Tessellation transformations

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I. Introduction. Notations

In this paper we present some results on mappings induced by cellular automata. These mappings will be called here tessellation transformations. The notions and notations may be found partly in [2] and [7], but for the sake of the convenience of the reader, all necessary definitions will be given in this introductory part and at the beginning of the third section.

In the second section we deal with semigroups consisting of all tessellation transformations in a given tessellation array, under a fixed number of internal states. They will be characterized up to isomorphism by two parameters. Some inclusion theorems for these semigroups are also proved. The third section of our paper concerns cellular automata with a quiescent state. The investigations are related to and may be considered as a continuation of Moore's and Myhill's results in this area. Among others, it will be shown that the density of the tessellation transformations which are one-to-one on the finite configurations is equal to zero in the set of all such transformations. This solves a problem raised by Moore in [5].

Now we shall list the basic concepts.

A d-dimensional cellular automaton (shortly: CA) is a quadruple $\mathfrak{A}=(A, E^d, X, f)$, where

1. A is a finite set called the state alphabet. Its cardinality is supposed to be at least two.
2. E^d, called the d-dimensional tessellation array, is the set of all d-tuples of integers called cells. E^d is an Abelian group with respect to the componentwise sum of the d-tuples. E^d can be visualized as a Euclidean d-space subdivided into cells which are d-cubes of unit dimensions and whose centers have integer coordinates.
3. X, called the neighbourhood template, is an n-tuple of distinct elements of E^d (n is a positive integer), i.e.,

$$X = (\xi_1, \ldots, \xi_n), \quad \xi_i \in E^d, \qquad i = 1, \ldots, n.$$

For any $\alpha \in E^d$, $N(X, \alpha)=(\alpha+\xi_1, \ldots, \alpha+\xi_n)$ is said to be the neighbourhood of the cell α.

4. f is an arbitrary function from A^n into A called the local transition function.

We shall refer to a mapping $c\colon E^d \to A$ as configuration (more precisely: d-dimensional configuration over the alphabet A). The set of all configurations is denoted by $C_{A,d}$. The image $c(\alpha)$ of $\alpha \in E^d$ will be called the contents of the cell α under the configuration c. The restriction of c to $N(X, \alpha)$ is denoted by $c(N(X, \alpha))$, i.e.,

$$c(N(X, \alpha)) = (c(\alpha+\xi_1), \ldots, c(\alpha+\xi_n)).$$

The global transition function $\Phi_{\mathfrak{A}}: C_{A,d} \to C_{A,d}$ of the CA is defined by

$$(c\Phi_{\mathfrak{A}})(\alpha) = f(c(N(X, \alpha))) \quad \text{for all} \quad \alpha \in E^d.$$

II. Full semigroups of tessellation transformations

Let A be a finite nonempty set ($|A| \geqq 2$) and let d be a positive integer. A mapping $\Phi: C_{A,d} \to C_{A,d}$ is said to be a tessellation transformation if there exists a CA $\mathfrak{A}$ such that its global transition function $\Phi_{\mathfrak{A}}$ is equal to Φ. The set of all such mappings will be denoted by $M_{A,d}$.

Theorem 1. If $\Phi, \Psi \in M_{A,d}$, then $\Phi\Psi \in M_{A,d}$, i.e. $M_{A,d}$ is a semigroup.

Proof. According to the assumption of the theorem there are two CA $\mathfrak{A}^{(1)} = (A, E^d, X^{(1)}, f^{(1)}) (X^{(1)} = (\xi_1^{(1)}, \ldots, \xi_n^{(1)}))$ and $\mathfrak{A}^{(2)} = (A, E^d, X^{(2)}, f^{(2)}) (X^{(2)} = (\xi_1^{(2)}, \ldots, \xi_m^{(2)}))$ such that $\Phi_{\mathfrak{A}^{(1)}} = \Phi$ and $\Phi_{\mathfrak{A}^{(2)}} = \Psi$. Let us consider a CA $\mathfrak{A} = (A, E^d, X, f)$, where

1. The set of the components of X is $\langle \xi_i^{(1)} + \xi_j^{(2)} | 1 \leqq i \leqq n, 1 \leqq j \leqq m \rangle$.
2. We obtain f in the following way: Consider the function $f' = f^{(2)}(f^{(1)}(x_{11}, \ldots, x_{1n}), \ldots, f^{(1)}(x_{m1}, \ldots, x_{mn}))$. Identify x_{ij} with $x_{i'j'}$ if $\xi_i^{(1)} + \xi_j^{(2)} = \xi_{i'}^{(1)} + \xi_{j'}^{(2)}$, $1 \leqq i, i' \leqq n$, $1 \leqq j, j' \leqq m$. Then we obtain from the function f' a new function f''. Finally, writing the variables of f'' in the order which corresponds to X we get f. It is easy to see that $\Phi_{\mathfrak{A}} = \Phi\Psi$.

We call $M_{A,d}$ the full semigroup of tessellation transformations.

