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Stabilization and set-point regulation of underactuated mechanical systems

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Abstract. Mechanical systems are referred to as underactuated if the number of independent actuators are fewer than the number of degrees of freedom, a general encountered problem in engineering applications. The considered mechanical systems belong to the class of Euler-Lagrange systems where both kinetic energy and potential energy are modeled in their most general form and energy dissipation is modeled according to the dissipation function of Rayleigh, i.e. viscous damping forces are assumed. The control objectives are stabilization and set-point regulation. The structure of the controller is a parallel combination of static output feedback with dynamic output feedback where nonlinear amplifiers are included. An energy based approach with Liapunov functions and the Kalman-Yacubovich-Popov main lemma yields alternative stability theorems. A number of conditions are introduced with respect to the controller's structure in order to guarantee stability. However, sufficient design freedom is left to choose a proper tuning principle and obtain the specified control objectives such as fast convergence to a set-point combined with disturbance rejection. A restriction on the control input energy can be incorporated as well. The applicability of the method will be illustrated with planar manipulators. The main contribution is that a methodology is developed which incorporates many controllers and tuning facilities.

1. Introduction

The control of underactuated mechanical systems is and has been the focus of many researchers because of the broad range engineering applications. Among others, R. Ortega has an impressive list of contributions in the field of controlling underactuated mechanical systems. Passivity based control is a common approach of his in earlier work [1], [2], [3] as well of the more recent approach of interconnection and damping assignment passivity-based control (IDA-PCB)[4], [5], [6] which inspired many researchers, [7], [8]. The IDA-PCB approach regulates the position of an underactuated system by total energy reshaping to include set point stability and other desired control features. The implementation requires the solution of a set of partial differential equations, a cumbersome task for mechanical systems where the number of actuators is quite smaller than number of degrees of freedom. It is not the author's intention to provide a literature overview. A nice overview can be found in [9], where many methods and benchmark systems are classified and supported with appropriate references. Considering this classification it can be stated that we present a nonlinear control design frame work based on Liapunov functions to stabilize the closed loop systems. The control objective is set point regulation, the method used belongs to the class of energy methods. Most recent contributions of Liapunov based controllers focus on dedicated applications [10], [11]. The Liapunov function used here has a quite general form and therefore comprises several types of controllers and control objectives such as control effort constraints. It is shown that, among others, the Euler-Lagrange controller can be implemented. Although many Euler-Lagrange controllers exist, a different property is introduced here by adding a strong nonlinearity to the controller. An idea inspired by the field of passive vibration control. The paper is organized as follows. First, the class of mechanical systems to be controlled is introduced in section 2. Second, the structure of the control loop is explained in Section 3. Section 4 explains and derives the general Liapunov frame work to obtain set point regulation. Different controllers are addressed to illustrate the design freedom offered by the Liapunov-based control strategy. Some control features are illustrated with a two-link manipulator in Section 5. Especially, it is shown that a strong nonlinearity can be included, a challenge mentioned in [9]. Finally some conclusion is formulated in Section 6.

2. Class of systems

We consider mechanical Euler-Lagrange systems configured with a set of m generalized coordinates $q \in R^m$. The equations of motion can be written as [1] :

$$\frac{\partial \mathcal{L}}{\partial q}(q, \dot{q}) - \frac{d}{dt}\,[\frac{\partial \mathcal{L}}{\partial \dot{q}}(q, \dot{q})] - \frac{\partial \mathcal{F}}{\partial \dot{q}}(\dot{q}) + Mu = 0 \quad (1)$$

$$w = M'q \quad (2)$$

The function $\mathcal{L}$ is termed Lagrangian function and defined as

$$\mathcal{L}(q, \dot{q}) \stackrel{\triangle}{=} \mathcal{T}(q, \dot{q}) - \mathcal{V}(q) \quad (3)$$

$\mathcal{T}(q, \dot{q}) \in C^1$ and $\mathcal{V}(q) \in C^1$ are scalar quantities which represent the kinetic energy and the potential energy. The kinetic energy can be written in the form [12] :

$$\mathcal{T}(q, \dot{q}) = \frac{1}{2}\dot{q}'D(q)\dot{q} + b'(q)\dot{q} + c(q) \quad (4)$$

We have kinetic energy due to inertia type forces, coriolis type forces and centrifugal type forces.This nonnatural system is most frequently encountered in rotating mechanical systems. Two kinds of non-conservative forces are introduced : dissipative forces and control forces. The mechanical system is assumed to be lightly damped. The dissipative forces stem from Rayleigh's dissipation function $\mathcal{F}(\dot{q}) \in C^1$:

$$\dot{q}'\frac{\partial \mathcal{F}}{\partial \dot{q}}(\dot{q}) \geq 0\,,\ \forall \dot{q} \in R^m \quad (5)$$

The control forces u enter linearly with a constant matrix $M \in R^{m \times r}$. The rank of $M = r < m$, the system is underactuated. We will focus on collocated control. As a consequence the measurement vector w is written as $w = M'q$ with $w \in R^r$.

