Università di Pisa
Facoltà di Scienze Matematiche, Fisiche e Naturali
Corso di Laurea Magistrale in Fisica

Anno Accademico 2011/2012

Elaborato Finale

T-duality and Generalized Complex Geometry

Candidato:
Francesco Muia

Relatore:
Dr. Jarah Evslin
Correlatore:
Prof. Kenichi Konishi

Contents

Conventions

We denote by I_n a finite set of indices $i \in \{1, \ldots, n\}$. I denotes an infinite set of indices, while $I_n^0 = \{0, 1, \ldots, n\}$.

With the expression $f : U \to V$ is *smooth* or *differentiable* (where $U \in \mathbb{R}^n, V \in \mathbb{R}^m$) we will always mean that it is differentiable an infinite number of times in its definition domain. A differentiable map in U is denoted by $f \in S^\infty(U)$. A diffeomorphism is a map which is smooth, invertible and such that the inverse map is also smooth.

Let n indicates the dimension of the space considered, which can be a smooth manifold, a linear space, etc. Italic letters from the middle of the alphabet like $i, j, k, \ldots$ are usually referred to real coordinates indices. They take values in I_n. Latin letters from the middle of the alphabet like $\mu, \nu, \rho, \ldots$ are usually referred to complex coordinates indices. Since they are used on complex space, usually there are n coordinates x^μ and their complex conjugates $\overline{x}^\mu$. Then μ takes their values in I_n but can be referred to two different set of coordinates which are conjugate among each other. Italian letters from the beginning of the alphabet like $a, b, c, \ldots$ are usually referred to vielbeins directions. They take values in I_n.

Latin letters from the beginning of the alphabet like α, β, γ are usually referred to open sets. They take values in I. For example U_α is an open set on a smooth manifold M. $U_{\alpha\beta}$ denotes the overlap $U_\alpha \cap U_\beta$. More generally $U_{\alpha_1 \ldots \alpha_n}$ denotes the overlap of the n open sets $U_{\alpha_1} \cap U_{\alpha_2} \cap \cdots \cap U_{\alpha_n}$.

We will always denote the identity over a generic space X by 1_X.

We will always denote the transpose of the matrix A by A^T. The transposition of the invertible matrix A^{-1} is denoted by A^{-T}.

1 Introduction

String Theory was born in 60's to explain the strong interactions, but it was soon superseded by QCD. The massless state with spin two which String Theory possesses in its own spectrum has been considered a problem for a long time.

However since 1974 the two-spin massless state was recognized to have the same properties of the graviton, String Theory rapidly became the most promising theory in trying to unify all the fundamental interactions in a unique framework [1, 2].

The introduction of fermionic matter in String Theory brings to consider the supersymmetric extension of this, also called Superstrings Theory [3]. The number of dimensions in which Superstring Theory is consistently defined is $d = 10$. The discrepancy with phenomenology, which provides only four dimensions is filled by one of the most interesting theoretical aspects of Superstring Theory, that is the *compactification* of the 6 extra dimensions.

The most common way to compactify the extra dimensions is a generalization of the *dimensional reduction*. This procedure was first used by T. Kaluza and O. Klein [4] [5]. They succeded in unifying the gravity and the electromagnetism in four dimensions by deriving both interactions from a five dimensional theory of pure gravity. The idea is both simple and surprising. As an example let us consider the five dimensional action for a real massless scalar φ

$$S = \int d^5x \; \partial_\mu \varphi \partial^\mu \varphi \tag{1.1}$$

where we took the flat metric on the five dimensional space M. Let us we compactify a direction of M such that it decomposes as

$$M = M_4 \times S^1 \tag{1.2}$$

where M_4 is a four dimensional manifold while S^1 is a circle of radius R. Moreover let x^μ be the set of cordinates which locally parametrize M_4, while let x be the coordinate which parametrizes the circle, such that $x \sim x + 2\pi$. Then the Klein-Gordon equation reads

$$\Box\varphi = 0 \quad \Rightarrow \quad \partial_\mu \partial^\mu \varphi + \partial_x^2 \varphi = 0 \tag{1.3}$$

so that by using the periodicity in x we can write the Fourier expansion

$$\varphi(x^\mu, x) = \frac{1}{\sqrt{2\pi R}} \sum_{n=-\infty}^{\infty} \varphi_n(x^\mu) e^{-i\frac{nx}{R}} \tag{1.4}$$

By substituting in Equation (1.3) we obtain

$$\partial_\mu \partial^\mu \varphi_n - \frac{n^2}{R^2} \varphi_n = 0 \tag{1.5}$$

which includes the actual idea of the compactification procedure: the compactified directions give rise to the mass term for the real scalar φ_n. In particular a tower of states is obtained, each of which has mass proportional to $\frac{n}{R}$. The main point here is that at low energies the only observable states are the massless ones. This amount to take the limit for $R \mapsto 0$, which is physically amounts to to think about the size of the compact direction to be of the order $l_S \sim \frac{1}{M_P}$, where $M_P \cong 10^{19}\,Gev$ is the Planck mass. In this limit only φ_0 remains light, while all the other modes φ_n with $n \neq 0$ become increasingly heavy and can be discarded. The dimensional reduction is precisely the limit in which only the zero mode is kept. Its name is due to the fact that we would have obtained the same results by taking $\varphi \equiv \varphi(x^\mu)$.

This procedure is generalizable to the ten dimensional case [7]. The manifold is decomposed as

$$M = M_4 \otimes K \tag{1.6}$$

where M_4 is a four dimensional maximally symmetric manifold, while K is a six-dimensional manifold called the *internal space*. This form of the decomposition is forced by the requirement that the Poincarè invariance is preserved in four dimensions.

In this Section we will indicate by $M, N, \ldots$ indices which refer to the ten-dimensional space, by $\mu, \nu, \ldots$ indices which refer to the four-dimensional space and by $i, j, \ldots$ indices which refer to the internal space.

The dimensional reduction of the fields in ten dimensions brings to the following results

- A gauge field which transforms as a vector in the $SO(1,9)$ decomposes under $SO(1,3) \otimes SO(6)$ as

$$\mathbf{9} = (\mathbf{4}, \mathbf{1}) \otimes (\mathbf{1}, \mathbf{6}) \tag{1.7}$$

  so that we can recognize a four-dimensional gauge field A_μ and six scalars $\{A_i\}_{i \in I_6}$.

- The metric tensor g_{MN} which decomposes in the components $g_{\mu\nu}$, $g_{i\mu}$ and g_{ij}, where $g_{\mu\nu}$ is the four-dimensional metric tensor, while form the four-dimensional point of view $g_{i\mu}$ are spin one fields and g_{ij} are scalar fields.

- The spinor fields decomposed in a non-trivial way that we will study in Section 3.1.4. They play a key role since as it is well known they constitute the matter of our Universe.

It's remarkable to probe the consequences of Dirac equation in ten dimensions. Let us denote by Γ^M the ten-dimensional Dirac matrices, and let us consider the Dirac field Ψ. The d-dimensional Dirac operator is denoted by D_d Then we can write

$$0 = i\not{D}_{10}\Psi = i\sum_{M=1}^{10} \Gamma^M D_M \Psi = i\sum_{\mu=1}^{4} (\Gamma^\mu D_\mu \Psi) + i\sum_{i=5}^{10} \Gamma^i D_i \Psi \equiv i(\not{D}_4 + \not{D}_6)\Psi \tag{1.8}$$

From

$$i\not{D}_4 = -\not{D}_6\Psi \tag{1.9}$$

we can immediately see that the term $-\not{D}_6\Psi$ plays the role of a mass operator whose eigenvalues are the masses as seen in four dimensions. However, as we have mentioned before in the dimensional reduction we have to neglect the massive terms. This means that the zero modes of the six-dimensional Dirac operator $\not{D}_6$ corresponds to the massless fermions in four dimensions. Massless fermions are those we are interested in, since the observed fermions are massless in this approssimation. In fact they acquire their small masses as a consequence of a symmetry breaking.

The fact that observed fermions are expected to correspond to zero modes of the operator $\not{D}_6$ allows us to say that the way in which fields appear in the four-dimensional world is strictly related to the topology and geometry of the internal space K. The present work concerns the study of different aspects of the geometry of K.

As a consequence of what we have just said, phenomenology puts strong constraints on the geometry of K [7]. The most accredited phenomenological models are currently those which provide for a $N = 1$ supersymmetric extension of the Standard model. In fact one of the major concerns of String Theory in the last two decades has been to find a realistic compactification which brings to a Standard model sector in four dimensions at low energies.

Strikingly, it turns out that the (NS, NS) groundstates of Superstrings Theory are described by a set of objects, namely (g, H, ϕ), where g is the Riemannian metric on the internal space, H is a three-form also called the *Neveu-Schwarz flux*, while ϕ is the *dilaton*. Moreover, these three objects can be inserted in the *Polyakov* *action*, which describe exactly the propagation of a string in the background defined by (g, H, ϕ). Compactifications with vanishing H-flux have been intensively studied until the first half of the '90s, and they brought to the study of a particular kind of complex manifold, also called Calabi-Yau. Calabi-Yau are simply a kind of manifolds which admits the existence of a covariantly constant well defined spinor. On the contrary, the geometry of the manifolds involved in compactifications with H-flux turned on has been unknown for long time.

Recently, the interest for compactifications with H-flux turned on has grown since it has been proven that non-vanishing vev for H can be used to partially break the $N = 2$ supersymmetry of Calabi-Yau compactifications to $N = 1$ [8]. In fact the advent of the so-called G-structures technique (reviewed in Section 2.1.5) to study complex structures with additional structures has solved many problems. In particular now a complete classification of this kind of manifolds is given. If the H-flux is turned on then the internal space geometry is no longer Kähler: it is called **generalized Kähler** [10]. The first part of the present work is devoted to the study of the G-structures. In particular we will see that the generalized Kähler structures are $SU(3)$ structures, and how they can be described in terms of spinors on a manifold.

T-duality is a non-local symmetry of String Theory related to duality with respect to the inversion of the compactification radius $R \longmapsto \frac{1}{R}$. In the case of compactifications with H flux, T-duality consists of a map T which associate to a background (g, H, ϕ) its dual background (g', H', ϕ'). At the level of local supergravity backgrounds, there exists a standard way to find the dual background, which is given by what is called the Buscher rules. These consist in introducing a gauge field by gauging the non-linear sigma model defined by (g, H, ϕ). The dual background can be simply obtained by integrating the gauge field out.

One of the aspects of the present work is to understand under which conditions a dual background can be defined in a global manner. C. Hull [11] has furnished general arguments to understand if the non-linear sigma model associated to a global background can be gauged in a way which defines a global dual background. It's in this context that the *double field theory* was born [12].

In the present work we will explicitly study the non-physical example of the three-torus T^3. Even if this example can't be used as an actual background (its dimension is 3!) it is very useful since it allows us to highlight the mathematical details of the question. Moreover, even if a global treatment is possible in this case, we will see explicitly that the results locally agree with those given by Buscher rules. In particular we will explicitly show that the three-torus represents the simplest example in which an ungaugeable isometry can actually be gauged by using what is known as the *double space* technique. In particular, as it was formalized by P. Bouwknegt, J. Evslin and V. Mathai [13] the topology of the background can change after T-duality. We will explicitly see this phenomenon in the T^3 example.

The main point of the present work is however the systematic study of the Generalized Complex Geometry [14] [15]. It turns out to be the natural framework to describe generalized Kähler structures. Since it provides a doubling of the degrees of freedom due to the fact that tangent space and cotangent spaces are merged together, it can be used to describe the doubled space in a natural way. In particular T-duality map takes a very simple form when written in terms of generalized structures [16].

There are various versions of Superstring Theory. We will deal only with a couple of these, and we will concentrate on the geometric aspects of their backgrounds. It will be shown that Generalized Complex Geometry provides the right way to describe type II superstrings backgrounds at low energy, and in this context we will consider two explicit examples which are $SU(3)$ structures. In particular we will study the form of the T-duality map written in terms of pure spinors for these examples, and we will see explicitly that the local form of such a map is equivalent to that prescribed by Buscher rules.

Doubtless the most interesting point is to understand if such *local* dual supergravity backgrounds can be extended to *global* Superstring backgrounds. We will see explicitly that the examples considered are T-folds according to the definition given in [11] and we will study the mathematical details which descend from it. In particular we will concentrate on the generalized geometry consequences for T-folds.

The thesis is organized as follows:

- In Chapter 2 we will give the basic notions in differential geometry which are needed to work with Riemannian manifolds, with fiber bundles and with G structures.
- In Chapter 3 we will review the basics notions on spinors. In particular we will focus on their algebraic

nature as elements of a Clifford algebra and on conditions needed to exist over a smooth manifold.

- In Chapter 4 we will study the complex geometry. The final purpose of this Chapter is to describe $SU(3)$ structures.
- In Chapter 5 we enter the topic of Generalized Complex Geometry. We will focus on aspects which are useful to study examples in the following Chapter, as for example the definition of generalized metric and vielbeins.
- In Chapter 6 we will study various aspects of T-duality.

2 Geometry background

Real differential geometry is the most immediate attempt to generalize our innate geometrical vision of the world. It is exactly half way between the linear algebra, which talks about lines, plans, etc, and the topology, which permits us to classify and to study objects of any shape.

Both the linear algebra and the topology are not completely satisfactory to describe the real world. In fact, if on the one hand the linear algebra is too rigid to describe the enormous variety of objects that make up the world and their complexity, on the other the topology is too little. Roughly speaking and following the topological classification, one could say that a bottle is equivalent (homeomorphic) to a couch, since neither has holes (this is true only in three dimensions). Of course in a large variety of situations this classification turns out to be too little restrictive, and then it must be avoided.

Real differential geometry is just an attempt to strike a balance between the linear algebra and the topology by mixing them into a single structure: *a manifold*. It can globally assume any form, but it locally seems like a real vector space $\mathbb{R}^n$. One of the most important feature of a manifold is that we can define some way to perform differential calculus on it.

In Section 2.1.5 we will introduce the *G-structures*. They provide a useful tool to describe the mathematical structures which play a fundamental role in the present work, namely the $SU(3)$-structures. We will study their mathematical details in Chapter 4 where we will explain also the physical motivation to introduce them.

In the present Chapter we briefly recall some basic concepts in differential geometry on Riemannian manifolds and fiber bundles.

2.1 Basics in real geometry

2.1.1 Real manifolds

Differential structures

A smooth manifold is a set which locally looks like a subset of $\mathbb{R}^n$, and in which the gluing of all this kind of subsets is smooth. More precisely

Definition 2.1.1. Let $U \subseteq M$ and $p \in U$. Let $\varphi : U \to \varphi(U) \subseteq \mathbb{R}^n$ be a bijective map, where $\varphi(U)$ is an open set in $\mathbb{R}^n$. The pair (U, φ) is an **n-chart** over M. Two n-charts (U, φ) and (V, ψ) over M are **compatible** if $U \cap V = \{\varnothing\}$ or if $U \cap V \neq \{\varnothing\}$, the sets $\varphi(U \cap V)$ and $\psi(U \cap V)$ are open sets in $\mathbb{R}^n$ and the map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is a diffeomorphism. The map $\psi \circ \varphi^{-1}$ is a **chart's change**, while the inverse map $\varphi^{-1} : \varphi(U) \to U$ is a *local parametrization*.

Since $\varphi(U) \subseteq \mathbb{R}^n$, if we consider the canonical basis of $\mathbb{R}^n$, we can write in coordinates

$$\varphi(p) = (x^1(p), \ldots, x^n(p)) \equiv x^i(p) \tag{2.1}$$

$\{x^i(p)\}_{i \in I_n}$ are the *local coordinates in the given n-chart* (U, φ).

In order to define a smooth manifold we have to consider a set of charts which forms a covering.

Definition 2.1.2. A collection $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in I}$ of n-charts over a set M is a **smooth n-atlas** if $M = \bigcup_{\alpha\in I} U_\alpha$ and if the n-charts are compatible two by two. A smooth n-atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in I}$ is a *smooth n-structure* if each n-chart compatible with all the elements in $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in I}$ is already contained in $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in I}$ itself.

One can prove that each smooth atlas is contained in a unique maximal smooth atlas which is the union of all the charts compatible with the charts given [17]. Eventually

Definition 2.1.3. Let M be a set endowed with a smooth n-structure. Then M is a **smooth manifold** of dimension n, i.e. $dim(M) = n$.

Needless to say, if a set M allows for a smooth n-structure, it can't admit a smooth m-structure with $n \neq m$ (Theorem of the invariance of the dimension [17]). From now on we will leave understood the dimension of the charts and of the manifolds, assuming that it is always equal to n, unless differently specified.

It is interesting to notice that in many books the initial requirement is not for an arbitrary set M, but for a topological space. In that case each set U defining a chart has to be an open set in the topology of M and each map defining a chart has to be a homeomorphism with the image. It's amazing to observe that this is an unnecessary requirement, since each smooth atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in I}$ defines uniquely a topological structure over the set M [17]. In fact it suffices to state

$$A \subseteq M \text{ is an open set} \quad \Leftrightarrow \quad \forall\, \alpha \in I \quad \varphi_\alpha(A \cap U_\alpha) \text{ is an open set in } \mathbb{R}^n$$

In this way we have defined the *topology induced by the smooth structure* over M. Naturally, if we will define a smooth structure over a topological space, we will assume that the induced topology be exactly the given topology.

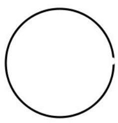
Although it is clear that given a point on a manifold $p \in M$, we can always find a neighbourhood containing p which locally looks like an open set of $\mathbb{R}^n$, the concept which makes a smooth manifold really interesting and efficient is the charts' compatibility, which allows us to move among charts smoothly.

Let us give two simple examples of smooth manifolds, which are useful to our purposes.

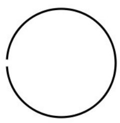
Example 2.1.1. The circle $\mathbf{S^1}$
S^1 can be defined as a subset of $\mathbb{R}^2$

$$S^1 = \{(x, y) \in \mathbb{R}^2 | \quad x^2 + y^2 = 1\} \tag{2.2}$$

It can be equipped with a differentiable structure in the following way. Let us consider the two open sets in Figure (2.1). Let us suppose that the length of the circle is equal to 1. Then we can define the local



(a) U_1.



(b) U_2.

Figure 2.1: A covering $\{U_1, U_2\}$ for a circle S^1.

parametrizations such that

$$\varphi_1 : U_1 \to (0, 1) \qquad \varphi_2 : U_2 \to \left(-\frac{1}{2}, \frac{1}{2}\right) \tag{2.3}$$

There are two connected components for the intersection $U_{12} = U_1 \cap U_2$. We will call them the *upper component* U_{12}^+ and the *lower component* U_{12}^-. It turns out that a convenient choice for the transition functions is

$$\begin{aligned} \varphi_{12}^+ : U_{12}^+ \quad &\to \quad U_{12}^+ \\ x \quad &\mapsto \quad x \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} \varphi_{12}^- : U_{12}^- \quad &\to \quad U_{12}^- \\ x \quad &\mapsto \quad x + 1 \end{aligned} \tag{2.5}$$

We observe that if M, N are smooth manifolds with dimension respectively $dim(M) = m$ and $dim(N) = n$, then $M \times N$ has a natural structure of smooth manifold with $dim(M \times N) = m + n$. If $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$ is an atlas for M and $\mathcal{B} = \{(V_\beta, \psi_\beta)\}$ is an atlas for N, then an atlas for $M \times N$ is simply given by the product atlas

$$\mathcal{A} \times \mathcal{B} = \{(U_\alpha \times V_\beta, \varphi_\alpha \times \psi_\beta)\} \tag{2.6}$$

where the map $\varphi_\alpha \times \psi_\beta : U_\alpha \times V_\beta \to \mathbb{R}^{m+n}$ is defined by

$$\varphi_\alpha \times \psi_\beta(x, y) = (\varphi_\alpha(x), \psi_\beta(y)) \tag{2.7}$$

Example 2.1.2. The n-**dimensional torus** is defined as $\mathsf{T}^n = \underbrace{S^1 \times \cdots \times S^1}_{n \text{ times}}$.

The case $n = 3$ will be diffusely studied in Chapter 6.

It's natural to generalize the concept of differentiability to maps between two smooth manifolds.

Definition 2.1.4. Let M and N be two smooth manifolds such that $dim(M) = m$ and $dim(N) = n$. Let $F : M \to N$ be a map. F is **differentiable** or **smooth** in $p \in M$ if there exist two charts (U, φ) in $p \in M$ and (V, ψ) in $F(p) \in N$ such that $F(U) \subseteq V$ and there exists a neighborhood of p, $U' \subset U$ such that the composition $\psi \circ F \circ \varphi^{-1} : U \supseteq U' \to V$ is a smooth map. If F is smooth in each $p \in M$ then it is **smooth over M**. A smooth bijection, with smooth inverse is a **diffeomorphism**.

It's immediate to notice that a map φ defining a chart (U, φ) over a smooth manifold M is automatically a diffeomorphism between U and $\varphi(U) \subseteq \mathbb{R}^n$.

The power of Definition 2.1.4 resides in the fact that the differentiability concept is completely independent of the chosen chart [17, 19]. Moreover, if $F : M \to N$ is smooth in some $p \in M$, then it is continuous in p [17]. Finally, if $F : M \to N$ and $G : N \to S$ are two smooth maps between manifolds, then also their composition $G \circ F : M \to S$ is smooth.

If there exists a diffeomorphism between M and N, they are said to be **diffeomorphic**. If M is diffeomorphic to N then $dim(M) = dim(N)$. Exactly as homeomorphisms classify spaces according to whether it is possible to continuously deform one of them to the other, in the same way diffeomorphisms classify spaces according to whether it's possible to smoothly deform one of them to the other: they define an equivalence class. Smooth functions $f : M \to M$ form a group called the *diffeomorphism group of* M.

Vectors and one-forms

Vectors on a manifold M can be induced from vectors which are tangent to some curve on M [19, 20] as the intuition suggests us.

Let for example $\gamma : \mathbb{R} \supseteq I \to M$ $(0 \in I)$ be a smooth curve which intersects a chart (U, φ) and such that $p = \gamma(0) \in U$. Let $\{x^i\}_{i \in I_n}$ be the coordinates induced by φ on U. The coordinates of γ on U are $x^i(\gamma(t))$ and the tangent vector to this curve is defined as

$$\frac{d}{dt}(x^i(\gamma(t))) \tag{2.8}$$

Let $f \in C^\infty(M)$ be a smooth map. In $t = 0$, the change of f is given by

$$\frac{d}{dt}(f(\gamma(t)))\bigg|_{t=0} \tag{2.9}$$

or, in local coordinates

$$\frac{d(f \circ \varphi^{-1})}{dx^i}\bigg|_{x^i = \varphi(\gamma(t))} \frac{dx^i}{dt}\bigg|_{t=0} \tag{2.10}$$

Then defining

$$X = X^i \left.\frac{\partial}{\partial x^i}\right|_p \quad \text{where} \quad X^i = \left.\frac{dx^i(\gamma(t))}{dt}\right|_{t=0} \tag{2.11}$$

we obtain

$$\frac{d}{dt} f(\gamma(t))\bigg|_{t=0} = X[f] \tag{2.12}$$

The differential operator X is a tangent vector to the manifold at the point $p \in M$, since it is the tangent vector to the curve γ in $t = 0$. If now we consider the following equivalence class of curves

$$[\gamma(t)] = \left\{ \widetilde{\gamma}(t) \quad \text{such that} \quad \widetilde{\gamma}(0) = \gamma(0) \quad \text{and} \quad \frac{dx^i(\widetilde{\gamma}(t))}{dt} = \frac{dx^i(\gamma(t))}{dt} \right\} \tag{2.13}$$

All the equivalence classes of curves passing through $p \in M$, namely all tangent vectors at $p \in M$ span a real vector space that is the **tangent space $\mathbf{T_pM}$**. In local coordinates the set of vectors

$$\left\{ \frac{\partial}{\partial x^i}\bigg|_p \right\}_{i \in I_n} \tag{2.14}$$

form the *coordinate basis* for the tangent space T_pM.

However, a basis for T_pM doesn't need to be induced by the local coordinates. In fact we can take a set of matrices $\{A_i^i \in GL(n, \mathbb{R})\}$ and define a basis for T_pM such that $e_a|_p = A_a^i \frac{\partial}{\partial x^i}\big|_p$ which is called a *non-coordinate basis*.

It is well known that for each finite dimensional vector space there exists a relative dual. The dual space of the tangent space is the **cotangent space $\mathbf{T_p^*M}$**, which is spanned in the coordinate basis by

$$\{dx^i|_p\}_{i \in I_n} \tag{2.15}$$

and the non-coordinate basis $\{e^a|_p\}_{a \in I_n}$ can be defined in the same way of the tangent space.

A non-degenerate scalar product $(,) : T_pM \times T_pM \to \mathbb{R}$ is defined on the vector space T_pM such that $(e_a|_p, e_b|_p) = \delta_{ab}$.

The duality relation (which is also denoted by $(,)$) naturally holds

$$(e^a|_p, e_b|_p) = (e_b|_p, e^a|_p) = \delta_b^a \qquad \forall\, a, b \in I_n \tag{2.16}$$

where $(,)$ is the natural interior product between a vector space and its dual, induced by the map

$$\begin{aligned} T_p^*M \ni e^a : T_pM \quad &\rightarrow \quad \mathbb{R} \\ e_b \quad &\mapsto \quad (e_a, e_b) = \delta_b^a \end{aligned} \tag{2.17}$$

A dual vector $\omega|_p = \omega_a e^a|_p \in T_p^*M$, is a **one-form on** M.

Submanifolds

As it seems to be intuitive we can define the concept of *submanifold*.

Definition 2.1.5. Let M, N be smooth manifolds, and $p \in M$, where $m = dim(M)$ and $n = dim(N)$. Let $f : M \to N$ be a smooth map. f induces the **differential map** f_* if $\forall$ smooth map $g \in C^\infty(N)$

$$f_* : T_pM \to T_{f(p)}(N)$$

such that

$$f_* X[g] = X[g \circ f] \tag{2.18}$$

f_* is a *pushforward* of vectors. If $h : N \to L$, with L a smooth manifold, then the naturality condition

$$(h \circ f)_* = h_* \circ f_* \tag{2.19}$$

holds.

If we consider a chart (U, φ) in $p \in M$, and a chart (V, ψ) in $f(p) \in N$, which respectively establish the coordinates $\{x^i\}_{i \in I_m}$ and $\{y^i\}_{i \in I_n}$, then we can compute the expression in components of the pushforward. In fact Equation (2.18) means

$$f_* X[g \circ \psi^{-1}](y) = X[g \circ f \circ \varphi^{-1}](x) \tag{2.20}$$

where $\varphi(p) = x$ and $\psi(f(p)) = y$. This gives

$$f_* X^j \frac{\partial}{\partial y^j}[g \circ \psi^{-1}](y) = X^i \frac{\partial}{\partial x^i}[g \circ f \circ \varphi^{-1}](x) \tag{2.21}$$

Now putting $g = y^j$ we obtain the expression in components

$$f_* X^j = X^i \frac{\partial y^j}{\partial x^i} \tag{2.22}$$

Finally we can explore the concept of submanifold.

Definition 2.1.6. Let $f : M \to N$ be a smooth map, and let $dim(M) \leq dim(N)$. The map f is an **immersion** of M into N if $f_* : T_pM \to T_{f(p)}N$ is an injection, namely if $rk(f_*) = dim(M)$. The map f is an **embedding** if f is an injection and an immersion. Usually we will denote an embedding i by $\overset{i}{\hookrightarrow}$. Finally, if i is an embedding, then $i(M)$ is a **submanifold** of N, and $i(M)$ is naturally diffeomorphic to M.

2.1.2 Fiber bundles

T_pM and T_p^*M characterize the manifold only in a neighborhood of the point $p \in M$. However, if we are interested in the global properties of a manifold, it's much more convenient to introduce a new object: *a bundle*.

The simplest example of bundle we can study is the trivial one. In fact we can always endow a smooth manifold M with a bundle structure simply by taking the product of M with another smooth manifold F. We have to define also the smooth map

$$\begin{array}{rcll} \pi_1 : M \times F & \to & M & \\ (p, x) & \mapsto & p & \forall\, p \in M, \quad \forall\, x \in F \end{array} \tag{2.23}$$

which projects on the first factor of the pair $(\cdot, \cdot)$. Then

Definition 2.1.7. The quadruple $(M \times F, M, \pi_1, F)$ is a **trivial bundle**.

Definition 2.1.7 is introductory to the following

Definition 2.1.8. The quadruple (E, M, π, F) is a **fiber bundle** if the following conditions hold

1. E, M and F are smooth manifolds called the *total space*, the *base space* and the *standard fiber* respectively. The smooth map $\pi : E \to M$ is surjective and is called the *projection*.
2. There exists an open covering $\{U_\alpha\}_{\alpha \in I}$ of M such that $\forall\, \alpha \in I$ there exists a diffeomorphism $t_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ and a commutative diagram such as in Figure 2.2. The pair (U_α, t_α) is a *local trivialization for the bundle.* The set of all local trivializations $\{(U_\alpha, t_\alpha)\}_{\alpha \in I}$ is a *trivialization for the bundle.*

If the fiber F is a real (complex) vector space then the fiber bundle is a **real (complex) vector bundle**. The *rank* of a vector bundle is the dimension of F as a vector space.

In absence of ambiguities, instead of a quadruple, we will often denote a vector bundle by its projection $\pi : E \to M$, leaving F implicit.

$$\begin{tikzcd} \pi^{-1}(U_\alpha) \arrow[r, "t_\alpha"] \arrow[dr, "\pi"'] & U_\alpha \times F \arrow[d, "\pi_1"] \\ & U_\alpha \end{tikzcd}$$

Figure 2.2: Local structure of a fiber bundle.

Let $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ be a smooth atlas for the smooth manifold M, and let $\pi : E \to M$ be a fiber bundle. We want to underline that it's unnecessary that $\{U_\alpha\}_{\alpha \in I}$ be the covering of an atlas of the differentiable structure of M. In that case $\{\pi^{-1}(U_\alpha)\}_{\alpha \in I}$ would constitute the covering of a differentiable atlas on E, called a *fiber* *atlas for* E, and we will say that the smooth atlas $\{(U_\alpha, \varphi_\alpha)\}$ *trivializes* the fiber bundle $\pi : E \to M$. We will always consider smooth atlases which trivialize $\pi : E \to M$ unless differently specified.

So far, a fiber bundle seems to be a differentiable way to associate a fiber to each point of a manifold. Locally, the way of doing it is trivial: it's just the topological product. However, the way in which fibers are glued together is the really interesting point, which clarifies the global topological properties of the bundle. With this purpose we give the following [17, 18]

Proposition 2.1.1. Let M be a smooth manifold and let E be a set. Let $\pi : E \to M$ be a surjective map. Let $\{(U_\alpha, t_\alpha)\}_{\alpha \in I}$ be a trivialization of E. If the following conditions hold $\forall\, \alpha, \beta \in I$

1. $\pi_1 \circ t_\alpha = \pi|_{\pi^{-1}(U_\alpha)}$
2. $\forall\, U_{\alpha\beta} \neq \{\varnothing\}$ there exists a smooth map

$$g_{\alpha\beta} : U_{\alpha\beta} \to GL(n, \mathbb{R}) \tag{2.24}$$

such that the composition $t_\alpha \circ t_\beta^{-1} : U_{\alpha\beta} \times F \to U_{\alpha\beta} \times F$ is of the form

$$t_\alpha \circ t_\beta^{-1}(p, x) = (p, g_{\alpha\beta}(p)(x)) \qquad p \in U_{\alpha\beta}, \quad x \in F \tag{2.25}$$

then E admits a unique structure (up to isomorphisms [20]) of fiber bundle, for which $\{(U_\alpha, t_\alpha)\}_{\alpha \in I}$ is a trivialization.

$\{g_{\alpha\beta}\}_{\alpha,\beta \in I}$ are the **transition functions** for the bundle $\pi : E \to M$. As Equation (2.25) shows, they act on the fiber by a left translation. $\{(U_{\alpha\beta}, g_{\alpha\beta})\}_{\alpha,\beta \in I}$ form a *cocycle on* M, namely they obey the *cocycle conditions*: $\forall\, p \in U_{\alpha\beta\gamma}$

1. $g_{\alpha\alpha}(p) = 1_F$
2. $(g_{\alpha\beta}(p))^{-1} = g_{\beta\alpha}(p)$
3. $g_{\alpha\beta}(p) \circ g_{\beta\gamma}(p) \circ g_{\gamma\alpha}(p) = 1_F$

Naturally, if we can choose all the transition functions of a bundle E to be the identity, then the bundle E is trivial. Moreover one can show that a fiber bundle over a contractible space is trivial [20].

Every time we introduce a new structure, we have to introduce also a class of maps which preserve the new structure. With this purpose we give the following

Definition 2.1.9. Let $\pi_1 : E_1 \to M_1$ and $\pi_2 : E_2 \to M_2$ be two fiber bundles. A pair of maps (Φ, ϕ) such that $\Phi : E_1 \to E_2$ and $\phi : M_1 \to M_2$ is a **bundle morphism** if

$$\pi_2 \circ \Phi = \phi \circ \pi_1 \tag{2.26}$$

namely if the diagram in Figure 2.3 is commutative. If $\phi : M_1 \to M_2$ is a diffeomorphism, then the bundle morphism is a *strong morphism.* If $M_1 = M_2 = M$ and $\phi \equiv 1_M$, then the bundle morphism is a *vertical morphism.*

A bundle morphism is a map between bundles which preserves both the differentiable structure and the bundle structure.

Remarkably, if $M_1 = M_2$, $\phi = 1_B$ and Φ is injective, then $\pi_1 : E_1 \to M_1$ is a **subbundle** of $\pi_2 : E_2 \to M_2$.

$$\begin{tikzcd}
E_1 \arrow[r, "\Phi"] \arrow[d, "\pi_1"'] & E_2 \arrow[d, "\pi_2"] \\
M_1 \arrow[r, "\phi"] & M_2
\end{tikzcd}$$

Figure 2.3: The pair (Φ, ϕ) represents a *bundle morphism* between E_1 and E_2.

Example 2.1.3. The tangent bundle

Let us try to apply Proposition 2.1.1 to tangent spaces. In particular let M be a smooth manifold such that $dim(M) = n$. Let us define

$$T = \coprod_{p\in M} T_pM \tag{2.27}$$

where $\coprod$ indicates the disjoint union, and define the natural projection

$$\begin{array}{rcl}\pi : T & \to & M \\ T_pM & \mapsto & p\end{array} \tag{2.28}$$

Let us consider, for simplicity, a coordinate basis $\{x^i_\alpha\}_{i\in I_n}$ induced by a smooth atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in I}$ on M. The trivialization functions $t_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$ can be defined as follows

$$t_\alpha\left(\sum_{i=1}^n v^i \left.\frac{\partial}{\partial x^i_\alpha}\right|_p\right) = (p, v) \tag{2.29}$$

where $v = (v^1, \ldots, v^n) \in (\mathbb{R}^n)^*$, and $\left\{\left.\frac{\partial}{\partial x^i_\alpha}\right|_p\right\}_{i\in I_n}$ is the coordinate basis for T_pM. Naturally we can obtain a non-coordinate basis as described in Section 2.1.1. Then

$$t_\alpha \circ t_\beta^{-1}(p, v) = t_\alpha\left(\sum_{i=1}^n v^i \left.\frac{\partial}{\partial x^i_\beta}\right|_p\right) = t_\alpha\left(\sum_{j=1}^n \left[\sum_{i=1}^n \frac{\partial x^j_\alpha}{\partial x^i_\beta}(p)\, v^i\right] \left.\frac{\partial}{\partial x^j_\alpha}\right|_p\right) = \left(p, \frac{\partial x_\alpha}{\partial x_\beta}(p)\, v\right) \tag{2.30}$$

where $\frac{\partial x_\alpha}{\partial x_\beta}$ is the Jacobian matrix of the coordinate change $\varphi_\alpha \circ \varphi_\beta^{-1}$. Then Proposition 2.1.1 is satisfied with transition functions

$$g_{\alpha\beta} = \frac{\partial x_\alpha}{\partial x_\beta} \tag{2.31}$$

and T has the structure of a vector bundle with rank n. T is the **tangent bundle**. Where the notation creates some ambiguities about the base space, we denote the tangent bundle over M by TM.

Example 2.1.4. The cotangent bundle

Let us define

$$T^* = \coprod_{p\in M} T^*_pM \tag{2.32}$$

and define the natural projection

$$\begin{array}{rcl}\pi : T^* & \to & M \\ T^*_pM & \mapsto & p\end{array} \tag{2.33}$$

Again, for simplicity, consider the coordinate basis induces by a smooth atlas $\{(U_\alpha, \varphi_\alpha)\}$ on M. Such a basis will be dual respect to the tangent space coordinate basis in Example 2.1.3: $\{dx^i_\alpha|_p\}_{i\in I_n}$. Then we can define trivialization functions on each U_α: $t_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$ by imposing

$$t_\alpha\left(\sum_{i=1}^n w_i dx^i_\alpha|_p\right) = (p, w^T) \tag{2.34}$$

where $w^T \in \mathbb{R}^n$. We obtain

$$t_\alpha \circ t_\beta^{-1}(p, w^T) = t_\alpha\left(\sum_{i=1}^n w_i dx^i_\beta|_p\right) = t_\alpha\left(\sum_{j=1}^n \left[\sum_{i=1}^n \frac{\partial x^i_\beta}{\partial x^j_\alpha}(p) w_i\right] dx^j_\alpha|_p\right) = \left(p, \left(\left[\frac{\partial x_\beta}{\partial x_\alpha}(p)\right]\right)^T w^T\right) \tag{2.35}$$

Thus using Proposition 2.1.1, the transition functions

$$g_{\alpha\beta} = \left(\left[\frac{\partial x_\beta}{\partial x_\alpha}\right]\right)^T \tag{2.36}$$

define the structure of a vector bundle on T^* with rank n. T^* is the **cotangent bundle**. Where the notation creates some ambiguities about the base space, we denote the cotangent bundle over M by T^*M.

Let us observe that given two vector bundles E_1, E_2 on the same base M the algebraic operations on vector spaces (see Appendix A) can be extended to define vector bundles such as $E_1 \oplus E_2$, $E_1 \otimes E_2$ [21]. In this way we can endow a manifold with several useful structures. This observation is crucial for the development of the Generalized Complex Geometry in Chapter 5.

In fact, recall that $\forall x \in M$ the tangent and the cotangent spaces T_xM and T_x^*M are vector spaces, and define

$$T_q^p(M) = \coprod_{x\in M} T_q^p\big|_x M \tag{2.37}$$

where $\coprod_{x\in M}$ denotes the disjoint union. Let us define the natural projection $\pi : T_q^p(M) \to M$ which maps $T_q^p\big|_x M$ into $x \in M$. Using Proposition 2.1.2 is straightforward to prove that $T_q^p(M)$ is a fiber bundle. It is the $\binom{p}{q}$**-tensor fiber bundle over** M. A basis for this fiber bundle is trivially given by making the 6tensor product of the elements of the basis of T_pM and T_p^*M (see Examples 2.1.3 and 2.1.4):

$$\left\{ \frac{\partial}{\partial x_\alpha^{i_1}}\bigg|_p \otimes\cdots\otimes \frac{\partial}{\partial x_\alpha^{i_p}}\bigg|_p \otimes dx_\alpha^{j_1}|_p \otimes\cdots\otimes dx_\alpha^{j_q}|_p \right\}_{i_i,j_j\in I_n} \tag{2.38}$$

Sometimes it's udeful to observe that each element as in Equation (2.38) can be identified as a multilinear map which acts as follows

$$T_p^*M|_{U_\alpha} \otimes\cdots\otimes T_p^*M|_{U_\alpha} \otimes T_pM|_{U_\alpha} \otimes\cdots\otimes T_pM|_{U_\alpha} \quad\to\quad \mathbb{R} \tag{2.39}$$

It's extremely useful to introduce a sort of inverse map with respect to the projection π, since it allows us to interpret tensors as functions on the base space of a bundle.

