Topological analysis of linear systems

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Abstract

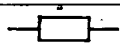
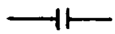
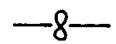
The author deals with the solution of the input—output analysis problem of the linear system models. Beside the traditional elements a suggestion is presented for introducing degenerate elements, hereby a more general class of practical linear systems can be taken into account. For the analysis by topological formulas the author gives k-trees generation procedures which are essentially applications of his earlier methods written in papers [3] and [4]. Finally concrete examples are presented from the area of the electrical networks model as well.

Introduction

In this paper we are going to deal with models of linear systems which can be described by differencial equations in general, and can disjoin two terminal tools connecting two points or domains. In practice we can find such systems at electrical, mechanical, pneumatic, thermodynamic, etc. networks. After defining the system elements let the structure of the linear system be given by an abstract graph. Beside the conservative system elements well-known from reference [9], degenerate elements will be introduced by which the model of more general linear systems can be given. For example, it can be shown that any two terminal or two part electrical network consisting of passive elements, mutual inductances and controlled generators may be modelled as a network consisting only of degenerate (nullator and norator) and passive elements [8]. The network determinant is set in the centre of the formal solution of the system equations, a suggestion will be presented for its calculation by a topological formula with application of the generation of k-trees suggested by the author in earlier papers [3] and [4]. Finally, the order of the topological analysis will be shown by concrete examples for the calculations of the electrical linear systems.

The elements of the linear system

In the model of the linear system variables of two type are allowed; namely, through and *across*, which are characterized by the usual measuring directions [9]. (In practice through variables are electrical current, power, flood, convection of heat, etc., across variables are voltage, rotation, pressure, temperature, etc.). In general a variable is a function of the time, and we assume that there exists its Laplace-transform. The system elements are defined by relations between the through and the across variables in the following manner:

Type of the relation	Name of the element	Sign of the element	Relation between through and across variables
Inductive	Inductance		$L.\ s.\ J(s)=V(s)$
Resistive	Resistance		$R.\ J(s)=V(s)$
Capacitive	Capaticy		$J(s)=C.\ s.\ V(s)$
Over determined	Nullator		$J(s)=0,\ V(s)=0$
Undetermined	Norator		$J(s)=$ arbitrary, $V(s)=$ arbitrary

The symbol s is the complex value, J and V are the Laplace-transforms of the corresponding functions (through and across variables). The equations between the Laplace-transforms are valid under zero initial conditions. L, R and C are arbitrary real numbers differing from zero (they are the parameters of the corresponding system elements). In case of a concrete system $V(s)$ and $J(s)$ belonging to a norator are defined by the other elements of the system, the word "arbitrary" is to be understood in this manner.

The first three system elements are to be regarded as classical ones from the reference. These are the so called "passive elements". Now we give a reason for the introduction of degenerate elements (nullator and norator).

The through and across source variables driving the linear systems (the independent generators) are given by their functions. Beside such ideal source variables other ones may occur as well, the function of which depend on through or across variable between two vertices of the system graph (controlled generators). For example, it may occur that the controlled through source variable between vertices i and k is the multiple of the across variable between vertices l and k. This is the situation with electrical linear networks in case of voltage controlled circuit generators. The conventional sign of the controlled source variable occurring in the present example and its nullator—norator pair equivalent network are shown by Fig. 1. According to Fig. 1 such an equivalent network may be produced from an unique passive element (resistance) and from a nullator—norator pair.

After the introduction of the nullator—norator pairs there is a possibility for producing the models of the controlled generators of all types and the ideal transformator by electrical networks [7]. In the nullator—norator pair model of a general linear network only elements with parameters R, L and C, nullators and norators, (the latters as a pair) occur [6]. Thus by the introduction of degenerated element suggested in the present paper, a practically larger class of the linear systems may be described than by the set of passive elements.