3. Control loop

Let

$$\left[\begin{array}{c} q \\ \dot{q} \end{array}\right] \stackrel{\triangle}{=} x \in R^{2m} \quad (6)$$

be the state vector of the Euler-Lagrange system and $\left[\begin{array}{c} w \\ \dot{w} \end{array}\right] \in R^{2r}$ the output available for feedback. The control scheme is presented on Fig. 1. The control loop shows a parallel loop of

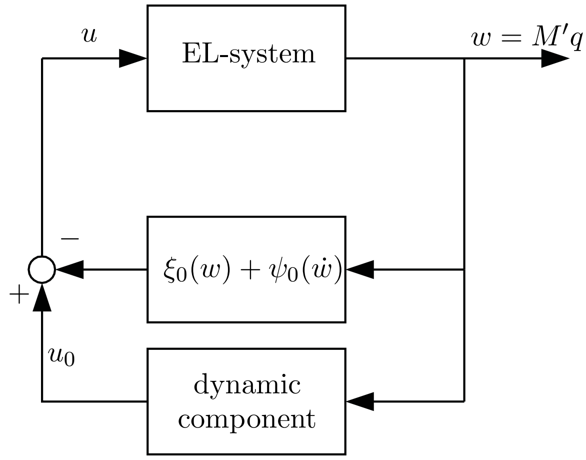

Figure 1. Control loop: static and dynamic control.

a static and a dynamic feedback control. The control forces read :

$$u = -\left[\xi_0(w) + \psi_0(\dot{w})\right] + u_0 \tag{7}$$

The static feedback loop corresponds to the well known technique of potential energy shaping and damping injection. The dynamic component is introduced to enlarge the control design freedom as only part of the system's state x is available for feedback. The dynamic component has the following structure :

$$\dot{z} = Az - Bf(\sigma) + \eta(w, \dot{w}) \tag{8}$$

$$u_0 = \varphi(z, \dot{z}) \tag{9}$$

The controller state is $z \in R^n$, $\sigma \stackrel{\triangle}{=} C'z \in R^s$, $f(\sigma) \stackrel{\triangle}{=}$ col$[f_i(\sigma_i);\ i = 1 \ldots s]$ and $\sigma \stackrel{\triangle}{=}$ col$[\sigma_i;\ i = 1 \ldots s]$. The state of the closed loop system reads $x_c \stackrel{\triangle}{=} \begin{bmatrix} x \\ z \end{bmatrix} \in R^{2m+n}$. A number of assumptions are introduced to be used in the stability analysis of Section 3. The matrix A is a Hurwitz matrix, the pair (A, B) is controllable and the pair (A, C) is observable. The matrices B and C are chosen an equal rank $s \leq n$. To avoid undesired equilibria, it holds that $Az - Bf(C'z) = 0 \ \Leftrightarrow\ z = 0$ The operating point is $x_c = 0$.

4. Stability

The stability of the closed loop is analyzed based on Liapunov theory and the invariance principle of LaSalle [13]. A Liapunov function is defined as a real scalar function $V(x) \in C^1$ such that along the solutions of the dynamical system

$$\dot{x} = f(x) \tag{10}$$

$$\dot{V}(x) = \left[\frac{\partial V(x)}{\partial x}\right]' f(x) \leq 0 \quad ; \quad \forall x \in \bar{G} \tag{11}$$

$\bar{G}$ is the closure of a set G in R^n. We define $\mathcal{E} \stackrel{\triangle}{=} \{x \in \bar{G};\ \dot{V}(x) = 0\}$ and $\mathcal{M} \in \mathcal{E}$ as the largest invariant subset of $\mathcal{E}$. The invariance principle of LaSalle reads :

Theorem 4.1 *If $V(x)$ is a Liapunov function of (10) in G then every solution $x(t, x_0)$ of (10), resting in G for all $t > 0$, will converge to $\mathcal{M}^* \triangleq \mathcal{M} \cup \{\infty\}$ if $t \to +\infty$. If $\mathcal{M}$ is bounded, then $x(t, x_0) \to \mathcal{M}$, or $|x(t, x_0)| \to +\infty$ if $t \to +\infty$.*

Stability conditions based on the existence of Liapunov functions are sufficient conditions. The work of [14] offers several options to choose Liapunov functions and obtain the corresponding stability conditions. The stability conditions are rather weak such that sufficient design freedom is available to imply closed loop specifications on transient behavior and disturbance rejection. We will focus on two different techniques: a modified circle criterion and a criterion to incorporate control effort constraints.