Definition 2.1.10. Let E be a fiber bundle, and let $U \subseteq M$ be an open set. A smooth map $\sigma : U \to \pi^{-1}(U)$, such that $\pi\circ\sigma|_U = 1_U$ is a *local section of* π. If $U \equiv M$ then σ is a *global section of* π or simply a **section** of E. The space of local sections defined on U is denoted by $\Gamma(U, E)$, while the space of global sections is denoted simply by $\mathfrak{X}(E)$. Moreover we can see smooth functions $f : M \to \mathbb{R}$ as global sections and write $f \in \Gamma(\mathbb{R}) \equiv C^\infty(M)$.

Example 2.1.5. Let M be a smooth manifold. It is well known that the tangent bundle $\pi : T \to M$ is a vector bundle. $\Gamma(T) \equiv \mathfrak{X}(M)$ is the space of **smooth vector fields** over M. Similarly we can speak about the cotangent bundle T^* over M. The sections of this bundle $\Gamma(T^*) \equiv \Omega^1(M)$ are the **exterior 1-forms** over M. In particular, let (U,φ) a chart in $p \in M$ which determines the coordinates $\{x^i\}_{i\in I_n}$. We can define a set of local sections of the tangent bundle $\{\partial_1,\dots,\partial_n\}$ such that

$$\begin{aligned} \partial_i : M \quad &\to \quad \mathfrak{X}(M) \\ p \quad &\mapsto \quad \partial_i(p) \equiv \frac{\partial}{\partial x^i}\bigg|_p \in T_pM \end{aligned} \tag{2.40}$$

In particular, if $X \in \mathfrak{X}(M)$, then it is a linear combination of $\partial_1(p),\dots,\partial_n(p)$, so that we can find n functions $X_1,\dots,X_n : U \to \mathbb{R}$ such that

$$X(p) = \sum_{l=1}^n X^l(p)\partial_l(p) \tag{2.41}$$

An similar reasoning can be repetead for exterior one-forms. Moreover, let (V,ψ) be another chart in $p \in M$, and let us denote by $\{\widetilde{\partial}_q,\dots,\widetilde{\partial}_p\}$ the associated local sections of T. We find that on $U\cap V$

$$\widetilde{\partial}_i = \sum_{j=1}^n \frac{\partial x^j}{\partial x^i}\partial_j \tag{2.42}$$

Since $X = \sum X^j\partial_j = \sum \widetilde{X}^k\widetilde{\partial}_k$ we get that

$$X^i = \sum_{l=1}^n \frac{\partial x^i}{\partial x^j}\widetilde{X}^j \tag{2.43}$$

One can find that there exists a set of local sections $\{dx^1(p),\dots,dx^n(p)\}$ which form a local frame for the cotangent bundle, and are defined as the dual vectors of the basis vectors $\{\partial_1(p),\dots,\partial_n(p)\}$.

Furthermore

Definition 2.1.11. Smooth sections in $\Gamma(T^p_q(M)) \equiv T^p_q$ are the **smooth $\binom{p}{q}$-tensors on M**.

In presence of ambiguities we will write $\Gamma(T^p_q(M)) \equiv T^p_q M$.

Every construction seen in this Section can be generalized from the tangent bundle T to a generic vector bundle E, obtaining for example the space of smooth vector fields on E - $\mathfrak{X}(E)$) - or the space of one-forms on E - $\Omega^1(E)$ - as well as the space of the $\binom{p}{q}$-tensor fields on E, namely $T^p_q E$.

2.1.3 Exterior forms

Needless to say, the exterior forms are one of the most powerful tools in differential geometry. Since their principal feature is the antisymmetry, as we can easily imagine they are strictly related to anticommuting objects as spinors, as we will see.

Basics

For the linear algebra underlying the present Section we refer to Appendix A.

We can define the **exterior algebra over M**

$$\Lambda(T^*) = \bigoplus_{p \leq n} \Lambda^p(T^*) \tag{2.44}$$

The space of its smooth sections (namely all forms over M) is denoted by $\Lambda T^* \equiv \Gamma(M, \Lambda(T^*))$.

It's obviously possible to build up by analogy the space of sections $\Lambda T = \Gamma(M, \Lambda(T))$, where $\Lambda(T) = \bigoplus_{p \leq n} \Lambda^p(T)$ and the elements of $\Lambda^p(T)$ are the alternating p-vectors over M. In general we can repeat the same constructions of for a general vector bundle E to obtain for example the space of the p-forms $\Lambda^p E^*$ or the space of alternating p-multivectors $\Lambda^p E$ (see Section 2.1.2).

Obviously the exterior algebra ΛT^* inherits from $\Lambda(V^*)$ its algebra structure. In particular, it inherits the wedge product. We define the **exterior product** between two forms $\omega, \eta \in \Lambda T^*$ as the form

$$(\omega \wedge \eta)(p) = \omega(p) \wedge \eta(p) \in \Lambda T^* \qquad \forall\, p \in M \tag{2.45}$$

The exterior product obeys the following properties $\forall\, \omega, \eta, \lambda \in \Lambda T^*$, $\forall\, a \in \mathbb{R}$

1. It is associative, namely $(\omega \wedge \eta) \wedge \lambda = \omega \wedge (\eta \wedge \lambda)$.
2. It is distributive with respect to the sum, namely $\omega \wedge (\eta \wedge \lambda) = \omega \wedge \eta + \omega \wedge \lambda$.
3. It commutes with the product with scalars $\omega \wedge (a\eta) = a(\omega \wedge \eta) = (a\omega) \wedge \eta$.
4. It is graded, namely if $\omega \in \Lambda^p T^*$ and $\eta \in \Lambda^q T^*$ then $\omega \wedge \eta \in \Lambda^{p+q} T^*$.
5. It is anticommutative, namely $\omega \wedge \eta = (-1)^{pq} \eta \wedge \omega$.

where we left implicit the point $p \in M$ in which the forms take values. Hereafter we will use this convention. Properties from 1. to 5. mean that ΛT^* is a *graded*, *associative* and *anticommutative* algebra.

Needless to say ΛT^* inherits an inner product which acts fiberwise, from that defined in Equation (A.16).

Let us choose a chart $(U_\alpha, \varphi_\alpha)$ over M which induces coordinates $\{x^i_\alpha\}_{i \in I_n}$, where $dim(M) = n$. A r-form $\phi \in \Lambda^r T^*$ is locally expressed by

$$\phi|_{U_\alpha}(p) = \frac{1}{r!} \sum_{\{i_i\}_{i \in I_r}} \phi_{i_1 \dots i_r} dx^{i_1}_\alpha \wedge \dots \wedge dx^{i_r}_\alpha \tag{2.46}$$

where $p \in U_\alpha \subseteq M$ and $\phi_{i_1 \dots i_r} \in C^\infty(U_\alpha)$. In other words a basis for the exterior algebra $\Lambda^p T|^*_{U_\alpha}$ is simply given by the set

$$\{dx^{i_1}_\alpha \wedge \dots \wedge dx^{i_p}_\alpha\} \tag{2.47}$$

so that the rank of $\Lambda^r T^*$ as a vector bundle is given by $\binom{n}{p}$. Consequently the rank of the whole exterior algebra is $\sum_p \binom{n}{p}$. Moreover we get $dim(\Lambda^p T^*) = dim(\Lambda^{n-p} T^*)$.

As we have seen, there are some difficulties to transport vector fields by means of differentiable maps between manifolds. One of the most interesting properties of differential forms is that they are easily transportable. In fact

Definition 2.1.12. Let M, N be smooth manifolds, and $p \in M$. Let $F : M \to N$ be a smooth map. F induces the **pullback map** F^* such that $\forall X \in T_pM$, $\forall \omega \in T^*_{F(p)}M$

$$F^* : T^*_{F(p)}N \to T^*_pM \tag{2.48}$$

such that

$$(F^*\omega)(X) = \omega(F_*X) \tag{2.49}$$

In components we obtain [20]

$$F^*\omega_i(x) = \omega_j(y(x))\frac{\partial y^j}{\partial x^i} \tag{2.50}$$

where $\{x^i\}_{i\in I_n}$ are the coordinates on $U \subset M$ and $\{y^i\}_{i\in I_n}$ are the coordinates on $F(U) \subset N$.

It's straightforward to generalize the map for generic q-forms. In fact

Definition 2.1.13. The map $F^* : \Lambda^q T^*N \to \Lambda^q T^*M$ such that

$$F^*\omega(X_1, \dots, X_q) = \omega(F_*X_1, \dots, F_*X_q) \tag{2.51}$$

is the pullback of a q-form.

In coordinates we can write

$$F^*\omega_{i_1\dots i_q}(x) = \omega_{j_1\dots j_q}(y(x))\frac{\partial y^{j_1}}{\partial x^{i_1}}\cdots\frac{\partial y^{j_q}}{\partial x^{i_q}} \tag{2.52}$$

Moreover we can sum up some of the main properties of the pullback map F^* [20]

1. $(G \circ F)^* = F^* \circ G^*$ where $F : M \to N$ and $G : N \to P$ are smooth maps
2. $F^*(\omega \wedge \tau) = (F^*\omega) \wedge (F^*\tau)$ $\forall \omega \in \Lambda^p T^*$, $\forall \tau \in \Lambda^q T^*$

We have to notice that pushforward is defined only for vectors, while pullback is defined only for forms. However if the map which induces them is a diffeomorphism between manifolds, we can define both pushforward and pullback on vectors and forms. In fact it suffices to note that in the case of diffeomorphism $(F^{-1})_* = F^*$ and $(F^{-1})^* = F_*$ [23, 24].

A fundamental tool is given in the following

Definition 2.1.14. Let $X \in \mathfrak{X}(M)$ be a vector field on the smooth manifold M. Let $\omega \in \Lambda^p T^*$. The **contraction** $C^\infty(M)$-linear map

$$i_X : \Lambda^p T^* \quad \to \quad \Lambda^{p-1} T^* \tag{2.53}$$

such that

$$(i_X\omega)(Y_1, \dots, Y_{p-1}) = \omega(X, Y_1, \dots, Y_{p-1}) \qquad \forall Y_1, \dots, Y_{p-1} \in \mathfrak{X}(M) \tag{2.54}$$

for each $p \geq 1$, with the convention that $i_X(\Lambda^0 T^*) = 0$.

We can see the map i_X as a sort of generalization of the inner product in Equation (2.16).

Definition 2.1.15. Let M be a smooth manifold such that $dim(M) = n$ and let $i : \mathfrak{X}(M) \to End(\Lambda T^*)$ such that $\forall X = X^j \frac{\partial}{\partial x^j} \in \mathfrak{X}(M)$

$$i(X) = i_X \tag{2.55}$$

i_X has the following properties

1. $i_X f = 0$ $\forall f \in C^\infty(M)$
2. $i_X dx^j = X^j$ $\forall dx^j \in \Omega^1(M)$

3. $i_X^2 = 0 \qquad \forall X \in \mathfrak{X}(M)$

4. $i_X(\omega \wedge \eta) = i_X\omega \wedge \eta + (-1)^p \omega \wedge i_X\eta \qquad \forall \omega \in \Lambda^p T^*, \forall \eta \in \Lambda^q T^*$

For example if $\omega \in T^*$ we obtain

$$i_X\omega = X^j\omega_j = \omega(X) \tag{2.56}$$

since we recall that a one-form can be seen as a linear map $\omega : T \to \mathbb{R}$. If $\xi \in \Lambda^2 T^*$ then

$$i_X\xi = i_X\xi_{ij}(dx^i \wedge dx^j) = \frac{1}{2!}\xi_{ij}\left(X^i dx^j - X^j dx^i\right) = \xi(X) \tag{2.57}$$

from which we recall that a two-form can be seen as a linear map $\xi : T \to T^*$. In general if $\omega \in \Lambda^p T^*$

$$i_X\omega = \frac{1}{p!}\sum_{i=1}^{p}(-1)^{i-1}X^{i_i}\omega_{i_1\dots i_i\dots i_p}dx^{i_1}\wedge\dots\wedge d\hat{x}^{i_i}\wedge\dots\wedge dx^{i_p} =$$

$$= \frac{1}{(p-1)!}X^{i_i}\omega_{i_i i_2\dots i_p}dx^{i_2}\wedge\dots\wedge dx^{i_p} \tag{2.58}$$

where the hatted index denotes the absence of the element.

At this point it seems quite natural to wonder if there is a map which is the inverse of i_X. It turns out that not only such a map exists, but it is one of the most important tools in the differential geometry. In fact

Theorem 1. *Let M be a smooth manifold such that $dim(M) = n$. Then $\exists!\, \mathbb{R}-$linear map $d : \Lambda T^* \to \Lambda T^*$, the **exterior differential**, such that the following properties*

1. $d(\Lambda^p T^*) \subseteq \Lambda^{p+1}T^* \qquad \forall p \in \mathbb{N}$

2. *If $f \in \Lambda^0 T^* \equiv C^\infty(M)$, then $df \in \Lambda^1 T^*$ is the differential of f*

3. *If $\omega \in \Lambda^p T^*$, $\eta \in \Lambda^q T^*$ then*

$$d(\omega \wedge \eta) = (d\omega)\wedge\eta + (-1)^p\omega\wedge(d\eta) \tag{2.59}$$

4. $d^2 = 0$

The exterior differential is the backbone on which is based the cohomology theory, and it is immediately related to a series of objects with important geometrical meaning. We will explore these questions in some details in Sections 2.3. The starting point of the cohomology theory is the definition of *closed* and *exact* forms

$$\Lambda^p T^* \supseteq Z^p(T^*) = \{\phi \in \Lambda^p T^* | \quad d\phi = 0\} = Ker(d) \quad \forall p \geq 0 \tag{2.60}$$

Elements in $Z^p(T^*)$ are the **closed p-forms over M**, also called the *p-cocycles*. Also, $d\Lambda^{p-1}T^* \subseteq Z^p(T^*)$. Elements in $d\Lambda^p T^* = Im(d)$ are the **exact p-forms over M**, also called the *p-coboundary*.

The exterior differential satisfies several important properties [17], for example $\forall \omega \in \Lambda T^*$

- d is local, namely if $\omega = \omega'$ on the open set $U \subset M$ then $d\omega|_U = d\omega'|_U$.
- d commutes with the restriction, namely if $U \subseteq M$ is an open set, then $d(\omega|_U) = (d\omega)|_U$.
- d commutes with the pull-back map, namely if $F : M \to N$ is a smooth map, then

$$d(F^*\omega) = F^*(d\omega) \tag{2.61}$$

Sometimes it's useful to speak about a more general class of objects, namely the $\binom{\mathbf{p}}{\mathbf{q}}$**-tensor valued r-forms**. As it seems intuitive they are simply sections of the bundle $T^p_q(M) \otimes \Lambda^r(T^*)$, which we denote by

$$t \in T^p_q \otimes \Lambda^r T^* \tag{2.62}$$

Integration

Exterior form provides a useful as well as convenient framework to perform integrations over a smooth manifold M. For what concerns manifolds with boundaries we refer to Appendix B.

The first important concept is that of orientability.

Definition 2.1.16. Let M be a connected manifold such that $dim(M) = n$. Then M is **orientable** if there exists a smooth atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in I}$ such that the transition functions $\{\varphi_\alpha\}_{\alpha\in I}$ have positive jacobian determinant.

The condition of orientability is equivalent to the existence of a consistent choice of oriented basis on the tangent bundle T. Moreover it is also equivalent to the existence of a nowhere vanishing $\nu \in \Lambda^n T^*$. If there exists two nowhere vanishing forms $\nu_1, \nu_2 \in \Lambda^n T^*$ and a smooth positive map $f \in C^\infty(M)$ such that $\nu_1 = f\nu_2$, then ν_1 and ν_2 define the same orientation over M.

Definition 2.1.17. A nowhere vanishing form $\nu \in \Lambda^n T^*$ is a **volume form**.

ν is called a volume form because as we will see in the following it allows us to integrate forms on a smooth manifold.

Definition 2.1.18. Let $\omega \in \Lambda T^*$. The closure of the set $\{p \in M|\omega(p) \neq 0\}$ is the **support of** ω and is denoted by $supp(\omega)$. A form $\omega \in \Lambda T^*$ such that $supp(\omega) \subset K \subset M$ where K is a compact subset of M is a **form with compact support**.

The form with compact support are integrable over a smooth manifold M. In particular it can be shown [17] that if M is orientable, $\omega \in \Lambda^n T^*$ with compact support contained in the overlap of two charts (U, φ) and (V, ψ), then

$$\int_{\varphi(U)} (\varphi^{-1})^*\omega = \int_{\psi(V)} (\psi^{-1})^*\omega \tag{2.63}$$

This result allows us to give the following [17]

Proposition 2.1.2. Let M be an orientable smooth manifold. Let $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in I}$ be an oriented atlas and let $\{\rho_\alpha\}$ be a partition of unity subject to this atlas. Then for each $\omega \in \Lambda^n T^*$ with compact support we can define the integral

$$\int_M \omega = \sum_{\alpha\in I} \int_M \rho_\alpha \omega \tag{2.64}$$

which is independent from both the atlas and the partition of unity.

We can simply generalize the Definition of an integral over a manifold to a function $f \in C^\infty(M)$. In fact if $j \in \Lambda^n T^*$ is a volume form we can write

$$\int_M f = \int_M fj \tag{2.65}$$

Eventually, if M is compact we define the **j-volume** as

$$vol_j(M) \int_M j \tag{2.66}$$

$vol_j(M)$ is always positive.

We can list some useful properties of the integration over a manifold M, in fact

1. Let M be an oriented manifold. Let $-M$ the manifold with opposite orientation. Then

$$\int_{-M} \omega = -\int_M \omega \tag{2.67}$$

2. Let M and N be two oriented manifolds such that $dim(M) = dim(N) = n$. Let $F : M \to N$ be a diffeomorphism. Let us suppose that F preserves the orientation, then

$$\int_M F^*(\omega) = \int_N \omega \tag{2.68}$$

while if F inverts the orientation

$$\int_M F^*(\omega) = -\int_N \omega \tag{2.69}$$

Eventually the fundamental

Theorem 2. ***Stokes' Theorem***
Let M be a smooth oriented manifold with boundary such that $dim(M) = n$, and let ∂M be its $n-1$-dimensional boundary. Let ω be a n-form over M. Then

$$\int_M d\omega = \int_{\partial M} \omega \tag{2.70}$$

2.1.4 Flows and Lie derivatives

In this Section we want to introduce a way to compare vectors and one-forms lying on different tangent spaces of the tangent bundle, without using any metric. The Lie derivative is the tool which allow us to compare vectors (as well as one-forms) computed in different but near points on the manifold. Remarkably, it is an intrinsic object on a manifold.

Let $\gamma(t,p) : I \times M \to M$ be an integral curve ($I \subseteq \mathbb{R}$ and $0 \in I$) on a smooth manifold M. This means that it is a curve whose tangent vector is given at each point $p \in M$ by a vector field $X \in \mathfrak{X}(M)$.

We choose a chart (U, φ) in p, such that $\varphi(p) = x \in \varphi(U) \subseteq \mathbb{R}^n$ and coordinates $\{x^i\}_{i \in I_n}$. So locally we can write

$$\frac{d\gamma^i(t,x)}{dt} = X^i(\gamma(t,x)) \quad \text{with} \quad \gamma^i(0,p) = x^i \tag{2.71}$$

γ is the **flow** generated by the vector $X \in \mathfrak{X}(M)$. A flow satisfies the following

Proposition 2.1.3. Let $X \in \mathfrak{X}(M)$. Then $\forall\, p \in M$ $\exists$ an integral curve, a flow $\gamma : I \times M \to M$ such that $\gamma(t,p)$ is a solution of the differential Equation (2.71).

Proposition 2.1.4. A flow satisfies the group property

$$\gamma(t, \gamma(s,p)) = \gamma(t+s, p) \qquad \forall\, t, s \in I \subseteq \mathbb{R} \tag{2.72}$$

Definition 2.1.19. Let $\gamma(t,p)$ be a flow over the smooth manifold M. While keeping t fixed, we can rewrite

$$\gamma(t,p) \equiv \gamma_t(p) \tag{2.73}$$

The map $\gamma_t : M \to M$ is a diffeomorphism and represents the commutative **one parameter group**, which satisfies

1. $\gamma_0 = 1_M$
2. $\gamma_t^{-1} = \gamma_{-t}$
3. $\gamma_t \gamma_s = \gamma_{t+s}$

Choosing the parameter t infinitesimal, we find the *infinitesimal flow* from Equation (2.71)

$$\gamma_t^i(p) = x^i + tX^i(p) \tag{2.74}$$

and X is the *infinitesimal generator of the flow group* γ_t. Recall that a finite flow can be expressed throughout the exponentiation

$$\gamma^i(t,p) = \exp(tX)x^i \tag{2.75}$$

The commutator between two vector fields is a very common tool among physicists

Definition 2.1.20. Let $X, Y \in \mathfrak{X}(M)$. The bilinear map $[,] : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ is the **Lie bracket** or the **commutator**. The vector field $[X,Y] = XY - YX$ is defined by

$$[X,Y]f = X(Y[f]) - Y(X[f]) \qquad \forall\, f \in C^\infty(M) \tag{2.76}$$

We will say that X and Y commute if $[X,Y] = 0$.

Moreover [17]

Proposition 2.1.5. Let $X, Y, Z \in \mathfrak{X}(M)$, $a, b \in \mathbb{R}$, $f, g \in C^\infty(M)$ and $F : M \to N$ smooth map. Then

1. $[,]$ is anticommutative, namely $[X, Y] = -[Y, X]$
2. $[,]$ is $\mathbb{R}$-linear in both the entries, namely
$$[aX + bY, Z] = a[X, Z] + b[Y, Z] \quad \text{and} \quad [X, aY + bZ] = a[X, Y] + b[X, Z] \tag{2.77}$$
3. $[,]$ satisfies the Jacobi identity, namely $J(X, Y, Z) = [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$
4. $[,]$ satisfies the following Leibniz rule $[fX, gY] = fg[X, Y] + f(X[g])Y - g(Y[f])X$
5. The push-forward map F_* acts naturally on the Lie bracket, namely $F_*[X, Y] = [F_*X, F_*Y]$
6. If (U, φ) is a local chart on M which induces coordinates $\{x^i\}_{i \in I_n}$ on U, then we can locally write
$$[X, Y] = \sum_{i,j=1}^{n} \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} \right) \partial_j \tag{2.78}$$
and in particular $[\partial_i, \partial_j] = 0$.

The function $J(X, Y, Z)$ is the *Jacobiator*.

As we mentioned at the beginning of the Section we are interested in the change of a vector field X along a flow γ. Since we can't compare vectors in different tangent spaces, thus we have to define an operator which allows us to quantify the difference between vectors. This kind of operator is the **Lie derivative**.

Definition 2.1.21. We define the **Lie derivative** of a vector field $Y \in \mathfrak{X}(M)$ along the vector field $X \in \mathfrak{X}(M)$ as
$$\mathfrak{L}_X Y = \lim_{t \to 0} \quad \frac{1}{t}[\gamma_{-t*} Y(\gamma_t(p)) - Y(p)] = \left[X^i \frac{\partial Y^j}{\partial X^i} - Y^i \frac{\partial X^j}{\partial x^i} \right] \frac{\partial}{\partial x^j} \tag{2.79}$$

For the last equality see [19, 20]. Then
$$\left[X^i \frac{\partial Y^j}{\partial X^i} - Y^i \frac{\partial X^j}{\partial x^i} \right] \frac{\partial}{\partial x^j} \equiv [X, Y]^k \frac{\partial}{\partial x^k} \tag{2.80}$$

and we we can simply write, $\mathfrak{L}_X Y = [X, Y]$. Moreover, since if $Y \in \mathfrak{X}(M)$ and $\omega \in \Omega^1(M)$, then the interior product $(\omega, Y) \in C^\infty(M)$, and by imposing that $\mathfrak{L}_X(\omega, Y) = (\mathfrak{L}_X \omega, Y) + (\omega, \mathfrak{L}_X Y)$, we can find the action of the Lie derivative on one-forms.

Definition 2.1.22. Let $\omega \in \Lambda^q T^*$. The **Lie derivative** of ω along the vector field $X \in \mathfrak{X}(M)$ is defined as
$$\mathfrak{L}_X \omega = \lim_{t \to 0} \quad \frac{1}{t}[\gamma_t^* \omega(\gamma_t(p)) - \omega(p)] \quad \forall\, p \in M \tag{2.81}$$

Explicitly we find
$$\mathfrak{L}_X \omega = \frac{1}{q!} \left[X^j \partial_j \omega_{i_1 \dots i_q} + q \omega_{j i_2 \dots i_q} \partial_{i_1} X^j \right] dx^{i_1} \wedge \cdots \wedge dx^{i_q} \tag{2.82}$$
In particular, if ω is a one-form
$$\mathfrak{L}_X \omega = \left[X^j \partial_j \omega_i + \omega_j \partial_i X^j \right] dx^i \tag{2.83}$$
While if $f \in \Lambda^0 T^* \equiv C^\infty(M)$, then
$$\mathfrak{L}_X f = X[f] \tag{2.84}$$
It's useful to rewrite the Lie derivative of a q-form ω in a more compact manner as follow
$$\mathfrak{L}_X \omega = (i_X d + d i_X)\, \omega \tag{2.85}$$
which is the *Cartan formula*. It is convenient to immediately see an application of the Cartan formula, which we will use several times in the work

Lemma 2.1.1. *Let* $\omega \in \Omega^1(M)$*. Then*
$$d\omega(X, Y) = i_X \omega(Y) - i_Y \omega(X) - \omega([X, Y]) \qquad \forall\, X, Y \in \mathfrak{X}(M) \tag{2.86}$$

In fact

$$
\begin{aligned}
d\omega(X,Y) &= i_Y(i_X)d\omega = i_Y(\mathfrak{L}_X\omega) - i_Y(di_X\omega) = \\
&= \mathfrak{L}_X(i_Y\omega) + i_{[Y,X]}\omega - i_Y(di_X\omega) = di_Xi_Y\omega + i_Xd\omega(Y) - i_Yd\omega(X) + i_{[Y,X]}\omega = \\
&= i_Xd\omega(Y) - i_Yd\omega(X) - \omega([X,Y])
\end{aligned}
\tag{2.87}
$$

where we have used that $i_Xi_Y\omega = 0$ and that $[i_Y, \mathfrak{L}_X] = i_{[Y,X]}$, as stated in the Lie derivative properties listed below.

It's interesting to notice that by knowing the action of the Lie derivative on the tensor product of tensors $T_1 \in T^{p_1}_{q_1}$ and $T_2 \in T^{p_2}_{q_2}$

$$
\mathfrak{L}_X(T_1 \otimes T_2) = (\mathfrak{L}_X T_1) \otimes T_2 + T_1 \otimes (\mathfrak{L}_X T_2) \tag{2.88}
$$

one can deduce the action of the Lie derivative on a general tensor $T \in T^p_q$ from its action on smooth functions $f \in C^\infty(M)$, on vectors $X \in \mathfrak{X}(M)$ and on one-forms $\omega \in \Omega^1(M)$.

We can sum up some of the main properties of the Lie derivative, that is $\forall\, f \in C^\infty$, $\forall\, X, Y \in \mathfrak{X}(M)$ [20]:

1. $\mathfrak{L}_X fY = [X, fY]$
2. $\mathfrak{L}_{fX} Y = [fX, Y]$
3. $[\mathfrak{L}_X, i_X] = 0$
4. $[\mathfrak{L}_X, d] = 0$
5. $[\mathfrak{L}_X, i_Y] = i_{[X,Y]}$
6. $[\mathfrak{L}_X, \mathfrak{L}_Y] = \mathfrak{L}_{[X,Y]}$

2.1.5 G-structures

As we will see in Chapter 4, the condition on geometry arising from the supersymmetric compactifications can be successfully studied in terms of G-structures. In the present Section we introduce them.

In physical applications, a fiber bundle often come with a preferred group of transformations, which is a subgroup of $GL(n, \mathbb{R})$. This is due to the fact that it is often necessary to restrict the allowed transition functions on the overlappings of an atlas. These restrictions can be encoded by a new structure: the *structure group*.

Definition 2.1.23. Let $(E, M, \pi, F; \lambda, G)$ be a sextuple such that

1. (E, M, π, F) is a fiber bundle. G is a Lie group called the *structure group* and $\lambda : G \to GL(n, \mathbb{R})$ defines a left action on the standard fiber F.
2. There exists a family of preferred trivializations $\{(U_\alpha, t_\alpha)\}_{\alpha \in I}$ such that the following holds. Let $g_{\alpha\beta} : U_{\alpha\beta} \to GL(n, \mathbb{R})$ define transition functions. There exists a family of functions $h_{\alpha\beta} : U_{\alpha\beta} \to G$ such that if $p \in U_{\alpha\beta\gamma}$ the following relations hold (see Figure 2.4)
   - $h_{\alpha\alpha}(p) = 1_G$
   - $(h_{\alpha\beta}(p))^{-1} = h_{\beta\alpha}(p)$
   - $h_{\alpha\beta}(p) \circ h_{\beta\gamma}(p) \circ h_{\gamma\alpha}(p) = 1_G$

   Then $(E, M, \pi, F; \lambda, G)$ is a **fiber bundle with structure group G** or simply a **G-structure**. $h_{\alpha\beta}$ are the *transition functions with values in* G, and depend on the trivializations chosen. The set $\{(U_{\alpha\beta}, h_{\alpha\beta})\}_{\alpha,\beta \in I}$ forms a *cocycle with values in* G. The preferred trivializations are said to be *compatible with the structure*. We will often denote a G-structure simply by specifying the structure group G in addition to its fiber bundle structure $\pi : E \to M$.

The diagram in Figure 2.4 explains how a G-structure works. In fact starting from the group G and the transition functions $g_{\alpha\beta}$, troughout the upper part of the diagram we can manipulate the G-action on the fiber, for example selecting only a subgroup of G by means of $h_{\alpha\beta}$, and then implementing the action on the fiber F by means of λ. In this regard, given a G-structure it's possible to make a pair of operations: to enlarge the structure group, or to reduce it [21]. The latter operation is very common and we have just described it. It's

$$\begin{tikzcd}
G \arrow[dr, "\lambda"] & \\
U_{\alpha\beta} \arrow[u, "h_{\alpha\beta}"] \arrow[r, "g_{\alpha\beta}"'] & GL(n,\mathbb{R})
\end{tikzcd}$$

Figure 2.4: The structure of a structure bundle with its cocycle.

equivalent to the existence of some extra structure over the base space M. For example in General Relativity the presence of a metric on a orientable manifold reduces the structure group to $SO(n, R)$.

In many physical applications, it is often necessary to specialize more the structure of a fiber bundle. This fact brings us to the following

Definition 2.1.24. Let $\pi : P \to M$ be a G-structure. If the fiber is taken equal to the structure group itself, then it is a **principal bundle**.

Remember that, from Proposition 2.1.1, transition functions act locally on the fiber by a left translation. In the case of the principal bundles, it is also important to define a right action on the fiber. It's intuitive that, if $\{(U_\alpha, t_\alpha)\}$ is a trivialization for the G-structure $\pi : E \to M$ (with fiber F), then the right action can be defined locally on $t_\alpha(\pi^{-1}(U_\alpha))$ as

$$\begin{array}{rcll} R_g : t_\alpha(\pi^{-1}(U_\alpha)) & \to & t_\alpha(\pi^{-1}(U_\alpha)) & \\ (p, x) & \mapsto & (p, x \cdot g) & \forall\, p \in U_\alpha, \quad \forall\, x \in F, \forall\, g \in G \end{array} \tag{2.89}$$

One of the important features of a principal bundle is that the right action is preserved by the transition functions. This is a natural consequence of the fact that transition functions act by left translations on the fibers. Thus we have

Proposition 2.1.6. [18] Let $\pi : P \to M$ be a principal bundle with structure group G. There exists a global right action on $R_G : P \times G \to P$ such that $\forall\, p, q \in P$

1. R_g is *free*, i.e. if $R_g(p) = p$ then $g = 1$
2. R_g is *transitive*, i.e. if $\pi(p) = \pi(q)$ then $\exists\, g \in G$ such that $q = R_g(p)$
3. R_g is *vertical* on the fibers, i.e. $\pi(R_g(p)) = \pi(p) \quad \forall\, p \in P$

In other words 2. and 3. dictates that the fibers of a principal bundle are the orbits of the group G. The local expression of R_g is given in Equation (2.89).

One can see that it isn't possible to define a left action preserved by transition functions [18]. Moreover it can be proved that the existence of global sections is equivalent to a strong constraint over principal bundles. In fact [19]

Proposition 2.1.7. Let $\pi : P \to M$ be a principal bundle with structure group G. Then it admits global sections if and only if it is trivial.

In fact let $s \in \Gamma(P)$ a global section, i.e. a map $s : M \to P$ where M is the base space of the bundle. Each element of the form $R_g s(p)$ where $g \in G$ belongs to the fiber in p. Since the right action is free and transitive, then there exist $p \in M$ and $g \in g$ such that each element $u \in P$ is uniquely written as $R_g s(p)$. Eventually we can define an homeomorphism

$$\begin{array}{rcl} \Phi : P & \to & M \times G \\ R_g s(p) & \mapsto & (p, g) \end{array} \tag{2.90}$$

which assures that $P \simeq M \times G$. Conversely, let us assume that $P \simeq M \times G$. Then let $t : M \times G \to P$ be a trivialization function and let $g \in G$. Then the map

$$\begin{array}{rcl} s_g : M & \to & P \\ p & \mapsto & t(p, g) \end{array} \tag{2.91}$$

is a global section. In this sense the principal bundles are different from vector bundles, on which it is always possible to define at least the global null section.

Again, we have to fix what kind of maps preserve the principal bundle structure.

Definition 2.1.25. Let $\pi : P \to M$ and $\pi' : P' \to M'$ be two principal bundles with structure group respectively G and G'. Let $\theta : G \to G'$ be a Lie group homomorphism. The maps $\Phi : P \to P'$ and $\phi : P \to P'$ form a **principal morphism with respect to** θ if

$$\Phi \circ R_{\theta(g)} = R_g \circ \Phi \qquad \forall\, g \in G \tag{2.92}$$

namely if the diagram in Figure 2.5 is commutative. If $G = G'$ and $\theta_g = 1$ then the pair (Φ, ϕ) is a **principal morphism**.

$$\begin{tikzcd} P \arrow[r, "\Phi"] \arrow[d, "R_g"'] & P' \arrow[d, "R_{\theta(g)}"] \\ P \arrow[r, "\Phi"] & P' \end{tikzcd}$$

Figure 2.5: Principal morphisms with respect to θ_G.

A very useful result is the following [17]

Proposition 2.1.8. Let $\pi : P \to M$ a surjection. Let $\theta : P \times G \to M$ be a free action of the Lie group G on the manifold M such that the orbits of θ coincide with the fiber of $\pi : P \to M$. Then $\pi :\to M$ si a principal bundle with structure group G.

One of the most important examples of principal bundles is

Example 2.1.6. [18] **The frame bundle**
Let M be a smooth manifold. A **frame** at $p \in M$

$$\hat{e}_a(p) = (\hat{e}_1(p), \ldots, \hat{e}_n(p)) \tag{2.93}$$

is an ordered basis of the tangent space T_pM. Let us define

$$L_pM = \{\hat{e}_a(p) = \{\hat{e}_1(p), \ldots, \hat{e}_n(p)\} | \quad \hat{e}_a(p) \quad \text{is a frame at } p \in M\}\} \tag{2.94}$$

and then consider the union of every $L_p(M)$

$$LM = \bigcup_{p \in M} L_pM \tag{2.95}$$

We can define a projection in the natural way

$$\begin{aligned} \pi : LM \quad &\to \quad M \\ \hat{e}_a(p) \quad &\mapsto \quad p \end{aligned} \tag{2.96}$$

and in addition we can define a $GL(n, \mathbb{R})$-right action which acts freely on the elements of LM, that is

$$\begin{aligned} LM \times GL(n, \mathbb{R}) \quad &\to \quad LM \\ \hat{e}_a(p) \times h \quad &\mapsto \quad \hat{e}'_a(p) = (\hat{e}_a h^a{}_1, \ldots, \hat{e}_a h^a{}_n) \end{aligned} \tag{2.97}$$

It can be shown that it is a smoothly varying well defined right action, so that $\pi : LM \to M$ is a principal bundle with structure group G. It is called the **frame bundle**.

LM represents an immediate way to associate a principal bundle to a vector one like the tangent bundle T.

We can also consider the dual of the frame bundle. It is immediately given by the set of all frames of the cotangent bundle T^*, which we can write

$$e^a(p) = (e^1(p), \ldots, e^n(p)) \tag{2.98}$$

The construction of the **coframe bundle** is identycal to that of LM, and one can choose the frames such that

$$e^a(e_b) = \delta^a{}_b \tag{2.99}$$

so that e^a can be interpreted as the inverse of $\hat{e}_a$. Frame and coframe bundles are crucial since they allow us to define the *vielbein* on a manifold, as we will see in Section 3.2.3.

A G-structure can always be interpreted as the result of a *reduction of the structure group of the frame bundle*, to one of its subgroups $G \subset GL(n, \mathbb{R})$. For example, if it's possible to find a global section σ of the frame bundle LE, then it's possible to choose the local frames $\{\hat{e}_a(p)\}$ such that σ has the same form everywhere. This brings to the fact that the only transition function which preserves it is the identity, so that the structure group is the trivial subgroup of $GL(n, \mathbb{R})$ consisting of only the identity element. In that case the manifold is **parallelizable**.

In the general case a useful way to describe a G-structure is in terms of one or more G-invariant tensors (or spinors, as we will see in Chapter 3), which are globally defined and non-degenerate. If for example, is is possible to define a nowhere vanishing, positive definite, symmetric tensor $g \in T^2_0$ (that is a Riemannian metric, as we will see in Section 2.2.1) then we see that the structure group is reduced from $GL(n, \mathbb{R})$ to $O(n, \mathbb{R})$. Let us work out explicitly this example.

Example 2.1.7. The Riemannian structure

Let T be the tangent bundle, and let $g \in T^2_0$ be a symmetric, positive definite and nowhere vanishing tensor. We require that g be globally defined, namely that in each overlap $U_{\alpha\beta}$ we get

$$g^\alpha = g^\beta \tag{2.100}$$

where g^α and g^β are the restriction of the tensor respectively on U_α and U_β. This means that

$$g^\alpha_{ij}\, dx^i_\alpha \otimes dx^j_\alpha = g^\beta_{ij}\, dx^i_\beta \otimes dx^j_\beta \tag{2.101}$$

where $\{x^\alpha\}$ and $\{x^\beta\}$ are the coordinates respectively on U_α and U_β. This implies that

$$g^\beta_{ij} = g^\alpha_{kl} \frac{\partial x^k_\alpha}{\partial x^i_\beta} \frac{\partial x^l_\alpha}{\partial x^j_\beta} \tag{2.102}$$

from which follows that the transition functions $U^l{}_j = \frac{\partial x^l}{\partial x^j}$ have to obey

$$g^\beta = U^T g^\alpha U \tag{2.103}$$

or in other words they have to belong to $O(n, \mathbb{R}) \subset GL(n, \mathbb{R})$. The principal bundle obtained by reducing the set of allowed transition functions is the **orthonormal frame bundle O(M)**. If we take g to be defined over a generic vector bundle E instead of to be defined on the tangent bundle T, we can repeat the same argument to obtain the orthonormal frame bundle over E which is denoted by $O(E)$.

If in addition the manifold is orientable, so that we can find a globally defined volume form $j \in \Lambda^n T^*$, then the structure group is further reduced to $SO(n, \mathbb{R})$. The resulting principal bundle is the **special orthonormal frame bundle SO(E)**.

