined by

$$f(v)(q) := \mathop{\overset{\frown}{\underset{\smile}{\nless}}}\limits_{(x,r)\in t(q)} v(x)(r) \quad (v\in Z^T,\ q\in Q).$$

Then, we have for any $k \in \mathbf{N}$, any $w \in A^{(k+1)T}$ and any $q \in Q$:

$$g(w)(q) = \sigma_{\overline{(h_q^{k+1}}(w|_{M_q^{k+1}}))} = \sigma_{\left(\underset{(x,r)\in t(q)}{\widehat{\underline{\mathcal{A}}}} \overline{h_r^k}((w \ominus x)|_{kT|M_r^k})\right)} = \underset{(x,r)\in t(q)}{\widehat{\underline{\mathcal{A}}}} \sigma_{\overline{(h_r^k}((w\ominus x)|_{kT|M_r^k}))} =$$

$$= \underset{(x,r)\in t(q)}{\widehat{\underline{\mathcal{A}}}} \bar{\bar{g}}(w)(x)(r) = f(\bar{\bar{g}}(w))(q).$$

Thus, the entities Z, F, g and f fulfill the conditions of Theorem 2 and, therefore, the T-recognizability of $\hat{h}(R)$ is shown. Because the construction of h has not depended on R, Theorem 11 is proved. □

Theorem 12. There is a context-free language, C_2, with the following property: Let T be any template with exactly 2 extreme points,[7] e_1 and e_2. Let $h_T=(h_T^k)$ be the T-wrap such that h_T^k begins at point ke_1, moves strictly toward ke_2 and ends there (ke_1 and ke_2 are the extreme points of $kT(k \geqq 0)$). Then $\hat{h}_T(C_2)$ is not T-recognizable.

Proof. Let T be a template with exactly two extreme points. We will give some context-free language C_2 such that any T-wrap, h, beginning with the one extreme point of kT and moving strictly toward the other one yields a non-T-recognizable T-language. Without loss of generality we may assume that $\mathbf{0} \in T$ and, therefore, that $T \subseteq \mathbf{Z}$. Moreover, let $\mathbf{0}$ be the left extreme point of T, i.e.: we take $T=\{0=x_1, x_2, \ldots, x_s=m\}$ with $x_1<x_2<x_3<\ldots<x_s$. Furthermore, let us assume that $\gcd(x_2, x_3, \ldots, x_s)=1$[8]. (These restrictions are without loss of generality, because they correspond to certain affine transformations.) Then, according to [5], there exist two natural numbers, 1 and r, such that for all $k \geqq k_0$ ($k_0 := m^2 \cdot s$) it holds

$$kT = \underline{\underline{M}} \cup [l, km-r] \cup (km - \overline{\overline{M}}) \tag{10}$$

where $\underline{\underline{M}} \subseteq [0, l-2]$, $\overline{\overline{M}} \subseteq [0, r-2]$ and $[i,j]$ denotes the set of all integers between and including i and j.

Now, define the context-free language C_2 ($\subseteq A^*$ with $A=\{a, b, c, d, \S\}$) by

$$C_2 := \bigcup_{i,j\in\mathbf{N}} aa^i \S (a, b, c, d)^* ca^i db^j c(a, b, c, d)^* \S b^j b.$$

Because C_2 is quasi-symmetric we restrict our considerations to T-wraps h from left to right. In the sequel let, for any word $p \in A^*$, $\bar{p}$ denote $\hat{h}(p)$ (if it is defined for that p). The proof that $\hat{h}(C_2)$ is not T-recognizable is carried out using Theorem 5.

[7] For the reader who is not familiar with convex sets we recapitulate the notion of convex hulls and extreme points: Let $\mathbf{R}$ denote the set of all real numbers and $\mathbf{R}^d$ the set of all d-tuples of real numbers. For any finite, not empty set $M \subseteq \mathbf{Z}^d$ ($\subseteq \mathbf{R}^d$) let $\tilde{M}$ denote the *convex hull* of M, defined by

$$\tilde{M} := \{\Sigma_{y\in M} a_y \cdot y / 0 \leqq a_y \leqq 1 (y \in M) \text{ and } \Sigma_{y\in M} a_y = 1\}.$$

A point $x \in M$ is called an *extreme point* of M if any representation $x=\Sigma_{y\in M} a_y \cdot y$ with $0 \leqq a_y \leqq 1$ ($y \in M$) and $\Sigma_{y\in M} a_y = 1$ implies $a_x=1$ and $a_y=0 (y \neq x)$. It is matter of triviality that a point e is an extreme point of template T iff ke is an extreme point of kT ($k \geqq 1$).