The transformations θ_δ $(\delta \in E^d)$ defined by $(c\theta_\delta)(\alpha) = c(\alpha - \delta)$ for all $\alpha \in E^d$ are called translations. They are obviously tessellation transformations. A $c' \in C_{A,d}$ is said to be a copy of $c \in C_{A,d}$ if there is a translation θ_δ such that $c' = c\theta_\delta$. Clearly, if $c = c\theta_\delta$ for all $\delta \in E^d$, then there exists an $a \in A$ such that $c(\alpha) = a$ for all $\alpha \in E_d$. It is evident that $\theta_{\delta_1}\theta_{\delta_2} = \theta_{\delta_1+\delta_2}$ for any $\delta_1, \delta_2 \in E^d$, and if T_d denotes the set of all translations we get $T_d \cong E^d$.

The transformations Ω_a $(a \in A)$ defined by $(c\Omega_a)(\alpha) = a$ for all $\alpha \in E^d$ are called constant transformations (briefly: constants). They are tessellation transformations. It is trivial that the number of all constants is $|A|$.

Lemma 1. A tessellation transformation Φ is a central element of $M_{A,d}$ if and only if Φ is a translation.

Proof. Let $Z(M_{A,d})$ denote the center of $M_{A,d}$. The set of all translations $T_d \subset Z(M_{A,d})$ is trivial. Now suppose that $\Phi \in Z(M_{A,d})$ and suppose $\mathfrak{A} = (A, E^d, X, f)$ $(X = (\xi_1, \ldots, \xi_n))$ is a CA such that $\Phi_{\mathfrak{A}} = \Phi$. Consider a CA $\mathfrak{B} = (A, E^d, Y, g)$ $(Y = (\eta_1, \ldots, \eta_m))$ such that every element of the set $\langle \xi_i + \eta_j | 1 \leqq i \leqq n, 1 \leqq j \leqq m \rangle$ has a unique representation as a sum of components of X and Y. (For any X there exists such a

Y with an arbitrary number of componens.) Under such choice of Y, $\Phi_{\mathfrak{A}}\Phi_{\mathfrak{B}}=\Phi_{\mathfrak{B}}\Phi_{\mathfrak{A}}$ implies

$$f(g(x_{11}, \ldots, x_{1m}), \ldots, g(x_{n1}, \ldots, x_{nm})) = g(f(x_{11}, \ldots, x_{n1}), \ldots, f(x_{1m}, \ldots, x_{nm}))$$

for all $x_{11}, \ldots, x_{nm} \in A$ (see 2. in the proof of Theorem 1). Accordingly we obtain that f commutes with all functions defined on A. Hence it follows that f is a projection ([3] pp. 128, Prop. 3.2.), consequently $\Phi_{\mathfrak{A}}=\Phi$ is a translation.

Lemma 2. A tessellation transformation Φ is a right zero if and only if Φ is a constant.

Proof. The sufficiency is trivial. Assume that Φ is a right zero and let $\mathfrak{A}=(A, E^d, X, f)$ (with n-ary f) be a CA such that $\Phi_{\mathfrak{A}}=\Phi$. Let $c \in C_{A,d}$ be a configuration for which

$$A^n = \langle c(N(X, \alpha)) | \alpha \in E^d \rangle \tag{$*$}$$

holds. Since Φ is a right zero, we obtain $(c\Phi)\theta_\delta = c(\Phi\theta_\delta) = c(\theta_\delta\Phi) = c\Phi$ for all translations θ_δ, whence there exists an $a \in A$ such that $(c\Phi)(\alpha)=a$ for all $\alpha \in E^d$. According to $(*)$ it follows that $f(x_1, \ldots, x_n)=a$ for any $x_1, \ldots, x_n \in A$ showing that Φ is a constant.

Theorem 2. $M_{A_1,d_1} \cong M_{A_2,d_2}$ if and only if $|A_1|=|A_2|$ and $d_1=d_2$.

Proof. The sufficiency is trivial. $M_{A_1,d_1} \cong M_{A_2,d_2}$ implies that the numbers of the right zeros of M_{A_1,d_1} and M_{A_2,d_2} are equal, i.e., in view of Lemma 2 we have $|A_1|=|A_2|$. Furthermore $Z(M_{A_1,d_1}) \cong Z(M_{A_2,d_2})$, whence by Lemma 1 we get $T_{d_1} \cong T_{d_2}$, which implies $E^{d_1} \cong E^{d_2}$. Hence $d_1=d_2$.

According to Theorem 2 a full semigroup of tessellation transformations is determined up to isomorphism by two positive integers $l(=|A|)$ and d. Therefore we shall denote this semigroup also by $M_{l,d}$.

Theorem 3. For any positive integer l_1, l_2 ($\geqq 2$) and d_1, d_2 such that $d_1 \leqq d_2$, the semigroup M_{l_1,d_1} is the homomorphic image of a subsemigroup of M_{l_1,d_1}.