We do not define exactly the rules of the connections between the system elements. But we assume in the present paper that the graph of the system is connected,

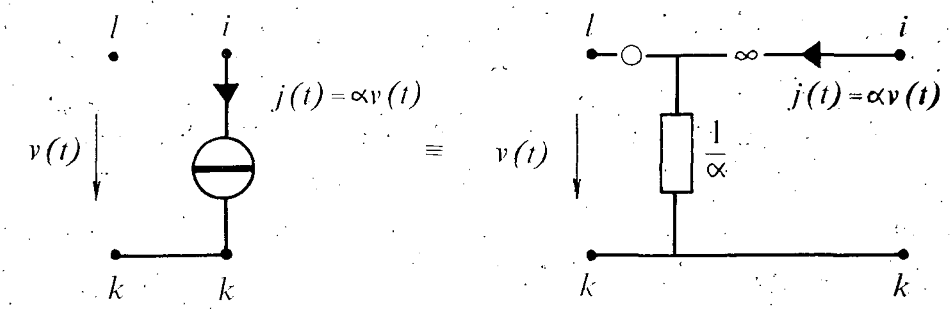


Fig. 1

an element does not contain a loop, the degenerated elements occur only in pairs and degenerated element cannot be parallel with any passive one. According to [8] the latter condition does not break the general case. Practically, concrete linear systems obviously hold these conditions.

After this we draw up the program of the input—output analysis of linear systems in the following manner. Consider a model of a linear system by its graph, the edges of which are system elements, through and across source variables driving the system. Determine the concrete values of the through and across variables in each passive elements.

System equations and their formal solution

Let the system graph consist of n vertices, l edges containing a passive element and N pairs of nullator—norator edges. The equivalent network of an edge of the graph containing a passive element is shown in Fig. 2 (Fig. 2 takes into account the generalized case).

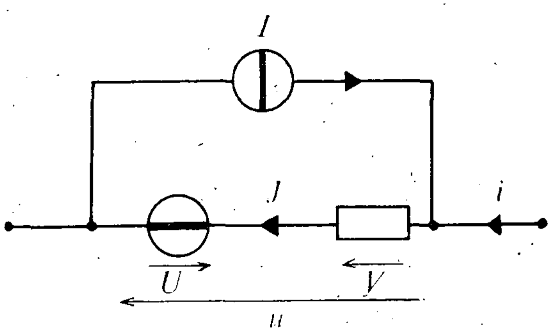


Fig. 2

Concerning the passive element the vector equations are the following:

$$\mathbf{V} = \mathbf{u}+\mathbf{U} \tag{1}$$

$$\mathbf{J} = \mathbf{i}+\mathbf{I}, \tag{2}$$

where $\mathbf{U}$ and $\mathbf{I}$ are vectors of the across and through source variables, respectively, the components of which can be found in the equivalent networks of the edges of the graph, $\mathbf{u}$ and $\mathbf{i}$ are column vectors of size l made from the Laplace-transforms of the across and through variables of the edges.

After suitable numbering of the edges, let $\mathbf{A}$ be the incidence matrix of the graph concerning a reference vertex, and divide it into the following parts:

$$\mathbf{A} = [\mathbf{A}_p\, \mathbf{A}_0\, \mathbf{A}_\infty]$$

where submatrix $\mathbf{A}_p$ corresponds to the edges containing a passive element, $\mathbf{A}_0$ to the nullator and $\mathbf{A}_\infty$ to the norator edges.

It is true [cf. 8] that

$$\mathbf{J} = \mathbf{y}\mathbf{V} \tag{3}$$

where $\mathbf{y}=\langle y_1, y_2, \ldots, y_l\rangle$ is a diagonal matrix and y_i is the operator admittance of the i-th passive element. Namely,

$$y_i = \frac{J_i(s)}{V_i(s)}.$$

After this we define the incidence matrices of the modified graphs.

Let ${}_0\mathbf{A}_p$ represent the reduced incidence matrix of the graph which is determined by the edges containing a passive element after short circuiting all nullator edge endpoints. We use matrix ${}_\infty\mathbf{A}_p$ in a similar sense, i.e., let ${}_\infty\mathbf{A}_p$ be the reduced incidence matrix of the graph which is determined by the edges containing a passive element after short circuiting all norator edges endpoints.

To describe the linear system we write the law of the node [8] in the following form:

$${}_\infty\mathbf{A}_p \cdot \mathbf{i} = \mathbf{O}. \tag{4}$$

Finally, let us introduce the vector $\mathbf{P}$ of size $(n-N)$ by the equation

$$\mathbf{u} = {}_0\mathbf{A}_p^t \cdot \mathbf{P} \tag{5}$$

the components of which are sum of the across variables along the path connecting the suitable vertex of the graph with the reference vertex [9].

(1)—(5) are the basic equations of the examined system.