[8] gcd = greatest common divisor.

In order to do this, set $N_k := \{(i,j)/0 \leqq i, j \leqq k\}$ $(k \in \mathbf{N})$. For any $k \in \mathbf{N}$ and any function $f: \mathbf{N}^2 \to \{0, 1\}$ let w_f^k denote the word

$$w_f^k := \overline{aa^k \S \big(\underset{(i,j)\in N_k}{\lesssim} v_{f,i,j}^k \big)(u_f^k) \S\, b^k b}$$

where

$$v_{f,i,j}^k := \begin{cases} ca^i db^j c & \text{iff } f(i,j) = 1 \\ cc & \text{iff } f(i,j) = 0 \end{cases} \quad (i,j) \in N_k$$

and u_f^k is chosen as a sequence of c's such that w_f^k fits into some KT (e.g.: the next smallest) with $(K-k) \geqq k_0$ $(K = K(k,f))$; let $<$ be any fixed total ordering of set $\mathbf{N}^2$.

For any $k \in \mathbf{N}$ with $km \geqq \max(l, r)$, we represent the words w_f^{2km} as

$$w_f^{2km} = \overline{aa^{2km} \S\, q_f^{2km} \S\, b^{2km} b}$$

where q_f^{2km} is appropriately chosen from A^*. Then, for any i $(0 \leqq i \leqq k)$ and any $x \in iT$ (which implies $0 \leqq x \leqq im$), the word ${}_{i,x}w_f^{2km}$, defined by

$${}_{i,x}w_f^{2km} := (w_f^{2km} \ominus x|_{(D(w_f^{2km})-i)T}),$$

has the form

$${}_{i,x}w_f^{2km} = \overline{aa^{F(k,i,x)} \S\, q_f^{2km} \S\, b^{G(k,i,x)} b}$$

where

$$F(k, i, x) = 2km - x$$

and

$$G(k, i, x) = 2km - (im - x) = (2k - i)m + x.$$

(For an illustration see Fig. 4; note that we have $K - i \geqq k_0$ $(K = K(2km, f))$ which, in turn, implies that ${}_{i,x}w_f^{2km}$ has domain $\underline{\underline{M}} \cup [l, (K-i)m - r] \cup (K-i)\, m - \overline{\overline{M}}$. Thus, ${}_{i,x}w_f^{2km}$ is taken from w_f^{2km} by only removing x a's from left and $im - x$ b's from right.)

Thus, we have ${}_{i,x}w_f^{2km} \in \hat{h}(C_2)$ iff $ca^{F(k,i,x)}\, db^{G(k,i,x)} c$ is contained in q_f^{2km} which, in turn, holds iff $f\big(F(k, i, x),\ G(k, i, x)\big) = 1$. Therefore, for any two functions f, f': $\mathbf{N}^2 \to \{0, 1\}$ which differ on at least one point of $\big\{\big(F(k, i, x),\ G(k, i, x)\big)/$ $/0 \leqq i \leqq k,\ x \in iT\big\}$ $(=: R_k)$, we get that w_f^{2km} and $w_{f'}^{2km}$ are not $k - \hat{h}(C_2)$-equivalent. Now, clearly, $0 \leqq i, i' \leqq k$, $x \in iT$, and $x' \in i'T$ with $(i, x) \neq (i', x')$ implies that $\big(F(k, i, x),\ G(k, i, x)\big) \neq \big(F(k, i', x'),\ G(k, i', x')\big)$ which, in turn, yields $\operatorname{card}(R_k) =$ $= \operatorname{card}\big(\{(i, x)/0 \leqq i \leqq k,\ x \in iT\}\big)$. Therefore we get at least $2^{(e_{k,T})}$ $k - \hat{h}(C_2)$-equivalence classes. Furthermore, from (10) we get that $d_{k,T}$ equals asymptotically km and $e_{k,T}$ equals asymptotically $k^2m/2$. Thus $E_{k-\hat{h}(C_2)}$ cannot be bounded by any $C^{(d_{k,T})}$ which proves our claim that $\hat{h}(C_2)$ cannot be T-recognized. □