Proof. The reader can easily verify that, if $d_1 \leqq d_2$ then M_{l,d_1} may be embedded in M_{l,d_2} for any l. Therefore it is sufficient to prove the statement for $d_1=d_2=d$. Let A_1 and A_2 be two sets with cardinality l_1 and l_2. We have to prove that $M_{A_1,d}$ is the homomorphic image of a subsemigroup of $M_{A_2,d}$.

1. First suppose that $|A_1| \leqq |A_2|$. We may assume without loss of generality that $A_1 \subset A_2$. Thus we get $C_{A_1,d} \subset C_{A_2,d}$. Let us consider the subsemigroup M of $M_{A_2,d}$ defined by

$$M = \langle \Phi | \Phi \in M_{A_2,d} \quad \text{and} \quad c\Phi \in C_{A_1,d} \quad \text{for all} \quad c \in C_{A_1,d} \rangle,$$

and let ϱ be a congruence on M defined by

$$\Phi_1 \varrho \Phi_2 \quad \text{if and only if} \quad c\Phi_1 = c\Phi_2 \quad \text{for any} \quad c \in C_{A_1,d}.$$

It is easy to see that

$$M/\varrho \cong M_{A_1,d}.$$

2. Now assume that $|A_1|>|A_2|$. For the sake of easier perspicuity we prove only in the case $d=2$. The proof for an arbitrary d is similar. On the base of the first part of the proof we may assume that $|A_2|=2$, and we may also assume that $A_2=\langle 0, 1\rangle$. Let n be a positive integer such that $2^{(n-4)^2}>l_1$. Let us subdivide the tessellation array E^2 into square blocks of size $n\times n$. Every block is designated by an element of E^2, as shown in Fig. 1. For any $\alpha\in E^2$, B_α denotes the block designated by α. The subdivided tessellation array will be referred to as block structure of E^2 denoted by E^2_B. It may be considered a tessellation array whose cells are blocks.

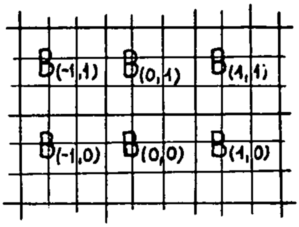

Fig. 1

Let S_{n-4} be a square block of size $(n-4)\times(n-4)$ and let A_2' be the set of all mappings from S_{n-4} into A_2. (E.g., Fig. 2 shows an element of A_2'). Since $|A_2'|=2^{(n-4)^2}>l_1$, there is a one-to-one mapping $\tau: A_1\to A_2'$. Now we define a one-to-one mapping ϑ: $C_{A_1,2}\to C_{A_2,2}$. For any $c\in C_{A_1,2}$, $c\vartheta\in C_{A_2,2}$ is a configuration whose restriction to an arbitrary block $B_\alpha(\in E^2_B)$, denoted by $c\vartheta/B_\alpha$, is defined by the following way:

1. The restriction of $c\vartheta$ to the inner array of size $(n-4)\times(n-4)$ of the block B_α equals $(c(\alpha))\tau$.

2. Each cell belonging to the outside layer of size 1 of the block B_α contains state 1.

1	0	1	
0	0	1	$n-4$
1	0	0	

Fig. 2

		$n-4$			
1	1	1 ... 1	1	1	
1	0	0 ... 0	0	1	
1 ⋮ 1	0 ⋮ 0	$(c(\alpha))\tau$	0 ⋮ 0	1 ⋮ 1	n
1	0	0 ... 0	0	1	
1	1	1 ... 1	1	1	

Fig. 3

3. Each cell belonging to the layer of size 1 around the inner array of size $(n-4)\times(n-4)$ of the block B_α contains state 0 (see Fig. 3).

Let C be the subset of $C_{A_2,2}$ defined by

$$C = \langle c | c = c'\vartheta, \quad c' \in C_{A_1,2}\rangle.$$

(It may be seen from the definition of the elements of C that the block structure of c can be uniquely recognized for all $c\in C$.) Now we associate a mapping Ψ: $C\to C$ with every $\Phi\in M_{A_1,2}$ defined by $c\Psi=((c\vartheta^{-1})\Phi)\vartheta$ for all $c\in C$. Let M be denote the set of such mappings, i.e.,

$$M = \langle \Psi | \Psi = \vartheta^{-1}\Phi\vartheta, \quad \Phi \in M_{A_1,2}\rangle.$$

It is obvious from the definitions that $M \cong M_{A_1, 2}$. Thus it is enough to prove that M is a homomorphic image of a subsemigroup of $M_{A_2, 2}$. For this it is sufficient to show that for any $\Psi \in M$ there is a $\Psi' \in M_{A_2, 2}$ such that the restriction of Ψ' to C is equal to Ψ.

Let $\Psi = \vartheta^{-1}\Phi\vartheta \in M(\Phi \in M_{A_1, 2})$ and let $\mathfrak{A} = (A^1, E^2, X, f)$ $(X = (\xi_1, \ldots, \xi_n))$ be a CA such that $\Phi_{\mathfrak{A}} = \Phi$. We associate with X a neighbourhood template X_B in E_B^2 defined by $X_B = (B_{\xi_1}, \ldots, B_{\xi_n})$. The sequence $(B_{\alpha+\xi_1}, \ldots, B_{\alpha+\xi_n})$ $(\alpha \in E^2)$ is denoted by $N(X_B, B_\alpha)$.