4.1. The modified Circle criterion

The candidate Liapunov function has the following structure :

$$V_c(x_c) \triangleq \mathcal{H}_0(q, \dot{q}) + V_1(z) \tag{12}$$

where

$$\mathcal{H}_0(q, \dot{q}) \triangleq \mathcal{H}(q, \dot{q}) + \int_0^w \xi_0'(\tau) d\tau \quad \text{and} \quad \mathcal{H}(q, \dot{q}) \triangleq \frac{1}{2}\dot{q}' D(q)\dot{q} + \mathcal{V}(q) - c(q) \tag{13}$$

and

$\mathcal{H}$ is the Hamiltonian of the mechanical system. The scalar function $V_1(z)$ reads :

$$V_1(z) = z'Pz + f'(C'z)Sf(C'z) + z'Lf(C'z) + \int_0^{C'z} f'(\theta)\overline{\alpha} d\theta \tag{14}$$

with $P = P' > 0$, $S = S' \in R^{s \times s}$, $L \in R^{n \times s}$ and $\overline{\alpha} = \text{diag}[\alpha_i;\ i = 1 \dots s]$. We simplify the derivations for stability by choosing $\mathcal{F}(\dot{q}) \equiv 0$ (no damping in the mechanical system) and $\psi_0(\dot{w}) \equiv 0$ (no damping injection) such that $u = -\xi_0(w) + u_0$. As a consequence, a number of non-negative terms in $\dot{V}_c(x_c)$ will be omitted. We introduce restrictions on the slope of the nonlinear amplifier :

$$\lambda \triangleq \dot{z}'C\left[I - K^{-1} f_d(C'z)\right]\overline{\gamma} f_d(C'z)C'\dot{z} \geq 0\,;\ \forall z, \dot{z} \in R^n \tag{15}$$

$K = \text{diag}(k_i) > 0$ and $\overline{\gamma} = \text{diag}(\gamma_i) \geq 0$ are diagonal $(s \times s)$-matrices and $f_d(\sigma) \triangleq \text{diag}\left[\frac{df_i(\sigma_i)}{d\sigma_i};\right.$ $\left. i = 1 \dots s\right]$. The derivative of $\dot{V}_c(x_c)$ is calculated along the solutions of the closed loop system.

$$\dot{V}_c(x_c) = \dot{w}' u_0 + \frac{\partial V}{\partial z}\dot{z} \tag{16}$$

First, we calculate $\frac{\partial V_1}{\partial z}\dot{z}$.

$$\dot{V}_1(z, \eta) = \dot{z}'\left\{PA^{-1} + A^{-1\prime}P - A^{-1\prime}\left(C\overline{\alpha} + 2PA^{-1}B\right) f_d(C'z)C'\right\}\dot{z} + \eta'\zeta \tag{17}$$

L and S are substituted as

$$\begin{aligned} L &= -C\overline{\alpha} - 2PA^{-1}B \\ S &= B'A^{-1\prime}PA^{-1}B + \frac{1}{2}B'A^{-1\prime}C\overline{\alpha} \end{aligned} \tag{18}$$

while

$$\zeta \triangleq \left[A^{-1\prime}C\overline{\alpha} + 2WB\right] f_d(C'z)C'\dot{z} - 2WA\dot{z} \tag{19}$$

where $W \stackrel{\triangle}{=} A^{-1\prime}PA^{-1} \in R^{n\times n}$. $S = S'$ requires that

$$\overline{\alpha}C'A^{-1}B = (\overline{\alpha}C'A^{-1}B)' \tag{20}$$

with $\varepsilon \in R$ and $Q \in R^{n\times s}$ we rewrite (17) as :

$$\begin{aligned} \dot{V}_1(z,\eta) &= -\lambda - \left[Q'\dot{z} + K^{-\frac{1}{2}}\overline{\gamma}^{\frac{1}{2}}f_d(C'z)C'\dot{z}\right]' \left[Q'\dot{z} + K^{-\frac{1}{2}}\overline{\gamma}^{\frac{1}{2}}f_d(C'z)C'\dot{z}\right] \\ &\quad -\varepsilon\dot{z}'\dot{z} + \eta'\zeta \end{aligned} \tag{21}$$

Therefore

$$\left.\begin{aligned} A'W + WA &= -QQ' - \varepsilon I \\ 2WB + A^{-1\prime}C\overline{\alpha} - C\overline{\gamma} &= 2QK^{-\frac{1}{2}}\overline{\gamma}^{\frac{1}{2}} \end{aligned}\right\} \tag{22}$$

has to be solved. According to the Kalman-Yakubovich-Popov main lemma [16], the matrix equation will have a real solution $W = W' > 0$, Q, $\varepsilon > 0$ sufficiently small if the following criterion is satisfied :

$$\mathrm{He}\left[\overline{\gamma} - \frac{1}{j\omega}\overline{\alpha}\right]\left[K^{-1} + G(j\omega)\right] > 0\,;\ \forall\omega \in R \tag{23}$$

and

$$\lim_{\omega\to\infty} \omega^2 \mathrm{He}\left[\overline{\gamma} - \frac{1}{j\omega}\overline{\alpha}\right]\left[K^{-1} + G(j\omega)\right] > 0 \tag{24}$$

He is the Hermitian part of a matrix and

$$G(s) \stackrel{\triangle}{=} C'\left(sI - A\right)^{-1}B \tag{25}$$

We choose $\eta = N\dot{w}$ with rank $N = r$ and $\varphi = -N'\zeta$ in (8)-(9). It can be shown that

$$\dot{V}_c(x_c) \ \leq\ -\varepsilon\dot{z}'\dot{z} \ \leq\ 0\,; \quad \forall x_c \in R^{2m+n} \tag{26}$$