Theorem 13. There is a context-free language, C_3, with the following property: Let T be any template with exactly 3 extreme points, e_1, e_2 and e_3. (This implies that kT has also exactly 3 extreme points, namely ke_1, ke_2 and ke_3.) Let $h = (h^k)_{k \geqq 0}$ be any T-wrap such that h^k begins with the (possibly sparsely filled) "line" $ke_1 \to ke_2$, fills that row completely (in any order), moves then to the next parallel row, fills

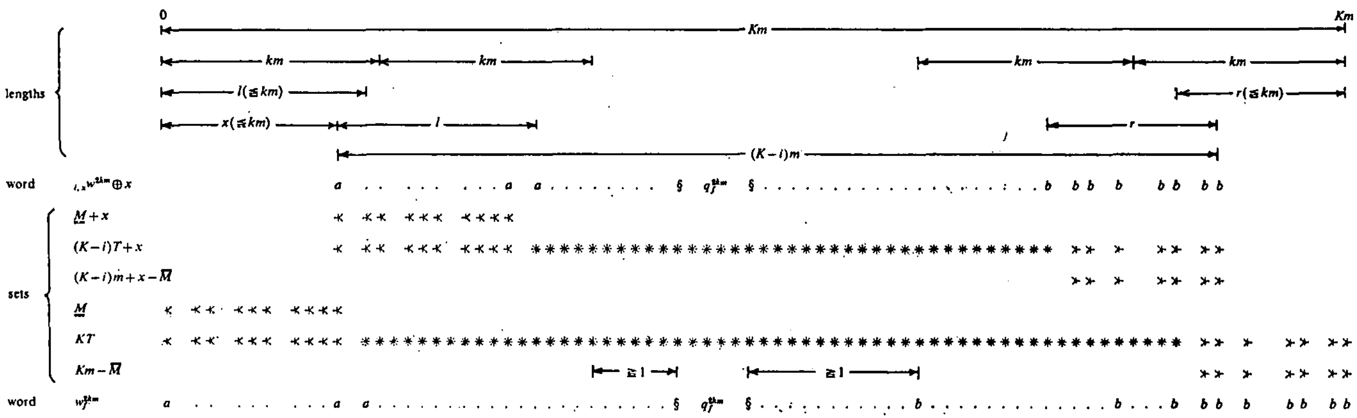

Fig.4

Illustration of the words w_f^{2km} and ${}_{i,x}w_f^{2km}$ $(0\leqq i\leqq k,\ x\in iT\ (\Rightarrow 0\leqq x\leqq km))$. Note that $(K-i)T+x\subseteqq KT$ and $K=K(2km,f)$. $\ast\ast\ldots\ast$ $(\ast\ast\ldots\ast)$: possible pattern of set $\underline{\underline{M}}\,(-\overline{\overline{M}})$

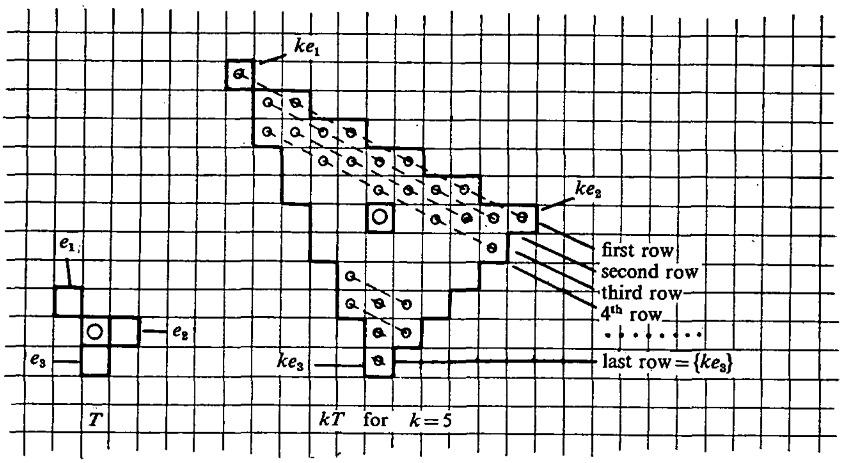

Fig. 5
Illustration to help understanding the T-wraps mentioned in Theorem 13 (for some 2-dimensional template T with exactly 3 extreme points e_1, e_2 and e_3)

this row (in any order), goes then to the next row etc. until it reaches the third extreme point, ke_3 (sc. Fig. 5). Then $\hat{h}(C_3)$ is not T-recognizable.