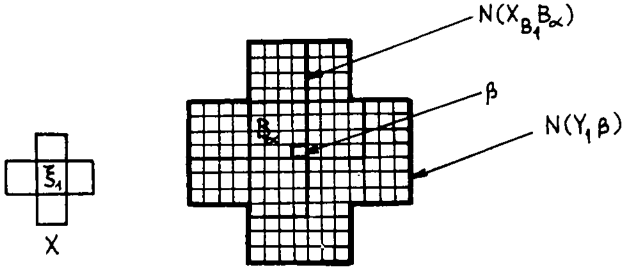

Fig. 4

Now we define a CA $\mathfrak{B} = (A_2, E^2, Y, g)$ such that the restriction of $\Phi_{\mathfrak{B}}$ to C is equal to Ψ.

1. Y is a neighbourhood template for which $N(Y, \beta)$ contains all blocks which belong to $N(X_B, B_\alpha)$ $(B_\alpha \in E_B^2)$ for all $\beta \in B_\alpha$ (see Fig. 4).

2. Let c be an arbitrary element of C. We now show how we may determine $g(c(N(Y, \beta)))$ for any $\beta \in E^2$. Let $\beta \in B_\alpha (\in E_B^2)$. Since $N(Y, \beta)$ contains all blocks belonging to $N(X_B, B_\alpha)$ and the block structure of c can be uniquely recognized we know $c|B_{\alpha+\xi_1}, \ldots, c|B_{\alpha+\xi_n}$ and so we also know $c\vartheta^{-1}(\alpha+\xi_1), \ldots, c\vartheta^{-1}(\alpha+\xi_n)$, i.e., $c\vartheta^{-1}(N(X, \alpha))$. From this we can determine $f(c\vartheta^{-1}(N(X, \alpha))) = ((c\vartheta^{-1})\Phi)(\alpha)$ and $((c\vartheta^{-1})\Phi)\vartheta|B_\alpha = c\Psi|B_\alpha$ as well. But we can uniquely determine the position of β in the block B_α, thus we can also determine $g(c(N(Y, \beta))) = (c\Psi)(\beta)$ as the state contained in the cell β under $c\Psi|B_\alpha$.

Theorem 4. If $d_1 \leq d_2$, then $M_{l_1, d_1} \times M_{l_2, d_2}$ can be embedded in $M_{l_1 l_2, d_2}$.

Proof. In wiew of the remark at the beginning of the preceding proof we prove the statement for $d_1 = d_2 = d$ only. Let A_1 and A_2 be two sets with cardinalities l_1 and l_2. We have to show that $M_{A_1, d} \times M_{A_2, d}$ can be embedded in $M_{A_1 \times A_2, d}$. Let $c \in C_{A_1 \times A_2, d}$. Since $c(\alpha) = (c_1(\alpha), c_2(\alpha))$ $(\alpha \in E^d)$ we may write (c_1, c_2) instead of c. Let $\vartheta: M_{A_1, d} \times M_{A_2, d} \to M_{A_1 \times A_2, d}$ a mapping defined by

$$c(\Phi\vartheta) = (c_1, c_2)(\Phi\vartheta) = (c_1\Phi_1, c_2\Phi_2) \quad (c = (c_1, c_2) \in C_{A_1 \times A_2\ d})$$

for all $\Phi = (\Phi_1, \Phi_2) \in M_{A_1, d} \times M_{A_2, d}$. One can easy verify that $\Phi\vartheta$ is a tessellation transformation and ϑ is an isomorphism.

Remark. $M_{l_1, d_1} \times M_{l_2, d_2}$ is not isomorphic to $M_{l_1 l_2, d_2}$ because $Z(M_{l_1 l_2, d_2}) \cong E^{d_2}$, $Z(M_{l_1, d_1} \times M_{l_2, d_2}) \cong E^{d_1+d_2}$ and $E^{d_2} \cong E^{d_1+d_2}$.

For a CA $\mathfrak{A}=(A, E, X, f)$ we shall say that the global transition function $\Phi_{\mathfrak{A}}$ has speed p, if the maximum of the absolute values of the coordinates of the components belonging to X is p, i.e.,

$$p = \max_{\substack{1 \leqq i \leqq n \\ 1 \leqq j \leqq d}} |i_j|$$

where $X=(\xi_1, \ldots, \xi_n)$ and $\xi_i=(i_1, \ldots, i_d)$, $i=1, \ldots, n$.

Lemma 3. Let $C=\langle c_1, \ldots, c_k \rangle$ be a finite set of distinct configurations for which $R_{ij}=\langle \alpha | \alpha \in E^d$ and $c_i(\alpha) \neq c_j(\alpha) \rangle$, $1 \leqq i, j \leqq k$, is a finite set and no element of C is a copy of another element of C. For any transformation $\Phi: C \to C$ there exists a tessellation transformation Ψ such that $c_i \Phi = c_i \Psi$, $i=1, \ldots, k$.