The function $\dot{V}_c(x_c)$ is a global Liapunov function for (1)-(2)-(8)-(9). The set of states where $\dot{V}_c(x_c) \equiv 0$ correspond to $\dot{z} \equiv 0$. [14] shows that consequently u and w remain constant as well. If the mechanical system does not possess solutions for constant u and constant w except constant q then the set of equilibria of the closed loop system is globally convergent. If $H_0(q,\dot{q})$ of (13) possesses a local minimum at $q = \dot{q} = 0$ then the operating point is locally asymptotically stable. If $H_0(q,\dot{q})$ is radially unbounded in q and $\dot{q}$ all closed loop solutions are bounded for $t \geq 0$.

It should be noted that ε can be chosen equal to zero. As a consequence, (24) is no longer needed and $\dot{V}_c(x_c)$ reduces to $\dot{V}_c(x_c) = -\dot{z}'QQ'\dot{z}$.

We include two different controller choices to illustrate the variety of controllers the modified circle criterion comprises.

4.1.1. PD-control with nonlinearity in parallel A PD-controller is extended with a dynamic component where the nonlinearity is expressed as

$$\begin{aligned} f(y) &= f_1 y + f_2 \arctan\frac{(f_0 - f_1)}{f_2}y\,;\ f_0 > 0,\ f_1 > 0,\ f_2 > 0 \\ &\to f_0 y \text{ for } y \to 0 \\ &\to f_1 y + \text{constant for } |y| \to \infty \end{aligned}$$

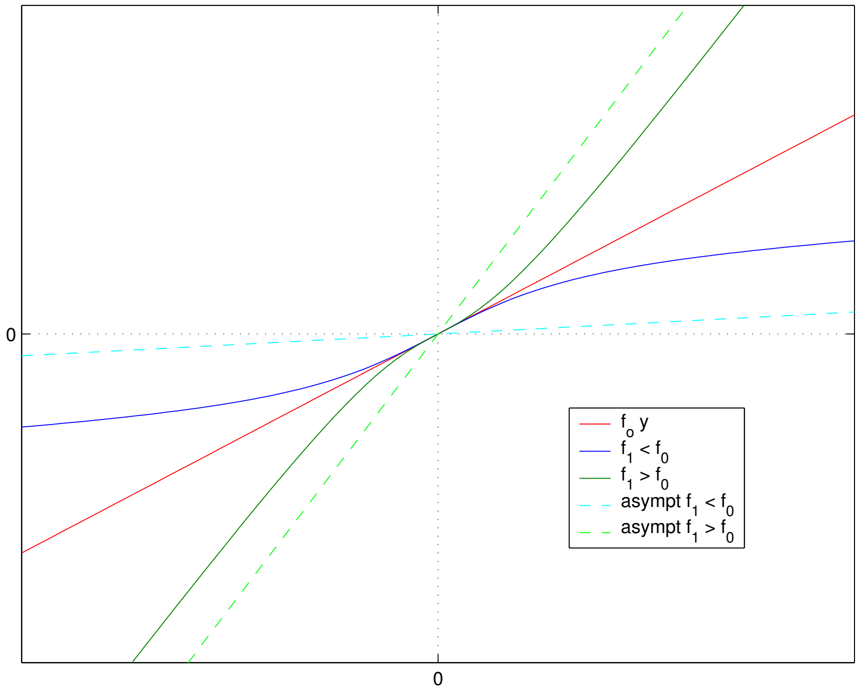

Figure 2. Characteristic of $f(.)$.

and plotted on Fig. 2. This type of characteristic has different slopes in different regions of the state space. In case of large values of $|C'z|$, the nonlinear characteristic behaves as

$$f(y) = f_1 y$$

The choice of f_1 could be used to maximize the linearized system's damping by shifting the most critical poles to more negative real parts and as such reduce the settling time. A maximum damping policy however could reduce the closed loop system's capability to deal with high frequent noise in the neighborhood of the operating point. The tuning of f_0 in the nonlinear gain can be used for this purpose. The tuning of this type of nonlinearity is based on the linearization of the closed loop dynamics in several regions of state space.

