Differentiability properties of computable functions — a summary

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To the memory of Professor László Kalmár

Contemporary computing machinery includes many analog devices — machines which, by a direct process, produce continuous functions of a real variable as their output. These functions appear to be computable by virtue of the fact that there are real existing devices which generate them. One might attempt to understand these functions by means of traditional computer — oriented techniques. For example, one might begin with an effectively generable set of functions each of which is "obviously computable". One might then consider the class of functions obtained from this set by finite programs (=interpreted schemes=flow charts=finite algorithms), [3]. However, as Shepherdson has pointed out [8], no procedure of this kind can encompass all the computable functions of a real variable. Fortunately the literature of recursion theory provides a precise definition of this concept — Grzegorczyk [1]. We give this definition below.

Our paper is concerned with the differentiability properties of computable functions of a real variable. Unless stated otherwise, we restrict our attention to functions defined on compact intervals. Grzegorczyk raised the question [1, p. 201] whether differentiation and integration are computable processes. The indefinite integral of a computable function is computable ([4], [7]). Lacombe [5] stated a negative result for differentiation, although he gave no proof. He made no mention of higher derivatives. In 1971, Myhill [6] showed that the derivative of a computable continuously differentiable function need not be computable. He suggested in a footnote that the same should hold for infinitely differentiable functions. This seems at first glance to follow from a modification of his construction — a modification so obvious that it need not be written down. However, the result turns out to be false. We prove that if $f(x)$ is infinitely differentiable and computable, then all of its derivatives are computable. This follows from the stronger statement:

Proposition. If $f(x)$ is computable and of class C^2 (twice continuously differentiable) on a compact interval $[-M, M]$ (with M a positive integer), then $f'(x)$ is computable.

This proposition is best possible. For by modifying Myhill's counterexample [6] slightly, we can construct a computable function which is twice differentiable (but not continuously), and whose derivative is not computable. Using a completely different construction we can show:

Example. There is a computable continuously differentiable function $f(x)$ on $[0, 1]$ whose derivative $f'(x)$ is not computable, but such that $f'(x)$ is "Banach—Mazur computable" (definition below).

As an immediate consequence of the proposition we have:

Corollary. If $f(x)$ is computable and C^∞ (infinitely differentiable) on a compact interval $[-M, M]$ (M a positive integer), then the n-th derivative $f^{(n)}(x)$ is computable for each n.

We now give Grzegorczyk's definition of a computable function of a real variable. The fundamental definition is phrased in terms of general recursive functionals. In [2], Grzegorczyk presented seven definitions all of which were proved equivalent to the fundamental definition. In this paper we find it convenient to use one of these other definitions. First we need:

Definition 1. A sequence of reals $\{x_n\}$ is *computable* if there exist recursive functions $a(n, k)$, $b(n, k)$, and $s(n, k)$ such that

$$\left| x_n - (-1)^{s(n,k)} \frac{a(n,k)}{b(n,k)} \right| < \frac{1}{k+1}$$

for all n, k (with $b(n, k) \neq 0$).

Roughly, this means that there is a recursive double sequence of rationals r_{nk} which converge effectively to x_n as $k \to \infty$.

Definition 2. A function $f(x)$ from a compact interval of $\mathbf{R}$ into $\mathbf{R}$ is computable if:

(i) f maps every computable sequence of reals into a computable sequence of reals (the Banach—Mazur property);

(ii) f is "effectively uniformly continuous", i.e. there is a recursive function $g(n) > 0$ such that

$$|x-y| < \frac{1}{g(n)} \quad \text{implies} \quad |f(x)-f(y)| < \frac{1}{n+1}.$$

(This is Grzegorczyk's definition reduced to the case of a compact interval. Grzegorczyk considered functions from $\mathbf{R}$ to $\mathbf{R}$, and used a more complicated version of condition ii) to take account of the noncompactness of the domain.)

Two further results follow from our work. 1) The example above shows that there exists a computable continuously differentiable function $f(x)$ whose derivative satisfies condition i) of definition 2 (the Banach—Mazur condition), but not condition ii). By contrast, there is no case where $f(x)$ is computable and $f'(x)$ satisfies ii) but not i).

2) An attempt to extend the corollary leads to the following counterexample. There is a computable infinitely differentiable function $f(x)$ on $[0, 1]$ such that the sequence of n-th derivatives is not uniformly computable as a function of n. In other words, although by the corollary each derivative is computable, the sequence of derivatives need not be.

We now consider the proofs. The proposition is fairly easy, and so we shall give a sketch. However, the counterexamples are rather intricate. For the sake of brevity we omit an account of the constructions involved.

To prove the proposition we proceed as follows: Since $f''(x)$ is continuous on a compact set, it is bounded. Thus $|f''(x)| \leqq K$, an integer. Now by the mean value theorem, for any $x, y \in [-M, M]$ with $x<y$, there exists a ξ with $x<\xi<y$ such that:

$$f'(y)-f'(x) = f''(\xi)(y-x).$$

Hence $f'(x)$ is effectively uniformly continuous — in fact, $|f'(y)-f'(x)| \leqq K|y-x|$. Now applying the mean value theorem again (this time to f and f') we have:

For all $x, y \in [-M, M]$ with $x<y$, there exists a ξ with $x<\xi<y$ such that:

$$f'(\xi) = \frac{f(y)-f(x)}{y-x}.$$

The difference quotient $[f(y)-f(x)]/(y-x)$ is computable since f is. And the effective uniform continuity of f' means that $f'(\xi)$ converges effectively to $f'(x)$ as $\xi \to x$. □

The result proved above does not hold for functions defined on noncompact intervals such as the real line. (Here we return to Grzegorczyk's original definition [1], with its more complicated condition (ii).) By modifying Myhill's counterexample in an obvious way, we can show that: There is a computable infinitely differentiable function on the real line whose first derivative is not computable.

A detailed account of the results discussed in this note is planned for a forthcoming paper.

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