Proof. By the assumption $R=\bigcup_{i,j=1}^{k} R_{ij}$ is a finite set. Therefore there is a positive integer p such that R can be included in a d-cube of size $p \times p$.

Let Φ be a transformation of C and let $\mathfrak{A}=(A, E^d, X, f)$ be a CA, where the set of all components of X is equal to the square of size $(2p+1) \times (2p+1)$ with center $(0, \ldots, 0) \in E^d$. Since $N(X, \alpha)$ contains R for all $\alpha \in R$, one can easily define the local transition function f such that the restriction of $\Phi_{\mathfrak{A}}$ to C equals Φ.

Theorem 5. Every finite semigroup is a homomorphic image of a subsemigroup of $M_{l,d}$.

Proof. It is enough to prove the statement for the full transformation semigroup on a finite set with arbitrary cardinality. This is trivial from Lemma 3.

Corollary. Any $M_{l,d}$ generates the variety of all semigroups.

Proof. Indeed, from Theorem 5 it follows that non-trivial identities do not hold on $M_{l,d}$ (for this Corollary consult [3]).

III. Tessellation transformations with a distinquished state

A CA (A, E^d, X, f) is said to be an initial cellular automaton (shortly: ICA), if there is a state $a_0 \in A$ called the quiescent state, such that $f(a_0, \ldots, a_0)=a_0$. In this case we shall use the notation (A, a_0, E^d, X, f).

For a set A $(|A| \geqq 2)$ and $a_0 \in A$, the symbol M_{A,d,a_0} denotes the set of all tessellation transformations in $M_{A,d}$ induced by ICA with quiescent state a_0. It is evident that M_{A,d,a_0} is a subsemigroup of $M_{A,d}$.

A $c \in C_{A,d}$ is said to be a finite configuration if

$$\sup(c) = \langle \alpha | \alpha \in E^d \quad \text{and} \quad c(\alpha) \neq a_0 \rangle$$

is a finite set. C_F denotes the set of such configurations. In the sequel, if estimates are given for the number of configurations with some properties we shall not distinquish between two configurations if one is a copy of the other. Clearly, if $\Phi \in M_{A,d,a_0}$ and $c \in C_F$ than $c\Phi \in C_F$.

A $c \in C_F$ is said to be an n-configuration if $\sup(c)$ may be included in a d-cube of size $n \times n$, i.e., there is an $\alpha=(i_1, \ldots, i_d) \in E^d$ such that $c(\beta)=a_0$ for all

$\beta=(j_1, \ldots, j_d)\in E^d$ with $j_k<i_k$ or $j_k>i_k+n$ for at least one k, $1\leqq k\leqq d$. Every $c\in C_F$ is an n-configuration for some n, and if c is an n-configuration then c is also an m-configuration for all $m\geqq n$.

For any set $R\subset E^d$ and $c\in C_{A,d}$, $c|R$ denotes the restriction of c to R. A $c'\in C_F$ is said to be a subconfiguration of $c\in C_F$, if

$$c|\sup(c') = c'|\sup(c').$$

For a $\Phi\in M_{A,d,a_0}$, a $c\in C_F$ will be referred to as image configuration if c may be written in the form $c'\Phi(c'\in C_F)$ and c will be called a Garden-of-Eden configuration if no image configuration containing c as a subconfiguration.

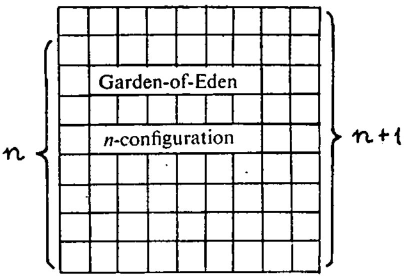

Fig. 5

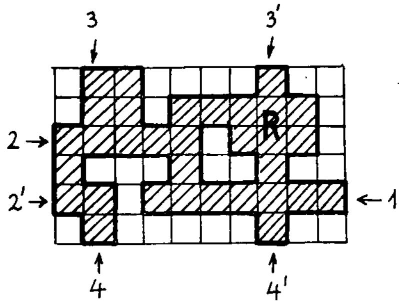

Fig. 6

A compact formulation of Moore's and Myhill's results proved in [5] and [6] (which may be found in [4]) is the following:

Theorem 6. The restriction of a $\Phi\in M_{A,d,a_0}$ to C_F is one-to-one if and only if there exists no Garden-of-Eden configuration.

Let G_n denote the number of Garden-of-Eden n-configurations and let H_n denote the number of all n-configurations

Theorem 7. For any $\Phi\in M_{A,d,a_0}$, $\lim\limits_{n\to\infty} G_n/H_n=0$ or 1 according to whether the restriction of Φ to C_F is one-to-one or not.