4.1.2. Euler-Lagrange controller The controller equation reads :

$$D_0\ddot{p} + C_0\dot{p} + K_0 p + C_1 f(C_1' p) + N_0\dot{w} = 0 \tag{27}$$

with $D_0 = D_0' > 0$; $C_0 = C_0' > 0$; $K_0 = K_0' > 0$. The Euler-Lagrange controller has generalized coordinates p, kinetic energy $\tilde{\mathcal{T}}(\dot{p}) = \frac{1}{2}\dot{p}' D_0 \dot{p}$, potential energy $\tilde{\mathcal{V}}(p) = \frac{1}{2}p' K_0 p + \int_0^{C_1' p} f'(\theta)d\theta$ and dissipation function $\tilde{\mathcal{F}}(\dot{p}) = \frac{1}{2}\dot{p}' C_0 \dot{p}$. We write (27) as (8) with $z \triangleq \begin{bmatrix} p \\ \dot{p} \end{bmatrix}$; $A \triangleq \begin{bmatrix} 0 & I \\ -D_0^{-1}K_0 & -D_0^{-1}C_0 \end{bmatrix}$; $B \triangleq \begin{bmatrix} 0 \\ D_0^{-1}C_1 \end{bmatrix}$; $C \triangleq \begin{bmatrix} C_1 \\ 0 \end{bmatrix}$; $\eta(\dot{w}) = N\dot{w} \triangleq \begin{bmatrix} 0 \\ -D_0^{-1}N_0 \end{bmatrix}\dot{w}$, we choose rank $N =$ rank $N_0 = r$. We choose $\overline{\alpha} = I$, $\overline{\gamma} = 0$ and $\varepsilon = 0$. The matrices P and Q can easily be calculated :

$$P = \frac{1}{2}\begin{bmatrix} K_0 & \\ & D_0 \end{bmatrix}; \quad Q = \begin{bmatrix} C_0^{\frac{1}{2}} \\ 0 \end{bmatrix};$$

It holds that

$$\dot{V}_c(x_c) = -\dot{z}'QQ'\dot{z} = -\left[\begin{array}{c}\dot{p}\\ \ddot{p}\end{array}\right]'\left[\begin{array}{cc}C_0 & 0\\ 0 & 0\end{array}\right]'\left[\begin{array}{c}\dot{p}\\ \ddot{p}\end{array}\right] = -\dot{p}'C_0\dot{p} \leq 0$$

For $\dot{V}_c(x_c) \equiv 0$ it holds that $\dot{p} \equiv 0$ and $\ddot{p} \equiv 0$ such that $\dot{z} \equiv 0$ of $z \equiv$ constant.With $\varphi = -N'\zeta$ and ζ defined according to (19), $V_c(x_c)$ remains a global Liapunov function although $\varepsilon = 0$.
If an Euler-Lagrange system is controlled with another Euler-Lagrange system the typical structure holds for the closed loop as well.

4.2. Control effort constraint
The candidate Liapunov function is altered to :

$$V_c(x_c) \stackrel{\triangle}{=} \mathcal{H}(q,\dot{q}) + z'Pz + V_2(z,w) - \left(\frac{\partial V_2}{\partial z}\right)' z \quad ; \quad P = P' > 0 \tag{28}$$

The derivative $\dot{V}_c(x_c)$ along the closed loop system's solution can be written as :

$$\begin{aligned}\frac{d}{dt}\left[\mathcal{H}(q,\dot{q}) + V_1(z) + V_2(z,w)\right] &= \dot{w}'u - \lambda - z'QQ'z - \varepsilon z'z + 2\eta'Pz + \\ &\left(\frac{\partial V_2}{\partial z}\right)'\dot{z} + \left(\frac{\partial V_2}{\partial w}\right)'\dot{w} - \left(\frac{\partial V_2}{\partial z}\right)'\dot{z} - \left[\frac{d}{dt}\left(\frac{\partial V_2}{\partial z}\right)\right]' z\end{aligned} \tag{29}$$

where

$$\left.\begin{aligned} A'P + PA &= -QQ' - \varepsilon I \\ 2PB &= C \end{aligned}\right\} \tag{30}$$

(30) can be solved according to the Kalman-Yacubovich-Popov- main lemma [16] with solution $P = P' > 0$, Q, $\varepsilon > 0$ and sufficiently small if :

$$\mathrm{He}\, G(j\omega) > 0\,; \forall\omega \in R \tag{31}$$

We choose

$$u = \varphi(z,w) = -\frac{\partial V_2}{\partial w}(z,w) \tag{32}$$

and

$$\eta = \frac{1}{2}P^{-1}\frac{d}{dt}\left[\frac{\partial V_2}{\partial z}(z,w)\right] \tag{33}$$

As a result, the derivative $\dot{V}_c(x_c)$ can be written as

$$\dot{V}_c(x_c) = -\lambda - z'QQ'z - \varepsilon z'z \leq 0\,; \quad \forall x_c \in R^{2m+n} \tag{34}$$

The function $V_c(x_c)$ is a global Liapunov function. The controller structure changes to

$$\dot{z} = Az - Bf(C'z) + \frac{1}{2}P^{-1}\frac{d}{dt}\left[\frac{\partial V_2}{\partial z}(z,w)\right] \tag{35}$$

$$u = -\frac{\partial V_2}{\partial w}(z,w) \tag{36}$$

The choice of $V_2(z,w)$ should guarantee the control effort to be within predefined limits. $V_2(z,w)$ is chosen as

$$V_2(z,w) = \int_0^{C'z+D'w} \psi'(\theta)d\theta \tag{37}$$

where $\theta \triangleq \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_s \end{bmatrix}$; $\psi(\theta) \triangleq \begin{bmatrix} \psi_1(\theta_1) \\ \vdots \\ \psi_s(\theta_s) \end{bmatrix}$; $C \in R^{n\times s}$; $D \in R^{r\times s}$, rank $D = r$ It follows that