Definition 2.1.25 allows us to to enlarge the structure group of a G-structure. Let $K(H)$ be the center of the Lie group H

$$K(H) = \{h \in H| \quad hk = kh \quad \forall\, k \in H\} \tag{2.104}$$

Then

Definition 2.1.26. Let $f : G \to H$ be a surjective, covering homomorphism such that $Ker(f) \subseteq K(H)$. A bundle morphism $\hat{f}$ between $\pi : P \to M$ with structure group G adn $\pi' : Q \to N$ with structure group G' is a **lift of P to Q** if it is a principal morphism with respect to f. If f is the universal covering of the Lie group G, then $\hat{f}$ is the **universal lift of G to H**.

Unfortunately it is not always possible to lift a principal bundle, because topological obstructions can occur, as we will see in Chapter 3.

Eventually we introduce a tool which is important in the context of spinors, as we will see in Section 3.1.4. We have seen that the frame bundle can be interpreted as a tool which allows us to associate a principal bundle to a vector bundle. Now we can see a way to go in the opposite direction. There is in fact a way to canonically associate a vector bundle to the principal bundle $\pi : P \to M$ with structure group G, provided that a continuous homomorphism

$$\rho : G \to GL(n, \mathbb{R}) \tag{2.105}$$

is fixed. If F is a vector space, the map ρ allows us to define a free right action over the bundle $P \times F$ in the following way

$$R_g(p, f) = (R_g(p), \rho(g^{-1})f) \qquad \forall\, (p, f) \in P \times F \tag{2.106}$$

and let us denote by

$$O_{(p,f)} = \{R_g(p, f)| \quad g \in G\} \tag{2.107}$$

the *G-orbit* of the point (p, f). Then we can give the following

Definition 2.1.27. Let (P, G) be a principal bundle and let ρ be a linear representation of G over the n-dimensional vector space F, as in Equation (2.105). Next define an equivalence relation $\sim$

$$(p, f) \sim (p', f') \quad \Leftrightarrow \quad (p', f') \in O_{(p,f)} \tag{2.108}$$

Then the quotient

$$P \times_\rho F = {}^{(P \times F)}\!/\!_{\sim} \tag{2.109}$$

is a fiber bundle, called the **associated bundle to P by** ρ.

The projection of the associated bundle $\pi' : P \times_\rho F \to M$ is inherited from the projection $\pi : P \to M$ of the starting bundle P, so that the associated bundle is a bundle over M. If F is a vector space, then the associated bundle is a vector bundle over M.

2.2 Riemannian geometry

So far we have studied the basics concepts in differential geometry, which allow us to define a differentiable structure on a topological manifold, and hence to perform differential calculus on it. We have not yet addressed the question of how to measure the distance between two points on a smooth manifold. This is exactly the question dealt with by the Riemannian geometry.

2.2.1 Riemannian manifold

In this Section we will set up the whole apparatus of the Riemannian geometry. The first principal novelty we will introduce is the concept of *connection*, which allows us to give a sort of generalization of the directional derivative studied in analysis. Next we will start the study of the notion of *metric* which gives us a way to compute distances between points on a manifold. The last fundamental object which we will introduce is the *curvature*, which tells us how much a space is curved, changing significantly the geometrical intuition suggested by the Euclidean geometry.

Connections

Let M be a smooth manifold such that $dim(M) = n$ and let $\pi : E \to M$ be a vector bundle of rank $dim(E) = r$.

Definition 2.2.1. Let the following map

$$\begin{aligned} \nabla : \mathfrak{X}(M) \times \mathfrak{X}(E) &\quad\to\quad \mathfrak{X}(E) \\ (X, V) &\quad\mapsto\quad \nabla_X V \end{aligned} \tag{2.110}$$

obeys the following

1. ∇ is C^∞-linear in the first argument, namely

$$\nabla_{fX+gY} V = f\nabla_X s + g\nabla_Y V \qquad \forall\, X, Y \in \mathfrak{X}(M), \quad \forall\, f, g \in C^\infty(M), \quad \forall\, V \in \mathfrak{X}(E) \tag{2.111}$$

2. ∇ is $\mathbb{R}$-linear in the second argument, namely

$$\nabla_X(aV + bV') = a\nabla_X V + b\nabla_X V' \qquad \forall X \in \mathfrak{X}(M), \quad \forall V, V' \in \mathfrak{X}(E), \quad \forall a, b \in \mathbb{R} \tag{2.112}$$

Then ∇ is a **connection over E**.

The section $\nabla_X V \in \mathfrak{X}(E)$ is the **covariant derivative of V along X**. Finally, if $E \equiv T$, ∇ is a **linear connection**.

The simplest obvious example is the connection on a trivial bundle $E = M \times \mathbb{R}^n$. Let us recall that in this case, the general section of the vector bundle is of the form $V = \sum_j V^j e_j$, where $\{e_j\}_{j\in I_r}$ is the global frame of the trivial bundle. It's straightforward to see that

$$\nabla_X V = \sum_j X[V^j]e_j \tag{2.113}$$

is a connection over E. It is called the **flat connection**.

The connections over vector bundles obey several interesting properties, and in particular it can be shown that each vector bundle admits a connection. A curious fact is that the linear combination of connections is far to be a connection again. It happens to be only in the case of an affine linear combination of connections.

However the most important feature on which we shold focus is the local behaviour of the connections, which is fundamental if one wants to think about connections as a generalization of the directional derivatives. In particular it can be easily shown [17] that the value of $\nabla_X V(p)$ depends only on the direction $X(p)$ of the derivative at p and on the behaviour of the section V restricted to a curve throughout p which has $X(p)$ as tangent vector in p.

We can give a local characterization of the connection by writing

$$\nabla_i e_j = \sum_k^r \Gamma^k{}_{ij} e_k \qquad i \in I_n, \quad j \in I_r \tag{2.114}$$

where we have written ∇_i instead of $\nabla_{\partial/\partial x^i}$, and $\{x^j\}_{j\in I_n}$ are the coordinates induced by the local chart. The functions $\Gamma^k{}_{ij} \in C^\infty(M)$ are the **Christoffel symbols of** ∇ with respect to the local frame and to the local chart chosen. The Christoffel symbols uniquely determine the connection. In particular we can write

$$\nabla_X V = \sum_j^n X^j \nabla_j \left(\sum_k^r V^k e_k \right) = \sum_k^r X(V^k) e_k + \sum_j^n \sum_{kl}^r X^j V^k \Gamma^l{}_{jk} e_l \tag{2.115}$$

Let us notice that for example the flat connection has vanishing Christoffel symbols.

The significance of the the r.h.s. of Equation (2.115) is evident: the first term $\sum_k^r X[V^k]e_k$ indicates the change of the section V along the direction of the derivative X, while the second term $\sum_j^n \sum_{kl}^r X^j V^k \Gamma^l{}_{jl} e_l$ measures the change of the section V due to the fact that local frame $\{e_j\}_{j\in I_r}$ change from point to point.

We can associate to each locally defined connection a new tensor.

Proposition 2.2.1. Let ∇ be a linear connection over the smooth manifold M. The map $\tau : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ such that

$$\tau(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] \tag{2.116}$$

is the **torsion of** ∇. Then τ is a tensor $\tau \in T^1_2$. ∇ is **symmetric** if $\tau = 0$.

The following Proposition impose an important constraint on the Christoffel symbols of a symmetric connection

Proposition 2.2.2. ∇ be symmetric if and only if for each choice of the coordinates we have

$$\Gamma^k{}_{ij} = \Gamma^k{}_{ji} \tag{2.117}$$

Next we can introduce the *parallel transport*. Let $X \in \mathfrak{X}(M)$ and let $V \in \mathfrak{X}(E)$. V is **parallel in the direction of X at p $\in$ M** if

$$\nabla_X V(p) = 0 \tag{2.118}$$

If $\gamma : \mathbb{R} \supset I \to M$ is a smooth curve, then V is **parallel along** γ if

$$\nabla_{\dot{\gamma}} V(\gamma(t)) \quad \forall\, t \in I \tag{2.119}$$

Moreover, the parallel transport condition in Equation (2.157) can be locally rewritten

$$\frac{d}{dt} V^k + \sum_i^n \sum_j^r {\Gamma^k}_{ij} X^i V^k = 0 \qquad \forall\, k \in I_r \tag{2.120}$$

An interesting point is that the theorem of existence and unicity of the solution of a Cauchy's problem allows us to extend the local definition of parallel trasnport, given in Equation (2.120). In particular one can find that the **parallel transport along** γ with respect to ∇ is the map [17]

$$\widetilde{\gamma} : \pi^{-1}(p) \quad \rightarrow \quad \pi^{-1}(q) \tag{2.121}$$

such that $\widetilde{\gamma}(v) = V(1)$. $\widetilde{\gamma}$ is an isomorphism. Moreover if $\gamma : [0,1] \to M$ is a closed curve, then the map $\widetilde{\gamma} \in Aut\big((\pi^{-1}(p))\big)$. The set of such automorphisms is called the **holonomy group of M at p**. We will explore these arguments in detail in the next Section.

An interesting observation is that, given a connection ∇ over a vector bundle E and a smooth curve $\gamma : I \to M$, there always exists a local parallel frame, namely a r-ple of sections $\{e_i \in \mathfrak{X}(M)|_\gamma\}_{i \in I_n}$, each of which is parallel along γ and such that $\{e_i(\gamma(t))\}_{i \in I_n}$ is a basis for $\pi^{-1}(\gamma(t))$. In fact, it's sufficient to choose a point $t_0 \in I$ and a basis $\{e_j\}$ of $\pi^{-1}(t_0)$, and the to use the parallel extension of each element of the basis.

Riemannian metrics

In the present Section we study the consequensces of introducing a tensor such in Example 2.1.7 on a smooth manifold M. It is called a *metric*.

Definition 2.2.2. Let M be a smooth manifold such that $dim(M) = n$. Let g be a positive definite quadratic form $g : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathbb{R}$, such that

1. $g(X,Y) = g(Y,X) \quad \forall\, X,Y \in \mathfrak{X}(M)$
2. $g(X,Y) > 0 \quad \forall\, X,Y \in \mathfrak{X}(M)$

Then g is said to be a **Riemannian metric** over M. A manifold on which a Riemannian metric is defined is a **Riemaniann manifold**. g can be seen also as a $\binom{0}{2}$-tensor. If g is such that 1. holds, while instead of 2. only the condition of non-degeneracy holds

$$2'. \quad g(X,Y) = 0 \quad \forall\, X \in \mathfrak{X}(M) \quad \Rightarrow \quad Y = 0 \tag{2.122}$$

In this case g is a **pseudo-Riemannian metric** over M. A **σ-manifold** is a manifold on which a pseudo-Riemannian metric with signature $\sigma = (r,s)$ is defined. In particular if $\sigma = (1, n-1)$ then we speak of a **Lorentzian manifold**.

It can be proved that on each smooth manifold M a Riemannian metric exists [22]. On the contrary, there may be some topological obstructions which prevent the existence of a pseudo-Riemannian metric on M. For example a compact n-dimensional smooth manifold M admits the existence of a Lorentzian metric if and only if its Euler characteristic vanishes. In fact the presence of a Lorentzian metric means that a globally defined and nowhere vanishing vector field can be chosen (it refers to the time direction). This condition holds if and only if the Euler characteristic vanishes. We will deal with these issues also in Section D.

It's important to define the pullback of the metric along a map $f : M \to N$

$$g'(p) = f^* g(f(p)) \tag{2.123}$$

since it is strictly related with symmetries on a smooth manifold M.

Let (U, φ) be a chart in $p \in M$ and (V, ψ) be a chart in $f(p) = q$. Let be $\varphi(p) = x^i$ and $\psi(q) = y^i$. Then we obtain

$$g'_{ij}(x) = g_{km}(y(x)) \frac{\partial y^k(x)}{\partial x^i} \frac{\partial y^m(x)}{\partial x^j} \tag{2.124}$$

If $f : M \to M$, we can define the **isometry group** as the group of transformations such that

$$g(p) = f^* g(f(p)) \tag{2.125}$$

Isometry transformations preserve the length of the vectors. In particular, if we take as map between manifolds an infinitesimal flow

$$x'^i = x^i + t\xi^i \tag{2.126}$$

where $\xi = \xi^i \frac{\partial}{\partial x^i}$ is the vector which generates the flow, then we can rewrite Equation (2.124) imposing the isometry condition, and obtain

$$g_{ij}(x) = g_{km}(x + t\xi) \frac{\partial (x^k + t\xi^k)}{\partial x^i} \frac{\partial (x^m + t\xi^m)}{\partial x^j} \tag{2.127}$$

from which, expanding, we can obtain the **Killing equation**

$$\xi^k \partial_k g_{ij}(x) + g_{kj}(x) \partial_i \xi^k + g_{ik}(x) \partial_j \xi^k = 0 \tag{2.128}$$

and its solution $\xi = \xi^i \frac{\partial}{\partial x^i}$ is called the **Killing vector**. Equation (2.128) is central in the study of the isometry transformations of a manifold. If we remember the Definition 2.79, we can rewrite the isometry condition in Equation (2.125) as [23]

$$\mathfrak{L}_X g(p) = 0 \tag{2.129}$$

Let us notice that the last consideration is completely independent from the existence of a connection over the manifold M: we are allowed to speak about isometries over a manifold M, once a metric is defined over it. Further conditions can be imposed such that a compatibility relation between metric and connection is established. In fact, take a smooth manifold M endowed with a metric g. We can put the restriction that g be *covariantly constant*, i.e.

$$\nabla_l \, g_{ij} = 0 \tag{2.130}$$

It's easy to find [19] that a covariantly constant metric is a metric which keeps the scalar product between parallel transported vectors constant. Equation (2.130) can be rewritten [19, 23]

$$\partial_l g_{ij} - \Gamma^k{}_{li} g_{kj} - \Gamma^l{}_{kj} g_{ki} = 0 \tag{2.131}$$

The condition in Equation (2.130) is called **metric compatibility**.

The parallel transport can be strictrly related to the metric by defining a **geodesic curve** $\gamma : \mathbb{R} \supseteq I \to M$ by the following

$$\nabla_X X = \alpha(\gamma(t)) X \tag{2.132}$$

where $X \in \mathfrak{X}(M)$ is the tangent vector field to the curve γ. In a chart (U, φ) which establishes the set of local coordinates $\{x^i\}_{i \in I_n}$, the curve is $x(t)$ and the tangent vector takes the form $X^i = \frac{dx^i}{dt}$. Then after some manipulations in order to reabsorb α we obtain in components

$$\frac{d^2 x^i}{dt^2} + \Gamma^i{}_{jk} \frac{dx^j}{dt} \frac{dx^k}{dt} = 0 \tag{2.133}$$

which is the **geodesic Equation**.

As it is well known, the two cornerstones of the Euclidean geometry are that parallel lines never cross and that the sum of the angles of a triangle always udd up to π. These two statements are consequences of the implicit Euclidean hypotesis of the space's flatness. However we know that the space can be curved. In fact let us think about a sphere, which is the most intuitive curved space, and take for example two non coincident longitudinal lines. When viewed from the equator, they appear to be parallel. But if you follow them in either direction, they eventually converge at the poles. Moreover, if you take a triangle over the sphere's surface, it's easy to see that its angles sum up to more that π. This is because the sphere curvature is positive. If the curvature is taken to be negative (as in the case of a saddle), then the angle of a triangle over its surface sum to less that π. Moreover, in an Euclidean space the parallel transport of a vector along two different paths which end at the same point returns the same vector. We will see that this is not true in general, on a manifold with non-zero curvature. Let us try to formalize these concepts.

We can introduce

Definition 2.2.3. Let ∇ be a linear connection, and let M be a smooth manifold. Let $X, Y \in \mathfrak{X}(M)$ and let $p, q \in \mathbb{N}$. The map

$$\begin{aligned} R_{XY} : T^p_q &\rightarrow T^p_q \\ R_{XY} &= \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \end{aligned} \tag{2.134}$$

is the **curvature endomorphism**.

It turns out that R_{XY} is C^∞-linear with respect to all the entries. Then

Definition 2.2.4. Let ∇ be a linear connection, and let M be a smooth manifold. The tensor field $R \in T^1_3$ such that $\forall X, Y, Z \in \mathfrak{X}(M)$

$$R(X, Y, Z) \equiv R_{XY} Z \tag{2.135}$$

is the **curvature tensor**.

If ∇ is the Levi-Civita connection of the Riemannian manifold (M, g) then we can consider also the tensor field $R \in T^0_4$ such that $\forall\, X, Y, Z, T \in \mathfrak{X}(M)$.

$$R(X, Y, Z, T) = g(R_{XY} Z, T) \tag{2.136}$$

A remarkable point is that the curvature tensor of a Riemannian manifold is invariant under local isometries [17].

The most important properties of the curvature tensor are listed in the following

Proposition 2.2.3. Let $R \in T^1_3$ be the curvature tensor of a Levi-Civita connection on the smooth manifold M. If $X, Y, Z, T \in \mathfrak{X}(M)$ then the following properties hold

- R is antysimmetric: $R_{XY} = -R_{YX}$.
- R satisfies the **first Bianchi identity**

$$R_{XY} Z + R_{YZ} X + R_{ZX} Y = 0 \tag{2.137}$$

and if in particular ∇ is the Levi-Civita connection of a Riemannian manifold, then

- $g(R_{XY} Z, T) = g(Z, R_{XY} T)$
- $g(R_{XY} Z, T) = g(R_{ZT} X, Y)$

In a chart (U, φ) on M we can write the local form of the curvature tensor explicitly. Let us fix the coordinate $\{x^i\}_{i \in I_n}$. If we write $R_{\partial_i \partial_j} \partial_k = R^l{}_{ijk} \partial_l$, then we can wirte

$$R^l{}_{ijk} = \frac{\partial \Gamma^l{}_{jk}}{\partial x^i} - \frac{\Gamma^l{}_{ik}}{\partial x^j} + \Gamma^m{}_{jk} \Gamma^k{}_{im} - \Gamma^m{}_{ik} \Gamma^k{}_{jm} \tag{2.138}$$

If we write

$$R_{ijkl} = g_{ml} R^m{}_{ijk} \tag{2.139}$$

then the properties in Proposition 2.2.3 can be rewritten

$$R_{ijkl} = -R_{jikl} \qquad R_{ijkl} + R_{jkil} + R_{kijl} = 0 \qquad R_{ijkl} = -R_{ijlk} \qquad R_{ijkl} = R_{klij} \tag{2.140}$$

Finally we define the **Ricci tensor** $\mathfrak{R}_{ij} \in T^0_2$ obtained by contraction of two indices

$$\mathfrak{R}_{ij} = R^l{}_{ilj} \tag{2.141}$$

and the **Ricci scalar** $\mathfrak{R}$

$$\mathfrak{R} = g^{ij} \mathfrak{R}_{ij} \tag{2.142}$$

2.2.2 Geometry of fiber bundles

In order to build up a gauge field theory it is not sufficient to limit our geometrical description to the Riemannian geometry over a smooth manifold M. In fact so far we don't know how to describe very general tools as gauge connections or field strenghts, which are very common objects in QFT. In this Section we focus over the geometry of pricipal bundles. They are central in the study of gauge theories, since the structure group can immediately be indentified with the *gauge group*.

Gauge connections

The first object that we want to define is a connection over a principal bundle. Since it takes values in the Lie algebra of the structure group it generalizes the connections studied in the previous Section, and it can be used to introduce general holonomies on a manifold.

Let $\pi : P \to M$ be a principal bundle with structure group G. Let $u \in P$ such that $\pi(u) = p$ and let us naturally denote the tangent space in u by T_uP. $\mathfrak{g}$ is the Lie algebra of the Lie group G, and remember that $\mathfrak{g} \simeq T_eG$. Next let $A \in \mathfrak{g}$, and define the following curve through $u \in P$

$$\begin{array}{rcl} \gamma : \mathbb{R} \supset I & \to & P \\ t & \mapsto & R_{\exp(tA)}u \end{array} \tag{2.143}$$

where as it is well known if $A \in \mathfrak{g}$ then $\exp(tA) \in G$. Since the right action over a principal bundle acts locally as in Equation (2.89), we can conclude that $\pi(u) = \pi(R_{\exp(tA)}u) = p$ and in particular that if $f \in C^\infty(P)$ then the one parameter group defined by the map $t \mapsto \exp(tA)$ defines the following vector field

$$A^\sharp f(u) = \frac{d}{dt}\, f(R_{\exp(tA)}u)\big|_{t=0} \qquad \forall\, A \in \mathfrak{g} \tag{2.144}$$

which is the **fundamental vector field**. Notice that $A^\sharp$ is contained in a subspace of T_uP which is parallel to the fiber G, namely it is tangent to the orbit of G through u. In particular, by varying $A \in \mathfrak{g}$ we obtain the basis of a vector space V_uP such that $dim(V_uP) = dim(\mathfrak{g})$. Formally

Definition 2.2.5. The vector space

$$V_uP = \{X \in T_uP | \quad \pi_*(X) = 0\} \equiv ker(\pi_*) \subset T_uP \tag{2.145}$$

is the **vertical subspace**. An element $X \in V_uP$ is a **vertical vector field**. The complement of V_uP is $H_uP \subset T_uP$ in T_uP and is called the **horizontal subspace**. An element $X \in H_uP$ is a **horizontal vector field**.

Definition 2.2.5 is well explained in Figure 2.6. The map $\sharp : \mathfrak{g} \to V_uP$ defines an isomorphism $\mathfrak{g} \cong V_uP$ which is uniquely defined [19, 20]. Moreover the vertical subspace is invariant under the G-action. In fact since for the transitivity property of R_g we have that $\pi \circ R_g = \pi$, then from the properties of the pushforward map we get that $\pi_* \circ R_{g*} = \pi_*$.

The map $\sharp$ preserves the Lie algebra structure, namely

$$[A^\sharp, B^\sharp] = [A, B]^\sharp \qquad \forall\, A, B \in \mathfrak{g}. \tag{2.146}$$

or in other words, the Lie bracket of two vertical vector fields is in turn a vector field. Eventually we arrive at the

Definition 2.2.6. Let (P, G) be a principal bundle. A **(Ehresmann) connection over** P is a unique splitting of $T_uP \quad \forall\, u \in P$ such that

1. $T_uP = V_uP \oplus H_uP$
2. A smooth vector field $X \in P$ can be uniquely decomposed as $X = X^H + X^V$, where $X^H \in H_uP$, while $X^V \in V_uP$.
3. $R_{g*}H_uP = H_{R_g(u)}P \qquad \forall\, u \in P, \quad \forall\, g \in G$

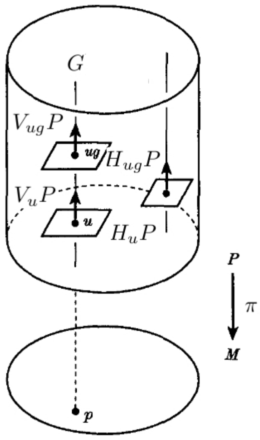


Figure 2.6: V_uP is the vertical subspace, while H_uP is the horizontal subspace.

Properties 1. and 2. can be resumed by saying that $TP = VP \oplus HP$ where VP and HP are respectively the collections of all the V_uP and H_uP, by smoothly varying $u \in P$. They are also called *distributions*. As shown in Figure 2.6 and as 3. in the last Definition dictates, the horizontal subspace obtained from the G-action over H_uP is again a horizontal subspace, $H_{R_g(u)}$. In other words the horizontal tangent bundle HP is G-invariant. Recall that instead, in the vertical case, each vertical subspace V_uP is G-invariant.

The following step is to reconnect Definition 2.2.6 with Definition 2.2.1 already seen in Section 2.2.1. In fact, according to those Definitions, we expect that the connection is representable through a one-form. This is easily achieved by introducing the following [19]

Definition 2.2.7. Let $\omega \in \mathfrak{g} \otimes T^*P$ be a Lie algebra valued one-form over P such that

1. $\omega(A^\sharp) = A \qquad \forall\, A \in \mathfrak{g}, \quad A^\sharp \in V_uP$
2. $R_g^*\omega = Ad_{g^{-1}}\omega \qquad \forall\, g \in G$

ω is the **connection one-form**.

As we expect Definition 2.2.7 is equivalent to Definition 2.2.6. This is easily proven, by noticing that we can redefine the horizontal subspace H_uP as

$$H_uP \equiv \{X \in T_uP| \quad \omega(X) = 0\} = ker(\omega) \tag{2.147}$$

Since from 3. in Definition 2.2.6 $\forall\, X \in H_uP$ then $R_{g*}X \in T_{R_g(u)}P$, from the Definition of pullback in Equation (2.48)

$$\omega(R_{g*}X) = R_g^*\omega(X) = Ad_{g^{-1}}\omega(X) = g^{-1}\omega(X)g = 0 \tag{2.148}$$

because $\omega(X) = 0 \quad \forall\, X \in H_uP$. It follows that $R_{g*}X \in H_{R_g(u)}P$. In this way we have proven that a connection as defined in Definition 2.2.7 implies the existence of an Ehresmann connection. Now we have to prove the inverse. Consider a given Ehresmann connection, and a $\mathfrak{g}$-valued one-form such that 1. and 2. in Definition 2.2.7 hold. If $X \in H_uP$ then 3. in Definition 2.2.7 holds trivially. If $A^\sharp \in V_uP$, then

$$R_g^*\omega(A_u^\sharp) = \omega(R_{g*}A_u^\sharp) = \omega\left((Ad_{g^{-1}}A)^\sharp_{R_g(u)}\right) = (Ad_{g^{-1}}A)_{R_g(u)} = (Ad_{g^{-1}}\omega(A^\sharp))_{R_g(u)} \tag{2.149}$$

which implies 3. Notice that we have used that the following relation holds

$$R_{g*}A_u^\sharp = \frac{d}{dt}R_g(R_{\exp(tA)}(u))\bigg|_{t=0} = \frac{d}{dt}\left(R_g(u)Ad_{g^{-1}}(\exp(tA))\right)\bigg|_{t=0} =$$

$$= \frac{d}{dt}\left(R_g(u)\exp(tAd_{g^{-1}}A)\right)\bigg|_{t=0} = \left(Ad_{g^{-1}}A\right)_{R_g(u)} \tag{2.150}$$

It is convenient to pullback the connection ω in order to obtain a connection defined over the base space manifold M of the principal bundle $\pi : P \to M$ with structure group G. Let $\{U_\alpha\}_{\alpha\in I}$ be a covering of M. Let us define a set of local sections $\sigma_\alpha : U_\alpha \to P$. We will call the set $\{\sigma_\alpha\}$ the *canonical trivialization* of the principal bundle if

$$\sigma_\alpha(p) = (p, e) \qquad \forall\, p \in M \tag{2.151}$$

Then each point $u \in P$ such that $\pi(u) = p$ can be reached by using the transitive action of R_g

$$R_g(\sigma_\alpha(p)) = (p, eg) = (p, g) \qquad \forall\, p \in M, \quad \forall\, g \in G \tag{2.152}$$

Then we can define the *local form of the connection* or **gauge connection**

$$\mathcal{A}_\alpha = \sigma_\alpha^*(\omega) \quad \in \mathfrak{g} \otimes \Lambda^1 T^* U_\alpha \tag{2.153}$$

If the whole set of couples $\{(U_\alpha, \mathcal{A}_\alpha)\}_{\alpha\in I}$ is given, then it is possible to reconstruct the Lie algebra valued one-form $\omega \in \mathfrak{g} \otimes T^*P$ [19].

A remarkable point is that the Lie algebra valued one-forms $\mathcal{A}_\alpha$ cannot be defined globally, since a principal bundle cannot have global sections (unless it is trivial) as we have seen in Section 2.1.5. Therefore, in order to make ω defined globally, we have to impose some constraints over the transformation of $\mathcal{A}_\alpha$ on the overlappings $U_{\alpha\beta}$. Such constraint is the defining property of a connection, and it is the analogous of Equation (**??**)

$$A_\beta = g_{\alpha\beta}^{-1} \circ A_\alpha \circ g_{\alpha\beta} + g_{\alpha\beta}^{-1} \circ dg_{\alpha\beta} \tag{2.154}$$

where $g_{\alpha\beta}$ are the transition functions from U_α and U_β. Again we stress on the fact that ω carries the global informations of the principal bundle, as well as the whole set $\{(U_\alpha, \mathcal{A}_\alpha)\}_{\alpha\in I}$ satisfying the compatibility condition in Equation (2.154).

Holonomy

At this point we can extend the definition of parallel transport given in Equation (2.157) by introducing the following

Definition 2.2.8. Let $\pi : P \to M$ a principal bundle with structure group G and let $\gamma : [0,1] \to M$ be a curve over M. The curve $\widetilde{\gamma} : [0,1] \to P$ is a **horizontal lift** of γ if

- $\pi \circ \widetilde{\gamma} = \gamma$
- $\frac{d}{dt}\widetilde{\gamma}(t) \in H_{\widetilde{\gamma}(t)}P$

Let $\widetilde{X}$ be a vector tangent to $\widetilde{\gamma}$. If $\omega \in \mathfrak{g} \otimes T^*P$ is the connection one-form, then $\omega(\widetilde{X}) = 0$ by definition. An horizontal lift always exists, up to the initial condition. In particular [19]

Proposition 2.2.4. Let $\gamma : [0,1] \to M$ be a smooth curve and let $u_0 = \pi^{-1}(\gamma(0))$. Then there exists a unique horizontal lift $\widetilde{\gamma}$ in P such that $\widetilde{\gamma}(0) = u_0$.

It's interesting to notice the following result

Lemma 2.2.1. *Let γ be a smooth curve over the smooth manifold M. Let $\widetilde{\gamma}$, $\widetilde{\gamma}'$ be two horizontal lifts of γ, such that $\widetilde{\gamma}'(0) = R_g(\widetilde{\gamma}(0))$. Then $\widetilde{\gamma}'(t) = R_g(\widetilde{\gamma}(t)) \quad \forall\, t \in [0,1]$.*

In fact the map

$$\begin{array}{rcl} \widetilde{\gamma}_g : [0,1] & \to & P \\ t & \mapsto & R_g(\widetilde{\gamma})(t) \end{array} \tag{2.155}$$

is also a horizontal lift of γ, since the horizontal subspace is invariant under R_g: $R_g H_u = H_{R_g(u)}$. Furthermore, Proposition 2.2.4 tells us that it is the unique horizontal lift through $R_g(\widetilde{\gamma})(0)$.

We can extend the concept of parallel transport. In fact let $\gamma : [0,1] \to M$ and let $\widetilde{\gamma}$ and consider the point $u_0 \in \pi^{-1}(\widetilde{\gamma}(0))$. Let $(U_\alpha, \varphi_\alpha)$ be the chart which contains $\gamma(t)$. Proposition 2.2.4 tells us that there exists a unique horizontal lift $\widetilde{\gamma}$ through u_0, and thus a unique point $u_1 = \widetilde{\gamma}(1)$, which is the **parallel transport** of u_0 along the curve $\widetilde{\gamma}$. We can define a map

$$\begin{aligned} \Gamma(\widetilde{\gamma}) : \pi^{-1}(\widetilde{\gamma}(0)) &\rightarrow \pi^{-1}(\widetilde{\gamma}(1)) \\ u_0 &\mapsto u_1 \end{aligned} \tag{2.156}$$

such that

$$u_1 = \sigma_\alpha(\gamma(1)) \mathcal{P} \exp\left\{ - \int_0^1 \mathcal{A}_{\alpha i} \frac{dx^i(\gamma(t))}{dt} dt \right\} \tag{2.157}$$

where $\mathcal{P}$ indicates that the integral is path-ordered.

Lemma 2.2.1 allows us to show that

$$\Gamma(\widetilde{\gamma}) \circ R_g = R_g \circ \Gamma(\widetilde{\gamma}) \tag{2.158}$$

In fact let $u_0 \in P$. Then $R_g \circ \Gamma(\widetilde{\gamma})(u_0) = R_g(u_1)$ and $\Gamma(\widetilde{\gamma}) \circ R_g(u_0) = \Gamma(\widetilde{\gamma})(R_g(u_0))$. The curve $R_g(\widetilde{\gamma})(t)$ is a horizontal lift through $R_g(u_0)$ and $R_g(u_1)$. Since the horizontal lift through $R_g(u_0)$ is unique, from Proposition 2.2.4 we have that $R_g(u_1) = \Gamma(\widetilde{\gamma})(R_g(u_0))$, and then $R_g \circ \Gamma(\widetilde{\gamma})(U_0) = \Gamma(\widetilde{\gamma}) \circ R_g(u_0) \quad \forall\, u_0 \in \pi^{-1}(\gamma(0))$, from which follows the initial statement in Equation (2.158).

Next consider the parallel transport along a closed curve. Let $\gamma, \lambda : [0,1] \rightarrow M$, such that $\gamma(0) = \lambda(0) = p$ and $\gamma(1) = \lambda(1) = q$ be two curves. Let $\widetilde{\gamma}$, $\widetilde{\lambda}$ be two horizontal lifts of γ and λ, such that $\widetilde{\gamma} = \widetilde{\lambda} = u_0$. It turns out that $\widetilde{\gamma}(1)$ is not necessarily equal to $\widetilde{\lambda}(1)$. Much more, if we consider a loop α, automatically we have defined a transformation

$$\tau_\alpha : \pi^{-1}(p) \quad \rightarrow \quad \pi^{-1}(p) \tag{2.159}$$

which is compatible with R_g, that is

$$\tau_\alpha(R_g(u)) = R_g(\tau_\alpha(u)) \tag{2.160}$$

as an obvious consequence of Equation (2.158). Let us notice the fundamental point that τ_γ depends not only on the loop γ, but also on the connection, as it is evident from Equation (2.157).

Let $p \in M$ be such that $\pi(u) = p$, and consider the set of loops at p, namely

$$\mathsf{C}_p(M) = \{\alpha : [0,1] \rightarrow M | \quad \alpha(0) = \alpha(1) = p\} \tag{2.161}$$

Then the set

$$\Phi_u(M) = \{g_\alpha \in G| \quad \tau_\alpha(u) = R_{g_\alpha}(u), \quad \alpha \in \mathsf{C}_p(M)\} \subseteq G \tag{2.162}$$

is a subset of the structure group G, and is called the **holonomy group at u**. The family

$$\Phi(M) = \bigcup_{u \in P} \Phi_u \tag{2.163}$$

is the **holonomy group**. The group properties can be derived by noticing that two curves $\gamma, \lambda : [0,1] \rightarrow M$ can be "composed" into $\gamma * \lambda : [0,1] \rightarrow M$ if $\gamma(1) = \lambda(0)$. In fact we can write

$$\gamma * \lambda(t) = \begin{cases} \gamma(2t) & \text{if} \quad 0 \leq t \leq \frac{1}{2} \\ \lambda(2t-1) & \text{if} \quad \frac{1}{2} \leq t \leq 1 \end{cases} \tag{2.164}$$

and obviously $\gamma^{-1}(t) = \gamma(1-t)$. Then we also get

$$\Gamma(\widetilde{\gamma}^{-1}) = \Gamma^{-1}(\widetilde{\gamma}) \qquad \Gamma(\widetilde{\gamma * \lambda}) = \Gamma(\widetilde{\gamma})\Gamma(\widetilde{\lambda}) \tag{2.165}$$

In particular let us notice that two loops α, β at the same base point $p \in M$ can always be composed. Moreover, let $\alpha, \beta, \gamma = \alpha * \beta$ be three loops at $p \in M$. Then we have $\tau_\gamma = \tau_\beta \circ \tau_\alpha$, and thus

$$\tau_\gamma(u) = \tau_\beta \circ \tau_\alpha(u) = \tau_\beta \circ R_{g_\alpha}(u) = R_{g_\alpha} \circ \tau_\beta(u) = R_{g_\alpha} \circ R_{g_\beta}(u) = R_{g_\beta g_\alpha}(u) \tag{2.166}$$

namely $g_\gamma = g_\beta \circ g_\alpha$. Moreover the constant loop $c : [0,1] \mapsto p$ defines the identity transformation $\tau_c : u \mapsto u$. The inverse loop γ^{-1} induces the inverse transformation $\tau_{\gamma^{-1}} = \tau_\gamma^{-1}$, and then $g_{\gamma^{-1}} = g_\gamma^{-1}$.

Field strenghts

After having introduced a gauge connection we have to study what a field strenght is. With this purpose in mind let define the horizontal projection $h : TP \to HP$ over the horizontal distribution, which is a family of maps

$$h_u(X) = \begin{cases} X & \text{if} \quad X \in H_uP \\ 0 & \text{if} \quad X \in V_uP \end{cases} \tag{2.167}$$

The obvious relation

$$h \circ R_{g*} = R_{g*} \circ h \tag{2.168}$$

holds. Moreover we can define $h^* : T^*P \to H^*P$ such that if $\phi \in \Lambda^r T^*P$

$$h^*\phi(X_1, \dots, X_r) = \phi(h(X_1), \dots, h(X_2)) \quad \forall\, X_1, \dots, X_r \in TP \tag{2.169}$$

Let us notice that h^* is the dual map of h, but it is not the pushforward of any smooth map $h : P \to P$ and in particular it does not commute with the exterior differential d, as a pullback map does. A form $\phi \in \Lambda T^*$ such that $h^*\phi = \phi$ is an **horizontal form**. Finally

Definition 2.2.9. Let $\pi : P \to M$ be a principal bundle with structure group G, let $HP \subset TP$ be an Ehresmann connection, and let $\omega \in \mathfrak{g} \otimes \Lambda^1 T^*P$ be a connection one-form. Then we define the **curvature 2-form** as

$$\Omega = h^* d\omega \in \mathfrak{g} \otimes \Lambda^2 T^*P \tag{2.170}$$

By Definition and by Lemma 2.1.1 we get

$$\begin{aligned} &\Omega(X,Y) = h^*d\omega(X,Y) = d\omega(hX, hY) = \\ &= i_{hX}\omega(hY) - i_{hY}\omega(hX) - \omega([hX, hY]) = -\omega([hX, hY]) \quad \forall\, X, Y \in TP \end{aligned} \tag{2.171}$$

since $\omega(hX) = \omega(hY) = 0$, for $hX, hY \in HP$. It's evident that $\Omega(X, \cdot) = 0 \quad \forall\, X \in VP$, because in that case $\Omega(X, \cdot) = h^*d\omega(X, \cdot) = d\omega(hX, h\cdot) = d\omega(0, h\cdot) = 0$. Instead it's really interesting to notice that $\Omega(X,Y) = 0$ if and only if $[hX, hY] \in HP$. In other words the curvature two-form Ω measures the failure of the integrability of the horizontal distribution $HP \subset TP$.

The curvature 2-form satisfies

- The **Cartan structure Equation**

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega] \tag{2.172}$$

  where $[\omega, \omega] = [T_a, T_b] \otimes \omega^a \wedge \omega^b$

- The **Bianchi identity**

$$h^* d\Omega = 0 \tag{2.173}$$

- The transformation rule

$$R_g^*\Omega = Ad_{g^{-1}}\Omega \qquad \forall\, g \in G \tag{2.174}$$

It's really useful to pullback also the curvature Ω on the base space manifold M. Let $\{U_\alpha\}_{\alpha \in I}$ be a covering of M and let us consider the canonical trivialization $\sigma_\alpha : U_\alpha \to P$ of the principal bundle $\pi : P \to M$ with structure group G. Then

$$\mathcal{F} = \sigma_\alpha^*(\Omega) \quad \in \mathfrak{g} \otimes \Lambda^2 T|_{U_\alpha}^* \tag{2.175}$$

is the **local curvature**, and can be expressed as

$$\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} \tag{2.176}$$

Moreover we have an analogous of the Cartan structure Equation

$$\mathcal{F}(X,Y) = d\mathcal{A}(X,Y) + [\mathcal{A}(X), \mathcal{A}(Y)] \qquad \forall\, X, Y \in \mathfrak{X}(M) \tag{2.177}$$

Finally, using Equation (2.154) it's straightforward to prove that

$$\mathcal{F}_\beta = g_{\alpha\beta}^{-1} \circ \mathcal{F}_\alpha \circ g_{\alpha\beta} \tag{2.178}$$

where F_α is defined on U_α, $\mathcal{F}_\beta$ is defined on U_β ($U_{\alpha\beta} \neq \{\varnothing\}$) and $g_{\alpha\beta}$ are the transition functions from U_α to U_β.