Considering (1) and (2),

$$\mathbf{i}+\mathbf{I} = \mathbf{y}(\mathbf{u}+\mathbf{U}). \tag{6}$$

Let us multiply (6) by the matrix ${}_\infty\mathbf{A}_p$ from the left, and consider (4). Then

$${}_\infty\mathbf{A}_p \cdot \mathbf{I} = {}_\infty\mathbf{A}_p \cdot \mathbf{y} \cdot \mathbf{u} + {}_\infty\mathbf{A}_p \cdot \mathbf{y} \cdot \mathbf{U}. \tag{7}$$

Taking into account (5), after some rearrangement we get

$${}_\infty\mathbf{A}_p \cdot \mathbf{y} \cdot {}_0\mathbf{A}_p^t \cdot \mathbf{P} = {}_\infty\mathbf{A}_p(\mathbf{I}-\mathbf{y}\cdot\mathbf{U}). \tag{8}$$

Let us introduce the symbol $\mathbf{Y}$ by

$$\mathbf{Y} = {}_{\infty}\mathbf{A}_p \cdot \mathbf{y} \cdot {}_0\mathbf{A}_p^t.$$

If det $(\mathbf{Y}) \neq 0$ then there exists the inverse matrix $\mathbf{Y}^{-1}$. In this case, by [8], we can write

$$\mathbf{P} = \mathbf{Y}^{-1} \cdot {}_{\infty}\mathbf{A}_p \cdot (\mathbf{I} - \mathbf{y} \cdot \mathbf{U}). \tag{9}$$

One can consider (9) as the explicite solution of the input—output analysis. We remark that in this manner the calculation of det $(\mathbf{Y})$ is necessary for the solution, or rather, for determining the inverse of a matrix.

If the system contains also purely across or through source variable edges then the form of (9) is modified slightly. There is no problem that the graph contains only through variable edges. Namely, in this case the equivalent network of the generalized passive edge has a passive element with "operator admittance of zero". It can be seen that in the case of across source variable edges the right side of (9) is invariable, and the modification of (9) does not influence the calculation of det $(\mathbf{Y})$. Further more, det $(\mathbf{Y})$ will be called *system determinant*.

Calculation of the system determinant

It is known from [5] that

$$\det(\mathbf{Y}) = \sum F^{N+1}. \tag{10}$$

This is a topological formula, where F^{N+1} is an edge admittance product formed by an $(N+1)$-tree of the system graph which consists of only edges containing a passive element, and this $(N+1)$-tree turns into a circuitless connected graph after short-circuiting either the nullator or the norator endpoints. The sign of each product comes from the product of the corresponding majors of ${}_{\infty}\mathbf{A}_p$ and ${}_0\mathbf{A}_p^t$. The summation takes into account the same $(N+1)$-trees of the type in question. The generation of such $(N+1)$-trees can be given by the following algorithm.

Step 1. After missing all nullator edges produce and list trees of the remained graph which contain all the norator edges. This is possible by the suitable organization of the method written in [3].

Step 2. Leave the norator edges from the trees produced by step 1. We obtain $'F^{N+1}$ $(N+1)$-trees, and

$$\{'F^{N+1}\} \supseteqq \{F^{N+1}\} \tag{11}$$

is obviously fulfilled.

Step 3. Short-circuit the endpoints of the left nullator edges in each of the elements of set $\{'F^{N+1}\}$. Let us select from them the circuitless graphs according to method [4]. By this the generation of all F^{N+1} comes to the end.

Finally, we remark that the sign of any F^{N+1} edge admittance product can be determined by the application of the Davies rule.

Universal parameters

We use the method written in this paper in the analysis of an electrical linear network model consisting of nullator—norator pairs. It is known that any universal parameter of a two port network (if there exists any) can be obtained as system determinant of the two port network its input and output being closed by suitable nullator—norator pairs [2]. One can see the suitable closure in Fig. 3 together with

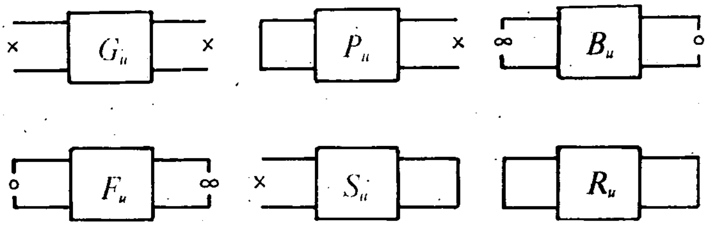


Fig. 3

the symbol of the universal parameter. Notice that the short circuit closure in Fig. 3 is equivalent to a parallel connected nullator—norator pair. Referring to the earlier, it is clear that for the production of any universal parameter by topological formulas we need the earlier k-trees of the suitable closed network graph, and now $N+1 \leqq k \leqq N+3$, where N is the number of the nullator—norator pairs in the original network model.