$$\frac{\partial V_2}{\partial z} = C\psi(C'z + D'w) \tag{38}$$

$$\frac{\partial V_2}{\partial w} = D\psi(C'z + D'w) \tag{39}$$

The control loop reads :

$$\dot{z} = Az - Bf(C'z) + B\frac{d}{dt}\psi(C'z + D'w) \tag{40}$$

$$u = -D\psi(C'z + D'w) \tag{41}$$

A saturating function as arctan can be chosen for $\psi(.)$. From a conceptual point of view, the stability problem with control effort constraints is solved. However, the equations of the controller request the derivation of the control effort which is from a practical point of view not advisable. Therefore, this type of controller will not be implemented in the example. [17] contains an alternative formulation, based on a more complex Liapunov function to avoid this derivation and applies it to a rotating pendulum.
Again it can be derived that $\dot{V}_c(x_c) \equiv 0$ implies that u and w remain constant. If the mechanical system does not possess solutions for constant u and constant w except constant q then the set of equilibria of the closed loop system is globally convergent. If $V_c(x_c)$ is radially unbounded, then all solutions remain bounded. If $V_c(x_c)$ possesses a local minimum at the operating point $x_c = 0$ then the operating point is locally asymptotically stable.

5. Application

We choose the flexible manipulator of two links, moving in a horizontal plane such that gravity has no part in the model. Fig. 3 shows the mechanical system. The equations of motion read

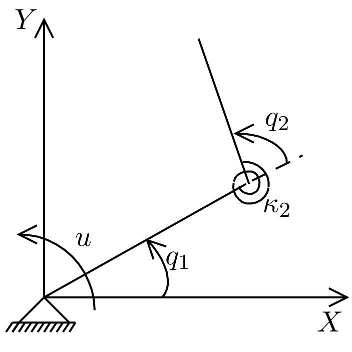

Figure 3. Two-link manipulator

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + K(q) = Mu$$

where

$$q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}; \quad D(q) = \begin{bmatrix} \chi_1 + \chi_2 + 2\chi_3\cos q_2 & \chi_2 + \chi_3\cos q_2 \\ \chi_2 + \chi_3\cos q_2 & \chi_2 \end{bmatrix};$$

$$C(q,\dot{q}) = \begin{bmatrix} -\chi_3(\sin q_2)\dot{q}_2 & -\chi_3(\sin q_2)\,(\dot{q}_2+\dot{q}_1) \\ \chi_3(\sin q_2)\dot{q}_1 & 0 \end{bmatrix}; \quad K = \begin{bmatrix} 0 & 0 \\ 0 & \kappa_2 \end{bmatrix};$$

$$M = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and

$$\chi_1 \triangleq m_1\frac{l_1^2}{4} + m_2 l_1^2 + I_1 \;\; \chi_2 \triangleq m_2\frac{l_2^2}{4} + I_2 \;\; \chi_3 \triangleq m_2 l_1 \frac{l_2}{2}$$

In Section 4 it was shown that the largest invariant set $\mathcal{M}$ of $\mathcal{E} \triangleq \{x_c \in R^{n+m};\ \dot{V}_c(x) = 0\}$ follows from all solutions of the mechanical system where

$$w(t) = q_1(t) \equiv c_0 = \text{constant} \quad ; \quad u(t) \equiv d_0 = \text{constant} \tag{42}$$

In [17] it has been shown that $x_c = 0$ is a globally asymptotically stable equilibrium point.
We will focus on systems with no control effort constraint and use the following numerical values :$\chi_1 = 0.0799$; $\chi_2 = 0.0244$; $\chi_3 = 0.0205$; $\kappa_2 = 1$
First we compare a PD-controller, with a PD-controller with parallel nonlinear controller of first order. The control input of the PD-controller reads :

$$u = -k_p q_1 - k_d \dot{q}_1 \tag{43}$$

The control input of the PD-controller with parallel nonlinear controller reads :

$$\left.\begin{aligned} \dot{z} &= -a_1 z - b f_2 \arctan \tfrac{f_0-f_1}{f_2} z + b\dot{q}_1 \\ u &= -k_p q_1 - k_d \dot{q}_1 - \left[a_1 + b\frac{f_0-f_1}{1+(\frac{f_0-f_1}{f_2}z)^2}\right]\dot{z} \end{aligned}\right\} \tag{44}$$

Based on 4.1.1 we tune the controllers as follows :

- de PD-controller

$$\begin{aligned} k_p &= 2.568 \\ k_d &= 0.921 \end{aligned}$$

  closed loop poles in the neighbourhood of the operating point: $-3.675 \pm j\,5.290$, $-3.675 \pm j\,5.193$.

- modified circle criterion

$$\begin{aligned} k_p &= 2.575 \\ k_d &= 0.922 \\ a_1 &= 9.136 \\ b &= 1.542 \times 10^{-4} \end{aligned}$$

  closed loop poles in the neighbourhood of the operating point: two pairs of complex conjugate poles: $-3.665 \pm j\,5.366$, $-3.663 \pm j\,5.114$ and one real pole -9.207.