Proof. If Φ is one-to-one the assertion is trivial by Theorem 6. Suppose that Φ is not one-to-one on C_F. Let $d=2$, in the case $d\neq 2$ we may proceed similarly. Take $|A|=l$; then $H_n=l^{n^2}$. First we show that G_n/H_n is a monotonous increasing sequence. If e.g. the left lower part of an $(n+1)$-configuration is a Garden-of-Eden n-configuration (as shown Fig. 5), then it is a Garden-of-Eden $(n+1)$-configuration. Using this fact we get $G_{n+1}\geqq G_n l^{2n+1}$. Thus

$$\frac{G_{n+1}}{H_{n+1}} \geqq \frac{G_n\cdot l^{2n+1}}{l^{(n+1)^2}} = \frac{G_n}{l^{n^2}} = \frac{G_n}{H_n},$$

hence G_n/H_n is a monotonous increasing sequence. Thus it is sufficient to prove the statement for a subsequence of G_n/H_n. Let $\bar{G}_n=H_n-G_n$. Since Φ is not one-to-one on C_F there is a Garden-of-Eden m-configuration for some m. Therefore $\bar{G}_m \leqq (l^{m^2}-1)$ and for any positive integer k, we have $\bar{G}_{km} \leqq (l^{m^2}-1)^{k^2}$. So we obtain that

$$0 \leqq \frac{\bar{G}_{km}}{H_{km}} \leqq \frac{(l^{m^2}-1)^{k^2}}{l^{k^2 m^2}} = \left(1-\frac{1}{l^{m^2}}\right)^{k^2} \to 0 \quad \text{if} \quad k \to \infty.$$

Finally we get

$$\lim_{k\to\infty} \frac{G_{km}}{H_{km}} = \lim_{k\to\infty} \left(1-\frac{\bar{G}_{km}}{H_{km}}\right) = 1.$$

Let us recall some definitions from [2]. Let $R \subset E^2$ be a nonempty finite set. Identifying the positive y axis with the direction north we call the set of cells in R with maximum abscissas the eastern perimeter of R. The northern western and southern perimeters of R are similarly defined. The following cells are called extremal cells of R:

1. The northernmost and southernmost cells in the eastern perimeter of R (cell 1 in Fig. 6).
2. The northernmost and southernmost cells in the western perimeter of R (cells 2, 2′).
3. The westernmost and easternmost cells in the nothern perimeter of R (cells 3, 3′).
4. The westernmost and easternmost cells in the southern perimeter of R (cells 4, 4′).

A function f: $A^n \to A$ is said to be cancellative with respect to its i-th variable, if $f(a_1, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_n) = f(a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n)$ implies $a_i = b$ for all $a_1, \dots, a_n, b \in A$.

Theorem 8. If for any *ICA* $\mathfrak{A} = (A, a_0, E^2, X, f)(X = (\xi_1, \dots, \xi_n))$, the local transition function f is cancellative with respect to its i-th variable and ξ_i is an extremal cell of X, then the restriction of $\Phi_{\mathfrak{A}}$ to C_F is one-to-one.

Proof. We may assume without loss of generality that the extremal cell mentioned in the theorem is the northernmost cell in the western perimeter of X. Suppose that c_1 and c_2 are distinct finite configurations. Then

$$R = \langle \alpha | \alpha \in E^2 \quad \text{and} \quad c_1(\alpha) \neq c_2(\alpha) \rangle$$

is a nonempty finite set. Let $\beta \in E^2$ be a cell such that the northernmost cell in the western perimeter of $N(X, \beta)$ and the southernmost cell in the eastern perimeter of R are equal to each other. Using the cancellativity of f we get

$$f\big(c_1(N(X, \beta))\big) \neq f\big(c_2(N(X, \beta))\big), \quad \text{i.e.,} \quad c_1\Phi_{\mathfrak{A}} \neq c_2\Phi_{\mathfrak{A}}.$$

The converse of Theorem 8 fails trivially. It may be expected, however, that if we restrict ourselves to considering local transition functions depending essentially on all variables associated with extremal cells of X, then the assumption is also necessary. The next counter-example shows that this is not true.

Let us consider two *ICA*: $\mathfrak{A}^{(1)} = (\langle 0,1 \rangle, 0, E^1, (-1, 0, 1), f^{(1)})$ and $\mathfrak{A}^{(2)} = (\langle 0,1 \rangle, 0, E^1, (-1, 0, 1), f^{(2)})$, where $f^{(1)}$ and $f^{(2)}$ are defined by Table 2. The restrictions of

Table 1.

x_1	x_2	x_3	x_4	x_5	$f(x_1, x_2, x_3, x_4, x_5)$	x_1	x_2	x_3	x_4	x_5	$f(x_1, x_2, x_3, x_4, x_5)$
0	0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	1	0	1	0	0	0	1	0
0	0	0	1	0	1	1	0	0	1	0	1
0	0	0	1	1	0	1	0	0	1	1	0
0	0	1	0	0	0	1	0	1	0	0	1
0	0	1	0	1	0	1	0	1	0	1	1
0	0	1	1	0	0	1	0	1	1	0	0
0	0	1	1	1	1	1	0	1	1	1	1
0	1	0	0	0	0	1	1	0	0	0	1
0	1	0	0	1	0	1	1	0	0	1	1
0	1	0	1	0	1	1	1	0	1	0	0
0	1	0	1	1	0	1	1	0	1	1	1
0	1	1	0	0	1	1	1	1	0	0	1
0	1	1	0	1	1	1	1	1	0	1	1
0	1	1	1	0	1	1	1	1	1	0	0
0	1	1	1	1	0	1	1	1	1	1	1