2.3 de-Rham cohomology

The cohomology group is a natural and intrinsic object which can be constructed over a smooth manifold M. It arises from the study of the exterior algebra, and it encodes the topological non-triviality of M. Many topological invariants, such as Chern classes, are elements of the de-Rham cohomology group.

The differential operator d induces the *de-Rham complex*

$$0 \overset{d}{\to} \Lambda^0 T^* \overset{d}{\to} \Lambda^1 T^* \overset{d}{\to} \dots \overset{d}{\to} \Lambda^{n-1} T^* \overset{d}{\to} \Lambda^n T^* \overset{d}{\to} 0 \tag{2.179}$$

And finally we can define the **p-th de Rham cohomology group over** M

$$H^p_d(M) = {Z^p(T^*)}\big/{d\Lambda^{p-1}T^*} \tag{2.180}$$

where $H^0_d(M) = Z^0(T^*)$ and $Z^0(T^*)$ is the space of constant functions over connected components of M. The space

$$H^*(M) = \bigoplus_{p=0}^{n} H^p_d(M) \tag{2.181}$$

is a ring with the wedge product $\wedge : H^* \to H^*$ induced by $\wedge : H^p_d(M) \times H^q_d(M) \to H^{p+q}_d(M)$, $\forall\, p,q$ such that $p+q \leq n$.

Next let us state the fundamental

Lemma 2.3.1. ***Poincarè Lemma***
Let M be a smooth manifold and let U be a contractible open set $U \subset M$. Then $\forall\, \omega \in \Lambda^p T|^*_U$ such that $d\omega = 0$ there exists a $\tau \in \Lambda^{p-1}T|^*_U$ such that $\omega = d\tau$.

In other words each closed form is locally exact, but the converse is in general not true.

Example 2.3.1. The circle bundle
The circle bundle is a principal bundle with structure group $U(1) \sim S^1$. Given a covering $\{U_{\alpha\beta}\}_{\alpha,\beta\in I}$ of the base space M the circle bundle can be defined as a set of transition functions

$$g_{\alpha\beta} : U_{\alpha\beta} \quad \to \quad S^1 \tag{2.182}$$

such that $g_{\alpha\alpha} = 1$, $g_{\alpha\beta} = g^{-1}_{\beta\alpha}$ and the cocycle condition is satisfied in each triple overlap $U_{\alpha\beta\gamma}$

$$g_{\alpha\beta} \circ g_{\beta\gamma} \circ g_{\gamma\alpha} = 1 \tag{2.183}$$

One of the most interesting point is that a circle bundle can be associated to each closed two-form $\frac{\mathcal{F}}{2\pi} \in H^2(M,\mathbb{Z})$ on the base space. In fact by using the Poincarè Lemma 2.3.1 we can find a descent chain of relations

$$\mathcal{F} = d\mathcal{A}_\alpha \qquad \mathcal{A}_\alpha \in \Lambda^1 T|^*_{U_\alpha} \tag{2.184}$$
$$\mathcal{A}_\alpha - \mathcal{A}_\beta = d\Lambda_{\alpha\beta} \qquad \Lambda_{\alpha\beta} \in C^\infty(U_{\alpha\beta}) \tag{2.185}$$
$$\Lambda_{\alpha\beta} + \Lambda_{\beta\gamma} + \Lambda_{\gamma\alpha} = d_{\alpha\beta\gamma} \qquad d_{\alpha\beta\gamma} \in 2\pi\mathbb{Z} \tag{2.186}$$

where the last relation is guaranteed from the fact that $\frac{\mathcal{F}}{2\pi} \in H^2(M,\mathbb{Z})$ [50]. Equation (2.186) permits us to exponentiate the transition functions

$$g_{\alpha\beta} = e^{i\Lambda_{\alpha\beta}} \tag{2.187}$$

so that Equation (2.185) takes the nice form

$$i\mathcal{A}_\alpha - i\mathcal{A}_\beta = g^{-1}_{\alpha\beta} \circ dg_{\alpha\beta} \tag{2.188}$$

in which we recognize the transformation rule of the gauge connection for a $U(1)$-bundle. This means that the set of local connections $\{\mathcal{A}_\alpha\}_{\alpha\in I}$ defines a connection-one-form on the bundle, and that $\mathcal{F}$ is the field strenght of the circle bundle. The choices of inequivalent connections with the same curvature are parametrized by the coset

$$H^1(M,\mathbb{R})\big/H^1(M,\mathbb{Z}) \tag{2.189}$$

An interesting generalization of this Example is given in Section 5.2.1.

The Poincarè lemma leads us to investigate the presence of a duality, relating $\Lambda^p T^*$ and $\Lambda^{n-p}T^*$.

Let $\omega \in \Lambda^p T^*$, let $\eta \in \Lambda^{n-p}T^*$, and let M be a smooth manifold such that $dim(M) = n$. If we note that $\omega \wedge \eta$ is a volume form, then we can define a bilinear inner product

$$\begin{aligned} \langle , \rangle : \Lambda^p T^* \times \Lambda^{n-p} T^* \to \mathbb{R} \\ \langle \omega, \eta \rangle = \textstyle\int_M \omega \wedge \eta \end{aligned} \tag{2.190}$$

Since $\langle , \rangle$ is non-degenerate, it defines the **Poincaré duality** between $\Lambda^p T^* \cong \Lambda^{n-p}T^*$. It can be naturally extended to cohomology groups: $H^p_d(M) \simeq H^{n-p}_d(M)$.

We can write explicitly the isomorphism given by the Poincarè duality $\Lambda^p T^*$ and $\Lambda^{n-p}T^*$. Surprisingly it involves the Riemannian metric, in fact

Definition 2.3.1. Let $*$ be the map

$$* : \Lambda^p T^* \to \Lambda^{n-p} T^* \tag{2.191}$$

such that on basis elements

$$*(dx^{j_1} \wedge \cdots \wedge dx^{j_p}) = \frac{1}{(n-p)!}\sqrt{g}g^{j_1 k_1} \dots g^{j_p k_p} \epsilon_{k_1 \dots k_p k_{p+1} \dots k_n} dx^{k_{p+1}} \wedge \cdots \wedge dx^{k_n} \tag{2.192}$$

The following relation

$$* * \omega = (-1)^{p(n-p)} \omega \tag{2.193}$$

holds, where $\omega \in \Lambda^p T^*$. $*$ is the **Hodge star**.

An inner product over the space of real forms is automatically defined

$$\begin{aligned} (,) : \Lambda^p T^* \times \Lambda^p T^* \to \mathbb{R} \\ \omega \times \xi \to \textstyle\int_M \omega \wedge * \xi \end{aligned} \tag{2.194}$$

It's straightforward to see that $(\omega, \xi) = (\xi, \omega)$ and that if $\omega, \xi \in \Lambda^p T^*$

$$(\omega, \xi) = \frac{1}{p!} \int \omega_{j_1 \dots j_p} \xi^{j_1 \dots j_p} \sqrt{g} dx^1 \wedge \cdots \wedge dx^n \tag{2.195}$$

$(,)$ gives us the chance to define the adjoint of the d operator:

$$d^\dagger : \Lambda^p T^* \to \Lambda^{p-1} T^* \tag{2.196}$$

such that $\forall \omega \in \Lambda^p T^*, \forall \xi \in \Lambda^{p-1} T^*$

$$(\omega, d\xi) = (d^\dagger \omega, \xi) \tag{2.197}$$

For boundaryless M ($\partial M = \{\varnothing\}$) we obtain that $d^\dagger = (-1)^{p(n-p+1)} * d *$.

A generalization of the concept of the laplacian in real analysis is given simply as follows

Definition 2.3.2. Let Δ be the map

$$\begin{aligned} \Delta : \Lambda^p T^* \quad \longrightarrow \quad \Lambda^p T^* \\ \Delta = d d^\dagger + d^\dagger d \end{aligned} \tag{2.198}$$

We will call this operator **laplacian**.

And naturally

Definition 2.3.3. Let $\omega \in \Lambda^p T^*$. If $\Delta\omega = 0$, then ω is said to be a **harmonic form**, and we will denote it by $\omega \in \Upsilon^p(M)$.

It's easy to see that $\Delta\omega = 0$ is equivalent to the condition that ω be *closed* $d\omega = 0$ and *coclosed* $d^\dagger\omega$ at the same time [29].

A generic r-form can always be decomposed in a closed form, plus a coclosed form, plus an harmonic one. In fact

Theorem 3. ***Hodge's theorem*** *[29, 19]*
Let (M, g) be a compact, boundaryless Riemannian manifold. Then $\Lambda^p T^*$ admits a unique orthogonal decomposition

$$\Lambda^r T^* = d\Lambda^{r-1}T^* \oplus d^\dagger \Lambda^{r+1}T^* \oplus \Upsilon^r(M) \tag{2.199}$$

namely $\omega \in \Lambda^p T^*$ is uniquely expressed as

$$\omega = d\alpha + d^\dagger \beta + \gamma \tag{2.200}$$

where $\alpha \in \Lambda^{p-1}T^*$, $\beta \in \Lambda^{p+1}T^*$, $\gamma \in \Upsilon^p(M)$.

The last Theorem allows us to define a couple of topological invariants. In fact if we take $\omega \in H^p(M)$ and $\beta \in \Lambda^{p+1}T^*$, thanks to Theorem 3, we can write

$$0 = (d\omega, \beta) = (dd^\dagger\beta, \beta) = (d^\dagger\beta, d^\dagger\beta) \tag{2.201}$$

and then $d^\dagger\beta = 0$, or in other words $\omega = d\alpha + \gamma$, where $\alpha \in \Lambda^{p-1}T^*$ and $\gamma \in \Upsilon^p(M)$. Repeating the same reasoning after having chosen ω to be coclosed $d^\dagger = 0$, we obtain $\omega = d^\dagger\beta + \gamma$, where $\beta \in \Lambda^{p+1}T^*$ and $\gamma \in \Upsilon^p(M)$. In addition, if ω is harmonic, then we obtain that $\omega = \gamma$. This implies that it is the harmonic component of a form which determines its cohomology class and as a consequence there exists an isomorphism

$$\Upsilon^p(M) \simeq H^p(M) \tag{2.202}$$

Then we can define the **Betti numbers**

$$b_p = dim(H^p(M)) \tag{2.203}$$

which represents the number of linearly independent harmonic p-forms. Thanks to Poincaré duality we can write

$$b_p = b_{n-p} \tag{2.204}$$

The Betti numbers are topological invariants.

Another topological invariant is the **Euler characteristic** defined as

$$\chi(M) = \sum_{p=0}^{n}(-1)^p b_p = \sum_{p=0}^{n}(-1)^p dim(Ker(\Delta_p)) \tag{2.205}$$

If we take a manifold such as $M = M_1 \times M_2$ then the cohomology can be decomposed as suggested by the **Künnet formula**

$$H^k(M) = \bigoplus_{p+q=k} [H^p(M_1) \otimes H^q(M_2)] \tag{2.206}$$

Hence the Betti numbers are related by

$$b^k(M) = \sum_{p+q=k} b^p(M_1) b^q(M_2) \tag{2.207}$$

and the Euler characteristic becomes

$$\chi(M) = \chi(M_1)\chi(M_2) \tag{2.208}$$

3 Spinors

In this Chapter we will try to briefly build the theory of spinors on curved manifolds, expressing it in the most useful way for the development of the Generalized Complex Geometry (GCG) in Chapter 5 and to understand its role in the supersymmetric string theories. Furthermore, we will study some characteristic classes which feature the topology of a manifold M in terms of e connections.

The spinors have different transformation rules with respect to tensor fields. In fact we know that under a coordinate change the components of a real vector field X on the real smooth manifold M obey the following rule

$$X^i \to X'^i = \frac{\partial x'^i}{\partial x^j} X^j \equiv U^i{}_j X^j \tag{3.1}$$

where the matrix $U^i{}_j = \frac{\partial x'^i}{\partial x^j} \in GL(n, \mathbb{R})$ as we have seen in Example 2.1.3. Since $SO(n, \mathbb{R}) \subset GL(n, \mathbb{R})$, it is obvious that properly choosing the U matrices, we can obtain the representations of $SO(n, \mathbb{R})$ as restrictions of the representations of $GL(n, \mathbb{R})$. Next, using the G-structures technique developed in Section 2.1.5 we can identify $SO(n, \mathbb{R})$ with the structure group of a G-structure and eventually build up a theory with bosonic fields coupled to g_{ij}.

A realistic field theory must include anticommuting spinor fields describing objects with half-integer spin and also the covariance must be preserved. However it is well known that $SO(n, \mathbb{R})$ doesn't allow for the existence of objects with half-integer spin. In order to obtain such kind of objects we need to use another technique which we mentioned in Section 2.1.5 - the *lift of the structure group* - whose peculiarity is to allow for an enlargement of the structure group. We will explore this in detail.

In addition, it is particularly important to study the spinors since realistic String theories are the supersymmetric ones. Supersymmetry is a of global symmetry which mixes bosonic and fermionic fields of a theory. Moreover, the compactification of six of the dimensions which arise in Superstring theory, together with the requirement that four-dimensional results are realistic, brings us to some important constraints on the spinor which can be constructed on the compactification space. We will explore this in Section **??**.

3.1 Clifford algebras

The idea that led to the study of Clifford algebras is the attempt to extend to vectors the multiplication $\cdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ operation which is well defined for the real numbers. Its main properties are distributivity, associativity and commutativity. Unfortunately there is no chance of succesfully mantain the request of commutativity in dimension $n \geq 3$ so that we have to resort to a generalization of it.

3.1.1 Basic notions

Let V be a vector space over the field $\mathbb{K}$ (we will consider only $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) such that $dim(V) = n$. Let $\eta : V \times V \to \mathbb{K}$ be an inner product with signature $\sigma = (r, s)$ $(r + s = n)$ defining a quadratic form $Q : V \to \mathbb{K}$

by $Q(v) = \eta(v, v) \quad \forall\, v \in V$. It is well known that for a quadratic form Q the *polarization* relation

$$Q(v+w) - Q(v) - Q(w) = 2\eta(v, w) \qquad \forall\, v, w \in V \tag{3.2}$$

holds. Let us begin with the

Definition 3.1.1. Let $J(Q)$ be the bilateral ideal generated in $T^\bullet(V)$ by the elements of the form $v \otimes v - Q(v)1_{\mathbb{K}}$, where $v \in V \hookrightarrow T^\bullet(V)$. In other words

$$J(Q) = \{x \otimes (v \otimes v - Q(v)1_{\mathbb{K}}) \otimes y | \quad x, y \in T^\bullet(V),\, v \in V\} \tag{3.3}$$

The quotient

$$C(V) = {T^\bullet(V)}\big/{J(Q)} \tag{3.4}$$

is the **Clifford algebra on V generated by Q**.

Let us notice that when we write $C(V)$, we leave understood the data Q. On the other hand once we have V and Q the Clifford algebra $C(V)$ is entirely defined. We can define a projection $\pi_Q : T^\bullet(V) \to C(V)$ such that $\forall\, x \in T^\bullet(V)$ it acts as $x + J(Q) \mapsto x$. The map

$$\pi_Q \circ i : T^k(V) \overset{i}{\hookrightarrow} T^\bullet(V) \to C(V) \tag{3.5}$$

is an injection only if $k \in \{0, 1\}$ since for $k \geq 2$ there are surely elements in $T^k(V)$ which are identified through elements in $J(Q)$. In this sense we can see V ($k = 1$) as sitting inside $C(V)$. For this reason we can write the images of a scalar λ or of a vector $v \in V$ in the Clifford algebra $C(V)$ simply as λ and v respectively. If $\eta = 1_V$, then the Clifford algebra simply becomes the exterior algebra $\Lambda(V)$, as we adverted in Section 2.1.3. From now on, in this Section we will write $1_{\mathbb{K}} \equiv 1$.

The tensor product $\otimes$ defined on $T^\bullet(V)$ induces the *Clifford product* on the Clifford algebra $C(V)$

$$T^\bullet(V) \ni \quad v \otimes w \quad \overset{\pi_Q \circ i}{\longmapsto} \quad vw \quad \in C(V) \tag{3.6}$$

Then for example for example that the image of the element $v \otimes v - Q(v)1 \in J(Q)$ is $[v^2 - Q(v)]$. Since by definition $[v^2 - Q(v)] = [0]$, then in the Clifford algebra we can write

$$v^2 = Q(v) \qquad \forall\, v \in C(V) \tag{3.7}$$

The interesting point is that in general, the Clifford product of two vectors doesn't return a degree-two object, as it seems to be intuitive since we are tensoring two vectors, but it operates a splitting (as in Equation due to the quotient which defines the Clifford algebra.

The Clifford algebra $C(V)$ is an associative unital $\mathbb{K}$ algebra with unity 1. The relation

$$vw + wv = 2\eta(v, w) \qquad \forall\, v, w \in V \subset C(V) \tag{3.8}$$

holds. Let us notice that it is required only the knowledge of Q, since η is uniquely defined from Equation (3.2). Again we see that the Clifford algebra $C(V)$ is uniquely determined by the data V and Q.

It's easy to show that if $\sigma = (0, 1)$ then the Clifford algebra obtained is isomorphic to $\mathbb{C}$, while for example if $\sigma = (0, 2)$, then the Clifford algebra is isomorphic to the algebra of quaternions [26, 18]. The key point is to fix the how the map $\pi_Q \circ i$ works. With this purpose let us choose a basis of the vector space V: $\{e_i\}_{i \in I_n}$. We write $\eta(e_i, e_j) = \eta_{ij} = \eta_{ji}$. Next let us define the image of the basis elements under the inclusion map $\pi_Q \circ i$ defined in Equation (3.5) simply by

$$\begin{array}{rcl} \pi_Q \circ i : V & \to & C(V) \\ e_i & \mapsto & e_i \end{array} \tag{3.9}$$

Since $\pi_Q \circ i|_V$ is an injection, then the elements e_i of the Clifford algebra are linearly independent in the image. Moreover the set of elements e_i satisfy the relation

$$e_i e_j + e_j e_i = 2\eta_{ij} \tag{3.10}$$

This is enough to write down the product of any two elements in $\{e_i\}_{i\in I_n}$, in fact for example

$$e_i e_j = e_{ij} + \eta_{ij} \tag{3.11}$$

where $e_{ij} = \frac{1}{2}(e_i e_j - e_j e_i)$. This is a new object, since it can't be reduced by using the Clifford algebra's defining relations. If we calculate the product $e_i e_{jk}$ we need to define another new object e_{ijk}. In general we can define

$$e_{i_1,\ldots,i_p} = \frac{1}{p!}\sum_{P\in\mathcal{P}} sgn(\sigma) e_{i_{P(1)}} \ldots e_{i_{P(p)}} \tag{3.12}$$

where $\mathcal{P}$ is the permutation group of the p indices $\{i_1,\ldots,i_p\}$. We see that $C(V)$ is generated by V and the identity 1, and is the linear span of $\{1, e_{i,ij}, \ldots, e_{i_1\ldots,i_n}\}_{n\in I_n}$ where $n = dim(V)$. In particular we see that

$$dim(C(V)) = \sum_{k=0}^{n}\binom{n}{k} = 2^n \tag{3.13}$$

Then for example, in the trivial case $\sigma = (0,0)$, we obtain that $C(V)$ is an associative algebra isomorphic to $\mathbb{R}$. If $\sigma = (0,1)$ there is one generator e such that $e^2 = -1$. This fact induces an isomorphism $C(V) \cong \mathbb{C}$

$$\begin{array}{ccc} C(V) & \to & \mathbb{C} \\ x1 + ye & \mapsto & x + iy \end{array} \tag{3.14}$$

As another example, we see that if $\sigma = (1,0)$, there is a unique generator e such that $e^2 = 1$. It's interesting to define a pair of projectors $p_\pm = \frac{1}{2}(1 \pm e)$, such that $p_+ + p_- = 1$, $p_+p_- = 0$ and $p_\pm^2 = p_\pm$. The induced isomorphism is $C(V) \cong \mathbb{R}\oplus\mathbb{R}$, that is

$$\begin{array}{ccc} C(V) & \to & \mathbb{R}\oplus\mathbb{R} \\ xp_+ + yp_- & \mapsto & (x,y) \end{array} \tag{3.15}$$

In particular we can easily recover the definition of the Clifford product in Equation (**??**). In fact, if $\sigma = (2,0)$ and $V = \mathbb{R}^2$ there are 4 generators $\{1, e_1, e_2, e_{12}\}$ where $\{e_1, e_2\}$ is an orthonormal basis of $\mathbb{R}^2$ and $e_{12} = e_1 e_2$. They are such that $e_1^2 = e_2^2 = 1$ and $e_{12}^2 = -1$. Moreover the relation $e_1e_2 + e_2e_1 = 0$ holds. Then take two generic vectors $v_1, v_2 \in \mathbb{R}^2$, which can obviously be written as

$$v_1 = x_1e_1 + x_2e_2 \qquad v_2 = y_1e_1 + y_2e_2 \tag{3.16}$$

Then the Clifford product is

$$\begin{aligned} v_1v_2 = (x_1e_1 + x_2e_2)(y_1e_1 + y_2e_2) = x_1y_1e_1^2 + x_2y_2e_2^2 + x_1y_2e_{12} + x_2y_1e_{21} = \\ = (x_1y_1 + x_2y_2)1 + (x_1y_2 - x_2y_1)e_{12} \end{aligned} \tag{3.17}$$

where we used that $e_{12} = -e_{21}$. Then we have recovered the Equation (**??**), since $v_1 \cdot v_2 = x_1y_1 + x_2y_2$ and $v_1 \wedge v_2 = (x_1y_2 - x_2y_1)e_{12}$ is the bivector which represents the oriented area segment build up with v_1 and v_2.

As the last two examples let us consider $V = \mathbb{R}^2$ and the signatures $\sigma = (0,2)$ and $\sigma(1,1)$. In the first case there are 4 generators $\{1, e_1, e_2, e_{12}\}$ such that $e_1^2 = e_2^2 = -1$ and $e_{12}^2 = -1$. Again the relation $e_1e_2 + e_2e_1 = 0$ holds and it can be easily shown that if $\sigma = (0,2)$ the map

$$\begin{array}{ccc} G : C(V) & \to & \mathbf{H} \\ a + be_1 + ce_2 + de_{12} & \mapsto & a + bi + cj + dk \end{array} \tag{3.18}$$

is an algebra isomorphism, where $\mathbf{H}$ is the algebra of the *quaternions* and as usual $i^2 = j^2 = k^2 = -1$. In the second case there are always 4 generators $\{1, e_1, e_2, e_{12}\}$ such that $e_1^2 = 1$, $e_2^2 = -1$ and $e_{12}^2 = 1$. It can be easily shown that if $\sigma = (1,1)$ the map

$$\begin{array}{ccc} H : C(V) & \to & M(2,\mathbb{R}) \\ a + be_1 + ce_2 + de_{12} & \mapsto & \begin{pmatrix} a+b & c+d \\ -c+d & a-b \end{pmatrix} \end{array} \tag{3.19}$$

is an algebra isomorphism, where $M(2,\mathbb{R})$ is the vector space of the 2-dimensional square matrices. Moreover, it's also easy to prove that also if $\sigma = (2,0)$ then $C(V) \cong M(2,\mathbb{R})$.

Eventually we can write that if $B = \{e_i\}_{i\in I_n}$ is an orthonormal basis of V with respect to η, then a basis for the Clifford algebra is given by the set

$$B_C = \{e_{i_1}e_{i_2}\dots e_{i_k} \equiv e_{i_1\dots i_k} | \quad i_1 \leq i_2 \leq \dots \leq i_k \quad \text{and} \quad \forall\, k \in I_n^0\} \tag{3.20}$$

and in particular the relation in Equation (3.8) holds. Let us notice that the indices run over all ordered sets of integers $k \leq n$, and we set $e_0 = 1$. Since $dim(C(V)) = 2^n = \Lambda(V)$, we know that a vector space isomorphism $\Lambda(V) \cong C(V)$ can be established.

Before to see how this isomorphism works in practice, let us notice that the Clifford algebra $C(V)$ inherits from the tensor algebra a natural filtration (see Section (2.1.2)). By placing $\mathcal{C}^p(V) = \pi_Q \circ i(\mathcal{T}^p(V))$ we get the Clifford algebra filtration

$$\mathcal{C}^0(V) \subset \mathcal{C}^1(V) \subset \mathcal{C}^2(V) \subset \dots \subset C(V) \tag{3.21}$$

which has the obvious property

$$\mathcal{C}^p(V)\mathcal{C}^q(V) \subseteq \mathcal{C}^{p+q}(V) \qquad \forall\, p, q \in \mathbb{N} \tag{3.22}$$

This makes the Clifford a **filtered algebra**. Finally we can construct the isomorphism mentioned before, and notice that it can defined in such a way to respect the filtration structure of the Clifford algebra $\mathbb{C}(V)$ [37]

Proposition 3.1.1. There exists a canonical vector space isomorphism $I : \Lambda(V) \overset{\cong}{\to} C(V)$ which preserves the filtrations, defined by the maps

$$\begin{array}{ccc} \Lambda^k(V) & \to & C(V) \\ v_1 \wedge \dots \wedge v_k & \mapsto & \dfrac{1}{p!} \displaystyle\sum_{P\in\mathcal{P}_p} v_{P(1)} \dots v_{P(p)} \end{array} \tag{3.23}$$

where P represents an element of the permutation group $\mathcal{P}$ of p-elements $\{1, \dots, p\}$.

We understand that the quadratic form Q plays a role in determining the relationship between $C(V)$ and $\Lambda(V)$ only at the moment in which the product defined on the algebra is involved. This is the reason why $C(V)$ and $\Lambda(V)$ are not isomorphic as algebras (unless $Q = 0$) but they are isomorphic as vector spaces. Moreover, since the map in Proposition 3.1.1 is canonical [37], we can think about each $\Lambda^p(V)$ as embedded in the Clifford algebra $\Lambda^p(V) \subset C(V) \quad \forall\, p \in \mathbb{N}$.

An important point is now to define certain kinds of automorphism of the Clifford algebra, which allow us to define the Spin group. For each $\lambda \in O(V)$ we can define the linear map

$$\begin{array}{rcl} j_\lambda : V & \to & C(V) \\ v & \mapsto & \lambda v \end{array} \tag{3.24}$$

is such that $(j_\lambda(v))^2 = (\lambda v)^2 = Q(\lambda v) = Q(v)$. It can be shown [26] that each map like j_λ can be extended to a K-algebra homomorphism

$$\begin{array}{rcl} J_\lambda : C(V) & \to & C(V) \\ v & \mapsto & \lambda v \end{array} \tag{3.25}$$

Moreover we get the important result that

$$J_{\lambda_1\lambda_2} = J_{\lambda_1}J_{\lambda_2} \qquad \forall\, \lambda_1, \lambda_2 \in O(V) \tag{3.26}$$

and that

$$J_{1_V} = 1_{C(V)} \tag{3.27}$$

where $1_{C(V)}$ is the identity on $C(V)$. This means that the map

$$\begin{array}{rcl} J : O(V) & \to & Aut(C(V)) \\ \lambda & \mapsto & J_\lambda \end{array} \tag{3.28}$$

is an injective group homomorphism.

An important case of Clifford algebra automorphism is for $\lambda = -1_V$. Firstly we define **the inversion**

$$\begin{array}{rcl} \alpha : C(V) & \rightarrow & C(V) \\ v_1 \dots v_n & \mapsto & j_{-1_V}(v_1) \dots j_{-1_V}(v_n) \end{array} \tag{3.29}$$

Under the action of α the Clifford algebra $C(V)$ decomposes into two eigenspaces

$$C(V) = C^+(V) \oplus C^-(V) \tag{3.30}$$

where $C^+(V)$ is said to be generated by *even elements* of $C(V)$, namely by elements which remain unchanged under an α-action. On the contrary $C^-(V)$ is said to be generated by *odd elements*, that is by elements in $C(V)$ which change their sign under an α-action. Naturally $dim(C^+(V)) = dim(C^-(V)) = 2^{n-1}$. Moreover let us notice that since $J(Q)$ isn't homogeneous, then $C(V)$ is a $\mathbb{Z}_2$**-graded algebra**, which is also called a **superalgebra**. This means that $C(V)$ can be decomposed in the direct sum of subalgebras $\{C_j\}_{j \in I_2}$ such that $C_i C_j \subseteq C_{ij}$ and $C_j C_i \subseteq C_{ji}$, where $i, j \in I_2$. In this case the decomposition which makes $C(V)$ into a superalgebra is exactly that in Equation (3.30).

Next, there is a second involutive anti-automorphism of the Clifford algebra, induced by the map

$$\begin{array}{rcl} \tau : T^k(V) & \rightarrow & T^k(V) \\ v_1 \otimes \cdots \otimes v_k & \mapsto & v_k \otimes \cdots \otimes v_1 \end{array} \tag{3.31}$$

and such that $\tau(x \otimes y) = \tau(y) \otimes \tau(x) \quad \forall\, x, y \in T^k(V)$. Then we can define the **transposition** as the map induced by τ on $C(V)$

$$\begin{array}{rcl} \bar{\cdot} : C(V) & \rightarrow & C(V) \\ v_1 \dots v_k & \mapsto & \overline{v_1 \dots v_k} = v_k \dots v_1 \end{array} \tag{3.32}$$

which is obviously an involution and doesnt'n depend on the basis chosen. Finally we can define the composition of the two involutions as the **conjugation**

$$\begin{array}{rcl} {}^* \equiv \overline{\alpha} : C(V) & \rightarrow & C(V) \\ v_1 v_2 \dots v_k & \mapsto & (v_1 v_2 \dots v_k)^* = (-1)^k (v_k v_{k-1} \dots v_1) \end{array} \tag{3.33}$$

3.1.2 Spin group and Spin algebra

We are able to define an inner product over the Clifford algebra $C(V)$ by using the isomorphism $\Lambda(V) \cong C(V)$ of Proposition 3.1.1. In fact using the inner product defined in Equation (A.16), we can define an inner product on $C(V)$ as the unique making the isomorphism $\Lambda(V) \cong C(V)$ an isometry. More in detail we can define the bilinear map

$$\begin{array}{rcl} (\cdot, \cdot) : C(V) \times C(V) & \rightarrow & \mathbb{R} \\ \alpha \times \beta & \mapsto & (\alpha, \beta) \equiv (1, \overline{\alpha}\beta) \end{array} \tag{3.34}$$

such that $(1, 1) = 1$. $(,)$ induces a norm on the Clifford algebra in the usual way

$$|\alpha| = \sqrt{(\alpha, \alpha)} \qquad \forall\, \alpha \in C(V) \tag{3.35}$$

Let us see how this scalar product works. Let $\{e_i\}_{i \in I_n}$ be an orthonormal basis for V. Let us denote by I the sequence of indices $(i_1, \dots, i_p)$. Let us take $I \neq J$. As we have seen if $e_I, e_J \in \Lambda^p(V)$ we have $(e_I, e_J) = 0$, while $(e_I, e_I) = Q(e_{i_1}) \dots Q(e_{i_p})$. For the corresponding vectors in the Clifford algebra $e_I, e_J \in C(V)$ we can write

$$(e_I, e_J) = (e_{i_1} \dots e_{i_p}, e_{j_1} \dots e_{j_p}) = (1, e_{i_p} \dots e_{i_1} e_{j_1} \dots e_{j_p}) = 0 \tag{3.36}$$

because $e_{i_p} \dots e_{i_1} e_{j_1} \dots e_{j_p}$ is not proportional to the identity 1. Otherwise

$$(e_I, e_I) = (e_{i_p} \dots e_{i_1} e_{i_1} \dots e_{i_p}) = Q(e_{i_1}) \dots Q(e_{i_p})(1, 1) = Q(e_{i_1}) \dots Q(e_{i_p}) \tag{3.37}$$

Given a Clifford algebra $C(V)$ we can define the **multiplicative group of units** as the subset

$$C^\times(V) = \{\varphi \in C(V) | \quad \exists\, \varphi^{-1} \in C(V) \quad \text{s. t.} \quad \varphi^{-1}\varphi = \varphi\varphi^{-1} = 1\} \tag{3.38}$$

It's evident that the group $C^\times(V)$ contains all the vectors $v \in V \overset{\pi_Q \circ i}{\hookrightarrow} C(V)$ such that $Q(v) \neq 0$. In fact the inverse for these elements is trivially

$$v^{-1} = \frac{v}{Q(v)} \qquad \forall\, v \in V \quad \overset{\pi_Q \circ i}{\hookrightarrow} \quad C^\times(V) \tag{3.39}$$

This extends to all the other elements of the group of units, namely

$$\varphi^{-1} = \frac{\overline{\varphi}}{|\varphi|^2} \qquad \varphi \in C^\times(V) \tag{3.40}$$

The group of units is a Lie group such that $dim(C^\times(V)) = 2^n$, where as usual $dim(V) = n$. It's interesting to see that the associated Lie algebra is given by the same Clifford algebra $\mathfrak{cl}^\times(V) = C(V)$, where the Lie bracket is defined simply by

$$[v, w] = vw - wv \qquad \forall\, v, w \in C(V) \equiv \mathfrak{cl}^\times(V) \tag{3.41}$$

Moreover, $C^\times(V)$ acts naturally as automorphisms of the Clifford algebra, that is we can define a homomorphism called the **adjoint representation**

$$\begin{array}{rcl} Ad : C^\times(V) & \longrightarrow & Aut(C(V)) \\ v & \longmapsto & Ad_v \quad \text{s.t.} \quad Ad_v(x) = vxv^{-1} \qquad \forall\, x \in C(V) \end{array} \tag{3.42}$$

The associated Lie algebra representation is given by the homomorphism

$$\begin{array}{rcl} ad : \mathfrak{cl}^\times(V) & \rightarrow & Der(C(V)) \\ y & \mapsto & ad_y \quad \text{s.t.} \quad ad_y(x) = [y, x] \quad \forall\, x \in C(V) \end{array} \tag{3.43}$$

where the space $Der(C(V))$ is the space of derivations of $C(V)$, i.e. the space of operators $\varphi : C(V) \rightarrow C(V)$ which obey the Leibniz rule, namely

$$\varphi(xy) = \varphi(x)y + x\varphi(y) \qquad \forall\, x, y \in C(V) \tag{3.44}$$

Let us recall the relation between Ad and ad. It is given by the **exponential map**

$$\begin{array}{rcl} exp : \mathfrak{cl}^\times(V) & \rightarrow & C^\times(V) \\ x & \mapsto & exp(x) = \displaystyle\sum_{j=0}^{\infty} \frac{1}{j!} x^j \end{array} \tag{3.45}$$

and one can verify that

$$\left. \frac{d}{dt} Ad_{exp(ty)}(x) \right|_{t=0} = ad_y(x) \tag{3.46}$$

As we can expect the orthogonal group of transformations

$$O(V) = \{\lambda \in GL(V) | \quad \lambda^* Q = Q\} \tag{3.47}$$

has a nice relationship with the group $C^\times(V)$. To probe this question, les us firstly investigate its Lie algebra, which as it is well known is generated by the skew matrices, namely

$$\mathfrak{so}(V) = \{X \in C(V) | \quad \eta(Xv, w) + \eta(v, Xw) = 0 \quad \forall\, v, w \in V\} \tag{3.48}$$

The vector space of the skew matrices is isomorphic to $\Lambda^2(V)$ and such isomorphism can be fixed by the map

$$\begin{array}{rcl} \Lambda^2(V) & \rightarrow & \mathfrak{so}(V) \\ u \wedge v & \mapsto & u \curlywedge v \end{array} \tag{3.49}$$

where

$$u \curlywedge v(x) = \eta(u, x)v - \eta(v, x)u \qquad \forall\, x \in V \tag{3.50}$$

Proposition 3.1.2. Let $v \in V \overset{\pi_Q \circ i}{\hookrightarrow} C(V)$ such that $Q(v) \neq 0$. Then V is invariant under the action of Ad_v, namely $Ad_v(V) = V$. In fact, $\forall\, w \in V$

$$Ad_v(w) = 2\frac{\eta(v,w)}{Q(v)}v - w \tag{3.51}$$

In fact

$$\begin{aligned} \text{Equation (3.8)} \quad &\Rightarrow \quad vwv^{-1} + wvv^{-1} = 2\eta(v,w)v^{-1} \quad \forall\, w \in V \quad \Rightarrow \\ &\Rightarrow \quad Ad_v(w) = 2\frac{\eta(v,w)}{Q(v)}v - w \quad \forall\, w \in V \end{aligned} \tag{3.52}$$

where we have used Equation (3.39) and the fact that $Ad_v(w) = vwv^{-1}$.

It's interesting to notice that the transformation Ad_v preserves the quadratic form $Q \quad \forall v \in V$ such that $Q(v) \neq 0$, in fact

$$\begin{aligned} &Ad_v^*(Q(w)) = Ad_v^*(\eta(w,w)) = \eta(Ad_v(w), Ad_v(w)) = \\ &= \eta(w,w) + 2\frac{\eta(v,w)}{Q(v)}\eta(v,w) + 2\frac{\eta(v,w)}{Q(v)}\eta(v,w) - 4\frac{\eta(v,w)}{Q(v)}\eta(v,w) = \eta(w,w) = Q(w) \quad \forall\, w \in V \end{aligned} \tag{3.53}$$

where we have used the bilinearity of η. Then we get that $Ad_v \in O(V) \quad \forall\, v \in V$ such that $Q(v) \neq 0$.

Definition 3.1.2. The set $Pin(V)$ generated by all vectors $v \in V \overset{\pi_Q \circ i}{\hookrightarrow} C(V)$ such that $v \in S(V)$ and by the identity 1 forms a group which is called the **Pin group**. In other words

$$Pin(V) = \{v_1 \dots v_r \in C(V) | \quad Q(v_i) = \eta(v_i, v_i) = \pm 1 \quad \forall\, v_i \in V \cap C^{\times}(V)\} \tag{3.54}$$

The group structure is immediately given by noticing that the norm induced by η on the Clifford algebra preserves the Clifford product, which means that

$$|\varphi\rho|^2 = |\varphi|^2 |\rho|^2 \qquad \forall\, \varphi, \rho \in C(V) \tag{3.55}$$

It's now interesting to notice that the r.h.s. of Equation (3.51) is nothing but a reflection with the wrong sign. In fact let us define, $\forall\, v \in V \cap C^{\times}(V)$

$$\begin{aligned} \rho_v : V \quad &\rightarrow \quad V \\ w \quad &\mapsto \quad w - \frac{\eta(v,w)}{Q(v)}v \end{aligned} \tag{3.56}$$

$\rho_v(w)$ is the reflection of the vector w across the hyperplane $v^{\perp} = \{w \in V | \quad \eta(v,w) = 0\}$. In particular it maps v in $-v$. Needless to say $\rho_v \in O(V)$.

In order to readjust the wrong sign in Equation (3.51), let us define the **twisted adjoint representation**

$$\begin{aligned} \lambda : C^{\times}(V) \quad &\rightarrow \quad Aut(C(V)) \\ \varphi \quad &\mapsto \quad \lambda_{\varphi} \end{aligned} \tag{3.57}$$

such that

$$\lambda_{\varphi}(v) = (\alpha(\varphi))v\varphi^{-1} \qquad \forall\, v \in C(V) \tag{3.58}$$

Let us notice that if $\varphi \in C^+(V)$ then $\lambda_{\varphi} = Ad_{\varphi}$ and that obviously $\lambda_{\varphi_1\varphi_2} = \lambda_{\varphi_1} \circ \lambda_{\varphi_2}$. In fact $\lambda_{\varphi_1\varphi_2}(w) = (\alpha(\varphi_1\varphi_2))w(\varphi_1\varphi_2)^{-1} = \alpha(\varphi_1)\alpha(\varphi_2)w(\varphi_2)^{-1}(\varphi_1)^{-1} = \alpha(\varphi_1)Ad_{\varphi_2}(w)\alpha(\varphi_1)^{-1} = Ad_{\varphi_1} \circ Ad_{\varphi_2}(w)$. In this way, $\lambda_v(w)$ represents exactly the reflection across $v^{\perp} \quad \forall v \in V \cap C^{\times}(V)$, and furthermore $Ad_{\varphi}(w)$ represents a composition of reflections

$$\lambda_{\varphi}(w) = \rho_{v_1} \circ \dots \circ \rho_{v_p} \qquad \forall\, \varphi = v_1 \dots v_p \in C(V) \tag{3.59}$$

Since the reflections are orthogonal maps, the restriction of λ to the subgroup $Pin(V)$ defines a homomorphism

$$\lambda : Pin(V) \rightarrow O(V) \tag{3.60}$$

which is surjective due to the following classical result

Theorem 4. ***Cartan-Dieudonnè***
Let $O(v)$ be the group of orthogonal transformation of the vector space V, endowed with the non-degenerate quadratic form Q. Then each $g \in O(V)$ can be written as the product of r reflections

$$g = \rho_1 \circ \cdots \circ \rho_r \tag{3.61}$$

where $r \leq n = dim(V)$.