For an initial condition $q_1(0) = \dot{q}_1 = \dot{q}_2 = 0$ and $q_2 = \frac{\pi}{2}$ we report the overshoot, settling time and maximum control input for both controllers, where the settling time is defined as the time it takes for the Liapunov function to settle to 1 % of its initial value. We notice a better performance for the PD controller in combination with the nonlinear controller for the set point regulation.

controller	overshoot	settling time	u_{max}
PD	q_1:no; q_2 : 55.91%	1.18	2.08
PD+nonlinear	q_1:no; q_2 : 50.20%	1.03	1.99

Table 1. Comparison performance PD-control; PD+nonlinear-control

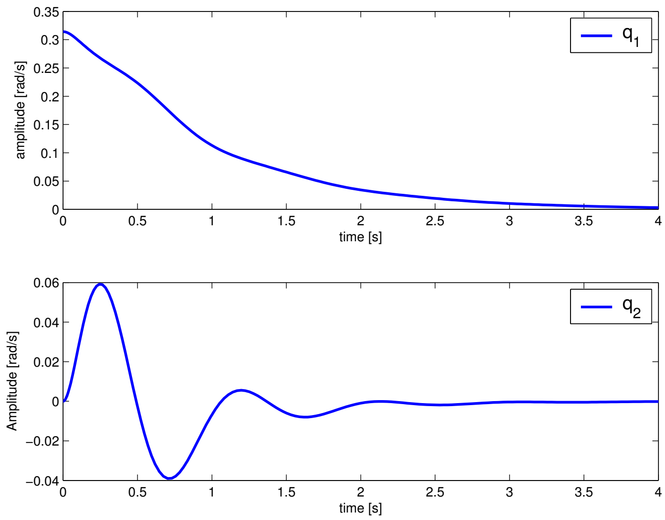

Figure 4. PD-control of the manipulator, $q_1(0) \neq 0$

Again the manipulator is controlled with a PD controller. The parameters of [18] are used this time. The control low reads $u = -k_p q_1 - k_d \dot{q}_1$ with $k_p = k_d = 1$. This set of parameters shows a well damped response of q_1 and some oscillations of q_2 for an initial state $q_1(0) \neq 0$ and all other initial states equal to zero. Fig.4 shows the responses.
Now an SDOF-Euler-Lagrange controller with generalized coordinate p is introduced. A single nonlinearity is considered, namely a cubic nonlinearity $f(p) = k_{na}p^3$. The Euler-Lagrange controller is chosen to be sensitive to the natural frequency $\sqrt{\frac{\kappa_2}{\chi_2}}$, i.e. $\frac{K_0}{D_0} = \frac{\kappa_2}{\chi_2}$. Fig. 5 compares the response of the PD-controller and the PD-controller extended the Euler-Lagrange controller in parallel for initial states $\dot{q}_2(0) \neq 0$. A hopeful result is noticed in the first cycle of vibration where the PD-controller in combination with the cubic nonlinearity performs better for q_1. If we consider the behavior of the generalized coordinate p of the Euler-Lagrange controller on Fig. 6 , we notice indeed a heavy reaction during this first cycle. However, after this first cycle, the energy is returned to manipulator introducing undesired oscillations. Proper, nonlinear energy dissipation should be added to stop the return of energy and dissipate the energy at the level of the controller. Including nonlinear damping into the Liapunov-based theory of Section 4 at the level of the Euler-lagrange controller is not a straightforward task which falls out of the scope of this contribution. Only the possibility of challenging the cubic nonlinear controller is shown and the fact that the system is asymptotically stable.

6. Conclusion

The underactuated control of mechanical systems was considered within a Liapunov-based frame work. Candidate Liapunov functions were presented to derive sufficient conditions for local or global asymptotic stability of the operating point. It was shown that the conditions and assumptions to guarantee stability are rather weak and offer quite some design freedom in the choice of the controller and its corresponding tuning procedure. The control action is a