$\Phi_{\mathfrak{A}^{(1)}}$ and $\Phi_{\mathfrak{A}^{(2)}}$ to C_F are one-to-one, because $f^{(1)}$ (resp. $f^{(2)}$) is cancellative with respect to its first (resp. third) variable. Then $\mathfrak{A}=(\langle 0,1\rangle, 0, E^1, (-2, -1, 0, 1, 2), f)$, where f is defined by Table 1 is the *ICA* whose global transition function $\Phi_{\mathfrak{A}}$ is equals to $\Phi_{\mathfrak{A}^{(1)}}\Phi_{\mathfrak{A}^{(2)}}$.

It can be seen that f depends on its first and fifth variables, but f is not cancellative with respect to any of them. Thus the restriction of $\Phi_{\mathfrak{A}}$ to C_F is one-to-one, but $\mathfrak{A}$ does not fulfil the assumptions of Theorem 8.

Lemma 4. If for an *ICA* $\mathfrak{A}=(A, a_0, E^d, X, f)$ (with n-ary f) the restriction of $\Phi_{\mathfrak{A}}$ to C_F is one-to-one, then all the classes of the partition $A^n/f\circ f^{-1}$ (partition on A^n induced by f) have the same cardinality.

Proof. Again we shall prove the statement for $d=2$ only. Take $|A|=l$. Suppose that $\Phi_{\mathfrak{A}}$ has speed p. Then X can be included in a square of size $(2p+1)\times(2p+1)$ with center $(0, 0)\in E^2$. We may assume without loss of generality that this square equals to X. In this case $n=(2p+1)^2$. If there are classes of $A^n/f\circ f^{-1}$ with different cardinalities, then there is an $a\in A$ such that $|af^{-1}|\geq l^{n-1}+1$. Let k be an arbitrary positive integer and let D_k denote the number of all $(k(2p+1)+2p)$-configuration which have $a\in A$ in the cells shown in Fig. 7. We have $D_k=l^{(k(2p+1)+2p)^2-k^2}$. Because $\Phi_{\mathfrak{A}}$ has speed p, $c\Phi_{\mathfrak{A}}$ is an $(m+2p)$-configuration for any m-configuration c. Since $\Phi_{\mathfrak{A}}$ is one-to-one with respect to C_F and $|af^{-1}|\geq l^{n-1}+1$ $(n=(2p+1)^2)$ we obtain that D_k is at least

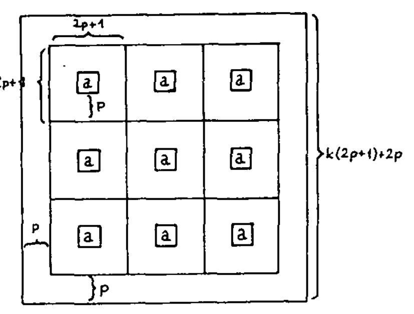

Fig. 7

$(l^{(2p+1)^2-1}+1)^{k^2}$. Thus we get

$$(l^{(2p+1)^2-1}+1)^{k^2} \leqq l^{(k(2p+1)+2p)^2-k^2},$$

whence

$$1+\frac{1}{l^{(2p+1)^2-1}} \leqq l^{\frac{4p(2p+1)}{k}+\frac{4p^2}{k^2}},$$

which is not true for all k.

Remark. The converse of Lemma 4 is not true. One can easily verify that the next *ICA* is a counter-example: $\mathfrak{A}=(\langle 0,1\rangle, 0, E^1, (-1,0,1), f^{(3)})$, where $f^{(3)}$ is defined by Table 2.

For a fixed neighbourhood template X and a state alphabet A ($|A|=n$) with a quiescent state $a_0 \in A$, the symbol K_n denotes the number of all tessellation transformations induced by *ICA* (A, a_0, E^d, X, f) whose restrictions to C_F are one-to-one. S_n will denote the number of all tessellation transformations induced by *ICA* (A, a_0, E^d, X, f).

Theorem 9. $\lim\limits_{n\to\infty} K_n/S_n=0$.