We give a block scheme of the k-trees generation in Fig. 4 to produce an arbitrary universal parameter. To realize this algorithm by a computer the procedure has all the advantages of methods written in [3] and [4] (i.e., "calculation of the trees" is possible "one by one", it is not necessary to reserve them in the storage capacity).

In the case of active networks the determination of the universal parameters makes possible to describe the system functions of the network model in question, which are quotiens of the suitable universal parameters in general [2].

Further on we are going to study the description of some system functions either passive or active networks.

Analysis of passive networks

First consider a two terminal network consisting of R, L and C elements and set ourselves an aim to write its operator admittance function in the general case. The task can be drawn up as the determination of the input admittance of a two terminal network closing its output by break (the output may be an arbitrarily choosen pair of the vertex in the network).

The first equation of the inverse hybrid characteristics is:

$$I_1 = D_{11} U_1 + D_{12} I_2$$

from which after considering $I_2=0$ we get

$$Y_{in}=\frac{I_1}{U_1}=D_{11}.$$

Taking into account the definitions of the universal parameters we can write

$$Y_{in}=\frac{G_u}{P_u}. \tag{12}$$

Taking into account the calculation of G_u and P_u by topological formulas (see Fig. 3) we can see that to produce the numerator of (12) all the trees of the network graph are needed, while the denominator needs all 2-trees which contain the input points in separate components of the graph. So k-trees needed to (12) can be generated by the somewhat modified method written in [3]. Next determine the transfer impedance function of an arbitrary RLC two port network, the scheme of which is represented in Fig. 5.

Now from the impedance characteristics of the network closed on the output by a break we obtain

$$Z_{tr}=\left.\frac{U_2}{I_1}\right|_{I_2=0}=Z_{21}, \tag{13}$$

and from (13), because of the definitions of the universal parameters,

$$Z_{tr}=\frac{B_u}{G_u} \tag{14}$$

follows.

To write the numerator of (14) all the 2-trees are needed, which separate either the input or the output vertices (i.e., the 2-trees contain these points in different components). From the latter statement it follows that to determine B_u the closed

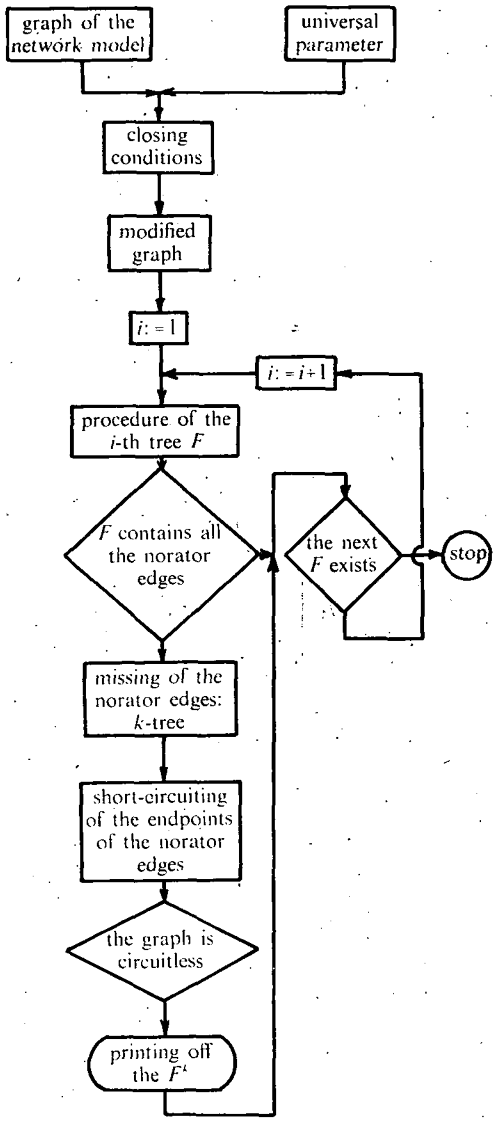


Fig. 4

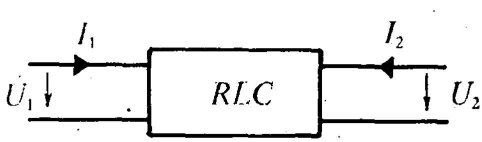


Fig. 5

network model contains a unique nullator—norator pair each degenerate element of which is connected to the input and output. It is not a problem to generate k-trees for the denominator of (14) (see the first example). We remark that similar k-trees are needed for producing F_u as in the case of B_u.