Moreover it can be shown [37] that $Ker(\lambda) = \{\pm 1\}$, and so that we immediately have the following exact sequence

$$1 \longrightarrow \{\pm 1\} \longrightarrow Pin(V) \xrightarrow{\lambda} O(V) \longrightarrow 1 \tag{3.62}$$

Finally we can define the **Spin group Spin(V)** as

$$Spin(V) = Pin(V) \cap C^+(V) \tag{3.63}$$

or also

$$Spin(V) = \{v_1 \dots v_{2r} \in C(V) | \quad Q(v_i) = \eta(v_i, v_i) = \pm 1 \quad \forall\, v_i \in V \cap C^\times(V)\} \tag{3.64}$$

There is an amazing end for the map λ defined in Equation (3.58). In fact let us notice that since a reflection $\rho_v \in O(v)$ is such that $\det(\rho_v) = -1$, then for an element $\varphi \in Pin(V)$

$$\det(\lambda_\varphi) = 1 \quad \Leftrightarrow \quad \varphi \in Spin(V) \tag{3.65}$$

This means that

$$Ker(\lambda) = Spin(V) \tag{3.66}$$

so that we immediately have the following exact sequence

$$1 \rightarrow \{\pm 1\} \rightarrow Spin(V) \xrightarrow{\lambda} SO(V) \rightarrow 1 \tag{3.67}$$

which shows us that the map $\lambda : Spin(V) \rightarrow SO(V)$ is a non-trivial covering of the group $SO(V)$ (at least for $n \geq 2$).

3.1.3 Clifford algebras classification

In order to study spinor representations, it's very useful to give a classification of the Clifford algebras. We will see that they are organized in a very nice vay, since a strong periodicity in the classification appears.

The idea is to classify the real Clifford algebras accordingly to the signature of the quadratic form from which they derive. Let us denote the signature $\sigma = (r, s)$, where as usual r is the dimension of the maximal positive definite subspace of V, while s is the dimension of the maximal negative definite subspace of V and $dim(V) = n = r + s$. In order to avoid confusion, where is needed we will denote the Clifford algebra $C(V)$ generated by the quadratic form with signature $\sigma = (r, s)$ by $C(r, s)$.

Moreover let us notice that in Section 3.1.1 we have already obtained some useful results, which we can resume in the following table

	$r=0$	$r=1$	$r=2$
$s=0$	$\mathbb{R}$	$\mathbb{R} \oplus \mathbb{R}$	$M(2,\mathbb{R})$
$s=1$	$\mathbb{C}$	$M(2,\mathbb{R})$	
$s=2$	$\mathbf{H}$		

Now the purpose is to complete this table for each $r, s \geq 0$. The first useful result is the following [37]

Proposition 3.1.3. For each $r, s \geq 0$ the following isomorphisms

$$C(0,n) \otimes C(2,0) = C(n+2,0) \tag{3.68}$$
$$C(n,0) \otimes C(0,2) = C(0,n+2) \tag{3.69}$$
$$C(r,s) \otimes C(1,1) = C(r+1,s+1) \tag{3.70}$$

hold.

By using the third relation in Proposition 3.1.3 we can easily obtain each element in the table of the form $C(1+i,i)$, $C(i,i)$, $C(i,1+i)$, where $i \geq 0$. For example $C(2,1) = C(1,0) \otimes C(1,1) \cong (\mathbb{R} \oplus \mathbb{R}) \otimes M(2,\mathbb{R}) = M(2,\mathbb{R}) \oplus M(2,\mathbb{R})$. As another example we can notice that $C(1,2) = C(0,1) \otimes C(1,1) \cong \mathbb{C} \otimes M(2,\mathbb{R}) = M(2,\mathbb{C})$.

Moreover by using the rimanent relations in Proposition 3.1.3 we can easily obtain each element in the table of the form $C(n,0)$ or $C(0,n)$. Let us give some examples

$$C(3,0) = C(0,1) \otimes C(2,0) \cong \mathbb{C} \otimes M(2,\mathbb{R}) \cong M(2,\mathbb{C}) \tag{3.71}$$
$$C(4,0) = C(0,2) \otimes C(2,0) \cong \mathbf{H} \otimes M(2,\mathbb{R}) \cong M(2,\mathbf{H}) \tag{3.72}$$
$$C(0,3) = C(1,0) \otimes C(0,2) \cong (\mathbb{R} \oplus \mathbb{R}) \otimes \mathbf{H} \cong \mathbf{H} \oplus \mathbf{H} \tag{3.73}$$
$$C(0,4) = C(2,0) \otimes C(0,2) \cong M(2,\mathbb{R}) \oplus \mathbf{H} \cong M(2,\mathbf{H}) \tag{3.74}$$

In this way, by using Proposition 3.1.3 and moving left and right, and then in diagonal on the table, we are able to obtain each element $C(r,s)$. One of the most interesting results is given in the following

Proposition 3.1.4. For each $r, s \geq 0$ the following isomorphisms

- $C(n+8,0) \cong C(n,0) \otimes M(16,\mathbb{R})$
- $C(0,n+8) \cong C(0,n) \otimes M(16,\mathbb{R})$
- $C(r+4,s+4) \cong C(r,s) \otimes M(16,\mathbb{R})$

hold. They are called **Bott periodicities**.

Thanks to Bott periodicities, we only need a table 8×8 to obtain $C(r,s)$ for each r, s. We give the complete table

	r = 0	r = 1	r = 2	r = 3	r = 4	r = 5	r = 6	r = 7
$s=0$	$\mathbb{R}$	$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbf{H}(2)$	$\mathbf{H}(2) \oplus \mathbf{H}(2)$	$\mathbf{H}(4)$	$\mathbb{C}(8)$
$s=1$	$\mathbb{C}$	$\mathbb{R}(2)$	$\mathbb{R}(2) \oplus \mathbb{R}(2)$	$\mathbb{R}(4)$	$\mathbb{C}(4)$	$\mathbf{H}(4)$	$\mathbf{H}(4) \oplus \mathbf{H}(4)$	$\mathbf{H}(8)$
$s=2$	$\mathbf{H}$	$\mathbb{C}(2)$	$\mathbb{R}(4)$	$\mathbb{R}(4) \oplus \mathbb{R}(4)$	$\mathbb{R}(8)$	$\mathbb{C}(8)$	$\mathbf{H}(8)$	$\mathbf{H}(8) \oplus \mathbf{H}(8)$
$s=3$	$\mathbf{H} \oplus \mathbf{H}$	$\mathbf{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	$\mathbb{R}(16)$	$\mathbb{C}(16)$	$\mathbf{H}(16)$
$s=4$	$\mathbf{H}(2)$	$\mathbf{H}(2) \oplus \mathbf{H}(2)$	$\mathbf{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$	$\mathbb{R}(16) \oplus \mathbb{R}(16)$	$\mathbb{R}(32)$	$\mathbb{C}(32)$
$s=5$	$\mathbb{C}(4)$	$\mathbf{H}(4)$	$\mathbf{H}(4) \oplus \mathbf{H}(4)$	$\mathbf{H}(8)$	$\mathbb{C}(16)$	$\mathbb{R}(32)$	$\mathbb{R}(32) \oplus \mathbb{R}(32)$	$\mathbb{R}(64)$
$s=6$	$\mathbb{R}(8)$	$\mathbb{C}(8)$	$\mathbf{H}(8)$	$\mathbf{H}(8) \oplus \mathbf{H}(8)$	$\mathbf{H}(16)$	$\mathbb{C}(32)$	$\mathbb{R}(64)$	$\mathbb{R}(64) \oplus \mathbb{R}(64)$
$s=7$	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	$\mathbb{R}(16)$	$\mathbb{C}(16)$	$\mathbf{H}(16)$	$\mathbf{H}(16) \oplus \mathbf{H}(16)$	$\mathbf{H}(32)$	$M(64,\mathbb{C})$	$\mathbb{R}(128)$

where we denoted $\mathbb{K}(n) \equiv M(n,\mathbb{K})$. Then the following

Theorem 5. ***Clifford algebras classification theorem***
The Clifford algebras $C(r,s)$ is isomorphic to different real associative algebras as explained in the following table

$(r-s) mod(8)$	$C(r,s)$
0, 6	$M(2^{\frac{n}{2}},\mathbb{R})$
7	$M(2^{\frac{(n-1)}{2}},\mathbb{R}) \oplus M(2^{\frac{(n-1)}{2}},\mathbb{R})$
1, 5	$M(2^{\frac{(n-1)}{2}},\mathbb{C})$
2, 4	$M(2^{\frac{(n-2)}{2}},\mathbf{H})$
3	$M(2^{\frac{(n-3)}{2}},\mathbf{H}) \oplus M(2^{\frac{(n-3)}{2}},\mathbf{H})$

where $n = r + s$.

In particular we notice the the case $(r,s) = (6,0)$ has Clifford algebra $C(6,0)$ which is isomorphic to the space of real 8×8 matrices

$$C(6,0) \cong M(8,\mathbb{R}) \tag{3.75}$$

and that the case $(r,s) = (6,6)$ has instead the Clifford algebra $C(6,6)$ isomorphic to the space of real 64×64 matrices

$$C(6,6) = M(64,\mathbb{R}) \tag{3.76}$$

Finally, in the study of spinor representation, it's important to identify the even subalgebra $C^+(r,s)$ of the Clifford algebra $C(r,s)$. Fortunately, $C^+(r,s)$ can be determined from the Clifford algebra of dimension $r+s-1$, in fact

Proposition 3.1.5. The following isomorphism

$$C(r,s) \cong C^+(r+1,s) \cong C^+(r,s+1) \tag{3.77}$$

holds. Moreover $C^+(r,s) \cong C^+(s,r)$.

Then the following Proposition on the classification of the even subalgebras holds

Proposition 3.1.6. The even Clifford algebras $C^+(r,s)$ is isomorphic to some real associative algebras as explained in the following table

$(r-s)mod(8)$	$C^+(r,s)$
$1,7$	$M(2^{\frac{(n-1)}{2}},\mathbb{R})$
0	$M(2^{\frac{(n-2)}{2}},\mathbb{R})\oplus M(2^{\frac{(n-2)}{2}},\mathbb{R})$
$2,6$	$M(2^{\frac{(n-2)}{2}},\mathbb{C})$
$3,5$	$M(2^{\frac{(n-3)}{2}},\mathbf{H})$
4	$M(2^{\frac{(n-4)}{2}},\mathbf{H})\oplus M(2^{\frac{(n-4)}{2}},\mathbf{H})$

In the particular cases which we will study in Chapter 4 and 5 we find

$$C(6,0)\cong M(4,\mathbb{C}) \tag{3.78}$$

and

$$C(6,6)\cong M(4,\mathbb{R})\oplus M(4,\mathbb{R}) \tag{3.79}$$

3.1.4 Spinor representations

As usual, the usefulness of algebras and groups becomes clear through the study of their representations. In particular we will be interested in the representations of the Spin group.

Let V be the usual vector space over $\mathbb{R}$, and let Q be the quadratic form with which we endow V. Then

Definition 3.1.3. Let $\mathbb{K}\supseteq k$ be a field containing the field k. Then a **$\mathbb{K}$-representation** of the Clifford algebra $C(V)$ is a k-algebra homomorphism

$$\rho : C(V)\to Hom_{\mathbb{K}}(W,W) \tag{3.80}$$

where $Hom_{\mathbb{K}}(W,W)$ is the algebra of linear transformations of the finite dimensional vector space W over $\mathbb{K}$. W is called **C(V)-module** over $\mathbb{K}$. We will simplify notation by simply writing

$$\rho(\varphi)(w)\equiv\varphi\cdot w \tag{3.81}$$

where $\varphi\in C(V)$ and $w\in W$. The product $\varphi\cdot w$ is called **Clifford multiplication**.

We recall that a $\mathbb{R}$-algebra homomorphism is a $\mathbb{R}$-linear map ρ such that $\rho(\varphi\psi)=\rho(\varphi)\circ\rho(\psi)\quad\forall\,\varphi,\psi\in C(V)$. The following Definition is a natural extension from the Lie algebras and Lie groups representation theory

Definition 3.1.4. A $\mathbb{K}$-representation $\rho : C(V)\to Hom_{\mathbb{K}}(W,W)$ is said to be **reducible** if the vector space W can be written as a non-trivial direct sum

$$W=W_1\oplus W_2 \tag{3.82}$$

such that W_i are invariant under the ρ-action, namely $\rho(\varphi)(W)\subseteq W\quad\forall\,\varphi\in C(V)$. In this case we can write also

$$\rho=\rho_1\oplus\rho_2 \tag{3.83}$$

where $\rho_i=\rho|_{W_i}$ for $i\in J_2$. A $\mathbb{K}$-representation is **irreducible** if it is not reducible.

In particular one finds that every $\mathbb{K}$-representation ρ of a Clifford algebra $C(V)$ can be decomposed into a direct sum

$$\rho=\rho_1\oplus\cdots\oplus\rho_n \tag{3.84}$$

and, as usual, if ρ_1 and ρ_2 are two $\mathbb{K}$-representations $\rho_j : C(V)\to Hom_{\mathbb{K}}(W_j)$ where $j\in J_2$, they are equivalent if there exists a $\mathbb{K}$-linear isomorphism $F : W_1\to W_2$ such that

$$F\circ\rho_1(\varphi)\circ F^{-1}=\rho_2(\varphi)\qquad\forall\,\varphi\in C(V) \tag{3.85}$$

As it seems to be intuitive, we give

Definition 3.1.5. A **spinor representation of Spin(V)** is the restriction to $Spin(V)$ of an irreducible representation of $C^+(V)\subset C(V)$.

It's amazing to see how spinor representations can be deduced only from the Clifford algebra classification and the following

Proposition 3.1.7. 1. Every irreducible $\mathbb{R}$-representations of the real algebra $M(n,\mathbb{R})$ is isomorphic to $\mathbb{R}^n$, where the representation matrices act on $\mathbb{R}^n$ via left multiplication.

2. Every irreducible $\mathbf{H}$-representations of the real algebra $M(n,\mathbf{H})$ is isomorphic to $\mathbf{H}^n$, where the representation matrices act on $\mathbf{H}^n$ via left multiplication.

3. Every irreducible $\mathbb{C}$-representation of the real algebra $M(n,\mathbb{C})$ is isomorphic either to $\mathbb{C}^n$ with the natural action given by the left matrix multiplication or to $\mathbb{C}^n$ via the complex conjugate action given by left matrix multiplication.

As a direct consequence of the last Proposition we can immediately give the next table, which follows from table in Theorem 5, and indicates the number of inequivalent spinor representations as a function of r and s

	r = 0	r = 1	r = 2	r = 3	r = 4	r = 5	r = 6	r = 7
s = 0	1	2	1	2	1	2	1	2
s = 1	2	1	2	1	2	1	2	1
s = 2	1	2	1	2	1	2	1	2
s = 3	2	1	2	1	2	1	2	1
s = 4	1	2	1	2	1	2	1	2
s = 5	2	1	2	1	2	1	2	1
s = 6	1	2	1	2	1	2	1	2
s = 7	2	1	2	1	2	1	2	1

where the cells with two inequivalent representations are associated either to even Clifford algebras isomorphic to $M(n,\mathbb{K})\otimes M(n,\mathbb{K})$ with $\mathbb{K}$ equal to $\mathbb{R}$ or equal to $\mathbf{H}$ or to even Clifford algebras isomorphic to $M(n,\mathbb{C})$. In fact from Proposition 3.1.7 we know that they have automatically two inequivalent representations. We can also understand better this argument in terms of the *volume form*. Let us give the following

Definition 3.1.6. Let us consider the vector space V endowed with a quadratic form with signature $\sigma=(r,s)$. Let us consider an orthonormal basis $\{e_i\}_{i\in I_n}$ of V. The **volume form** ω associated to $C(r,s)$ is the Clifford product of every element of the orthonormal basis

$$\omega = e_1 \dots e_n \tag{3.86}$$

Immediately we can give

Proposition 3.1.8. The volume form ω associated to $C(r,s)$, where $n=r+s$ satisfies the following properties

1. $\omega^2=(-1)^{s+\frac{n(n-1)}{2}}$
2. If $r+s$ is odd then ω is central.
3. If $r+s$ is even then $\forall v\in V$ we have $\omega v=-v\omega$.

From 1. it follows that the sign of ω^2 depends only on $(r-s)mod(4)$:

$$\omega^2=\begin{cases} 1 & r-s=0,3(mod(4)) \\ -1 & r-s=1,2(mod(4)) \end{cases} \tag{3.87}$$

Finally using the Bott periodicities we get the spinor representations in terms of $(r-s)mod(8)$. We denote the spinor representations space by S

1. $(r-s)=0mod(8)$: $S_\pm\cong\mathbb{R}^{2^{\frac{n-2}{2}}}$. $\omega^2=1$ and $S_\pm$ are its ±1-eigenspaces.
2. $(r-s)=1mod(8)$: $S\cong\mathbb{R}^{2^{\frac{n-1}{2}}}$.
3. $(r-s)=2mod(8)$: $S,\overline{S}\cong\mathbb{C}^{2^{\frac{n-2}{2}}}$.
4. $(r-s)=3mod(8)$: $S\cong\mathbf{H}^{2^{\frac{n-3}{2}}}$.
5. $(r-s)=4mod(8)$: $S_\pm\cong\mathbf{H}^{2^{\frac{n-4}{2}}}$.

6. $(r-s) = 5mod(8)$: $S \cong \mathbf{H}^{2^{\frac{n-3}{2}}}$.

7. $(r-s) = 6mod(8)$: $S, \overline{S} \cong \mathbb{C}^{2^{\frac{n-2}{2}}}$.

8. $(r-s) = 7mod(8)$: $S \cong \mathbb{R}^{2^{\frac{n-1}{2}}}$.

Hence we can study in particular the cases $(r,s) = (6,0)$ and $(r,s) = (6,6)$. $Spin(6,0) \equiv Spin(6)$ has spinor representations

$$S = \mathbb{C}^4 \qquad\qquad \overline{S} = \mathbb{C}^4 \tag{3.88}$$

which means that a $(6,0)$-spinor η is simply a vector of $\mathbb{C}^4$. The spinor representation allows to see that

$$Spin(6) \cong SU(4) \tag{3.89}$$

In fact the spinor representation $\rho : Spin(6) \to GL(n,\mathbb{C})$ since as we have seen $S_\pm \cong \mathbb{C}^4$. ρ is an injective homomorphism, so that the compactness of $Spin(6)$ has to be preserved. This means that $\rho(Spin(6)) \subset U(4) \subset GL(4,\mathbb{C})$, since $U(4)$ is the maximal compact subgroup of $GL(4,\mathbb{C})$. The restriction to $SU(4)$ is due to the simplicity of $Spin(6)$, while the isomorphicity follows by a simple dimensional analysis.

Otherwise, in the case with signature $(r,s) = (6,6)$ we find

$$S_+ = \mathbb{R}^{32} \qquad\qquad S_- = \mathbb{R}^{32} \tag{3.90}$$

Spinors in this real representation are called **Majorana-Weyl spinors**.

Finally, let us introduce a concept regarding spinors which will be mostly studied in the case of Generalized Complex Geometry in Chapter 5

Definition 3.1.7. Let S be a spinor representation space. A spinor $\eta \in S$ is a **pure spinor** if it is annihilated by half the gamma matrices.

Fortunately, it can be shown that in dimension $n \leq 6$, every spinor is a pure spinor. In the case $(r,s) = (6,6)$ the situation becomes much more involved, and we will see that the pure spinors play a fundamental role in the description of the geometric structures.

3.2 Spinors

As we have seen to be usual in differential geometry, once we have studied the linear formalism of the Clifford algebras, the next step is to transport it over the smooth manifolds. In fact, exactly as natural operations over linear spaces - such as sum, tensor product or exterior power - can be canonically carried over vector bundles, in the same way we expect that natural operations over linear spaces endowed with a quadratic form can be pushed up on vector bundles. However in the case of the Clifford algebras and of the Spin groups this step is far from trivial, due to several topological obstructions which can arise.

With this purpose in mind let us the standard representation on $\mathbb{R}^n$ of the special orthogonal group over a vector space V such that $dim(V) = n$, endowed with a quadratic from $Q : V \times V \to \mathbb{R}$

$$\rho_n : SO(V) \to Aut(\mathbb{R}^n) \tag{3.91}$$

As we have seen in Section 3.1.1 ρ_n induces a representation on the Clifford algebra $C(V)$, which we denote by

$$cl_{\rho_n} : SO(V) \to Aut(C(\mathbb{R}^n)) \tag{3.92}$$

Then we can give the following

Definition 3.2.1. Let cl_{ρ_n} be the Clifford algebra representation induced by the standard representation of the special orthogonal group $SO(V)$ on the vector space $\mathbb{R}^n$, where $dim(V) = n$. Let $SO(M)$ be the special orthonormal frame bundle of the vector bundle $\pi : E \to M$. Then the associated bundle

$$Cl(E) = SO(E) \times_{cl_{\rho_n}} C(\mathbb{R}^n) \tag{3.93}$$

is the **Clifford bundle**.

Since $C(\mathbb{R}^n)$ is a vector space, then $Cl(E)$ is a vector bundle, as we have seen in Section 2.1.5. Moreover it seems quite intuitive that $Cl(M)$ is nothing but a bundle of Clifford algebras over the base space M. The fiberwise Clifford multiplication in $Cl(M)$ gives to the space of section $\Gamma(Cl(M))$ an algebra structure.

It's also quite obvious [37] that each of the intrinsic notion about Clifford algebras can be transported over the Cliffor bundle $Cl(M)$. For example there exists a decomposition

$$Cl(M) = Cl(M)^+ \oplus Cl(M)^- \tag{3.94}$$

induced by the bundle automorphism

$$\alpha : Cl(M) \to Cl(M) \tag{3.95}$$

which extends the bundle morphism $\hat{\alpha} : T \to T$ such that $v \mapsto -v$. This is completely analogous to what already discuss in the linear framework in Section 3.1.1. In addition, there exists a vector bundle isometry which provides for a vector bundle isomorphism $\Lambda T^* \cong Cl(M)$. This bundle isometry, as it is predictable, preserves both the gradation structure and the filtration structure of the Clifford bundle $Cl(M)$.

So far the procedure seems to be quite straightforward. Unfortunately several complications arise if one asks for a vector bundle whose fiber is an irreducible module over $\pi^{-1}(p)$, as we will see in the next Section.

3.2.1 Spin structures

The purpose of the present Section is to fix what are the necessary conditions to build spinors over a vector bundle $\pi : E \to M$, where M is a smooth manifold. Let us endow E with a metric g whose signature is σ. In this Section we will denote the identity on the fiber by 1.

We denote the transition functions of the special orthonormal frame bundle $SO(E)$ by $g_{\alpha\beta} : U_{\alpha\beta} \to SO(V)$ (see Example 2.1.6). They have to obey the cocycle condition

$$g_{\alpha\beta}(p) \circ g_{\beta\gamma}(p) \circ g_{\gamma\alpha}(p) = 1 \qquad \forall\, p \in U_{\alpha\beta\gamma} \tag{3.96}$$

and in addition the trivial requests

$$(g_{\alpha\beta}(p))^{-1} = g_{\beta\alpha}(p) \qquad g_{\alpha\alpha}(p) = 1 \qquad \forall\, p \in U_{\alpha\beta} \tag{3.97}$$

Next recall that the homomorphism we defined in Proposition 3.58 $\lambda : Spin(V) \to SO(V)$ is a two-fold covering of the group $SO(V)$ as we have seen in Equation (3.67). In particular it is its universal covering and we can lift the orthonormal frame bundle $SO(E)$ to the principal bundle $Spin(E)$, which has $Spin(V)$ as structure group. The transition functions can be lifted by fixing the prescription

$$\lambda(\widetilde{g}_{\alpha\beta}(p)) = g_{\alpha\beta}(p) \qquad \forall\, p \in U_{\alpha\beta} \tag{3.98}$$

Since $Ker(\lambda) = \{\pm 1\}$, Equation (3.98) brings to a double possible choice of the lifted transition functions, in fact

$$\lambda(\pm\widetilde{g}_{\alpha\beta}(p)) = g_{\alpha\beta}(p) \qquad \forall\, p \in U_{\alpha\beta} \tag{3.99}$$

Needless to say they must satisfy

$$(\widetilde{g}_{\alpha\beta}(p))^{-1} = \widetilde{g}_{\beta\alpha}(p) \qquad \widetilde{g}_{\alpha\alpha}(p) = 1 \qquad \forall\, p \in U_{\alpha\beta} \tag{3.100}$$

Such a lift always exists locally. Moreover, since λ is an homomorphism, it follows that

$$\lambda(\widetilde{g}_{\alpha\beta}(p) \circ \widetilde{g}_{\beta\gamma}(p) \circ \widetilde{g}_{\gamma\alpha}(p)) = g_{\alpha\beta}(p) \circ g_{\beta\gamma}(p) \circ g_{\gamma\alpha}(p) = 1 \qquad \forall\, p \in U_{\alpha\beta\gamma} \tag{3.101}$$

and then we have that $\widetilde{g}_{\alpha\beta}(p) \circ \widetilde{g}_{\beta\gamma}(p) \circ \widetilde{g}_{\gamma\alpha}(p) \in Ker(\lambda) = \{\pm 1\}$. However if the transition functions $\widetilde{g}_{\alpha\beta}$ have to define a bundle, they must obey also the cocycle conditions

$$\widetilde{g}_{\alpha\beta}(p) \circ \widetilde{g}_{\beta\gamma}(p) \circ \widetilde{g}_{\gamma\alpha}(p) = 1 \qquad \forall\, p \in U_{\alpha\beta\gamma} \tag{3.102}$$

The bundle defined with the lift of the cocycle is the **Spin bundle $\mathbf{Spin(E)}$**, and can be represented as in Figure 3.1.

Moreover

$$\begin{tikzcd} Spin(E) \arrow[rr, "\widetilde{Ad}"] \arrow[dr] & & SO(E) \arrow[dl] \\ & \mathrm{M} & \end{tikzcd}$$

Figure 3.1: A Spin bundle.

Definition 3.2.2. Let $\pi : E \to M$ be a vector bundle, and let $Spin(E)$ be the spin bundle constructed by lifting the cocycle of the bundle $SO(E)$, with respect to the map defined in Equation (3.58). A **Spin structure on E** is given by a principal morphism which we also call $\lambda : Spin(E) \to SO(E)$ with respect to λ.

In dimension 2, the Spin group $Spin(V)$ has to be replaced by $SO(2) \cong U(1)$. Finally

Definition 3.2.3. A **Spin manifold** is an oriented smooth manifold M endowed with a Spin structure on its tangent bundle T.

If a lift exists, then it is not unique: a Spin manifold can admit many Spin structures.

3.2.2 Obstructions to Spin structures

A lift may not exist due to topological obstructions. This fact is encoded in the *Stiefel-Whitney classes*, which arise in the study of Čech cohomology.

Definition 3.2.4. Let M be a smooth manifold and let $\{U_\alpha\}_{\alpha\in I_r}$ be open sets in M such that $U_0\cap\cdots\cap U_r \neq \{\varnothing\}$. A map $f : U_0\cap\cdots\cap U_r \to \mathbb{Z}_2$ is a **Čech r-cochain** if $\forall\, P \in \mathcal{P}$, where $\mathcal{P}$ denotes the permutation group of the elements $\{0,\ldots,r\}$

$$f(i_0,\ldots,i_r) = f(i_{P(0)},\ldots,i_{P(r)}) \tag{3.103}$$

We will denote the multiplicative group of Čech r-cochains by $C^r(M,\mathbb{Z}_2)$.

We can define a coboundary operator $\delta : C^r(M,\mathbb{Z}_2) \to C^{r+1}(M,\mathbb{Z}_2)$ such that

$$(\delta f)(i_0,\ldots,i_{r+1}) = \prod_{\alpha=0}^{r+1} f(i_0,\ldots,\hat{i}_\alpha,\ldots,i_{r+1}) \tag{3.104}$$

where as usual the hat denotes the absence of the element. For example

$$(\delta f_0)(i_0,i_1) = f_0(i_1)f_0(i_0) \qquad f_0 \in C^0(M,\mathbb{Z}_2) \tag{3.105}$$

$$(\delta f_1)(i_0,i_1,i_2) = f_1(i_1,i_2)f_1(i_0 i_2)f_1(i_0,i_1) \qquad f_1 \in C^1(M,\mathbb{Z}_2) \tag{3.106}$$

δ is trivially nilpotent, namely $\delta^2 f = 1$. In fact

$$(\delta^2 f)(i_0,\ldots,i_{r+2}) = \prod_{j=0}^{r+2} \prod_{\substack{k=0\\ k\neq j}}^{k=r+2} f(i_0,\ldots,\hat{i}_k,\ldots,\hat{i}_j,\ldots,i_{r+2}) = 1 \tag{3.107}$$

since for each $\bar{j},\bar{k}$ such that $f(i_0,\ldots,\hat{i}_{\bar{k}},\ldots,\hat{i}_{\bar{j}},\ldots,i_{r+2})$ appears in the product, then also $f(i_0,\hat{i}_{\bar{j}},\ldots,\hat{i}_{\bar{k}},\ldots,i_{r+2})$ appears in the product. By using the symmetry of Equation (3.103) the latter has the same sign as the former, and then the final result is always $+1$.

The next step is to define

$$Z^r(M,\mathbb{Z}_2) = \{f \in C^r(M,\mathbb{Z}_2) \mid \quad \delta f = 1\} \tag{3.108}$$

$$B^r(M,\mathbb{Z}_2) = \{f \in C^r(M,\mathbb{Z}_2) \mid \quad \exists\, g \in C^{r+1}(M,\mathbb{Z}_2); \quad f = \delta g\} \tag{3.109}$$

$Z^r(M,\mathbb{Z}_2)$ is the *cocycle group*, while $B^r(M,\mathbb{Z}_2)$ is the *coboundary group*. As usual define the **Čech r-cohomology group** as

$$H^r(M,\mathbb{Z}_2) = {Z^r(M,\mathbb{Z}_2)}\big/{B^r(M,\mathbb{Z}_2)} \tag{3.110}$$

The Stiefel classes are equivalence classes of the Čech cohomology group $H^r(M,\mathbb{Z}_2)$. We will see that the first two of these classes are related to obstructions occurring in the orientability of a smooth manifold and in the presence of Spin structures.

Now consider an orthonormal frame bundle $O(E)$ over the smooth manifold M ($dim(M) = n$) endowed with a metric whose signature is σ. Let $\{U_\alpha\}_{\alpha\in I}$ be a simple covering of M, namely a covering such that $\forall U_1 \cap \cdots \cap U_r \neq \{\varnothing\}$ $(U_1,\dots,U_r \in \{U_\alpha\}_{\alpha\in I})$ $U_1 \cap \cdots \cap U_r$ is contractible. Let us denote by $\{e^\alpha_a(p)\}_{a\in I_n}$ a orthonormal frame in U_α. The transition functions are given as usual by functions $g_{\alpha\beta} : U_{\alpha\beta} \to O(\sigma,\mathbb{R})$ such that $e^\beta_a(p) = e^\alpha_b(p)(g_{\alpha\beta}(p))^b{}_a$.

Then we can define the Čech 1-cochain f as

$$f(\alpha,\beta) = \det(g_{\alpha\beta}) = \pm 1 \tag{3.111}$$

Since $f(\alpha,\beta) = f(\beta,\alpha)$, then $f \in C^1(M,\mathbb{Z}_2)$. Moreover as a consequence of the cocycle condition we find

$$(\delta f)(\alpha,\beta,\gamma) = f(\beta,\gamma)f(\alpha,\gamma)f(\alpha,\beta) = \det(g_{\beta\gamma})\det(g_{\alpha\gamma})\det(g_{\alpha\beta}) = \det(g_{\alpha\beta}\circ g_{\beta\gamma}\circ g_{\gamma\alpha}) = 1 \tag{3.112}$$

and then $f \in Z^1(M,\mathbb{Z}_2)$ defines an equivalence class $[f] = w_1(E) \in H^1(M,\mathbb{Z}_2)$, called the **first Stiefel-Whitney class of E**. It doesn't depend on the choice of the local orthonormal frame $\{e^\alpha_a(p)\}_{a\in I_n}$ in U_α. In fact let $\{\tilde e^\alpha_a(p)\}$ be another local orthonormal frame in U_α such that $\tilde e^\alpha_a(p) = e^\alpha_b(p)(h^\alpha)^b{}_a$ where $h^\alpha \in O(\sigma,\mathbb{R})$. Then there exists a set of new transition functions $\{\tilde g_{\alpha\beta}(p)\}$ such that $\tilde e^\beta_a(p) = (\tilde g_{\alpha\beta}(p))_a{}^b\tilde e^\alpha_b(p)$. By substituting the expression which gives $\tilde e^\alpha_a(p)$ as a function of $e^\alpha_a(p)$ we get that $\tilde g_{\alpha\beta} = (h^\alpha)^T g_{\alpha\beta}(h^\beta)^{-T}$. Now we can define the 0-cochain f_0 simply by $f_0(\alpha) = \det(h_\alpha)$ and then

$$\tilde f(\alpha,\beta) = \det((h^\alpha)^T g_{\alpha\beta}(h^\beta)^{-T}) = \det(h^\alpha)\det(h^\beta)\det(g_{\alpha\beta}) = (\delta f_0)(\alpha,\beta)f(\alpha,\beta) \tag{3.113}$$

where we used the fact that $h^\alpha, h^\beta \in O(n,\mathbb{R})$. Since $\tilde f$ changes by an exact term δf_0 under a change of the local orthonormal frame, then it defines the same cohomology class of f, namely $[f] \in H^1(M,\mathbb{Z}_2)$ [19].

Now we give the important result concerning the first Stiefel-Whitney class, which shows us that it is an obstruction for the orientability of the vector bundle E.

Proposition 3.2.1. Let $\pi : E \to M$ be a vector bundle. Then E is orientable if and only if the first Stiefel-Whitney class is trivial, i.e. $w_1(E) = 1$.

In fact, if the manifold M is orientable the structure group can be reduced to $SO(n,\mathbb{R})$. Then $\forall\,\alpha,\beta$ we have that $f(\alpha,\beta) = det(g_{\alpha\beta}) = 1$ and we can conclude that $w_1(M) = 1$. Conversely, if the first Stiefel-Whitney class is trivial $w_1(M) = f(\alpha,\beta) = 1$, then f is a coboundary, namely $f = \delta f_0$, where f_0 has been defined above. Since $f_0(\alpha) = \pm 1$, we can always choose $h_\alpha \in O(n,\mathbb{R})$ such that $det(h_\alpha) = f_0(\alpha)$ for each α. Then if we define a new local orthonormal frame in each U_α such that $\tilde e^\alpha_a(p) = e^\alpha_b(p)(h^\alpha)^b{}_a$, the new transition functions $\tilde g_{\alpha\beta}$ are such that $\tilde f(\alpha,\beta) = det(\tilde g_{\alpha\beta}) = +1$ for each α,β and then the manifold is orientable.

Moreover it can be shown [37] that if E is orientable, then the distinct orientations on E are in one-to-one corrispondence with elements of $H^0(M,\mathbb{Z}_2)$. This is a general property of Čech cohomology, as we will check below.

Now let us study when an orientable vector bundle E admits Spin structures. It is well known that the orientability allows us to reduce the structure group to $SO(V)$ and then we can consider the special orthonormal frame bundle $SO(E)$. Next define the Čech 2-cochain $f : U_{\alpha\beta\gamma} \to \mathbb{Z}_2$ as

$$\tilde g_{\alpha\beta}(p)\circ\tilde g_{\beta\gamma}(p)\circ\tilde g_{\gamma\alpha}(p) = f(\alpha,\beta,\gamma)1 \tag{3.114}$$

which is obviously symmetric and 1 represents the identity over the fiber of $SO(E)$. It is also closed, in fact

$$\begin{aligned}(\delta f)(\alpha,\beta,\gamma,\delta) &= f(\beta,\gamma,\delta)f(\alpha,\gamma,\delta)f(\alpha,\beta,\gamma) = \\ &= (\tilde g_{\beta\gamma}(p)\circ\tilde g_{\gamma\delta}(p)\circ\tilde g_{\delta\beta}(p))(\tilde g_{\alpha\gamma}(p)\circ\tilde g_{\gamma\delta}(p)\circ\tilde g_{\delta\alpha}(p))(\tilde g_{\alpha\beta}(p)\circ\tilde g_{\beta\gamma}(p)\circ\tilde g_{\gamma\alpha}(p)) = 1\end{aligned} \tag{3.115}$$

Then it defines an equivalence class $[f] = w_2(E) \in H^2(M,\mathbb{Z}_2)$, which is called the **second Stiefel-Whitney class of E**. As before, it can be shown that $w_2(E)$ is independent from the local orthonormal frame chosen. The second Stiefel-Whitney class represents an obstruction for a manifold to be a Spin manifold, as stated in the following [19]

Proposition 3.2.2. Let $\pi : E \to M$ be an orientable vector bundle. E admits Spin structures if and only if the second Stiefel-Whitney class is trivial, i.e. $w_2(E) = 1$.

In fact, let us suppose that E admits Spin structures. Then there are transitions functions $\widetilde{g}_{\alpha\beta}$ for each α, β such that

$$\widetilde{g}_{\alpha\beta} \circ \widetilde{g}_{\beta\gamma} \circ \widetilde{g}_{\gamma\alpha} = 1 \tag{3.116}$$

Then the second Stiefel-Whitney class is trivial $w_2(E) = f(\alpha, \beta, \gamma) = \widetilde{g}_{\alpha\beta} \circ \widetilde{g}_{\beta\gamma} \circ \widetilde{g}_{\gamma\alpha} = 1$. Conversely, let us suppose that $w_2(E)$ is trivial. Then it is a coboundary, namely

$$f(\alpha, \beta, \gamma) = (\delta f_1)(\alpha, \beta, \gamma) = f_1(\alpha, \beta) f_1(\beta, \gamma) f_1(\gamma, \alpha) \tag{3.117}$$

where $f_1 \in C^2(M, \mathbb{Z}_2)$ and $f_1(\alpha, \beta) = sign(g_{\alpha\beta})$. Now let us redefine the transition functions as $\widetilde{g}'_{\alpha\beta} = f_1(\alpha, \beta)\widetilde{g}_{\alpha\beta}$. Then the second Stiefel-Whitney class takes the form

$$w_2(E) = f(\alpha, \beta, \gamma) = ((\delta f_1)(\alpha, \beta, \gamma))^2 = +1 \tag{3.118}$$

and eventually we can conclude that the new transition functions define a Spin bundle.

It's important to notice that the existence of Spin structures on a vector bundle $\pi : E \to M$ doesn't depend on the presence of a metric on it, but only on its topological properties [37]. In particular, it strongly depends on the Holonomy of the vector bundle $\pi : E \to M$.

We conclude this Section by giving the following

Definition 3.2.5. Let $\pi : E \to M$ be a vector bundle such that $w_2(E) = 0$. A **real spinor bundle on E** is the associated bundle

$$S(M) = Spin(E) \times_\mu W \tag{3.119}$$

where $Spin(E)$ is the Spin bundle over E, W is a left module for $C(\mathbb{R}^n)$ and $\mu : Spin(V) \to SO(W)$ is the representation given by left multiplication by elements of $Spin(V) \subset C^+(V)$.