Proof. Let $T \subseteqq \mathbf{Z}^d$ be a neighbourhood template with exactly 3 extreme points e_1, e_2 and e_3. Let $\hat{h}=(h^k)_{k\in\mathbf{N}}$ be a T-wrap as described in Theorem 13. We will show that, for the context-free language $C_3:=\{w/w\in\{0, 1\}^*$ and $w=w^R\}$, i.e.: the set of all palindromes over alphabet $A=\{0, 1\}$, $\hat{h}(C_3)$ is not T-recognizable.

To pursue the proof we assume that $\hat{h}(C_3)$ is T-recognizable and apply Theorem 6 to get a contradiction. Without loss of generality we may presuppose that the origin $\mathbf{0}$ is one of the extreme points, e_3, say, and that dimension $d=2$. As one easily verifies, we get that $\mathbf{0}$, ke_1 and ke_2 are the (only) extreme points of kT and

$$kT \subseteqq \overline{\{\mathbf{0}, ke_1, ke_2\}}. \tag{11}$$

Now we look for entities which fulfill the conditions of Theorem 6. For any $k\in\mathbf{N}$, $0\leqq j\leqq k$, we define the sets N_k, $M_{k,j}$ and M_k and the number $\bar{k}$ in the following way:

$$\begin{aligned} N_k &:= \overline{\{ke_1, ke_2\}} \cap kT, \\ M_{k,j} &:= \overline{\{\mathbf{0}, je_1, je_2\}} \cap kT \\ \bar{k} &:= \min\{j/0 \leqq j \leqq k,\ \operatorname{card}(M_{k,j}) \geqq \operatorname{card}(N_k)\} \\ M_k &:= M_{k,\bar{k}}. \end{aligned} \tag{12}$$

Clearly, N_k and M_k are subsets of kT $(k\in\mathbf{N})$. Because N_k contains at least the points $je_1+(k-j)e_2$ $(0\leqq j\leqq k)$ which are all different and because $\overline{\{ke_1, ke_2\}}$ contains at most $k\sqrt{(e_1-e_2)^2}+1$ elements of $\mathbf{Z}^d$ there is a positive constant C_1 such that

$$k+1 \leqq \operatorname{card}(N_k) \leqq C_1\cdot k \quad (k \geqq 1).$$

On the other hand, $M_{k,j}$ contains at least the points $0+i_1e_1+i_2e_2$ $(0\leqq i_1, i_2\leqq j, i_1+i_2\leqq j)$ which are all different. Thus,

$$\operatorname{card}(M_{k,j}) \geqq j^2/2 \quad (0 \leqq j \leqq k).$$

Therefore we get $\bar{k}\leqq\lceil\sqrt{2C_1 k}\rceil$ which implies that there is a constant C_2 with

$$\bar{k} \leqq C_2\sqrt{k} \quad (k \geqq 1). \tag{13}$$

Particularly, this means that there is a $k_0\in\mathbf{N}$ such that $k>\bar{k}$ $(k\geqq k_0)$.

Now, for $k>k_0$, set $i(=i(k)):=k-\bar{k}-1$. Then we have

$$(N_k-iT)\cap(k-i)T \subseteqq \overline{\{(\bar{k}+1)e_1, (\bar{k}+1)e_2\}} \quad (k > k_0) \tag{14}$$

and

$$(M_k-iT)\cap(N_k-iT)\cap(k-i)T = \emptyset \quad (k > k_0). \tag{15}$$

This is shown in the following way: Obviously, according to (11) and (12), we have (for $k>k_0$)

$$(N_k-iT)\cap(k-i)T \subseteqq \left(\overline{\{ke_1, ke_2\}}-\overline{\{0, ie_1, ie_2\}}\right)\cap\overline{\{0, (k-i)e_1, (k-i)e_2\}}.$$

Furthermore, any element x of the right set has two representations $x=a_1ke_1+a_2ke_2-b_1ie_1-b_2ie_2$ and $x=c_1(k-i)e_1+c_2(k-i)e_2$ with $0\leqq a_1, a_2, b_1, b_2, c_1, c_2\leqq 1$ and $a:=a_1+a_2=1$, $0\leqq b:=b_1+b_2\leqq 1$ and $0\leqq c:=c_1+c_2\leqq 1$. Because e_1 and e_2 are linearly independent, we have $(a_1k-b_1i)=c_1(k-i)$ and $(a_2k-b_2i)=c_2(k-i)$ and, summing up both sides, $k-bi=c(k-i)$. Evaluating i yields $(1-b)k+b(\bar{k}+1)=c(\bar{k}+1)$. Because $\bar{k}+1\leqq k$, this is possible only if $c\geqq 1$, which yields $c_1+c_2=1$ and shows that x is a member of the right set of (14). Because $k-i=\bar{k}+1$, (14) is proved.