Network containing controlled generator

As a concrete example let us consider the two port network in Fig. 6 consisting of an ideal operational amplifier, and set as a task to generate k-trees necessary

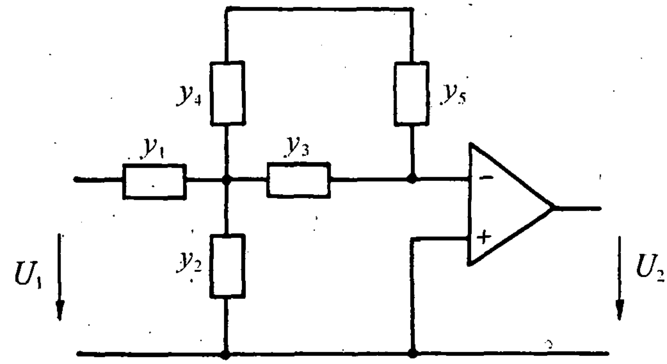


Fig. 6

to the calculation of the transfer voltage function of the network by topological formulas from the suitable network model.

From the inverse hybrid characteristics of a two port network whose output is closed by a break it follows:

$$A_u = \left.\frac{U_2}{U_1}\right|_{I_2=0} = D_{21}, \tag{15}$$

Taking into account the definitions of the universal parameters, from (15) we get

$$A_u = \frac{B_u}{P_u}. \tag{16}$$

After using the nullator—norator equivalent network of the ideal operational amplifier we can see the network model in Fig. 7. To determine the denominator of (16) close the input of the network model by a parallel connected nullator—norator pair. Thus we get the task discussed in paper [4], and the results are 3-trees

(01400), (03400), (04200), (04400) and (05400)

in order.

To determine the numerator of (16) close the network model shown in Fig. 7 by a norator on the input and by a nullator on the output. The modified network model is shown by Fig. 8. It can be seen, if we search the trees of the modified graph containing all the norator edges and left the norator edges from them, that the obtained 3-trees are equivalent to the 3-trees before the short circuiting listed in paper [4]:

(01200), (01400), (03400), (04200), (04400), (05200), (05400).

But at present the condition of the short circuiting differs from the earlier one. Namely,

$$a = 3 = 4 = 5$$

in respect to the unique symbol element.

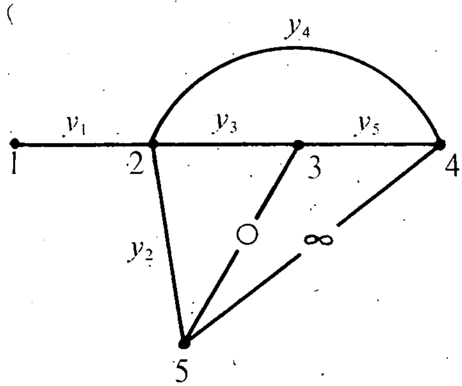


Fig. 7

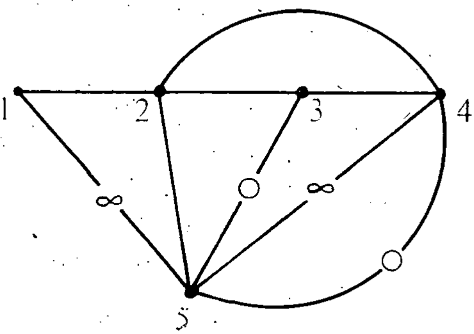


Fig. 8

To decide which subgraphs are circuitless write in a table the row vector representations. The first common row of the representations is

$$1 \quad 2 \quad a \quad a \quad a,$$

while the second rows are in order

$$\begin{array}{ccccc}
0 & 1 & 2 & 0 & 0 \\
0 & 1 & a & 0 & 0 \\
0 & a & a & 0 & 0 \\
0 & a & 2 & 0 & 0 \\
0 & a & a & 0 & 0 \\
0 & a & 2 & 0 & 0 \\
0 & a & a & 0 & 0
\end{array}$$

Performing their complete cycle check, only the first representation leads to a finite outcome and the suitable reduced graph is circuitless as well.

We get that the numerator of (16) needs only one 3-tree with representation (01200). According to topological formula (16), taking into account the sign of the edge admittance products, after some calculation we have got for the active network

$$\frac{U_2}{U_1} = -\frac{y_1 y_3}{y_5(y_1+y_2+y_3+y_4)+y_3 y_5}.$$

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