Similarly a **complex spinor bundle on E** is the associated bundle

$$S_{\mathbb{C}}(E) = Spin(E) \times_\mu (W \times \mathbb{C}) \tag{3.120}$$

where $W \times \mathbb{C}$ is a complex left module for $C(\mathbb{R}^n) \times \mathbb{C}$.

3.2.3 Vielbeins

The formalism of spinors developed so far will be used diffusely in the following of the work. However, in order to perform calculus with spinors it's useful to study also the *vielbein formalism*. Actually, it is not something new.

Let us consider the frame bundle LM on a smooth manifold M. We recall that a frame on a smooth manifold is just a basis of the tangent bundle

$$\hat{e}_a(p) = \{\hat{e}_1, \ldots, \hat{e}_n\} \tag{3.121}$$

and the coframe bundle is just a basis of the coframe bundle

$$e^a(p) = \{e^1, \ldots, e^n\} \tag{3.122}$$

such that

$$e^a(\hat{e}_b) = \delta^a{}_b \tag{3.123}$$

as we have seen in Example 2.1.6.

Now let us just reduce the structure group of the tangent bundle following the pattern

$$GL(n, \mathbb{R}) \quad \hookrightarrow \quad SO(n, \mathbb{R}) \tag{3.124}$$

so that a Riemannian structure is defined on M. Then a **vielbein** is just a frame like in Equation (2.93) whose vectors are orthonormal. As we will see in Section 5.3.2, if we consider a generic manifold with structure group $\hat{G}$ for the tangent bundle a vielbein can be obtained by reduction of $\hat{G}$ to its compact maximal.

The Riemannian metric can be easily recovered. We can write

$$e^a(p) = e^a{}_i dx^i \qquad e_a(p) = e^i{}_a \frac{\partial}{\partial x^i} \tag{3.125}$$

and then

$$g = e^T e \qquad g_{ij} = e^a{}_i e^b{}_j \delta_{ab} \tag{3.126}$$

$$g^{-1} = \hat{e}\hat{e}^T \qquad g^{ij} = \hat{e}^i_a \hat{e}^j_b \delta^{ab} \tag{3.127}$$

The introduction of the vielbeins allows us to better handle spinors. In fact we can immediately define the **spin connection** via

$$\nabla_i e^a{}_j = \partial_i e^a{}_j - \Gamma^k{}_{ij} e^a{}_k + \omega^a{}_{ib} e^b{}_j \tag{3.128}$$

which leads to the following expression for its components

$$\omega_i{}^{ab} = \frac{1}{2} \left(\Omega_{ijk} - \Omega_{jki} + \Omega_{kij} \right) e^{ja} e^{kb} \tag{3.129}$$

where

$$\Omega_{ijk} = \left(\partial_i e^a{}_j - \partial_j e^a{}_i \right) e_{ak} \tag{3.130}$$

The vielbeins allows us to define also *curved gamma matrices*

$$\Gamma^a = e^a{}_i \Gamma^i \qquad \Gamma_a = e_a{}^i \Gamma_i \tag{3.131}$$

And finally we ccan give an expression for the covariant derivative of the spinor fields

$$\nabla_i = \partial_i + \frac{1}{4} \omega_i{}^{ab} \Gamma_{ab} \tag{3.132}$$

This is very important because it gives us the possibility to define the **Killing vectors fields** η such that

$$\nabla \eta = 0 \tag{3.133}$$

Killing vector field are one of the most important objects in studying compactifications of Superstring theories.

3.3 Supersimmetry in Superstrings

It is well known that the realistic String theories are the supersymmetric ones, since they allow for the existence of fermions in their spectrum.

4 Complex geometry

In the present Chapter we will give a brief introduction to the complex geometry (CG).

We will analyze some special examples of complex manifolds: the *Hermitian manifolds*, the Kähler *manifolds* and the *Calabi-Yau* ones. Such classes of complex manifolds are distinguished by some particular constraints on the metric.

Calabi-Yau manifolds are a subclass of Kähler manifolds, which in turn are a subclass of Hermitian manifolds. Needless to say the Calabi-Yau constraints on the metric are more restrictive then the Kähler constraints, which in turn are more restrictive then the Hermitian ones. We will introduce also the symplectic manifolds.

Kähler and symplectic manifolds are only a subclass of the most important object we will study in the present Chapter: a *$SU(3)$-structure*. In fact as we will see, the $SU(3)$-structures allow us to fully classify the Superstring backgrounds with H-fluxes which preserve four dimensional minimal supersymmetry, namely $N = 1$.

It turns out that for all type IIB Superstrings vacua with $SU(3)$ structure, the internal manifold is complex, while for type IIA Superstrings vacua both complex and symplectic manifolds are allowable. This fact suggests that it would be far convenient to have a unifying description of these two kind of geometry. This idea leads naturally to the study of Generalized Complex Geometry in Chapter 5.

4.1 Complex manifolds

In analogy with the smooth case studied in Section 2, a complex manifold is a set which locally looks like an open set in $\mathbb{C}^n$. This time the gluing of charts has to be holomorphic. In fact

Definition 4.1.1. Let M be a smooth manifold such that $dim(M) = 2n$ is even ($n \in \mathbb{N}$). Let $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ be a smooth atlas over M. After having identified $\mathbb{R}^{2n} \cong \mathbb{C}^n$ via Equation (C.6), if $\forall\, \alpha, \beta \in I$ such that $U_{\alpha\beta} \neq \{\varnothing\}$ the homeomorphisms

$$\begin{aligned} \varphi_{\alpha\beta} &: \varphi_\beta(U_{\alpha\beta}) \to \varphi_\alpha(U_{\alpha\beta}) \\ \varphi_{\alpha\beta} &= \varphi_\alpha \circ \varphi_\beta^{-1} \end{aligned} \tag{4.1}$$

are holomorphic maps, $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ is the *holomorphic atlas*. If M admits a holomorphic atlas, it is a **complex manifold**. We define $dim_{\mathbb{C}}(M) = \frac{1}{2} dim(M) = n$ [30, 22].

The main difference between a smooth manifold of even dimension and a complex manifold is that on a complex manifold the transition functions $\varphi_{\alpha\beta}$ are holomorphic maps, while in a smooth manifold they are only smooth maps. This means that the transition functions don't mix holomorphic coordinates with antiholomorphic ones. Needless to say, each complex manifold is also a smooth manifold.

We generalize the concept of holomorphic maps

Definition 4.1.2. Let M be a complex manifold with holomorphic atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$. The function $f : M \to \mathbb{C}$ is an **holomorphic function** if $\forall\, \alpha \in I$ the map $f \circ \varphi_\alpha^{-1}$ is a holomorphic map.

Needless to say each holomorphic map $f : M \to \mathbb{C}$ is also a smooth map up to the identification in Equation (C.6), but the converse is not true.

A remarkable point is that each complex manifold is orientable. Let us study the simple case with $dim_{\mathbb{C}} M = 1$ since the general case is similar and only more complicated from the notational point of view.

Consider two arbitrary charts (U, φ) and (V, ψ) of the holomorphic structure on the manifold M, such that $U \cap V \neq \{\varnothing\}$. Let $\{x, y\}$ the set of local real coordinates determined by the first chart. After the identification in Equation (C.6) the transition functions are maps between open sets in $\mathbb{R}^2$, namely

$$\Phi \equiv \psi \circ \varphi^{-1} : \varphi(U \cap V) \to \varphi(U \cap V) \tag{4.2}$$

If we write $\Phi(p) = u + iv$, for each $p \in U \cap V$ the differential of Φ is given by

$$d\Phi(p) = \begin{pmatrix} \frac{\partial u}{\partial x}|_p & \frac{\partial u}{\partial y}|_p \\ \frac{\partial v}{\partial x}|_p & \frac{\partial v}{\partial y}|_p \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x}|_p & \frac{\partial u}{\partial y}|_p \\ -\frac{\partial u}{\partial y}|_p & \frac{\partial u}{\partial x}|_p \end{pmatrix} \tag{4.3}$$

for each $p \in U \cap V$. Then $\det(d\Phi(p)) = \left(\frac{\partial u}{\partial x}\Big|_p \right)^2 + \left(\frac{\partial u}{\partial y}\Big|_p \right)^2 > 0$, which assures the orientability as we know from Definition 2.1.16.

4.1.1 Almost complex structures

An almost complex manifold is an object which is halfway between a smooth manifold and a complex one. It has the virtue of introducing the almost complex structure, which is one of the most important objects in the whole CG, even if it needs an integrability condition in order to define a complex structure on a smooth manifold.

Definition 4.1.3. Let M be a smooth manifold such that $dim(M) = n$. A tensor $J \in T^1_1$ such that $J^2 = -1_T$ is an **almost complex structure**. If a smooth manifold M admits an almost complex structure J then (M, J) is an **almost complex manifold**.

We can see J as an endomorphism of the tangent bundle $J \in End(T)$, i.e.

$$J : T \to T \tag{4.4}$$

The condition $J^2 = -1_T$ means that J is nothing but the transposition over a smooth manifold of the *conjugation* map. Moreover it fixes a constraint on the dimension of an almost complex manifold M. In fact it implies that $(\det(J))^2 = \det(J^2) = \det(-1_T) = (-1)^n$. Since J is a real tensor we obtain that $\det(J)$ has to be real and then $(\det(J))^2 = (-1)^n$ has to be positive. It follows that n is even.

Next, let us define the almost complex structure J on a chart (U, φ) of an almost complex manifold M such that $dim(M) = 2n$. It is

$$J|_U = \varphi_*^{-1} \circ j \circ \varphi_*(X) \tag{4.5}$$

It's evident that it satisfies $J|_U^2 = -1_{T|_U}$, in fact

$$\begin{aligned} J|_U^2 &= (\varphi_*^{-1} \circ j \circ \varphi_*) \circ (\varphi_*^{-1} \circ j \circ \varphi_*) = \\ &= \varphi_*^{-1} \circ j \circ (\varphi_* \circ \varphi_*^{-1}) \circ j \circ \varphi_* = \varphi_*^{-1} \circ (j \circ j) \circ \varphi_* = \\ &= -\varphi_*^{-1} \circ \varphi_* = -1_{T|_U} \end{aligned} \tag{4.6}$$

From the form of j we know that $J|_U$ isn't diagonalizable over $T|_U$, since each T_pM (for each $p \in U$) is a real vector space. In order to be able to diagonalize $J|_U$ we have to complexify $T|_U$ and $T|_U^*$, obtaining respectively $T|_U^{\mathbb{C}}$ and $T|_U^{*\,\mathbb{C}}$. Then the action of $J|_U$ can be naturally extended on $T|_U M^{\mathbb{C}}$, so that it is still subject to the constraint $J|_U^2 = -1_{T|_U^{\mathbb{C}}}$, but it can now be diagonalized. The only allowed eigenvalues are $\pm i$, and they have the same multiplicity, so that extendin for all the open sets $\{U_\alpha\}$ of an atlas, J induces the decomposition

$$T^{\mathbb{C}} = T^{1,0} \oplus T^{0,1} \tag{4.7}$$

where $T^{1,0} = \{X \in T^{\mathbb{C}} | \ \ J(X) = +iX\}$, while $T^{0,1} = \{X \in T^{\mathbb{C}} | \ \ J(X) = -iX\}$. It is evident that the relations $\overline{T^{1,0}} = T^{0,1}$ and $\overline{T^{0,1}} = T^{1,0}$ hold. We can naturally define a pair of projectors on the eigenspaces of J

$$P^{1,0} = \frac{1}{2}(1 - iJ) \qquad P^{0,1} = \frac{1}{2}(1 + iJ) \tag{4.8}$$

satisfying

$$(P^{1,0})^2 = P^{1,0} \qquad (P^{0,1})^2 = P^{0,1} \qquad P^{1,0} + P^{0,1} = 1_{T^{\mathbb{C}}} \qquad P^{1,0}P^{0,1} = 0 \tag{4.9}$$

Obviously $P^{1,0}$ projects elements in the fiber of the tangent bundle T on $T^{1,0}$, while $P^{0,1}$ projects on $T^{0,1}$. Elements in $\mathfrak{X}^{1,0} \equiv \mathfrak{X}^{1,0}$ are called *holomorphic vectors*, while elements in $\mathfrak{X}^{0,1} \equiv \mathfrak{X}^{0,1}$ are called *antiholomorphic vectors*.

Let us remember that all the last relation are allowed only locally. This means that, for each U_α we can write the decomposition in Equation (4.7) but that the almost complex structure is defined only locally and in general cannot be patched from a chart to another.

Now we want to write the explicit matrix expression for $J|_U$. We can choose a real basis of $T|_U$

$$\left\{ \frac{\partial}{\partial x^\mu}\bigg|_U , \frac{\partial}{\partial y^\mu}\bigg|_U \right\}_{j \in I_n} \tag{4.10}$$

such that, as Equation (4.5) suggests

$$J|_U = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{4.11}$$

where 0 and 1 represent respectively the null and the identity $n \times n$ matrices. If we combine the basis vectors as in Equation (C.9) obtaining a complex basis of T_pM, $\left\{ \frac{\partial}{\partial z^\mu}\big|_p , \frac{\partial}{\partial \bar{z}^\mu}\big|_p \right\}_{\mu \in I_n}$, the matrix expression becomes

$$J|_U = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{4.12}$$

Again 0 and i represent respectively the null and the i times identity $n \times n$ matrices. In other words, J_p takes the nice forms, respectively in real and complex basis

$$J|_U = \frac{\partial}{\partial y^\mu} \otimes dx^\mu - \frac{\partial}{\partial x^\mu} \otimes dy^\mu \qquad \mu \in I_n \tag{4.13}$$

$$J|_U = i\frac{\partial}{\partial z^\mu} \otimes dz^\mu - i\frac{\partial}{\partial \bar{z}^\mu} \otimes d\bar{z}^\mu \qquad \mu \in I_n \tag{4.14}$$

Equation (4.14) shows the *standard form* of the almost complex structure. Remember that the last properties are allowed only locally, namely in a given chart. However, if M is a complex manifold, then the almos complex structure J encode the whole holomorphic structure of M. In fact if (U, φ) and (V, ψ) are two charts of M, we can write

$$\begin{aligned} J|_U(X) &= \varphi_*^{-1} \circ j \circ \varphi_*(X) = \varphi_*^{-1} \circ j \circ \varphi_* \circ \left(\psi_*^{-1} \circ \psi_*\right)(X) = \\ &= \varphi_*^{-1} \circ j \circ \left(\varphi_* \circ \psi_*^{-1}\right) \circ \psi_*(X) = \varphi_*^{-1} \circ \left(\varphi_* \circ \psi_*^{-1}\right) \circ j \circ \psi_*(X) = \\ &= \psi_*^{-1} \circ j \circ \psi_*(X) = J_V(X) \end{aligned} \tag{4.15}$$

where we have used that the transition function $\varphi \circ \psi^{-1}$ is a holomorphic map since M is a complex manifold. In other words we have seen that only if the manifold M is endowed with an holomorphic structure, then the almost complex structure is patchable to define a tensor on M.

Moreover, the last line shows us that

Proposition 4.1.1. Let M be a complex manifold. Then (M, J) is almost complex.

In fact, it is sufficient to define J in every charts as in Equation (4.11).

4.1.2 Integrability

What we are going to study is strictly related to a well known problem in General Relativity: the equivalence principle. In fact if (M, g) is a Riemannian manifold we know from the linear algebra that $\forall p \in M$, g can be diagonalized (it is a symmetric tensor). One may wonder whether g can take its standard form (namely the flat minkowskian metric $\eta_{\mu\nu}$) in a whole open neighbour U such that $p \in U$. It is well known that the necessary and sufficient condition is the vanishing of the curvature tensor R (we have to require also metric compatibility and vanishing of the torsion [23], which are two standard requirements in General Relativity) in the open set U. In this case we say that a *flat coordinate system* can be chosen in U. Coming back to our current task, we want to determine the necessary condition for M to be a complex manifold. We will see that it is equivalent to require that the the almost complex structure J can be written in its standard form in a whole open neighbour U.

Let us start with the concept of integrability in the complex case:

Definition 4.1.4. Let (M, J) be an almost complex manifold with almost complex structure J. If

$$[X, Y] \in \mathfrak{X}^{1,0} \qquad \forall\, X, Y \in \mathfrak{X}^{1,0} \tag{4.16}$$

then J is **integrable**.

Definition 4.1.5. Let $N \in T^1_2$. We can see it as a map $N : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, defined by

$$N(X, Y) = [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY] \qquad \forall\, X, Y \in \mathfrak{X}(M) \tag{4.17}$$

N is the **Nijenhuis tensor**.

We immediately notice that, if the manifold is complex, namely if we can put J in its standard form everywhere, then N vanishes.

Now we will give two important results. The first [19, 30] creates a link between the integrability and the Nijenhuis tensor

Proposition 4.1.2. Let (M, J) be an almost complex manifold with almost complex structure J. Then J is integrable if and only if $N \equiv 0$.

In fact, let $X, Y \in \mathfrak{X}(M)$, and let us define $Z = [\frac{1}{2}(1 - iJ)X, \frac{1}{2}(1 - iJ)Y]$. It's immediate to see that $\frac{1}{2}(1 + iJ)Z = \frac{1}{2}(1 + iJ)N(X, Y)$. Then $Z \in \mathfrak{X}^{1,0}$ if and only if $N(X, Y) = 0 \quad \forall\, X, Y \in \mathfrak{X}(M)$.

Finally, the theorem which furnishes the necessary condition for an almost complex structure to be complex

Theorem 6. *Newlander-Niremberg theorem*
Let (M, J) be an almost complex manifold. J is integrable if and only if M is a complex manifold.

4.1.3 Holomorphic forms

It's quite intuitive that also the cotangent bundle $\Omega^1(M)$ can be decomposed on a complex manifold M. In fact since $J \in T^1_1$, we can see it as an endomorphism of the tangent bundle ($J \in End(T)$) as well as an endomorphism of the cotangent bundle, namely $J \in End(T^*)$. As a consequence we can use the projectors in Equation (4.8) also to project on the cotangent bundle (now the projectors act by right multiplication), and the following decomposition is given

$$T^{*\mathbb{C}} = T^{*\,1,0} \oplus T^{*\,0,1} \tag{4.18}$$

The relations $\overline{T^{*\,1,0}} = T^{*\,0,1}$ and $\overline{T^{*\,0,1}} = T^{*\,1,0}$ hold. Elements in $\Gamma(T^{*\,1,0}) \equiv \Omega^{1,0}(M)$ are called *holomorphic one-forms* while elements in $\Gamma(T^{*\,0,1}) \equiv \Omega^{0,1}(M)$ are called *antiholomorphic one-forms*.

Let M be an almost complex manifold such that $dim(M) = 2n$. The **complexified forms** are elements $\phi \in \Lambda T^{*\mathbb{C}}$ where

$$\Lambda T^{*\mathbb{C}} = \{\phi = \omega + i\tau| \quad \omega, \tau \in \Lambda T^*\} \equiv \bigoplus_{k=0}^{2n} \Lambda^k T^{*\mathbb{C}} \tag{4.19}$$

In particular we can set $P^{1,0}(\Lambda^1 T^{*\mathbb{C}}) = \Lambda^{1,0} T^*$ and $P^{0,1}(\Lambda^1 T^{*\mathbb{C}}) = \Lambda^{0,1} T^*$ with the obvious identifications $\Lambda^{1,0} T^* \equiv T^{*\,1,0}$, $\Lambda^{0,1} T^* \equiv T^{*\,0,1}$ and we have that

$$T^{*\mathbb{C}} = \Lambda^1 T^{*\mathbb{C}} = \Lambda^{1,0} T^* \oplus \Lambda^{0,1} T^* \tag{4.20}$$

Next we can define

$$\Lambda^{k,0}T^* = \bigwedge_{i=0}^{k} \Lambda^{1,0}T^* \qquad \Lambda^{0,k}T^* = \bigwedge_{i=0}^{k} \Lambda^{0,1}T^* \tag{4.21}$$

and since the following result holds

$$\Lambda^k(V \oplus W) \cong \bigwedge_{i=0}^{k} \Lambda^i V \otimes \Lambda^{k-i} W \tag{4.22}$$

then we get

$$\Lambda^k T^{*\mathbb{C}} \cong \bigwedge_{p+q=k} \Lambda^{p,q}T^* \tag{4.23}$$

If $\phi \in \Lambda^{p,q}T^*$ then ϕ is a $(\mathbf{p},\mathbf{q})$**-form** and $\Lambda^{0,0}T^*$ is the space of smooth functions over M which takes value in $\mathbb{C}$, namely $C^\infty_{\mathbb{C}}(M)$. Let us notice that the relation $\overline{\Lambda^{p,q}T^*} = \Lambda^{q,p}T^*$ holds.

Let (M, J) be a complex manifold such that $dim_{\mathbb{C}}(M) = n$. Let $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ be a holomorphic atlas which determines the local coordinates $\{x^\mu_\alpha, y^\mu_\alpha\}_{\mu \in I_n}$. Since M is a complex manifold we can define the complex coordinates in each U_α as $z^\mu_\alpha = x^\mu_\alpha + iy^\mu_\alpha$, $\overline{z}^\mu_\alpha = x^\mu_\alpha - iy^\mu_\alpha$ and since we have $dz^\mu_\alpha = dx^\mu_\alpha + idy^\mu_\alpha$ and $d\overline{z}^\mu_\alpha = dx^\mu_\alpha - idy^\mu_\alpha$ then the sets

$$\{dz^\mu_\alpha\}_{\mu \in I_n} \qquad \{d\overline{z}^\mu_\alpha\}_{\mu \in I_n} \tag{4.24}$$

represent respectively a basis of $\Lambda^{1,0}T^*$ and of $\Lambda^{0,1}T^*$. More in general, a basis for $\Lambda^{p,q}T^*$ is given by the set

$$\{dz^{\mu_1}_\alpha \wedge \cdots \wedge dz^{\mu_p}_\alpha \wedge d\overline{z}^{\nu_1}_\alpha \wedge \cdots \wedge d\overline{z}^{\nu_q}_\alpha\}_{\mu_i, \nu_j \in I_n} \tag{4.25}$$

and then each $\phi \in \Lambda^{p,q}T^*$ locally on U_α takes the form

$$\phi = \frac{1}{p!q!} \sum_{\mu_i, \nu_j \in I_n} \phi_{\mu_1 \ldots \mu_p \nu_1 \ldots \nu_q} dz^{\mu_1}_\alpha \wedge \ldots dz^{\mu_p}_\alpha \wedge d\overline{z}^{\nu_1} \wedge \cdots \wedge d\overline{z}^{\nu_q} \tag{4.26}$$

Now we will begin to study how differential operators behaves on a complex manifold. Let us give immediately the following

Proposition 4.1.3. Let (M, J) be an almost complex manifold such that $dim(M) = 2n$. J is integrable if and only if $d\Lambda^{1,0}T^* \subset \Lambda^{2,0}T^* \oplus \Lambda^{1,1}T^*$.

In other words, the integrability condition is equivalent to the requirement that the $(0,2)$-component of $d\omega$ (where $\omega \in \Lambda^{1,0}T^*$) vanishes, namely if $d\omega(X,Y) = 0 \quad \forall X, Y \in \mathfrak{X}^{0,1}$. It follows from the fact that $d\omega(X,Y) = (i_X d\omega)(Y) = (\mathfrak{L}_X\omega)(Y) - d(i_X\omega)(Y) = \mathfrak{L}_X(\omega(Y)) - \omega(\mathfrak{L}_X Y) - (i_Y d)(\omega(X)) = (i_X d)(\omega(Y)) - \omega([X,Y]) - (i_Y d)\omega(X))$, from which we find that $d\omega(X,Y) = -\omega([X,Y]) = 0$ if and only if $[X,Y] \in \mathfrak{X}^{0,1}$, being $\omega \in \Lambda^{1,0}T^*$.

Using Proposition 4.1.3 we can define the **Dolbeault operators**

$$\partial : \Lambda^{p,q}T^* \to \Lambda^{p+1,q}T^*$$
$$\overline{\partial} : \Lambda^{p,q}T^* \to \Lambda^{p,q+1}T^*$$

where

$$d = \partial + \overline{\partial} \tag{4.27}$$

The following identities

$$\partial^2 = 0 \qquad \overline{\partial}^2 = 0 \qquad \partial\overline{\partial} + \overline{\partial}\partial = 0 \tag{4.28}$$

are obvious consequence of the nilpotence of the exterior differential operator, in fact

$$d^2 = (\partial + \overline{\partial})^2 = \partial^2 + \partial\overline{\partial} + \overline{\partial}\partial + \overline{\partial}^2 = 0 \tag{4.29}$$

and the operators ∂^2, $\overline{\partial}^2$, $\overline{\partial}\partial + \partial\overline{\partial}$ take value respectively in $\Lambda^{p+2,q}T^*$, $\Lambda^{p,q+2}T^*$, $\Lambda^{p,q}T^*$, so that they have to vanish indipendently. Moreover the following odd Leibniz rule

$$\overline{\partial}(\omega \wedge \tau) = (\overline{\partial}\omega) \wedge \tau + (-1)^{p+q}\omega \wedge (\overline{\partial}\tau) \tag{4.30}$$

holds, where $\omega \in \Lambda^{p,q}T^*$, $\tau \in \Lambda^{r,s}T^*$.

Finally we can give the analogous of the Poincarè Lemma for the complex case

Lemma 4.1.1. ***$\overline{\partial}$-Poincarè's lemma***
Let $\omega \in \Lambda^{p,q}T^*$ be $\overline{\partial}$-closed, namely $\overline{\partial}\omega = 0$. Then it is $\overline{\partial}$-exact. Analogously for a ∂-closed form.

and the important

Lemma 4.1.2. ***Local $\partial\overline{\partial}$ lemma***
Let M be a complex manifold, and let $\omega \in \Lambda^{1,1}T^* \cap \Lambda^2 T^*$ a real form. Then ω is closed if and only if $\forall\, p \in M \exists\, U$ open neighbour of p and $\exists\, u \in C^\infty(M)$ such that $\omega|_U = i\partial\overline{\partial}u$.

In fact let $\omega \in \Lambda^{1,1}T^* \cap \Lambda^2 T^*$ be a closed real form. Then from the Poincarè's Lemma 4.1.1 we get that locally exists a real $\tau \in \Lambda^1 T^*$ such that $\omega = d\tau$. Since we can write the decomposition $\tau = \tau^+ + \tau^-$, and since $\overline{\tau^+} = \tau^-$ then

$$\omega = d\tau = (\partial + \overline{\partial})(\tau^+ + \tau^-) = \partial\tau^+ + (\partial\tau^- + \overline{\partial}\tau^+) + \overline{\partial}\tau^- \in \Lambda^{1,1}T^* \tag{4.31}$$

from which we get that $\overline{\partial}\tau^- = \partial\tau^+ = 0$, since they respectively belong to $\Lambda^{0,2}T^*$ and $\Lambda^{2,0}T^*$. Moreover we get that $\omega = \partial\tau^- + \overline{\partial}\tau^+$. From the $\overline{\partial}$-Poincarè Lemma we know that locally exists a function f such that $\tau^- = \overline{\partial}f$. By complex conjugation we get $\tau^+ = \partial\overline{f}$, and then

$$\omega = \partial\overline{\partial}f + \overline{\partial}\partial\overline{f} = \partial\overline{\partial}(f - \overline{f}) = 2i\Im m(\partial\overline{\partial}f) \tag{4.32}$$

where we have used Equation (4.29). Conversely we have

$$d(\partial\overline{\partial}) = (\partial + \overline{\partial})(\partial\overline{\partial}) = (\partial^2\overline{\partial} + \overline{\partial}\partial\overline{\partial}) = (\partial^2\overline{\partial} - \partial\overline{\partial}^2) = 0 \tag{4.33}$$

4.2 Kähler manifolds

One of the motivations which make the Kähler manifolds worthy to be studied is that their structure allows to write some of the most important objects defined on a complex manifold - such as for example a metric and the associated curvature - by using a unique function defined on the manifold itself. Such a kind of function is called the *Kähler* potential.

4.2.1 Symplectic manifolds

We begin with the analysis of *symplectic manifold* which turn out to be special cases of Kähler manifolds.

Definition 4.2.1. Let M be a smooth manifold such that $dim(M) = n$ equipped with a nowhere vanishing two-form $\omega \in \Lambda^2 T^*$. Then ω is the **presymplectic form** and (M, ω) is a **presymplectic manifold**. If in addition ω is a closed two-form, then it is a **symplectic form** and M is a **symplectic manifold**.

The condition of non-degeneracy is equivalent to the condition that $\forall\, k \in I_n$

$$\omega^k \equiv \bigwedge_{i=0}^{k} \omega^i \tag{4.34}$$

is nowhere vanishing.

If now we choose a local chart (U, φ), which determines a set of coordinates $\{x^\mu\}_{\mu\in I_n}$, then we can write

$$\omega = \sum_{i,j=1}^{n} \omega_{ij} dx^i \wedge dx^j \tag{4.35}$$

and the non-degeneracy can be written as

$$\det(\omega_{ij}) \neq 0 \tag{4.36}$$

ω^{ij} is the inverse of ω_{ij}

$$\omega^{ik}\omega_{kj} = \omega_{ik}\omega^{kj} = \delta^i_j \tag{4.37}$$

ω_{ij} is an antisymmetric matrix. Since each invertible antisymmetric matrix has necessarily an even number of rows and coloumns, then symplectic manifolds are even dimensional.

Let M be a symplectic manifold and let $dim(M) = 2n$. Since ω^n is nowhere vanishing it represents a volume form for M. This means that each symplectic manifold is orientable.

The following Definition will be useful later

Definition 4.2.2. Let (M, ω) be a symplectic manifold such that $dim(M) = 2n$. A submanifold $L \subseteq M$ such that $dim(L) = n$, is a **Lagrangian submanifold** if $\omega|_L = 0$.

Now we state the result which makes clear the substantial difference between symplectic and Riemannian geometry.

Theorem 7. ***Darboux theorem***
Let (M, ω) be a symplectic manifold such that $dim(M) = 2n$. Then $\forall\, p \in M$ there exists a chart (U, φ) in p which determines the set of coordinates $\{x^i, y^i\}_{i \in I_n}$ such that the symplectic form takes its canonical form

$$\omega_0 = \sum_{i=0}^{n} dx^i \wedge dy^i \tag{4.38}$$

Theorem 7 makes manifest the profound difference between the Riemannian geometry and the symplectic one. In fact in the first case, as we have seen in Section 2.2.1, we can not in general reduce the metric to the standard form in an open neighborhood around each point $p \in M$. This is due to the presence of a non-vanishing curvature, which shifts the metric from its standard value as soon as we move from the point in which we have diagonalized it. In symplectic geometry instead, there is not an object analogous to the curvature, which obstructs the symplectic form to remain in its standard form in a whole neighborhood around each point $p \in M$.

The symplectic manifolds are the suitable space to build an object which is largely known to physicists, namely the Poisson bracket. Let us give some preliminary

Definition 4.2.3. Let $(M, \omega), (N, \tau)$ be two symplectic manifolds. Let $f : M \to N$ be a diffeomorphism such that

$$f^* \tau = \omega \tag{4.39}$$

Then f is a **symplectomorphism**.

If $(M, \omega) = (N, \tau)$, then f leaves the fundamental form invariant. This is the case of classical mechanics, where symplectomorphisms are diffeomorphisms of the phase space with itself. In that case, symplectomorphisms are simply the *canonical transformations*. Then we have

Proposition 4.2.1. Let (M, ω) be a symplectic manifold. Then every smooth function $H : M \to \mathbb{R}$ determines a vector field $X_H \in \mathfrak{X}(M)$ which generates a symplectomorphism in the sense that

$$\mathfrak{L}_{X_H} \omega = 0 \tag{4.40}$$

The function H is a *Hamiltonian*, while the vector X_H is a *Hamiltonian vector field*. It's straightforward to notice that the condition in Equation (4.39) is equivalent to the requirement in Equation (4.40).

As one can see in the proof of the Proposition 4.2.1 [30], X_H is determined by the relation $i_{X_H} = dH$.

Then we can introduce

Definition 4.2.4. Let M be a smooth manifold. Let $\{,\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ be a bilinear map such that the following properties hold $\forall\, a, b, c, d \in \mathbb{R}, \quad \forall\, f, g, h, k \in C^\infty(M)$

1. Bilinearity, namely
$$\{af + bg, ch + dk\} = ac\{f, h\} + ad\{f, k\} + bc\{g, h\} + bd\{g, k\} \tag{4.41}$$

2. Skew symmetry, namely
$$\{f, g\} = -\{g, f\} \tag{4.42}$$

3. $\{,\}$ obeys the Jacobi identity
$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0 \tag{4.43}$$

4. $\{,\}$ is a derivation with respect to the first argument
$$\{fg, h\} = f\{g, h\} + \{f, h\}g \tag{4.44}$$

Then $\{,\}$ is a **Poisson bracket**.

The set $(C^\infty(M), \{,\})$ is a *Poisson algebra*, while a smooth manifold equipped with a Poisson algebra is a **Poisson manifold**.

In particular one can show that a Poisson manifold is completely determined by a bivector $\omega \in \Lambda^2 T$.

Then if (M, ω) is a symplectic manifold, as a consequence of the non-degeneracy of its fundamental form $\omega \in Z^2(M)$, a Poisson manifold structure is naturally defined on M. In fact we can define the Poisson bracket as follows

$$\{f, g\} = \omega(df, dg) = \omega^{\mu\nu} \frac{\partial f}{\partial x^\mu} \frac{\partial g}{\partial x^\nu} \qquad \forall\, f, g \in C^\infty(M) \tag{4.45}$$

where in the last Equation ω represents the bivector built up with the inverse of the fundamental form

$$\omega = \omega^{\mu\nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu} \tag{4.46}$$

In addition, if $f \in C^\infty(M)$ is a Hamiltonian, then

$$\mathfrak{L}_{X_f}(g)\omega = \{g, f\} \tag{4.47}$$

Hence the following Proposition is quite obvious

Proposition 4.2.2. Let (M, ω) be a symplectic manifold. Then $(M, \{,\})$ is a Poisson manifold.

On the contrary a Poisson manifold is not always a symplectic manifold, since the bivector ω defining the Poisson bracket doesn't need to be non-degenerate. If it is then it can be used to define the fundamental form of the symplectic manifold associated to a Poisson one. Moreover [30]

Proposition 4.2.3. Let (M, ω) be a symplectic manifold. Then M is an almost complex manifold.

Finally

Definition 4.2.5. Let (M, ω) a symplectic manifold. An almost complex structure J is **compatible** with ω if

$$\omega(JX, JY) = \omega(X, Y) \qquad \omega(X, JX) > 0 \qquad \forall\, X, Y \in \mathfrak{X}(M) \tag{4.48}$$

4.2.2 Hermitian manifolds

An Hermitian manifold can be simply seen as an analog of a Riemannian manifold in the complex case. Its peculiarity is that an Hermitian scalar product is defined on the tangent space T_pM for each p of the manifold.

Definition 4.2.6. Let (M, J) be a complex manifold. Let g be a Riemannian metric over M. If

$$g(X, Y) = g(JX, JY) \qquad \forall\, X, Y \in \mathfrak{X}(M) \tag{4.49}$$

then g is a **Hermitian metric**. The pair (M, g) is a **Hermitian manifold**.

The following important result states that each complex manifold admits a Hermitian metric.

Proposition 4.2.4. Let M a complex manifold with complex structure J. Then M admits a Hermitian metric.

In fact just note that, if h is a Riemannian metric on M, then also

$$g(X, Y) = \frac{1}{2}(h(X, Y) + h(JX, JY)) \qquad \forall\, X, Y \in \mathfrak{X}(M) \tag{4.50}$$

is, and in addition it is Hermitian.

Moreover the extension by $\mathbb{C}$-linearity to the complexified tangent bundle $T^{\mathbb{C}}$ of the Hermitian metric satisfies

1. $g(\overline{X}, \overline{Y}) = \overline{g(X, Y)} \qquad \forall\, X, Y \in \Gamma(T^{\mathbb{C}})$
2. $g(X, \overline{X}) > 0 \qquad \forall\, X \in \Gamma(T^{\mathbb{C}}),\ X \neq 0$
3. $g(X, Y) = 0 \qquad \forall\, X, Y \in \mathfrak{X}^{1,0},\ \forall\, X, Y \in \mathfrak{X}^{0,1}$

This fact justifies the name of the *Hermitian metric*, since it represents a smoothly varying Hermitian product on the complexified tangent bundle. It can be shown that every metric satisfying 1., 2., 3. on $\Gamma(T^{\mathbb{C}})$ induces by restriction on $\mathfrak{X}(M)$ a Hermitian metric.

The Hermiticity is a geometric constraint on the metric, not on the manifold [30, 29], as the following results state.

Proposition 4.2.5. *Let (M, g) be an Hermitian manifold. Then holomorphic vectors $X \in \mathfrak{X}^{1,0}$ are orthogonal with respect to g.*

In fact let $X, Y \in \mathfrak{X}^{1,0}$. Then $g(X,Y) = g(JX,JY) = g(iX,iY) = -g(X,Y)$, from which we conclude that $g(X,Y) = 0$. The proof proceeds in the same way for the antiholomorphic vectors $X, Y \in \mathfrak{X}^{0,1}$.

From now on we will denote by a bar the indices which refer to antiholomorphic coordinates.

Moreover, let us consider a holomorphic atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in I}$ and the induced local coordinates $\{z^\mu_\alpha, \overline{z}^{\overline{\mu}}_\alpha\}_{\mu,\overline{\mu}\in I_n}$. Since if $X, Y \in \mathfrak{X}^{1,0}$ then

$$g_{\mu\nu}X^\mu Y^\nu = g(X,Y) = g(JX,JY) = g_{\lambda\rho}J_\mu{}^\lambda J_\nu{}^\rho X^\mu Y^\nu = g_{\lambda\rho}(i\delta^\lambda_\mu)(i\delta^\rho_\nu)X^\mu Y^\nu = -g_{\mu\nu}X^\mu Y^\nu \tag{4.51}$$

and we obtain that $g_{\mu\nu} = 0$. More in general, for a Hermitian metric terms with pure indices vanish

$$g_{\mu\nu} = g_{\overline{\mu}\overline{\nu}} = 0 \tag{4.52}$$

A Hermitian metric takes the local form in each chart [22]

$$g = g_{\mu\overline{\nu}}\, dz^\mu_\alpha \otimes d\overline{z}^{\overline{\nu}}_\alpha \tag{4.53}$$

where the coefficients $g_{\mu\overline{\nu}} \in C^\infty(U)$ obey the hermiticity condition

$$g_{\mu\overline{\nu}} = g_{\nu\overline{\mu}} \tag{4.54}$$

We can give the following

Definition 4.2.7. Let (M, J, g) be a Hermitian manifold. The two-form ω such that

$$\omega(X,Y) = g(JX,Y) \qquad \forall\, X, Y \in \mathfrak{X}(M) \tag{4.55}$$

is the **fundamental two- form**.

In other words

$$\omega_{\mu\nu} = J_\mu{}^\lambda g_{\lambda\nu} \tag{4.56}$$

Since the non-vanishing metric components are those with mixed indices, it turns out that $\omega \in \Lambda^{1,1}T^{*\mathbb{C}}$. The fundamental form is invariant under the action of J, in fact

$$\omega(JX,JY) = g(J^2X,JY) = g(J^2\,JX,J^2Y) = g(JX,Y) = \omega(X,Y) \qquad \forall\, X, Y \in \mathfrak{X}(M) \tag{4.57}$$

From the form of a Hermitian metric we easily find that in a chart (U, φ) which determines the local set of coordinates $\{z^\mu, \overline{z}^{\overline{\mu}}\}_{\mu,\overline{\mu}\in I_n}$

$$\omega = ig_{\mu\overline{\nu}}dz^\mu \wedge d\overline{z}^{\overline{\nu}} = -J_{\mu\overline{\nu}}dz^\mu \wedge d\overline{z}^{\overline{\nu}} \tag{4.58}$$

where $J_{\mu\overline{\nu}} = g_{\mu\overline{\rho}}J_{\overline{\nu}}{}^{\overline{\rho}} = -ig_{\mu\overline{\nu}}$. From the last Equation we can find that ω is a real form

$$\overline{\omega} = \omega \tag{4.59}$$

in fact $\overline{\omega} = (-i)g_{\overline{\mu}\nu}d\overline{z}^{\overline{\mu}} \wedge dz^\nu = ig_{\nu\overline{\mu}}dz^\nu \wedge d\overline{z}^{\overline{\mu}} = \omega$.