To prove (15) we use (14): Let $x\in M_k-iT$; then, using (11) and (12), we get that x is a member of $\overline{\{0, \bar{k}e_1, \bar{k}e_2\}}-\overline{\{0, ie_1, ie_2\}}$, too and, therefore, has representation $x=a_1\bar{k}e_1+a_2\bar{k}e_2-b_1ie_2-b_2ie_2$ with $0\leqq a_1, a_2, b_1, b_2\leqq 1$, $0\leqq a:=a_1+a_2\leqq 1$ and $0\leqq b:=b_1+b_2\leqq 1$. Because e_1 and e_2 are linearly independent, we get, evaluating i, that $x=d_1(\bar{k}+1)e_1+d_2(\bar{k}+1)e_2$ where $d_j=\left(a_j\bar{k}-b_j(k-\bar{k}-1)\right)/(\bar{k}+1)$ are uniquely determined $(j=1,2)$. Now we have $d_1+d_2=\left((a+b)\bar{k}+b(1-k)\right)/(\bar{k}+1)=$ $=b+a\bar{k}/(\bar{k}+1)-bk/(\bar{k}+1)\leqq a\bar{k}/(\bar{k}+1)<1$ (because $\bar{k}<k$). Thus x is not a member of $\overline{\{(\bar{k}+1)e_1, (k+1)e_2\}}$ and, because of (14), not a member of $(N_k-iT)\cap(k-i)T$. This proves (15).

Furthermore, because of $M_k\subseteqq\overline{\{0, \bar{k}e_1, \bar{k}e_2\}}$, $N_k\subseteqq\overline{\{ke_1, ke_2\}}$, e_1 and e_2 linearly independent and $k>\bar{k}$ for $k>k_0$ we conclude that

$$M_k\cap N_k = \emptyset \quad (k > k_0).$$

Now, for any $k>k_0$ and any word $p\in A^*$ of length $\operatorname{card}(kT)$, $\hat{h}(p)$ is constructed by filling N_k with the first $\operatorname{card}(N_k)$ symbols of p and filling M_k with the last $\operatorname{card}(M_k)$ $(\geqq \operatorname{card}(N_k))$ symbols of p. Let g_k: $N_k\to M_k$ be defined by

$$g_k(x) = x' \quad \text{iff} \quad h^k(x) = \operatorname{card}(kT)-h^k(x')+1 \quad (k > k_0, x\in N_k).$$

Clearly, g_k is injective. (Informally, for any $x\in N_k$, if the first j^{th} symbol of p is placed at point x, then the last j^{ht} symbol is placed at $g_k(x)$ $\left(1\leqq j\leqq\operatorname{card}(N_k)\right)$.)

Let $\underset{\sim}{M}_k$ and $\underset{\sim}{N}_k$ denote the set of all functions $n\colon N_k \to \{0, 1\}$ or $m\colon M_k \to \{0, 1\}$, resp. Furthermore, for any $k > k_0$, any $m \in \underset{\sim}{M}_k$ and $n \in \underset{\sim}{N}_k$ define the word $w^k_{m,n}$ ($\in \{0, 1\}^{kT}$) as follows:

$$w^k_{m,n|N_k} := n$$

$$w^k_{m,n|M_k} := m$$

$$w^k_{m,n|kT\setminus(M_k \cup N_k)} :\equiv 0.$$

Thus we have from the construction of $w^k_{m,n}$ that (1)—(4) are fulfilled. Now, $w^k_{m,n}$ is an element of $\hat{h}(C_3)$ iff $n(x) = m(g_k(x))$ for all $x \in N_k$ and $m|_{M_k \setminus g_k(N_k)} \equiv 0$. Therefore, for any two $n, n' \in \underset{\sim}{N}_k$ with $n \neq n'$ choose $m \in \underset{\sim}{M}_k$ such that $m(g_k(x)) = n(x)$ for $x \in N_k$ and $m|_{M_k \setminus g_k(N_k)} \equiv 0$. Then, clearly, we have

$$w^k_{m,n} \in \hat{h}(C_3) \quad \text{and} \quad w^k_{m,n'} \notin \hat{h}(C_3)$$

which establishes (5). Moreover, setting $i = i(k) := k - \bar{k} - 1$, we have $0 \leqq i(k) \leqq k$ and (15) which resembles (6).