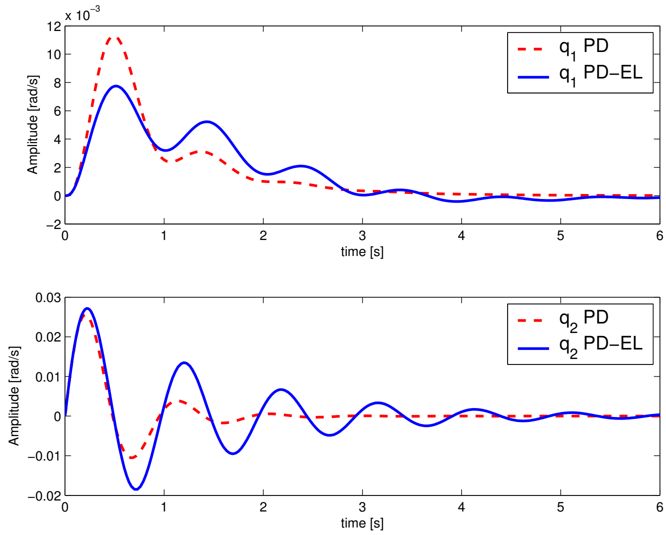

Figure 5. PD-EL-control of the manipulator, $\dot{q}_2(0) \neq 0$

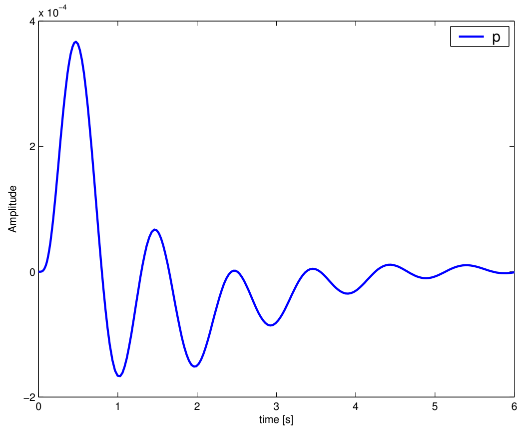

Figure 6. Generalized coordinate p of the EL-controller, $\dot{q}_2(0) \neq 0$

combination of static and dynamic feedback with a nonlinear gain. A general frame work to apply many types of controllers is developed.

Two different choices of the nonlinear controller function were used. The first approach focusses on different linearization models of the closed loop dynamics in different zones of the state space. The second choice of the nonlinear gain is a cubic nonlinearity. This type of Euler-Lagrange controller with the hardening 'stiffness' is able to capture the oscillations of the mechanical system. However, a proper nonlinear damping strategy should be added to force a settling of the mechanical system instead of the beating phenomenon which was observed.

7. References

[1] R. Ortega, A. Loria and P. Nicklasson and H. Sira-Ramirez 1998 *Passivity-based Control of Euler-Lagrange Systems* (Springer, Berlin)
[2] R. Ortega, A. J. Van der Schaft,I. Mareels and B. Maschke 2001 *I.E.E.E. Control Systems Magazine* **21** p18-33
[3] A. Astolfi, Romeo O. and R. Sépulchre 2002 *European Journal of Control* **8** p408-431
[4] R. Ortega, M. W. Spong, F. Gómez-Estern and G. Blankenstein 2002 *I.E.E.E. Transactions on Automatic Control* **47** p1218-1233

[5] J.A Acosta, R. Ortega, A. Astolfi and A.D. Mahindrakar 2005 *I.E.E.E. Transactions on Automatic Control* **50** p1936-1995

[6] A.D. Mahindrakar, J.A. Acosta and R. Ortega 2014 Constrained Stabilization of a Cart on an Asymmetric-Beam System through IDA-PBC *IEEE Conference on Control Applications, France* p1244-1248

[7] F. Valentinis, A.Donaire, and T. Perez, T 2015 Energy-based guidance of an underactuated unmanned underwater vehicle on a helical trajectory *Control Engineering Practice* **44** p138-156

[8] X.Z. Lai, C.Z.Pan, M. Wu, S.X. Yang, and W.H. Cao 2015 Control of an Underactuated Three-Link Passive-Active-Active Manipulator Based on Three Stages and Stability Analysis *Journal of Dynamic Systems Measurement and Control-Transactions of the ASME* **137** 021007

[9] Y. Liu and H. Yu A survey of underactuated mechanical systems 2013 *IET Control Theory and Applications* **7**, p921-935

[10] X. Xiu and Y.Liu A Set-Point Control for a Two-link Underactuated Robot With a Flexible Elbow Joint 2013 *Journal of Dynamic Systems, Measurement, and Control* **135** 051016-1

[11] S. Tahmasian and C. A. Woolsey 2015 A Control Design Method for Underactuated Mechanical Systems Using High-Frequency Inputs *Journal of Dynamic Systems, Measurement, and Control* **137** 071004-1

[12] L. Meirovitch 1980 Computational Methods in Structural Dynamics *S"ythoff and Noordhof, Alphen aan den R"yn*

[13] D. D. Siljak 1969 Nonlinear Systems *John Wiley, New York*

[14] L. Luyckx, M. Loccufier and E. Noldus 2001 On the design of nonlinear controllers for Euler-Lagrange systems *Nonlinear Dynamics and Systems Theory* **1** p99-110

[15] Y. S. Lee and G. Kerschen and A. F. Vakakis and P. N. Panagopoulos and L. A. Bergman and D. M. McFarland,Complicated dynamics of a linear oscillator with a light, essentially nonlinear attachment, *Physica D*, 204, 41-69, (2005).

[16] R. Lozano, B. Brogliato, O. Egeland and B. Maschke 2000 Dissipative Systems Analysis and Control *Springer, Berlin*

[17] L. Luyckx, M. Loccufier and E. Noldus 2003 Bounded output feedback systems for mechanical vibration control *Proc. 6th Nat. Congress on Theoretical and Applied Mechanics, Ghent*

[18] I. Fantoni and R. Lozano 2002 Nonlinear Control for Underactuated Mechanical Systems *Springer, Berlin*