Finally let us notice that, if $dim_{\mathbb{C}}(M) = n$, then the $2n$-form

$$\frac{\omega^n}{n!} = i^n(-1)^{n\frac{n-1}{2}}\det(g) \quad dz^1 \wedge \cdots \wedge dz^n \wedge d\overline{z}^1 \wedge \cdots \wedge d\overline{z}^n \tag{4.60}$$

is a good volume form on M.

4.2.3 Kähler manifolds

Definition 4.2.8. Let (M, J, g, ω) be a Hermitian manifold. If

$$d\omega = 0 \tag{4.61}$$

then (M, J, g, ω) is a **Kähler manifold**, g is the **Kähler metric** and ω is the **Kähler form**.

In other words, we can see a Kähler structure on a smooth manifold as a tern (g, J, ω), where g is a Riemannian metric, J is a complex structure and ω is a symplectic structure such that the diagram in Figure 4.1 commutes.

$$\begin{tikzcd} T \arrow[rr, "g"] & & T^* \\ & T \arrow[ul, "J"] \arrow[ur, "\omega"'] & \end{tikzcd}$$

Figure 4.1: A Kähler structure.

It's important to state the sufficient condition for a manifold to be Kähler , in terms of the complex structure [24]

Proposition 4.2.6. Let (M, g, J) be a Hermitian manifold. M is a Kähler manifold if and only if $\nabla J = 0$, where ∇ is the Levi-Civita connection associated to g.

Clearly all the Kähler manifolds are symplectic, since the Kähler form is closed and non-degenerate. In fact its inverse is simply given by

$$\omega^{\mu\nu} = -g^{\mu\rho} J_\rho{}^\nu \tag{4.62}$$

The converse isn't true in general, but we have the following

Proposition 4.2.7. Let (M, ω, J) be a symplectic manifold with compatible complex structure J. Then M is a Kähler manifold.

There is an additional feature of the Kähler manifold which makes it worthy to carefully study them. In fact it is well known that in each point p of a Riemannian manifold M we can define a set of coordinates - called the *normal coordinates* - such that the Riemannian metric osculates to the Euclidean one to the order 2 in a neighborhood of $p \in M$. The nice discovery is that on a Hermitian manifold, the requirement of the existence of a normal set of coordinates in each point, coincides with the requirement to be a Kähler manifold [24]

Proposition 4.2.8. Let (M, J) be a complex manifold, ad let g be a hermitian metric. Then g is Kähler if and only if $\forall\, p \in M\ \exists$ holomorphic coordinates $\{z^\mu, \overline{z}^{\overline{\mu}}\}$ $(z^\mu = x^\mu + iy^\mu)$ in which g can be written

$$g_{\mu\overline{\nu}}(p) = \frac{1}{2}\delta_{\mu\overline{\nu}} + \epsilon_{\mu\overline{\nu}}(p) \tag{4.63}$$

where

$$\epsilon_{\mu\overline{\nu}}(p) = \frac{\partial \epsilon_{\mu\overline{\nu}}}{\partial x^\lambda}(p) = \frac{\partial \epsilon_{\mu\overline{\nu}}}{\partial y^\lambda}(p) = 0 \qquad \mu, \overline{\nu}, \lambda \in I_{2n} \tag{4.64}$$

In other words the first non-vanishing correction to the standard form of a Hermitian metric is at order two.

The constraint on the fundamental form $d\omega = 0$ has important consequences on the geometry of a Kähler manifold. In fact, if we choose a chart (U, φ) which determines the set of local coordinates $\{z^\mu, \overline{z}^{\overline{\mu}}\}_{\mu, \overline{\mu} \in I_n}$ then

$$d\omega = (\partial + \overline{\partial})\omega = i\partial_\rho g_{\mu\overline{\nu}} dz^\rho \wedge dz^\mu \wedge dz^{\overline{\nu}} - i\partial_{\overline{\rho}} g_{\mu\overline{\nu}} dz^\mu \wedge dz^{\overline{\rho}} \wedge dz^{\overline{\nu}} = 0 \tag{4.65}$$

Each term in the last Equation must vanish indipendently, then

$$\partial_{[\rho}g_{\mu]\overline{\nu}} = 0 \qquad \partial_{[\overline{\rho}}g_{|\mu|\overline{\nu}]} = 0 \tag{4.66}$$

These equations can be translated in the fact that there exists $K \equiv K(z, \overline{z}) \in C^\infty(U)$, such that

$$g_{\mu\overline{\nu}} = \partial_\mu \partial_{\overline{\nu}} K \tag{4.67}$$

and then, on U

$$\omega = i\partial_\mu \partial_{\overline{\nu}} K dz^\mu \wedge dz^{\overline{\nu}} = i\partial\overline{\partial} K \tag{4.68}$$

The function K is the **Kähler potential**. The same result can be achieved using Lemma 4.1.2.

Moreover, since $d = \partial + \overline{\partial}$ and $d^2 = \partial^2 = \overline{\partial}^2 = 0$, we obtain that $\partial\overline{\partial} = -\frac{1}{2}d(\partial - \overline{\partial})$. This fact doesn't assure that ω is globally exact, because K is only locally defined on U. Instead the metric g is globally defined, but it takes the form in Equation (4.67) only in the chart U. Given another chart (V, φ), with the associated set of coordinates $\{w^\mu, \overline{w}^{\overline{\mu}}\}_{\mu,\overline{\mu}\in I_n}$ the Kähler potential doesn't need to be equal in the overlap $U \cap V$, but it has to obey the constraint

$$K|_V(w, \overline{w}) = K|_U(z, \overline{z}) + f(z) + \overline{f}(\overline{z}) \tag{4.69}$$

where $f(z)$ and $\overline{f}(\overline{z})$ are respectively a holomorphic and an antiholomorphic functions. Equation (4.69) defines the **Kähler trasformations**.

Finally, it's straightforward to prove that the metric is invariant under Kähler transformations of the Kähler potential. In fact, let $K|_U(z, \overline{z}) \mapsto K'|_U(z, \overline{z}) = K|_U + f(z) + g(\overline{z})$. The metric computed with $K'(z, \overline{z})$ will be

$$\begin{aligned} g'_{\mu\overline{\nu}} &= \partial_\mu \partial_{\overline{\nu}} K'(z, \overline{z}) = \partial_\mu \partial_{\overline{\nu}} \left(K(z, \overline{z}) + f(z) + g(\overline{z})\right) = \\ &= \partial_\mu \partial_{\overline{\nu}} K(z, \overline{z}) + \partial_\mu \partial_{\overline{\nu}} f(z) + \partial_\mu \partial_{\overline{\nu}} g(\overline{z}) = \partial_\mu \partial_{\overline{\nu}} K(z, \overline{z}) = g_{\mu\overline{\nu}} \end{aligned} \tag{4.70}$$

since $f(z)$ is holomorphic and $g(\overline{z})$ is antiholomorphic, namely $\partial_{\overline{\nu}} f(z) = 0$ and $\partial_\mu g(\overline{z}) = 0$.

Remember that on a smooth manifold a Levi-Civita connection is uniquely defined by two requirements: metric compatibility and vanishing of torsion. On a complex manifold it is natural to require also that the complex structure must be compatible. This requirement is equivalent to imposing that holomorphic vectors must remain holomorphic after parallel transport. Let us work in a coordinate basis and define the action of the covariant derivative on vectors basis

$$\nabla_\mu \frac{\partial}{\partial z^\nu} = \Gamma^\lambda{}_{\mu\nu} \frac{\partial}{\partial z^\lambda} \qquad \nabla_{\overline{\mu}} \frac{\partial}{\partial z^{\overline{\nu}}} = \Gamma^{\overline{\lambda}}{}_{\overline{\mu}\overline{\nu}} \frac{\partial}{\partial z^{\overline{\lambda}}} \tag{4.71}$$

The relation $\overline{\Gamma^\lambda{}_{\mu\nu}} = \Gamma^{\overline{\lambda}}{}_{\overline{\mu}\overline{\nu}}$ holds and these are the only non-vanishing components of the connection. On the dual basis, the action of ∇ is

$$\nabla_\mu dz^\nu = -\Gamma^\nu{}_{\mu\lambda} dz^\lambda \qquad \nabla_{\overline{\mu}} dz^{\overline{\nu}} = -\Gamma^{\overline{\nu}}{}_{\overline{\mu}\overline{\lambda}} dz^{\overline{\lambda}} \tag{4.72}$$

For example on a holomorphic vector $X^+ \in T^{1,0}$ and on an antiholomorphic vector $X^- \in T^{0,1}$, the action of ∇_μ is

$$\nabla_\mu X^+ = (\partial_\mu X^\lambda + X^\nu \Gamma^\lambda{}_{\mu\nu}) \frac{\partial}{\partial z^\lambda} \qquad \nabla_\mu X^- = (\partial_\mu X^{\overline{\nu}}) \frac{\partial}{\partial z^{\overline{\nu}}} \tag{4.73}$$

Notice that on a antiholomorphic vector field, ∇_μ acts exactly as an ordinary derivative. We can work analogously with $\nabla_{\overline{\mu}}$. Requiring also metric compatibility

$$\nabla_\rho g_{\mu\overline{\nu}} = 0 \qquad \nabla_{\overline{\rho}} g_{\mu\overline{\nu}} = 0 \tag{4.74}$$

we can rewrite

$$\partial_\rho g_{\mu\overline{\nu}} - g_{\lambda\overline{\nu}} \Gamma^\lambda{}_{\rho\mu} = 0 \qquad \partial_{\overline{\rho}} g_{\mu\overline{\nu}} - g_{\mu\overline{\lambda}} \Gamma^{\overline{\lambda}}{}_{\overline{\rho}\overline{\nu}} = 0 \tag{4.75}$$

from which we find the explicit expression for the connection components

$$\Gamma^\rho{}_{\mu\nu} = g^{\overline{\lambda}\mu} \partial_\nu g_{\nu\overline{\lambda}} \qquad \Gamma^{\overline{\rho}}{}_{\overline{\mu}\overline{\nu}} = g^{\lambda\overline{\rho}} \partial_{\overline{\mu}} g_{\overline{\nu}\lambda} \tag{4.76}$$

Definition 4.2.9. An affine connection compatible with the metric and such that all components with mixed indices are vanishing, is a **Hermitian connection**. It is unique by construction.

It is important that [19]

Proposition 4.2.9. The complex structure J is compatible with the Hermitian connection, i.e.

$$\nabla_\rho J_\mu{}^\nu = \nabla_{\overline{\rho}} J_\mu{}^\nu = \nabla_\rho J_{\overline{\mu}}{}^{\overline{\nu}} = \nabla_{\overline{\rho}} J_{\overline{\mu}}{}^{\overline{\nu}} = 0 \tag{4.77}$$

Now define the action of the torsion on a basis of vectors

$$\begin{aligned} T\left(\frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial z^\nu}\right) &= \Gamma^\rho{}_{[\mu\nu]} \frac{\partial}{\partial z^\rho} \\ T\left(\frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial z^{\overline{\nu}}}\right) = T\left(\frac{\partial}{\partial z^{\overline{\mu}}}, \frac{\partial}{\partial z^\nu}\right) &= 0 \\ T\left(\frac{\partial}{\partial z^{\overline{\mu}}}, \frac{\partial}{\partial z^{\overline{\nu}}}\right) &= \Gamma^{\overline{\rho}}{}_{[\overline{\mu\nu}]} \frac{\partial}{\partial z^{\overline{\rho}}} \end{aligned} \tag{4.78}$$

and thus the non-vanishing components are

$$\begin{aligned} T^\rho{}_{\mu\nu} &= \Gamma^\rho{}_{[\mu\nu]} = g^{\overline{\lambda}\rho}(\partial_{[\mu} g_{\nu]\overline{\lambda}}) \\ T^{\overline{\rho}}{}_{\overline{\mu\nu}} &= \Gamma^{\overline{\rho}}{}_{[\overline{\mu\nu}]} = g^{\lambda\overline{\rho}}(\partial_{[\overline{\mu}} g_{\overline{\nu}]\lambda}) \end{aligned} \tag{4.79}$$

Restrictions on the affine connection simplify the form of the Riemann tensor. In particular one can find that

$$R^\lambda{}_{\overline{\mu}lm} = R^{\overline{\lambda}}{}_{\mu lm} = R^l{}_{m\mu\lambda} = R^l{}_{m\overline{\mu}\overline{\lambda}} = 0 \tag{4.80}$$

where the indices l, m can take values from both holomorphic indices and antiholomorphic ones. Thanks to the trivial symmetry $R^\lambda{}_{\mu\overline{\nu}\rho} = -R^\lambda{}_{\mu\rho\overline{\nu}}$, the only independent components of the Riemann tensor are

$$\begin{aligned} R^\lambda{}_{\mu\overline{\nu}\rho} &= \partial_{\overline{\nu}} \Gamma^\lambda{}_{\rho\mu} = \partial_{\overline{\nu}}(g^{\lambda\overline{\alpha}} \partial_\rho g_{\mu\overline{\alpha}}) \\ R^{\overline{\lambda}}{}_{\overline{\mu}\nu\overline{\rho}} &= \partial_\nu \Gamma^{\overline{\lambda}}{}_{\overline{\rho\mu}} = \partial_\nu(g^{\alpha\overline{\lambda}} \partial_{\overline{\rho}} g_{\alpha\overline{\mu}}) \end{aligned} \tag{4.81}$$

Other important features of Riemann tensor are

$$\begin{aligned} R_{\overline{\mu}\nu\overline{\rho}\sigma} &= g_{\overline{\mu}\lambda} R^\lambda{}_{\nu\overline{\rho}\sigma} \\ R_{\mu\overline{\nu}\rho\overline{\sigma}} &= g_{\mu\overline{\lambda}} R^{\overline{\lambda}}{}_{\overline{\nu}\rho\overline{\sigma}} \end{aligned}$$

and the symmetries

$$R_{\overline{\mu}\nu\overline{\rho}\sigma} = -R_{\nu\overline{\mu}\overline{\rho}\sigma} \qquad R_{\mu\overline{\nu}\rho\overline{\sigma}} = R_{\overline{\nu}\mu\rho\overline{\sigma}} \qquad R_{\mu\overline{\nu}\rho\overline{\sigma}} = R_{\rho\overline{\nu}\mu\overline{\sigma}} = R_{\rho\overline{\sigma}\mu\overline{\mu}} \tag{4.82}$$

After, if we contracting indices of the Riemann tensor, we can define the **Ricci tensor** $\mathfrak{R}_{\mu\overline{\nu}}$

$$\mathfrak{R}_{\mu\overline{\nu}} = R^\lambda{}_{\lambda\mu\overline{\nu}} = -\partial_{\overline{\nu}}(g^{\lambda\overline{\rho}} \partial_\mu g_{\lambda\overline{\rho}}) = -\partial_{\overline{\nu}} \partial_\mu \log g \tag{4.83}$$

where $g = \det g_{\mu\overline{\nu}}$, and where we used the equality $\delta g = g g^{\mu\overline{\nu}} \delta g_{\mu\overline{\nu}}$. $\mathfrak{R}_{\mu\overline{\nu}}$ is explicitly antisymmetric, then we can define the **Ricci form**

$$\mathfrak{R} = i\mathfrak{R}_{\mu\overline{\nu}} dz^\mu \wedge dz^{\overline{\nu}} = i\partial\overline{\partial} \log g \tag{4.84}$$

From the equality $\partial\overline{\partial} = -\frac{1}{2} d(\partial - \overline{\partial})$ we find that $\mathfrak{R}$ is closed. But again $\mathfrak{R}$ isn't globally defined. So it isn't exact and then it defines a non trivial cohomology class

$$c_1(M) = \left[\frac{\mathfrak{R}}{2\pi}\right] \in H^2(M, \mathbb{R}) \tag{4.85}$$

As we will see in detail, $c_1(M)$ is the **first Chern class of M**.

The importance of $c_1(M)$ is that it is a topological invariant, i.e. it is invariant under smooth deformations of the metric $g_{\mu\overline{\nu}} \to g_{\mu\overline{\nu}} + \delta g_{\mu\overline{\nu}}$. In fact under this kind of deformation we find that [29]

$$\delta\mathfrak{R} = i\partial\overline{\partial}(g^{\mu\overline{\nu}} \delta g_{\mu\overline{\nu}}) = -\frac{i}{2} d[(\partial - \overline{\partial}) g^{\mu\overline{\nu}} \delta g_{\mu\overline{\nu}}] \tag{4.86}$$

which is exact, being $g^{\mu\overline{\nu}} \delta g_{\mu\overline{\nu}}$ a coordinate scalar. Thus smooth variations of the metric change $\mathfrak{R}$ but don't change $c_1(M)$.

4.2.4 Cohomology of Kähler manifolds

New differential objects can be defined in the case of complex manifold due to the splitting in Equation 4.27. Moreover, the Kähler condition in Equation (4.61) imposes very strong conditions on the cohomology of a Kähler manifold.

Applying Definition 2.3.1 to a complex form, we obtain that

$$* : \Lambda^{p,q}T^* \to \Lambda^{n-q,n-p}T^* \tag{4.87}$$

Then the operator

$$\overline{*} : \Lambda^{p,q}T^* \to \Lambda^{n-p,n-q}T^* \tag{4.88}$$

let us to define an inner product between two forms $\alpha, \beta \in \Lambda^{p,q}T^*$

$$(\alpha, \beta) = \int \alpha \wedge \overline{*}\beta \tag{4.89}$$

Then we can define adjoints of Dolbeault operators

$$(\alpha, \partial\beta) = (\partial^\dagger \alpha, \beta) \qquad (\alpha, \overline{\partial}\beta) = (\overline{\partial}^\dagger \alpha, \beta) \tag{4.90}$$

such that

$$\partial^\dagger : \Lambda^{p,q}T^* \to \Lambda^{p-1,q}T^* \qquad \overline{\partial}^\dagger : \Lambda^{p,q}T^* \to \Lambda^{p,q-1}T^* \tag{4.91}$$

Since a complex manifold is even dimensional if regarded as a real manifold, the relation $d^\dagger = - * d *$ holds. Then it's easy to prove that [19]

$$\partial^\dagger = - * \overline{\partial} * \qquad \overline{\partial}^\dagger = - * \partial * \qquad (\partial^\dagger)^2 = (\overline{\partial}^\dagger)^2 = 0 \tag{4.92}$$

After this, we can repeat exactly the same constructions done for the real case. Then define the **Laplacian** on a Hermitian manifold

$$\Delta_\partial = (\partial + \partial^\dagger)^2 = \partial\partial^\dagger + \partial^\dagger\partial \qquad \Delta_{\overline{\partial}} = \overline{\partial}\overline{\partial}^\dagger + \overline{\partial}^\dagger\overline{\partial} \tag{4.93}$$

Definition 4.2.10. Let M be a Hermitian manifold. Let $\omega \in \Lambda^{p,q}T^*$. If $\Delta_\partial\omega = 0$ ($\Delta_{\overline{\partial}}\omega = 0$) then ω is said to be ∂-harmonic ($\overline{\partial}$-harmonic) and we will write $\omega \in \Upsilon^{p,q}_{\partial}(M)$ ($\omega \in \Upsilon^{p,q}_{\overline{\partial}}(M)$).

Naturally, if $\Delta_\partial\omega = 0$ ($\Delta_{\overline{\partial}}\omega = 0$), then $\partial\omega = \partial^\dagger\omega = 0$ ($\overline{\partial}\omega = \overline{\partial}^\dagger\omega = 0$). Moreover

Theorem 8. ***Hodge's theorem** [19]*
Let M be a Hermitian manifold. Then $\Lambda^{p,q}T^*$ has a unique orthogonal decomposition

$$\Lambda^{p,q}T^* = \overline{\partial}\Lambda^{p,q-1}T^* \oplus \overline{\partial}^\dagger\Lambda^{p,q+1}T^* \oplus \Upsilon^{p,q}_{\overline{\partial}}(M) \tag{4.94}$$

namely a form $\omega \in \Lambda^{p,q}T^*$ is uniquely expressed as

$$\omega = \overline{\partial}\alpha + \overline{\partial}^\dagger\beta + \gamma \tag{4.95}$$

where $\alpha \in \Lambda^{p,q-1}T^*$, $\beta \in \Lambda^{p,q+1}T^*$, $\gamma \in \Upsilon^{p,q}_{\overline{\partial}}(M)$.

On a Hermitian manifold, Δ_∂, $\Delta_{\overline{\partial}}, \Delta$ don't have particular relationships. On a Kähler manifold instead they are essentially the same. In fact

Proposition 4.2.10. Let M be a Kähler manifold. Then

$$\Delta = 2\Delta_\partial = 2\Delta_{\overline{\partial}} \tag{4.96}$$

If ω is a holomorphic form, namely $\overline{\partial}\omega = 0$, then also $\overline{\partial}^\dagger = 0$ because ω doesn't contain factors $d\overline{z}^{\mu}$ in its expansion. Then we can notice that

$$\overline{\partial}\omega = \overline{\partial}^\dagger\omega = 0 \quad \Leftrightarrow \quad \Delta_{\overline{\partial}}\omega \quad \Leftrightarrow \quad \Delta_\partial\omega \quad \Leftrightarrow \quad \partial\omega = \overline{\partial}\omega = 0 \tag{4.97}$$

Then, according to Hodge's theorem, if ω is holomorphic, $\Delta\omega = 0$ holds, and since the converse is trivially true [19], then

$$\omega \text{ holomorphic} \quad \Leftrightarrow \quad \omega \text{ is harmonic}$$

At this point one can prove that [19]

Proposition 4.2.11. Let M be a Kähler manifold such that $dim_{\mathbb{C}}(M) = n$. Then

1. $b^k = \sum_{k=p+q} h^{p,q}$
2. $h^{p,q} = h^{q,p}$
3. $h^{p,q} = h^{n-p,n-q}$

and thus the Hodge diamond is symmetric respect to the vertical and the horizontal lines (see Figure 4.2).

$$\begin{array}{ccccccc}
 & & & h^{m,m} & & & \\
 & h^{m-1,m} & & \vdots & & h^{m-1,m} & \\
h^{0,m} & \cdots & & & & \cdots & h^{0,m} \\
 & h^{m-1,m} & & \vdots & & h^{m-1,m} & \\
 & & & h^{m,m} & & &
\end{array}$$

Figure 4.2: Hodge diamond for a Kähler manifold.

Due to these new symmetries, the number of independent Hodge's numbers becomes $(\frac{1}{2}n+1)^2$ if n is even, while it becomes $\frac{1}{4}(n+1)(n+3)$ if n is odd.

4.3 SU(n) structures

4.3.1 Motivation

4.3.2 Reduction of the structure group

An interesting point is to see how the structures introduced in the development of the complex geometry affect the structure group.

It is well known that the introduction of a Riemannian metric g on a smooth manifold M such that $dim(M) = n$ determines the reduction of the structure group following the pattern

$$GL(n, \mathbb{R}) \quad \hookrightarrow \quad O(n, \mathbb{R}) \tag{4.98}$$

When the tangent bundle is complexified, the dimension doubles, so that the tangent bundle has structure group $GL(2n, \mathbb{R})$. Let us assume that M is an almost complex manifold. Due to the splitting in Equation (4.20) we can construct the **canonical bundle**

$$\Lambda^{n,0}T^* \tag{4.99}$$

We will denote by Ω a local section of the canonical bundle which locally in U_α takes the form

$$\Omega = \theta^1 \wedge \cdots \wedge \theta^n \in \Lambda^{n,0}T|^*_{U_\alpha} \tag{4.100}$$

where $\{\theta^i\}_{i\in I_n}$ forms a local frame of holomorphic one-forms. A form which can be written as in Equation (4.100) is called a *decomposable form*.

Since the splitting depends only on the almost complex structure J, Ω can be build by using only J. Conversely from $\Omega \in \Lambda^{n,0}T|^*_{U_\alpha}$ we can build a subbundle as follows

$$\overline{L} = \{X \in T| \quad i_X\Omega = 0\} \tag{4.101}$$

We can then define J to be the operator such that $\overline{L}$ is its $-i$-eigenbundle. Its complement L in T will be the $+i$-eigenbundle of the operator J. In this way the almost complex structure J is reconstruct and we have shown that the information encoded by J itself is also encoded by a section Ω of the canonical bundle $\Lambda^{n,0}T^*$.

It is evident that Ω hides an ambiguity in its definition. In fact the one-forms $\{\theta^i\}_{i\in I_n}$ are determined up to a $GL(n,\mathbb{C})$ transformation. In other words Ω is determined up to an overall complex function which can be encoded by the determinant of the matrix representing a $GL(n,\mathbb{C})$ transformation. Needless to say, this framework makes explicit that the structure group has been reduced following the pattern

$$GL(2n,\mathbb{R}) \quad \hookrightarrow \quad GL(n,\mathbb{C}) \tag{4.102}$$

Let us now consider the fundamental form of a Hermitian manifold. Since it is non-degenerate but it doesn't need to be closed, ω defines a **Hermitian pre-symplectic structure on M**. It is a consequence of Equation (4.56) that

$$\omega \in \Lambda^{1,1}T^* \tag{4.103}$$

and moreover we know that it is real. The requirement that ω be globally defined implies the reduction of the structure group following the pattern

$$GL(n,\mathbb{C}) \quad \hookrightarrow \quad U(n) \tag{4.104}$$

Moreover since $\omega \in \Lambda^{n,0}T^*$ we get that

$$\omega \wedge \Omega = 0 \tag{4.105}$$

If Ω doesn't possess the ambiguity mentioned above the group structure can be further reduced as follows

$$U(n) \quad \hookrightarrow \quad SU(n) \tag{4.106}$$

We can give the following [15]

Proposition 4.3.1. Let (M,J) be an almost complex manifold such that $dim(M) = n$. Let Ω be a globally defined, decomposable, complex n-form Ω defined as in Equation (4.100), which is non-degenerate everywhere, namely

$$\Omega \wedge \overline{\Omega} \neq 0 \tag{4.107}$$

Let ω be a pre-symplectic two-form compatible with J. Then the structure group reduces to $SU(n)$.

Definition 4.3.1. Let (M,J,ω,Ω) be as in Proposition 4.3.1. It is a **SU(n)-structure**.

Usually the form Ω is normalized in the following way

$$\Omega \wedge \overline{\Omega} = \frac{(-2i)^n i^{\frac{n-1}{2}}}{n!}\omega^n \tag{4.108}$$

With this convention there exists a local frame $\{\theta^i\}_{i\in I_n}$ such that

$$\omega = -\frac{i}{2}\sum_{i,\bar{i}} \theta^i \wedge \overline{\theta}^{\bar{i}} \qquad\qquad \Omega = \theta^1 \wedge \cdots \wedge \theta^n \tag{4.109}$$

4.3.3 $SU(3)$ structures

In this Section we will specialize to the case $n = 3$.

In this particular case Equation (4.108) reduces to

$$\Omega \wedge \overline{\Omega} = \frac{4}{3}\omega^3 \tag{4.110}$$

We would like to express the $SU(3)$-structure in terms of spinors.

As we can see in Section 3.1.4, the Spin group $Spin(6)$ has two inequivalent spinor representations spaces

$$S \cong \overline{S} \cong \mathbb{C}^4 \tag{4.111}$$

which are associated to two different chiralities. In particular they are induced from the decomposition in eigenspaces of the volume form, which in the case $(r,s) = (6,0)$ is such that $\gamma_7^2 = -1$. Then we can write each spinor η as the sum

$$\eta = \xi + \chi \tag{4.112}$$

where the subscript $+$ denotes a positive chirality component such that $\gamma_7\xi = i\xi$, while the subscript $-$ denotes a negative chirality component such that $\gamma_7\chi = -i\chi$. In other words

$$\xi \in S \qquad\qquad \chi \in \overline{S} \tag{4.113}$$

It's amazing to see that the symplectic form ω and the canonical form Ω of a $SU(3)$ structure can be simply built by starting from a non-vanishing pure $Spin(6)$ spinor. As we have seen in Section 3.1.4 if the dimension $n \leq 6$ every spinor is a pure spinor, so that we don't have to worry about this requirement. As we know, $\eta_\pm$ is a locally a vector of $\mathbb{C}^4$. If η is nowhere vanishing, without loss of generality we can write it in a chart as

$$\eta_+ = \begin{pmatrix} \eta_0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{4.114}$$

and it is obvious that the subgroup of $SU(4)$ which leaves invariant η is $SU(3)$. Since it is non-vanishing, we can normalize η_+ so that

$$\overline{\eta}_+\eta_+ = 1 \tag{4.115}$$

Then the $SU(3)$-structure takes the nice form

$$\omega_{ij} = -i\overline{\eta}_+\gamma_{ij}\eta_+ \qquad\qquad \Omega_{ijk} = -i\overline{\eta}_-\gamma_{ijk}\eta_+ \tag{4.116}$$

where η_- is the charge conjugate of η_+, namely

$$\eta_- = C\eta^* \tag{4.117}$$

where the charge-conjugation matrix C is such that

$$\gamma_a^* = -C^{-1}\gamma_a C \tag{4.118}$$

4.3.4 Holonomy groups

We saw that the Riemann tensor over a Kähler manifold M has only few non-vanishing components. This fact was mainly due to the Kähler condition, which puts strong constraints on the metric.

We can reformulate these concepts in terms of the Holonomy group $\Phi(M)$, which is a more intuitive geometric tool. The fact that non-vanishing connection coefficients have pure indices, i.e. that parallel transport preserves the holomorphicity condition, tell us that the Holonomy group of a Kähler manifold is contained in $U(n)$.

Moreover we recall that the parallel transport of a vector with components X^l around an infinitesimal parallelogram with sides ϵ^m and τ^n lying along direction $\frac{\partial}{\partial x^m}$ and $\frac{\partial}{\partial x^n}$ gives

$$X'^l = X^l + X^r R^l{}_{rmn}\epsilon^m\tau^n \tag{4.119}$$

We require that under parallel transport holomorphicity is preserved, which means that the Riemann tensor is pure in (l, r) indices. Remembering all symmetries of the Riemann tensor, we find that the only non-vanishing components are those in Equation (4.81). The matrices $\epsilon^m\tau^n R^l{}_{rmn}$ are elements of $\Phi(M)$ (see Section 2.2.2) infinitesimally closed to the identity, i.e. are in the Lie algebra of $U(n)$, namely $\mathfrak{u}(n)$. In a neighbour of the identity we have that

$$U(n) \simeq SU(n) \times U(1) \tag{4.120}$$

which translates into the Lie algebras as

$$\mathfrak{u}(n) = \mathfrak{su}(n) \oplus \mathfrak{u}(1) \tag{4.121}$$

The Lie algebra $\mathfrak{su}(n)$ contains the traceless matrices, then the generator of the $\mathfrak{u}(1)$ part is

$$R^\lambda{}_{\lambda\mu\overline{\nu}}\epsilon^\mu\tau^{\overline{\nu}} = -4\mathfrak{R}_{\mu\overline{\nu}}\epsilon^\mu\tau^{\overline{\nu}} \tag{4.122}$$

Thus the Ricci tensor is the generator of the $U(1)$ part of $\Phi(M)$. We can now give

Proposition 4.3.2. Let M be a Kähler, Ricci-flat manifold. Then $\Phi(M) \subseteq SU(n)$.

Note that Proposition 4.3.2 isn't a direct consequence of the precedent reasoning, which prove the statement only locally. For a complete prove we remaind to [29].

There are a lot of ways to define Calabi-Yau manifolds. The first we will give is

Definition 4.3.2. Let M be a Kähler , Ricci-flat manifold. M is a **Calabi-Yau manifold**.

Let us remember that the condition of Ricci-flatness can be also written in terms of the first Chern-class, since

$$c_1(M) = \left[\frac{\mathfrak{R}}{2\pi}\right] \in H^2(M, \mathbb{R}) \tag{4.123}$$

5

Generalized complex geometry

After having introduced the CG, in this Chapter we will explore the Generalized Complex Geometry (GCG). This is a new kind of geometry in which we will find two fundamental novelties. The first is the fact that instead of the tangent T and the cotangent T^* bundles separately, we will consider them as a direct sum $T \oplus T^*$. This fact leads us to a natural generalization of the Lie bracket, i.e. the Courant bracket. The second innovation is the fact that the orthogonal group is enlarged by the so called B-action. We will explore this fact in detail.

5.1 Linear algebra of $V \oplus V^*$

Before to analyze the differential geometry of the GCG, let us study briefly the novelties which one obtains by studying the linear algebra of the direct sum $V \oplus V^*$.

5.1.1 Basic notions

Let V be a vector space such that $dim(V) = n$, and let V^* be its dual. $V \oplus V^*$ is endowed with the following natural and symmetric bilinear form [14]

$$\eta : V \oplus V^* \times V \oplus V^* \quad \rightarrow \quad \mathbb{R}$$

such that

$$\eta(X+\xi, Y+\eta) = \frac{1}{2}\left(\xi(Y) + \eta(X)\right) \qquad \forall\, X+\xi, Y+\eta \in V \oplus V^* \tag{5.1}$$

where $X \in V$ and $\eta \in V^*$. η is clearly symmetric, and its signature is (n, n). We will call it the *inner product on* $V \oplus V^*$.

Here and in the rest of the present Chapter, we will indicate with u, v, w elements lying in the direct sum $V \oplus V^*$, with X, Y, Z elements which belong to the "vector part" of $V \oplus V^*$, and with ξ, η, χ elements wich belong to the "form" part of $V \oplus V^*$. From the next Section $V \oplus V^*$ will turn into $T \oplus T^*$, but the convention will remain the same.

The natural pairing between $\Lambda(V)$ and $\Lambda(V^*)$ given by

$$(u^*, v) = \det(u_i^*(v_j)) \tag{5.2}$$

where $u^* = u_1^* \wedge \cdots \wedge u_n^* \in \Lambda^n(V^*)$ and $v = v_1 \wedge \cdots \wedge v_n \in \Lambda^n(V)$, allows us to identify

$$\Lambda^{2n}(V \oplus V^*) \cong \mathbb{R} \tag{5.3}$$

In this way, the unity $1 \in \mathbb{R}$ defines a canonical orientation on $V \oplus V^*$.

The isometry group of $V \oplus V^*$ is the *special orthogonal group*

$$SO(V \oplus V^*) \cong SO(n, n) \tag{5.4}$$

whose Lie algebra is

$$\mathfrak{so}(V \oplus V^*) = \{T \in End(V \oplus V^*)| \quad \langle Tv, w\rangle + \langle v, Tw\rangle = 0 \quad \forall\, v, w \in V \oplus V^*\} \tag{5.5}$$

The group $SO(n,n)$ will play a fundamental role in the developing of the work.

In Section 5.1.1 we have seen that $\mathfrak{so}(V \oplus V^*)$ naturally sits in the Clifford algebra $C(V \oplus V^*)$, since $\mathfrak{so}(V \oplus V^*) \simeq \Lambda^2(V \oplus V^*)$. Then we can write the decomposition

$$\mathfrak{so}(V \oplus V^*) = End(V) \oplus \wedge^2(V) \oplus \wedge^2(V^*) \tag{5.6}$$

Since $dim(V \oplus V^*) = dim(End(V)) = n^2$ we have the isomorphism $V \oplus V^* \cong End(V)$.

The decomposition in Equation (5.6) leads us to the conclusion that the most general transformation acting on $V \oplus V^*$, and leaving the inner product invariant is of the form

$$T = \begin{pmatrix} A & \beta \\ B & -A^T \end{pmatrix} \tag{5.7}$$

By imposing the defining property of the $\mathfrak{so}(n,n)$ Lie algebra it turns out that

$$B^* = -B \qquad \beta^* = -\beta \tag{5.8}$$

Let us see how the various components in the decomposition in Equation (5.6) are immersed into $\mathfrak{so}(n,n)$.

$End(V) \subset \mathfrak{so}(V \oplus V^*)$ acts as follows

$$\begin{array}{rcl} A : V \oplus V^* & \rightarrow & V \oplus V^* \\ X + \xi & \mapsto & A(X) - A^T(\xi) \end{array} \tag{5.9}$$

B acts naturally as a map

$$\begin{array}{rrcl} B: & V & \rightarrow & V^* \\ & X & \mapsto & i_X B \end{array} \tag{5.10}$$

so that it can be seen as $B \in \wedge^2(V^*)$. Also β acts as the map

$$\begin{array}{rrcl} \beta: & V^* & \rightarrow & V \\ & \xi & \mapsto & i_\xi \beta \end{array} \tag{5.11}$$

and then can be see as a map $\beta \in \Lambda^2(V)$.

In other words B is a dual bivector and β is a bivector, and we can make the group action explicit by writing

$$\begin{array}{rcl} e^B : V \oplus V^* & \rightarrow & V \oplus V^* \\ X + \xi & \mapsto & X + \xi + i_X B \end{array} \tag{5.12}$$

and

$$\begin{array}{rcl} e^\beta : V \oplus V^* & \rightarrow & V \oplus V^* \\ X + \xi & \mapsto & X + \xi + i_\xi \beta \end{array} \tag{5.13}$$

We will call the group action of B, β by B-action and β-action. G_B is the subgroup of elements which act as in Equation (5.12). The B-action fixes the direction parallel to V, while it acts by shearing in the V^* direction. Its action is described by the matrix

$$\mathcal{B} = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \tag{5.14}$$

On the other side we can denote by e^A the diagonal group action

$$\begin{array}{rcl} e^A : V \oplus V^* & \rightarrow & V \oplus V^* \\ X + \xi & \mapsto & e^A X \oplus e^{-A^T} \xi \end{array} \tag{5.15}$$

In order to describe spinors on $V \oplus V^*$ it's useful to give the following

Definition 5.1.1. Let $L \subseteq V \oplus V^*$. If $\eta(v, w) = 0 \quad \forall\, v, w \in L$, then L is an **isotropic subspace**. As it is well known from the linear algebra, if $dim(L) = n$ ($dim(V \oplus V^*) = n$), then the isotropic subspace is a maximal one. A maximal isotropic subspace is also called a **linear Dirac structure**.

In particular it's important to notice that we can write every linear Dirac structure L in the form [14]

$$L(E, \epsilon) = \{X + \xi \in E \oplus V^* | \quad \xi|_E = \epsilon(X)\} \tag{5.16}$$

where $E \subset V$, and $\epsilon \in \Lambda^2(E^*)$. Moreover

Definition 5.1.2. Let $\pi_V : V \oplus V^* \to V$ be the canonical projection on V. Let $E \subseteq V \oplus V^*$ be a linear subspace and let $L(E, \epsilon)$ the associated linear Dirac structure. Then the integer

$$t(L) = dim(Ann(E)) = n - dim(\pi_V(L)) \tag{5.17}$$

is **the type of** $\mathbf{L}(\mathbf{E}, \epsilon)$.

The most simple examples of Dirac structures are given by V and its dual V^*, respectively of type $t(V) = 0$ and $t(V^*) = n$.

The B-action doesn't affect the projection to V, but it only shifts the dual component $E \oplus V^* \supset X + \xi \mapsto X + \xi + i_X B$. This means that the B-action doesn't affect the type $t(L)$ of a linear Dirac structure $L(E, \epsilon)$. In other words the type of $L(E, \epsilon)$ is an invariant under the B-action and the linear Dirac structure transforms as follows

$$e^B L(E, \epsilon) = L(E, \epsilon + i^* B) \tag{5.18}$$

where $i : E \hookrightarrow V$ is the inclusion map. Moreover, it can be shown that by choosing B and E suitably, we can obtain every maximal isotropic of a given type as a B-transform of $L(E, 0)$.