Because we have assumed that $\hat{h}(C_3)$ is T-recognizable, Theorem 6 allows us to conclude that there is some constant C which does not depend on k such that

$$\operatorname{card}(\underset{\sim}{N}_k) \leqq C^{\operatorname{card}((N_k - iT)\cap(k-i)T)} \quad (k > k_0).$$

Clearly, $\operatorname{card}(\underset{\sim}{N}_k) \geqq 2^{k+1}$. On the other hand, because of (14) we have $\operatorname{card}((N_k - iT)\cap(k-i)T) \leqq (\bar{k}+1)\sqrt{(e_1-e_2)^2} + 1 \overset{(13)}{\leqq} C_2'\sqrt{k}$ (with appropriate constant C_2'). Thus, we would get

$$2^{k+1} \leqq C^{C_2' \cdot \sqrt{k}} \quad (k > k_0)$$

which, clearly, is impossible. This proves that $\hat{h}(C_3)$ is not T-recognizable. □

Theorem 14. There is a context-free language, C_4, such that, for any dimension d and any template $T (\subseteqq \mathbf{Z}^d)$ which contains more than 3 extreme points, there is no T-wrap at all such that $\hat{h}(C_3)$ is T-recognizable.

Proof. Let T be a neighbourhood template with 4 or more extreme points and $h = (h^k)_{k \in \mathrm{N}}$ be any T-wrap. We will show that, for the context-free language C_4 which contains all words p over the alphabet $A = \{a, b, c\}$ which exhibit (any number of c's and) exactly as many b's as a's, the language $\hat{h}(C_4)$ is not T-recognizable.

To carry out the proof, we assume that $\hat{h}(C_4)$ is T-recognizable and apply Theorem 6 to get a contradiction.

Let $E = \{e_1, e_2, e_3, \ldots, e_n\}$ $(n \geqq 4)$ be the set of all extreme points of T. Then, clearly, $\{ke_1, ke_2, \ldots, ke_n\}$ is the set of all extreme points of kT $(k \in \mathrm{N})$. Furthermore, we can choose two pairs of extreme points (e_1, e_2) and (e_3, e_4), say, such that $\overline{\{e_1, e_2\}}$ and $\overline{\{e_3, e_4\}}$ constitute disjoint extreme edges of $\bar{T}$, i.e.: for any point $x \in \overline{\{e_1, e_2\}}$ and any representation $x = \Sigma_{y \in T} a_y \cdot y$ with $0 \leqq a_y \leqq 1$ $(y \in T)$ and $\Sigma_{y \in T} a_y = 1$ we get $a_y = 0$ $(y \notin \{e_1, e_2\})$ and for any point $x \in \overline{\{e_3, e_4\}}$ and any representation $x = \Sigma_{y \in T} a_y \cdot y$ (with $0 \leqq a_y \leqq 1$ $(y \in T)$ and $\Sigma_{y \in T} a_y = 1$) we get $a_y = 0$ $(y \notin \{e_3, e_4\})$.

Moreover, $\overline{\{e_1, e_2\}}\cap\overline{\{e_3, e_4\}}=\emptyset$. This implies that $\overline{\{ke_1, ke_2\}}$ and $\overline{\{ke_3, ke_4\}}$ are the corresponding (disjoint) extreme edges of $\overline{kT}$ $(k\geqq 1)$ [9].

Now for any $k\geqq 1$, $0\leqq j\leqq k$ let $y_j^k:=je_1+(k-j)e_2$ and $x_j^k:=je_3+(k-j)e_4$. Clearly, x_j^k, $y_j^k\in kT$ and all points x_j^k and y_j^k, are different. Let $N_k:=\{y_j^k/0\leqq j\leqq k\}$ and $M_k:=\{x_j^k/0\leqq j\leqq k\}$. Set $\underset{\sim}{M}_k:=\underset{\sim}{N}_k:=\{0, 1, \ldots, k\}$ and for $m\in\underset{\sim}{M}_k$, $n\in\underset{\sim}{N}_k$ define $w_{m,n}^k$ $(\in A^{kT})$ by