Proof. If X has $k(\geqq 2)$ components, then $S_n=n^{(n^k-1)}$ and using the result of Lemma 4 we get

$$K_n \leqq \frac{(n^k-1)!}{(n^{k-1}!)^{n-1}(n^{k-1}-1)!} \quad (k \geqq 2).$$

Thus

$$\frac{K_n}{S_n} \leqq \frac{(n^k-1)!}{n^{(n^k-1)}(n^{k-1}!)^{n-1}(n^{k-1}-1)!} = \frac{n^k!}{n^{n^k}(n^{k-1}!)^n} \quad (k \geqq 2).$$

Using Stirling's formula $\left(n!=\left(\frac{n}{e}\right)^n \sqrt{2\pi n}\cdot e^{\frac{\Theta_n}{12n}},\ 0<\Theta_n<1\right)$ we get

$$\lim_{n\to\infty}\frac{n^k!}{n^{n^k}(n^{k-1}!)^n} = \lim_{n\to\infty}\frac{\left(\frac{n^k}{e}\right)^{n^k}\sqrt{2\pi n^k}\cdot e^{\frac{\Theta_{n^k}}{12n^k}}}{n^{n^k}\left(\left(\frac{n^{k-1}}{e}\right)^{n^{k-1}}\sqrt{2\pi n^{k-1}}\cdot e^{\frac{\Theta_{n^{k-1}}}{12n^{k-1}}}\right)^n} =$$

$$= \lim_{n\to\infty}\sqrt{\frac{1}{(2\pi)^{n-1}n^{n(k-1)-k}}}\cdot \lim_{n\to\infty} e^{\frac{\Theta_{n^k}}{12n^k}-\frac{\Theta_{n^{k-1}}}{12n^{k-2}}} = 0\cdot 1 = 0 \quad (k \geqq 2).$$

If $k=1$ then $S_n=n^{n-1}$ and $K_n \leqq (n-1)!$. Thus

$$\frac{K_n}{S_n} \leqq \frac{(n-1)!}{n^{n-1}} \to 0, \quad \text{if} \quad n\to\infty,$$

In the course of a conversation B. Csákány conjectured that for fixed A, d and a_0, the set S of all bijetive transformations of C_F induced by *ICA* form a group. We shall prove that this conjecture is false by giving an example which shows that S is not closed under the formation of inverses.

Let $\mathfrak{A}=(\langle 0, 1\rangle, 0, E^1, (-1, 0, 1), f^{(1)})$ be an ICA, where $f^{(1)}$ is defined by Table 2. The restriction of $\Phi_{\mathfrak{A}}$ to C_F is surjective (shown by Amoroso and Cooper in [3] pp. 163) and thus it is also bijective. Suppose that there is a $\Psi\in M_{\langle 0,1\rangle,1,0}$ such that $(c\Phi_{\mathfrak{A}})\Psi=c$ for all $c\in C_F$, and let $\mathfrak{B}=(\langle 0, 1\rangle, 0, E^1, X, f)$ be an ICA such that $\Phi_{\mathfrak{B}}=\Psi$. Assume that $\Phi_{\mathfrak{B}}$ has speed p. In this case $(c\Phi_{\mathfrak{B}})(\alpha)(\alpha\in E^1)$ is uniquely determined by $c(\alpha-p), \dots, c(\alpha), \dots, c(\alpha+p)$ for any configuration c. Let us consider the following two configurations and their image configurations under $\Phi_{\mathfrak{A}}$ (see. Fig. 8).

Table 2.

x_1 x_3 x_3	$f^{(1)}(x_1, x_2, x_3)$	$f^{(2)}(x_1, x_2, x_3)$	$f^{(3)}(x_1, x_2, x_3)$
0 0 0	0	0	0
0 0 1	0	1	0
0 1 0	1	1	0
0 1 1	0	0	1
1 0 0	1	0	0
1 0 1	1	1	1
1 1 0	0	0	1
1 1 1	1	1	1

$$
\begin{array}{llll}
 & \quad\;\; c_1(\alpha) & & \quad\;\; c_2(\alpha) \\
 & \quad\;\;\; \downarrow & & \quad\;\;\; \downarrow \\
c_1: & \dots 0\,\boxed{0}\,1\;0\;1\dots1\;0\;1\;0\;0\dots & c_2: & 0\;0\,\boxed{1}\,1\;1\dots1\;1\;1\;0\;0\dots \\
c_1\Phi_{\mathfrak{A}}: & \dots 0\,\boxed{0}\,1\;1\;1\dots1\;1\;1\;1\;0\dots & c_2\Phi_{\mathfrak{A}}: & 0\;0\,\boxed{0}\,1\;1\dots1\;1\;1\;1\;0\dots
\end{array}
$$

Fig. 8

It can be seen that $(c_1\Phi_{\mathfrak{A}})(\beta)=(c_2\Phi_{\mathfrak{A}})(\beta)$, $\alpha-p\leqq\beta\leqq\alpha+p$, but $((c_1\Phi_{\mathfrak{A}})\Phi_{\mathfrak{B}})(\alpha)= =c_1(\alpha)\neq c_2(\alpha)=((c_2\Phi_{\mathfrak{A}})\Phi_{\mathfrak{B}})(\alpha)$, which is a contradiction.

Мозаичные преобразования

Исследуется зависимость строения полугрупп мозаичных преобразований от размерности пространства-носителя, а также от числа состояний. Далее, рассматриваются преобразования конфигураций, индуцированные мозаичными автоматами с состоянием покоя. Отвечая на вопрос, поставленный Э. Ф. Муром доказывается, что почти все мозаичные автоматы обладают взаимно стираемыми конфигурациями.

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