$$w_{m,n}^k(x_j^k) := \begin{cases} a & \text{if} \quad 0 \leqq j < m \\ c & \text{if} \quad m \leqq j \leqq k \end{cases}$$

$$w_{m,n}^k(y_j^k) := \begin{cases} b & \text{if} \quad 0 \leqq j < n \\ c & \text{if} \quad n \leqq j \leqq k \end{cases}$$

$$w_{m,n|kT\setminus(N_k\cup M_k)}^k := c.$$

Thus $w_{m,n}^k$ depends on n only at N_k and on m only at M_k which implies conditions (1)—(4) of Theorem 6. Furthermore, for $n, n'\in\underset{\sim}{N}_k$ with $n\neq n'$, we get $w_{n,n}^k\in\hat{h}(C_4)$ whereas $w_{n,n'}^k\notin\hat{h}(C_4)$ because in any word $\tilde{w}_{m,n}^k$ the number of occurring a's differs from the number of occurring b's by exactly $|m-n|$. Hence, (5) is fulfilled, too.

Now, let $i(=i(k):=k-1)$ $(k\geqq 1\,(!))$. We have to ensure that

$$(N_k-iT)\cap(M_k-iT)\cap T = \emptyset. \tag{16}$$

This is true, because otherwise there would exist a point $x\in T$ with representations

$$x=je_1+(k-j)e_2-\Sigma_{y\in T}l_y\cdot y \;\; (0\leqq j\leqq k,\; l_y\in \mathbf{N},\; \Sigma_{y\in T}l_y=k-1)$$

and

$$x=j'e_3+(k-j')e_4-\Sigma_{y\in T}l'_y\cdot y \;\; (0\leqq j'\leqq k,\; l'_y\in \mathbf{N},\; \Sigma_{y\in T}l'_y=k-1).$$

This fact would imply that $(x+\Sigma_{y\in T}l_y\cdot y)/k = \frac{j}{k}e_1+\frac{k-j}{k}e_2$ $(\in\overline{\{e_1, e_2\}}$ and $(x+\Sigma_{y\in T}l'_y\cdot y)/k=\frac{j'}{k}e_3+\frac{k-j'}{k}e_4$ $(\in\overline{\{e_3, e_4\}})$. However, because $\overline{\{e_1, e_2\}}$ and $\overline{\{e_3, e_4\}}$ are extreme edges of $\overline{T}$, we might conclude that $x\in\overline{\{e_1, e_2\}}$ and $x\in\overline{\{e_3, e_4\}}$, which, obviously, is a contradiction to the assumption that these two extreme edges are disjoint. Thus, (16) and therefore (6) is fulfilled $(k\geqq 1)$.

Theorem 6 tells us that, in this case, there is a constant C such that card $(\underset{\sim}{N}_k)\leqq C^{\text{card}((N_k-iT)\cap(k-i)T)}$ $(k\geqq 1,\; i=i(k)=k-1)$. However, card $(\underset{\sim}{N}_k)=k+1$ whereas card $((N_k-iT)\cap(k-i)T)\leqq$card (T). Therefore the inequality just now mentioned can not be true. Thus $\hat{h}(C_4)$ is not T-recognizable. □

4. Conclusion and summary

Using new notions of (d-dimensional) languages and their recognition which seem to be more adequate to the phenomena occuring in d-dimensional cellular automata, we could generalize and improve the results of Seiferas [12] concerning the recognition speed of regular languages in such structures. Smith [11] raised the ques-

[9] This is an elementary fact which is easily proved using basic properties of convex set (cp. [4]).

tion whether contextfree languages, inscribed in a "natural" way into one-dimensional cellular automata, can be recognized in real-time or not. In our sense, this question is answered in the negative in a special case of dimension one as well as in a very general way for arbitrary dimensions ($d>1$). Thus we have found a further property, in which regular languages differ essentially from context-free ones.

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Abstract

This is a new approach to recognize formal languages by deterministic d-dimensional off-line cellular automata. It allows to exploit the parallelism inherent in such devices in a higher rate than this is done by two other approaches already known. Although the proposed notion of recognition turns out to be the strongest one known to date, the known results concerning real-time recognition of regular languages can be improved (for all dimensions). On the other hand, the strength of this notion allows us to show — under some very general assumptions — the non-recognizability of context-free languages in real-time (for all dimensions).

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