On the other side, as we can expect, the β-action modifies the type of a linear Dirac structure $L(E, \epsilon)$. In fact, let $\beta : V^* \to V$ and let L be a linear Dirac structure. If we define $V^* \supset F = \pi_{V^*} L$, $\gamma \in \Lambda^2(F^*)$ and $L(F, \gamma) = \{X + \xi \in V \oplus F| \quad X|_F = \gamma(\xi)\}$, then

$$e^\beta L(F, \gamma) = L(F, \gamma + i^* \beta) \tag{5.19}$$

where now $i : F \hookrightarrow V^*$. It can be shown that we can write the dimension of E as a function of γ

$$dim(E) = dim((L \cap V) + rk(\gamma)) \tag{5.20}$$

where $rk(\gamma) = dim(Im(\gamma))$. Since γ is an alternating bivector, its rank is even and since a β-action is such that $\gamma \mapsto \gamma + i^* \beta$ (which also has even rank) we obtain that the β-action can be used to change the type of $L(E, \epsilon)$ by an even number. Finally

Definition 5.1.3. A linear Dirac structure $L(E, \epsilon)$ whose type is $t(L)$ is said to have *even parity* if $t(L) = 0\, mod(2)$, while it has *odd parity* if $t(L) = 1\, mod(2)$.

It's intuitive that the generic even linear Dirac structure of even parity is a linear Dirac structure of type 0, which is V itself, while the generic odd linear Dirac structure is a linear Dirac structure of type 1. From these we can obtain linear Dirac structures of generic type by β-actions.

5.1.2 Spinors for $V \oplus V^*$

In this Section we want to extend the topics covered in Section 3.1 to the more general context of GCG.

Let us denote the Clifford algebra over $V \oplus V^*$ by $C(V \oplus V^*)$. The quadratic form which defines it is given in Equation (5.1). as we know it has signature $\sigma = (n, n)$, where $dim(V) = n$. The relation

$$v^2 = \eta(v, v) \qquad \forall\, v \in V \oplus V^* \tag{5.21}$$

defines $C(V \oplus V^*)$ together with the anticommutation relation

$$vw + wv = 2\eta(v, w) \qquad \forall v, w \in V \oplus V^* \tag{5.22}$$

The Clifford algebra has a natural representation on $\Lambda(V^*)$ defined by

$$(X + \xi) \cdot \varphi = i_X \varphi + \xi \wedge \varphi \qquad \forall\, X + \xi \in V \oplus V^*, \quad \forall \varphi \in \Lambda(V^*) \tag{5.23}$$

In fact $\forall\, X+\xi, Y+\eta \in V \oplus V^*$

$$\begin{aligned}(X+\xi)^2 \cdot \varphi = (X+\xi)\,(i_X\varphi + \xi \wedge \varphi) = i_X(i_X\varphi + \xi\wedge\varphi) + \xi \wedge (i_X\varphi + \xi\wedge\varphi) = \\ = (i_X\xi)\varphi - \xi \wedge (i_X\varphi) + \xi\wedge(i_X\varphi) = (i_X\xi)\wedge\varphi = \langle X+\xi, X+\xi\rangle\varphi \end{aligned} \tag{5.24}$$

and also

$$\begin{aligned}[(X+\xi)(Y+\eta) + (Y+\eta)(X+\xi)]\cdot\varphi = (X+\xi)(i_Y\varphi + \eta\wedge\varphi) + (Y+\eta)(i_X\varphi+\xi\wedge\varphi) = \\ = i_X i_Y\varphi + i_X(\eta\wedge\varphi) + \xi\wedge(i_Y\varphi) + \xi\wedge\eta\wedge\varphi + i_Y i_X\varphi + i_Y(\xi\wedge\varphi) + \eta\wedge(i_X\varphi) + \eta\wedge\xi\wedge\varphi = \\ = (i_X\eta)\varphi - \eta\wedge(i_X\varphi) + (i_Y\xi)\varphi - i_Y(\xi\wedge\varphi) + i_Y(\xi\wedge\varphi) + \eta\wedge(i_X\varphi) = \\ = 2\left(\frac{1}{2}(i_X\eta + i_Y\xi)\right) = 2\langle X+\xi, Y+\eta\rangle \end{aligned} \tag{5.25}$$

The decomposition in Equation (3.30) of the Clifford algebra $C(V\oplus V^*)$ immediately induces a decomposition of the representation space

$$\Lambda(V^*) = \Lambda^+(V^*) \oplus \Lambda^-(V^*) \tag{5.26}$$

where $\Lambda^+(V^*)$ includes all alternating (dual) multivectors of even order, while on the contrary $\Lambda^-(V^*)$ includes all alternating (dual) multivectors of odd order. This splitting isn't preserved by the whole Clifford algebra $C(V\oplus V^*)$, but $\Lambda^+(V^*)$ and $\Lambda^-(V^*)$ are separately irreducible representations of the Spin group.

We know that $\mathfrak{so}(V\oplus V^*) \cong \Lambda^2(V\oplus V^*)$. The next step is to determine how the Lie algebra components - namely the actions we studied in Equations (5.12), (5.13) and (5.15) - act on the spin representations.

We start with the B-action. Let $\{e_i\}_{i\in I_n}$ a basis for V and let $\{e^i\}_{i\in I_n}$ be its dual basis. As we have seen in Section 3.1.1, $B\in\Lambda^2(V^*)$ and we can write $B = \frac{1}{2}B_{ij}e^i\wedge e^j$ where $B_{ij} = -B_{ji}$. We recall that

$$X \quad \overset{B}{\longmapsto} \quad i_X B \qquad \forall\, X\in V \tag{5.27}$$

which means that, on the basis elements

$$i_{e_k}(e_i\wedge e_j) = \delta^i_k e^j - \delta^j_k e_i \tag{5.28}$$

Moreover, remember Proposition 3.43

$$\begin{aligned} ad_{e^j e^i}(e_k) = e^j e^i e_k - e_k e^j e^i = e^j e^i e_k + e^j e_k e^i - \delta^j_k e^i = \\ = e^j(e^i e_k + e_k e^i) - \delta^j_k e^i = \delta^i_k e^j - \delta^j_k e^i \end{aligned} \tag{5.29}$$

where we used the anticommutativity relation in Equation (5.22) and the associativity of the Clifford algebra $C(V\oplus V^*)$. Equation (5.29) provide the same result of Equation (5.28). In other words the image of $B = \frac{1}{2}B_{ij}e^i\wedge e^j$ in the Clifford algebra $C(V\oplus V^*)$ is $B = \frac{1}{2}B_{ij}e^j e^i$. Its action on the representation space $\Lambda(V^*)$ is then (see Equation (5.23))

$$B\cdot\varphi = \frac{1}{2}B_{ij}e^j\wedge(e^i\wedge\varphi) = -B\wedge\varphi \tag{5.30}$$

And the group action is given by exponentiating

$$e^{-B}\varphi = \left(1 - B + \frac{1}{2}B\wedge B + \dots\right)\wedge\varphi \tag{5.31}$$

As one can expect, since $B\in\mathfrak{so}(V\oplus V^*)$, then e^{-B} is an element of the Spin group $Spin(V\oplus V^*)$. In fact in calculating the norm of $e^{-B}\varphi$ one can see that each term of the form

$$(1,B) \qquad (B,B) \qquad (B^2,B) \qquad (B^3,B) \qquad (B^2,B^2) \qquad (B^3,B^2) \qquad \dots \tag{5.32}$$

vanishes. In fact, let us consider for simplicity the norm of the first order expansion $e^{-B}\cong 1-B$. The norm is given by

$$(1-B, 1-B) = (1, (\overline{1-B})(1-B)) = (1,1) - (1,\overline{B}) - (1,B) + (1,\overline{B}B) = 1 \tag{5.33}$$

where $(1,1) = 1$. The term $(1,B) = \frac{1}{2}B_{ij}(1, e^j e^i) = 0$. Analogously for the term $(1,\overline{B}) = 0$. Finally $(1,\overline{B}B) = \frac{1}{4}B_{ij}B_{lm}(1, \overline{e^i e^j}e^l e^m) = \frac{1}{4}B_{ij}B_{lm}(1, e^j e^i e^l e^m) = 0$. In fact there is no way to reduce the term $e^j e^i e^l e^m$

to some multiple of the identity, since the rule in Equation (5.22) tells us that elements in V^* anticommute in the Clifford algebra $C(V \oplus V^*)$. Consequently

$$|e^B|^2 = 1 \tag{5.34}$$

In addition $e^{-B} \in Spin_0(V \oplus V^*)$, which is the identity component of the Spin group $Spin(V \oplus V^*)$.

We can study the β-action case in a similar way. Let $\beta \in \Lambda^2(V)$, $\beta = \frac{1}{2}\beta^{ij} e_i \wedge e_j$ be the alternating bivector which defines the β-action

$$\xi \quad \stackrel{\beta}{\longmapsto} \quad i_\xi \beta \tag{5.35}$$

Its image in the Clifford algebra is given by $\frac{1}{2}\beta^{ij} e_j e_i$, and then the action on the representation space $\Lambda(V^*)$ is

$$\beta \cdot \varphi = \frac{1}{2}\beta^{ij} i_{e_j}(i_{e_i}\varphi) = i_\beta \varphi \tag{5.36}$$

Therefore, by exponentiating we obtain

$$e^\beta \varphi = \left(1 + i_\beta + \frac{1}{2} i_\beta^2 + \dots\right) \varphi \tag{5.37}$$

The case of the $GL(V)$-action is much more complicate. We will study it in Section **??**.

5.1.3 Pure spinors

There exists a pairing between spinors, which behaves well under spinor representations. In fact it remains invarinat under the action of the identity component of $Spin(V \oplus V^*)$.

We define a bilinear form on the spinor representation space by

$$\begin{array}{rcl} (,) : \Lambda(V^*) \times \Lambda(V^*) & \to & \det V^* \\ (\varphi, \psi) & \mapsto & (\overline{\varphi} \wedge \psi)\,|_{top} \end{array} \tag{5.38}$$

where $(,)\,|_{top}$ denotes that the top degree component of the alternating multivector is taken. It can be shown [14, 36] that

$$(v \cdot \varphi, v \cdot \psi) = \eta(v, v)(\varphi, \psi) \qquad \forall\, v \in V \oplus V^*, \quad \forall\, \varphi, \psi \in \Lambda(V^*) \tag{5.39}$$

so that in particular

$$(g \cdot \varphi, g \cdot \psi) = \pm(\varphi, \psi) \qquad \forall\, g \in Spin(V \oplus V^*), \quad \forall\, \varphi, \psi \in \Lambda(V^*) \tag{5.40}$$

which brings us to give the following

Proposition 5.1.1. The bilinear form in Equation (5.38) is invariant under the identity component of the Spin group $Spin(V \oplus V^*)$ namely

$$(x \cdot \varphi, x \cdot \psi) = (\varphi, \psi) \qquad \forall\, x \in Spin_0(V \oplus V^*) \tag{5.41}$$

For example we have

$$(e^B \cdot \varphi, e^B \cdot \psi) = (\varphi, \psi) \qquad \forall\, B \in \Lambda^2(V^*), \quad \forall\, \varphi, \psi \in \Lambda(V^*) \tag{5.42}$$

The bilinear form in Equation (5.38) is non-degenerate and it can be symmetric or skew symmetric depending on $n = dim(V)$. In fact

$$(\varphi, \psi) = (-1)^{\frac{n(n-1)}{2}} (\varphi, \psi) \qquad \forall\, \varphi, \psi \in \Lambda(V^*) \tag{5.43}$$

Now we are ready to study pure spinors. This concept is the one which allows us to study spinors by understanding the maximal isotropics. In fact, let $\varphi \in \Lambda(V^*)$.

Definition 5.1.4. The subspace $L_\varphi \subset V \oplus V^*$ defined by

$$L_\varphi = \{v \in V \oplus V^* | \quad v \cdot \varphi = 0\} \tag{5.44}$$

is the **null space of** φ.

Every null spaces are isotropic, in fact

$$2\eta(v,w)\varphi = (vw+wv)\cdot\varphi = 0 \quad \forall\, v,w \in L_\varphi \quad \Rightarrow \quad \eta(v,w) = 0 \quad \forall\, v,w \in L_\varphi \tag{5.45}$$

Definition 5.1.5. A spinor φ is a **pure spinor** if L_φ is a maximal isotropic, namely if $dim(L_\varphi) = n$.

It's interesting to notice that from a pure spinor we can easily obtain more pure spinors. For example let $1 \in \Lambda(V^*)$ be the unit spinor. It's a pure spinor because $L_1 = \{X+\xi \in V \oplus V^* | \quad (X+\xi)\cdot 1 = (i_X + \xi\wedge)1 = 0\} = V = L(V,0)$ is a maximal isotropic as we have seen in Section 5.1.1. From this we can obtain more pure spinors by simply applying a spin transformation to the spinor 1. For example, let $B \in \Lambda^2(V^*)$. Then $\varphi = e^{-B}\wedge 1 = e^{-B}$ is also a spinor. We can find its null space by noticing that, for $X \in V$

$$\begin{aligned}(X+i_XB)\cdot e^{-B} &= (X+i_XB)\cdot(1-B+\dots) = \\ &= -i_XB\wedge 1 + B\wedge i_X 1 + i_XB\wedge 1 - i_XB\wedge B \cong \\ &\cong B\wedge i_X 1 = B\wedge (X\cdot 1) = 0\end{aligned} \tag{5.46}$$

where we have used that 1 is annihilated by the maximal isotropic V. We have considered only first order terms in B, but one can verify that this result holds at higher degrees. So we can write the null space

$$L_{e^{-B}} = \{X + i_XB| \quad X \in V\} \tag{5.47}$$

It's obvious that $dim(L_{e^{-B}}) = n$, since there is an indipendent vector for each indipendent $X \in V$. Moreover it's quite evident that we can recover $L_{e^{-B}}$ by simply shifting the dual component of $L(V,0)$ with the B-action. We can eventually write

$$L_{e^{-B}} = \{X + i_XB| \quad X \in V\} = L(V,B) \tag{5.48}$$

Let us give another simple example. Let $\omega \in V^*$ be a non-zero dual vector. Its null space is given by

$$L_\omega = \{X+\xi \in V\oplus V^* | \quad X \in Ker(\omega) \quad \text{and} \quad \xi \in Span(\omega)\} = L(Ker(\omega),0) \tag{5.49}$$

since we can see $i_X\omega$ as a map $\omega : X \mapsto \mathbb{R}$, and $Span(\omega) = \{c\omega \in V^* | \quad c \in \mathbb{R}\}$. This is a maximal isotropic, then ω is a pure spinor and therefore also $e^{-B}\omega$ is.

Every maximal isotropic subspace of $V \oplus V^*$ is associated with a line bundle (lying in the representation space $\Lambda(V^*)$) which is that associated to the respective pure spinor. Let us be more precise

Proposition 5.1.2. Let $L(E,0) = E \oplus Ann(E)$ be the maximal isotropic associated with subspace $E \subset V$ such that $t(E) = k$. Then the data $L(E,0) = E\oplus Ann(E)$ is equivalent to the pure spinor line bundle

$$\det(Ann(E)) \subset \Lambda^k(V^*) \tag{5.50}$$

In fact, let $\varphi = \theta_1 \wedge \dots \wedge \theta_k$ be any non-zero element of $\det(Ann(E))$. Then $(X+\xi)\cdot\varphi = (i_X + \xi\wedge)\varphi = 0$ if and only if $X \in E$ and $\xi \in Ann(E)$. This is equivalent to say that $X + \xi \in L(E,0)$.

Now, as we have seen in Section 5.1.1, every maximal isotropics can be expressed as the B-transform of $L(E,0)$, once one chooses a $B \in \Lambda^2(V^*)$ such that $\epsilon = i^*B$. Remember that $i : E \hookrightarrow V$ is the natural inclusion, and then $\epsilon \in \Lambda^2(E)$. So, even if $\epsilon \notin \Lambda^2(V^*)$, with an abuse of notation we can write

$$L(E,\epsilon) = e^\epsilon(L(E,0)) \tag{5.51}$$

where ϵ represents just any $B \in \Lambda^2(V^*)$ such that $i^*B = \epsilon$. Finally we can give the obvious generalization of the Proposition 5.1.2

Proposition 5.1.3. Let $L(E,\epsilon)$ be any maximal isotropic. Then the pure spinor line U_L defining it is

$$U_L = e^{-\epsilon}\det(Ann(E)) \tag{5.52}$$

where, again ϵ represents any $B \in \Lambda^2(V^*)$ such that $i^*B = \epsilon$.

In other word, if $\{\theta_1,\dots,\theta_k\}$ is a basis for $Ann(E)$, and if $B \in \Lambda^2(V^*)$ such that $\epsilon = i^*B$, then the pure spinor associated to the maximal isotropic $L(E,\epsilon)$ is

$$\varphi_L = c\,e^{-B}\theta_1\wedge\dots\wedge\theta_k \qquad c \in \mathbb{R}/\{0\} \tag{5.53}$$

5.1.4 Complexification

We can extend by complexification all the results we have obtained until now in the present Chapter.

First of all we can extend the inner product η to the complexified one

$$\eta : (V \oplus V^*)^{\mathbb{C}} \times (V \oplus V^*)^{\mathbb{C}} \to \mathbb{C} \tag{5.54}$$

Let V be a real vector space such that $dim(V) = n$. A maximal isotropic subspace $L \subset (V \oplus V^*)^{\mathbb{C}}$ of type $t(L) \in I_n^0$ is equivalently specified by

1. A complex subspace $L \subset (V \oplus V^*)^{\mathbb{C}}$, maximal isotropic with respect to η and such that $E = \pi_{V^{\mathbb{C}}} L$ has $dim_{\mathbb{C}}(E) = n - k$.
2. A complex subspace $E \subset V^{\mathbb{C}}$ such that $dim_{\mathbb{C}}(E) = n-k$, together with a complex dual bivector $\epsilon \in \Lambda^2(E^*)$.
3. A complex spinor line $U_L \subset \Lambda(V^*)^{\mathbb{C}}$ generated by

$$\varphi_L = ce^{-(B+i\omega)}\theta_1 \wedge \cdots \wedge \theta_k \qquad c \in \mathbb{C}/\{0\} \tag{5.55}$$

where $\{\theta_i\}_{i \in I_k}$ are linearly independent complex dual vectors in $V^{*\mathbb{C}}$, while B and ω are the real and imaginary part of a complex dual bivector on $\Lambda^2(V^*)^{\mathbb{C}}$. As usual, when one complexifies a space, has to pay attention to the effects of the conjugation on it. The main consequence here is given by the following

Definition 5.1.6. Let $L \subset (V \oplus V^*)^{\mathbb{C}}$ be a maximal isotropic subspace. Then $L \cap \overline{L}$ is the complexification of some real subspace K , namely $L \cap \overline{L} = K^{\mathbb{C}}$, where $K \in V \oplus V^*$. The number

$$r(L) = dim_{\mathbb{C}}\left(L \cap \overline{L}\right) = dim(K) \tag{5.56}$$

is the **real index of the maximal isotropic L**.

5.2 Generalized Geometry

As usual in differential geometry the next step is to transport the linear algebra of $V \oplus V^*$ on a smooth manifold M. In this perspective we will define the generalized tangent bundle, which as the same name suggests is a generalization of the tangent bundle T. This is a delicate step, since it's the point in which we introduce an object which is central in the work, namely the closed three-form H, which plays a fundamental role in the theory of compactification developed in Chapter 6. Moreover, we have also to study the theory of pure spinors in the generalized geometry. They are central both in the definition of certain important structures on the generalized tangent bundle and because T-duality takes a particularly simple form if written in terms of pure spinors.

5.2.1 The generalized tangent bundle

The most immediate way to transport the machinery of $V \oplus V^*$ on a smooth manifold M is to consider the generalization of the tangent bundle T defined as

$$T \oplus T^* \tag{5.57}$$

This is a bundle over the smooth manifold M, with trivial projection $\pi : T \oplus T^* \to M$. However, as we will see, we are interested in incorporating a closed three-form $H \in H^3(M, \mathbb{R})$ in the construction of the generalization of the tangent bundle. Such a three form is used to *twist* the fibration of $T \oplus T^*$, and it plays a fundamental role in the compactification of the superstring theory with Neveau-Schwarz flux. Let us probe how it works.

We have seen in Chapter 2 that a closed two-form $F \in H^2(M, \mathbb{R})$ induces the definition of a $U(1)$-bundle. In the same way a closed three-form $H \in H^3(M, \mathbb{R})$ defines a more general object which is called a *gerbe.*

Let M be a smooth manifold, and let $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ be an atlas of M. Now consider a set of maps defined on triple overlaps

$$g_{\alpha\beta\gamma} : U_{\alpha\beta\gamma} \to S^1 \tag{5.58}$$

which satisfy the cocycle conditions

$$g_{\alpha\beta\gamma} = g_{\beta\gamma\alpha} = g_{\gamma\alpha\beta} = g^{-1}_{\alpha\gamma\beta} = g^{-1}_{\gamma\beta\alpha} = g^{-1}_{\beta\alpha\gamma} \tag{5.59}$$

as well as

$$g_{\alpha\beta\gamma} = g_{\beta\alpha\delta} = g_{\gamma\beta\delta} = g_{\delta\alpha\gamma} = 1 \tag{5.60}$$

in each quadruple overlap $U_{\alpha\beta\gamma\delta}$.

The important point however is a consequence of the Poincarè Lemma, which starting from a closed three-form $H \in H^3(M,\mathbb{R})$ allows us to to write the following chain of descent relations, also called the *connective structure of a gerbe*

$$H = dB_\alpha \qquad B_\alpha \in \Lambda^2 T^*|_{U_\alpha} \tag{5.61}$$
$$B_\alpha - B_\beta = dA_{\alpha\beta} \qquad A_{\alpha\beta} \in \Lambda^1 T^*|_{U_\alpha \cap U_\beta} \tag{5.62}$$
$$A_{\alpha\beta} + A_{\beta\gamma} + A_{\gamma\alpha} = d\Lambda_{\alpha\beta\gamma} \qquad \Lambda_{\alpha\beta\gamma} \in C^\infty(U_{\alpha\beta\gamma}) \tag{5.63}$$
$$\Lambda_{\alpha\beta\gamma} + \Lambda_{\beta\alpha\delta} + \Lambda_{\gamma\beta\delta} + \Lambda_{\delta\alpha\gamma} = d_{\alpha\beta\gamma\delta} \qquad d_{\alpha\beta\gamma\delta} \in \mathbb{Z} \tag{5.64}$$

From this chain of relations we can see that the transition functions

$$\begin{array}{rcl} g_{\alpha\beta\gamma} : U_{\alpha\beta\gamma} & \to & S^1 \\ p & \mapsto & e^{i\Lambda_{\alpha\beta\gamma}(p)} \end{array} \tag{5.65}$$

satisfy the cocycle conditions in Equations (5.59), (5.60) and then define a gerbe.

The descent relations in Equation (5.61) are particularly important to us, because their elements can be used to define the **twisted generalized tangent bundle** via the extension of the tangent bundle T

$$0 \longrightarrow T^* \longrightarrow E \xrightarrow{\pi} T \longrightarrow 0 \tag{5.66}$$

The fibration is specified by the patchings in the overlaps $U_{\alpha\beta}$

$$X_\alpha + \xi_\alpha \quad \mapsto \quad L_{\alpha\beta} X_\beta \, + \, L^{-T}_{\alpha\beta}\xi_\beta \, + \, i_{L_{\alpha\beta}X_\beta}(dA_{\alpha\beta}) \tag{5.67}$$

where $L \in GL(n,\mathbb{R})$ and $X_\alpha + \xi_\alpha \in E|_{U_\alpha}$. In other words E is a nontrivial fibration of the cotangent bundle T^* over T. In fact the twisting defined by the last term in Equation (5.67) must be added to the usual $GL(n,\mathbb{R})$ action on vectors and one-forms. The twisting term contains the gerbe data $dA_{\alpha\beta}$. Each section of the generalized tangent bundle can be written locally as the sum of a vector and a form, and it is called a *generalized vector*.

It's worthy to notice that if $H \in B^3(M,\mathbb{R})$, then $dA_{\alpha\beta} = 0$ and then the generalized tangent bundle E can be reduced to the trivial one $T \oplus T^*$.

The generalized tangent bundle encodes a natural $O(n,n)$ structure, which is inherited by the metric in Equation (5.1) so that in each open set U_α

$$\eta(v_\alpha, w_\alpha) = \frac{1}{2}(\xi_\alpha(X_\alpha) + \eta_\alpha(Y_\alpha)) \tag{5.68}$$

where $v_\alpha, w_\alpha \in E|_{U_\alpha}$ and $v_\alpha = X_\alpha + \xi_\alpha$, $w_\alpha = Y_\alpha + \eta_\alpha$. The central point here is that the $O(n,n)$ structure is preserved by the patchings in Equation (5.67), in fact one can easily find that for each $U_{\alpha\beta}$

$$\eta(v_\alpha, w_\alpha) = \eta(v_\beta, w_\beta) \tag{5.69}$$

It follows that the $O(n,n)$ actions which we studied in Section 5.1.1 are well defined locally on the fibers of E, and they preserve the metric η. Moreover, since the patchings in Equation (5.67) are actually $GL(n,\mathbb{R})$ actions on the fibers, followed by B-actions with a closed B, the structure group of the generalized tangent bundle E is reduced according to the pattern

$$O(n,n) \quad \longrightarrow \quad \Gamma(\mathbb{R}) \tag{5.70}$$

where $\Gamma(\mathbb{R})$ is the semidirect product defined by

$$\Gamma(\mathbb{R}) = \widetilde{G}_B \rtimes GL(n,\mathbb{R}) \tag{5.71}$$

where $\widetilde{G}_B$ is the subgroup of $O(n,n)$ of the B-actions with B a closed two-form. The last Equation means that each element in $\Gamma(\mathbb{R})$ is the product of two elements in $O(n,n)$ belonging respectively to $\widetilde{G}_B$ and to $GL(n,\mathbb{R})$. Moreover in the action over the fibers of the generalized tangent bundle, each element of $\Gamma(\mathbb{R})$ acts firstly by multiplication of the $GL(n,\mathbb{R})$ part.

Finally we can define a natural bracket on generalized vectors, which is a generalization of the Lie bracket. In fact

Definition 5.2.1. Let M be a smooth manifold and E the generalized tangent bundle induced by a three-form $H \in H^3(M, \mathbb{R})$. The **Courant bracket** is defined as

$$[X+\xi, Y+\eta]_C = [X,Y] + \mathfrak{L}_X\eta - \mathfrak{L}_Y\xi - \frac{1}{2}d\,(i_X\eta - i_Y\xi) \qquad \forall\, X+\xi, Y+\eta \in \mathfrak{X}(E) \tag{5.72}$$

It's obvious that the Courant bracket is defined locally on sections in $\mathfrak{X}(E)$, but we will omit the subscript α, β where this doesn't create confusion.

It's immediate that on vectors the Courant bracket reduces to the Lie bracket $[X, Y]$, while on forms the Courant bracket is 0. The Courant bracket isn't really a Lie bracket, since it fails to satisfy the Jacobi identity [14].

It is particularly important to study the symmetries of the Courant bracket, because as we will see they encode the differential structure of the generalized tangent bundle E. As we saw in Chapter 2, the Lie bracket is a canonically defined structure over a smooth manifold, namely it's invariant under diffeomorphisms. Moreover, it can be proved that there are no other symmetries of the tangent bundle preserving the Lie bracket.

For the generalzied tangent bundle E the situation is more involved because there is an additional symmetry, given by the *B-transformations* seen in Section 5.1.1. In fact

Proposition 5.2.1. The map $\exp(B)$ is an automorphism of the Courant bracket, namely

$$[e^B v, e^B w]_C = e^B[v,w]_C \qquad \forall\, v, w \in \mathfrak{X}(E) \tag{5.73}$$

if and only if B is closed, namely $dB = 0$.

In fact let $X+\xi, Y+\eta \in \mathfrak{X}(E)$ and let $B \in \Lambda^2 T^*$. Then

$$\begin{aligned}
[e^B(X+\xi), e^B(Y+\eta)]_C &= [X+\xi+i_XB, Y+\eta+i_YB]_C = [X+\xi, Y+\eta]_C + [X, i_YB]_C + [i_XB, Y]_C = \\
&= [X+\xi, Y+\eta]_C + \mathfrak{L}_X i_Y B - \tfrac{1}{2} d i_X i_Y B - \mathfrak{L}_Y i_X B + \tfrac{1}{2} d i_Y i_X B = \\
&= [X+\xi, Y+\eta]_C + \mathfrak{L}_X i_Y B - i_Y \mathfrak{L}_X B + i_Y i_X dB = [X+\xi, Y+\eta]_C + [\mathfrak{L}_X, i_Y B] + i_Y i_X dB = \\
&= [X+\xi, Y+\eta]_C + i_{[X,Y]}B + i_Y i_X dB = e^B([X+\xi, Y+\eta]_C) + i_Y i_X dB
\end{aligned} \tag{5.74}$$

where we have used the definition of the Courant bracket and the fact that $\{i_X, i_Y\} = 0$. Then $\exp(B)$ is an automorphism of the Courant bracket if and only if

$$i_X i_Y dB = 0 \quad \forall\, X, Y \in \mathfrak{X}(M) \quad \Leftrightarrow \quad dB = 0 \tag{5.75}$$

Then the group of transformations which preserve the Courant bracket is the same semi-direct product as in Equation (5.71). In this way the diffeomorphism group of the tangent bundle is substituted by the geometric group $\Gamma(\mathbb{R})$.

In the case of the tangent bundle T, the Lie derivative of a vector field is exactly the Lie bracket, which encodes the infinitesimal action of the diffeomorphism group. In the same way the infinitesimal action of the geometric group $\Gamma(\mathbb{R})$ is encoded in a generalization of the Lie derivative $\mathfrak{L}$. We define the **generalized Lie derivative**

$$\mathcal{L}_v w = \mathfrak{L}_X Y + \mathfrak{L}_\xi \eta - i_Y(d\xi) \tag{5.76}$$

where $v = X+\xi, w = Y+\eta \in \mathfrak{X}(E)$. We notice the misleading fact that the infinitesimal action of the geometric group doesn't translate in the action of the natural bracket on the generalized tangent bundle, but it is encoded in the **Dorfman bracket** $[,]_D$ defined as

$$[v,w] = \mathfrak{L}_X Y + \mathfrak{L}_\xi \eta - i_Y(d\xi) \tag{5.77}$$

Nevertheless it's easy to see that the Courant bracket isn't but the antisymmetrization of the Dorfman one [48], so that the information contained in one of them is encoded by the other one too.

5.2.2 Linear generalized complex structures

As the same name suggests, GCG is a generalization of the usual complex geometry seen in Chapter 4. More precisely is a generalization of the complex and symplectic geometry, which contains them as particular extreme cases. As usual, we will begin by studying the linear version of the structures we are going to study on a manifold.

Let V be a vector space, and let V^* be its dual space. We have to consider the endomorphisms of the vector space $V \oplus V^*$. Furthermore, it is important to remember that we can identify the dual space $(V \oplus V^*)^*$ with $V \oplus V^*$ itself. Then we give

Definition 5.2.2. Let $\mathfrak{J} : V \oplus V^* \to V \oplus V^*$. If $\mathfrak{J}$ is both a complex structure and a symplectic structure, then $\mathfrak{J}$ is a **generalized complex structure on** V.

In other words $\mathfrak{J}$ is a generalized complex structure if both the following relations hold

$$\mathfrak{J}^2 = -1 \qquad \mathfrak{J}^* = -\mathfrak{J} \tag{5.78}$$

as we remember from Sections 4.1 and 4.2.1. Moreover

Proposition 5.2.2. $\mathfrak{J} \in End(V \oplus V^*)$ is a generalized complex structure if and only if $\mathfrak{J}$ is a complex structure on $V \oplus V^*$ and it is orthogonal with respect to the inner product η, namely if $\mathfrak{J}^*\mathfrak{J} = \mathfrak{J}\mathfrak{J}^* = 1$.

In fact, if $\mathfrak{J}$ is a generalized complex structure then $\mathfrak{J}^* = -\mathfrak{J}$. Multiplying both sides for $\mathfrak{J}$ we get $\mathfrak{J}\mathfrak{J}^* = -\mathfrak{J}^2 = 1$, since $\mathfrak{J}$ is also a complex structure. This tells us that $\mathfrak{J}$ is orthogonal. Also the converse is quite obvious, in fact if $\mathfrak{J}$ is complex and orthogonal then we can write

$$\mathfrak{J}^*\mathfrak{J} = 1 \quad \Rightarrow \quad \mathfrak{J}^*\mathfrak{J}^2 = -\mathfrak{J}^* = \mathfrak{J} \tag{5.79}$$

The usual complex ($J \in End(T)$) and symplectic ($\omega \in End(T^*)$) structures are embedded in the notion of generalized complex structure in the following way. Consider the endomorphism whose matrix representation on $V \oplus V^*$ is

$$\mathfrak{J}_J = \begin{pmatrix} -J & 0 \\ 0 & J^T \end{pmatrix} \tag{5.80}$$

It's straightforward to see that $\mathfrak{J}_J^2 = -1$ and that $\mathfrak{J}_J^* = -\mathfrak{J}_J$, namely that $\mathfrak{J}_J$ is a generalized complex structure. Consider also the endomorphism

$$\mathfrak{J}_\omega = \begin{pmatrix} 0 & \omega^{-1} \\ -\omega & 0 \end{pmatrix} \tag{5.81}$$

where ω is the usual symplectic structure. Again, $\mathfrak{J}_\omega$ is a generalized complex structure, as can be straightforwardly shown. In other words, the diagonal and the antidiagonal generalized complex structures correspond to the complex and symplectic structures. As it is intuitive, there is a set of generalized complex structures that interpolate between these two extremal cases. The next goal is to understand how this mechanism works. The first point that we have to notice is given by the following [14]

Proposition 5.2.3. The specification of a generalized complex structure $\mathfrak{J}$ is completely equivalent to the specification of the complexification of a maximal isotropic subspace $L_\mathfrak{J} \subset (V \oplus V^*)^\mathbb{C}$ of real index $r(L) = 0$.

In fact, if $\mathfrak{J}$ is a generalized complex structure the condition $\mathfrak{J}^2 = -1$ implies that $(V \oplus V^*)^\mathbb{C}$ can be decomposed into the direct sum of a $+i$-eigenbundle, and a $-i$-eigenbundle, as in Equation 4.7. Let $L_\mathfrak{J}$ be the $+i$-eigenbundle. Then if $v, w \in L_\mathfrak{J}$ we have that $\eta(v,w) = \eta(\mathfrak{J}v, \mathfrak{J}w) = \eta(iv, iw) = -\eta(v,w)$ where we have used the orthogonality of $\mathfrak{J}$ as seen in Proposition 5.2.2 and the bilinearity of η. $\eta(v,w) = -\eta(v,w)$ implies that $\eta(v,w) = 0 \quad \forall\, v, w \in L_\mathfrak{J}$, and then $L_\mathfrak{J}$ is isotropic. Since the $+i$-eigenbundle has complex dimension equal to n, then $L_\mathfrak{J}$ is a maximal isotropic. Finally, since $\overline{L}_\mathfrak{J}$ will be the $-i$-eigenbundle, then we have that $L_\mathfrak{J} \cap \overline{L}_\mathfrak{J} = \{0\}$. Conversely, given a maximal isotropic $L_\mathfrak{J}$ such that $r(L_\mathfrak{J}) = 0$, we can simply define the generalized complex structure $\mathfrak{J}$ as the map which has $L_\mathfrak{J}$ as the $+i$-eigenbundle, and $\overline{L}_\mathfrak{J}$ as the $-i$-eigenbundle.

As it seems to be intuitive, a vector space V admits a generalized complex structure if and only if it is even dimensional. Moreover it can be shown that by equipping the $(V \oplus V^*)^\mathbb{C}$ bundle with a generalized complex structure is equivalent to make a reduction of its structure group from $SO(2n, 2n)$ to $U(n, n)$ [14]. This seems to be very similar to what happens when one equips a manifold with a complex structure (see Section 4.3.2).

Now we can see some examples of generalized complex structures [14]

Example 5.2.1. Symplectic type $\mathbf{t(L_{\mathfrak{J}_\omega}) = 0}$
The generalized complex structure $\mathfrak{J}_\omega$ over $(V \oplus V^*)^\mathbb{C}$ in Equation (5.81) determines a maximal isotropic

$$L_{\mathfrak{J}_\omega} = \{X + i\omega(X) | \quad X \in V^\mathbb{C}\} \tag{5.82}$$

which is also the $+i$-eigenbundle of the generalized complex structure $\mathfrak{J}_\omega$. In fact using orthogonality of $\mathfrak{J}_\omega$ we have $\eta(X+i\omega(X), X+i\omega(X)) = \eta(\mathfrak{J}_\omega(X+i\omega(X)), \mathfrak{J}_\omega(X+i\omega(X))) = \eta(iX-\omega(X), iX-\omega(X)) = -2i\omega(X,X) = 0 \quad \forall\, X \in V^{\mathbb{C}}$. Moreover, by using Proposition 5.1.3, we get that the spinor line $U_{L_{\mathfrak{J}_\omega}}$ is generated by the spinor

$$\varphi_{L_{\mathfrak{J}_\omega}} = e^{-i\omega} \tag{5.83}$$

This generalized complex structure has type $t(L_{\mathfrak{J}_\omega}) = 0$, since $dim\,(\pi_{V^{\mathbb{C}}}(L)) = n$. Remember that a B-transform doesn't change the type. Hence we can transform by a B-field and obtain another generalized complex structure of type $t = 0$. For example

$$e^{-B}\mathfrak{J}_\omega e^{B} = \begin{pmatrix} \omega^{-1}B & \omega^{-1} \\ -\omega - B\omega^{-1}B & -B\omega^{-1} \end{pmatrix} \tag{5.84}$$

$$e^{B}L_{\mathfrak{J}_\omega} = \{X + (B + i\omega)(X) | \quad X \in V^{\mathbb{C}}\} \tag{5.85}$$

$$\varphi_{e^B L_{\mathfrak{J}_\omega}} = e^{-B-i\omega} \tag{5.86}$$

This is a B-*symplectic structure*. Any generalized complex structure with vanishing type is the B-transform of a symplectic structure.

Example 5.2.2. Complex type $\mathbf{t(L_{\mathfrak{J}_J}) = n}$
The generalized complex structure $\mathfrak{J}_J$ over $(V \oplus V^*)^{\mathbb{C}}$ in Equation (5.80) determines a maximal isotropic

$$L_{\mathfrak{J}_J} = V^{0,1} \oplus V^{*1,0} \tag{5.87}$$

which is in the form $E \oplus Ann(E)$, where $E = V^{1,0} \subset V$ and $V^{0,1} \subset V^{\mathbb{C}}$ is the $-i$-eigenspace of J as we have seen in Section 4.1.1. In fact, using orthogonality and bilinearity we can easily get $\eta(v,w) = -\eta(v,w) = 0 \quad \forall\, v, w \in L_{\mathfrak{J}_J}$. Moreover, by using Proposition 5.1.3 one gets that the spinor line is obviously generated by $\det(Ann(V^{0,1}))$, namely

$$\varphi_{L_{\mathfrak{J}_J}} = \Omega^{n,0} \tag{5.88}$$

where $\Omega^{n,0}$ is a generator of $\Lambda^{n,0}(V^*)$ and $dim(V) = n$. If we make a B-transformation we obtain

$$e^{-B}\mathfrak{J}_J e^{B} = \begin{pmatrix} -J & 0 \\ BJ + J^T B & J^T \end{pmatrix} \tag{5.89}$$

$$e^{B}L_{\mathfrak{J}_J} = \{X + \xi + i_X B | \quad X + \xi \in V^{0,1} \oplus V^{*1,0}\} \tag{5.90}$$

$$\varphi_{e^B L_{\mathfrak{J}_J}} = e^{-B}\Omega^{n,0} \tag{5.91}$$

This generalized complex structure has type $t(L_{\mathfrak{J}_J}) = n$, and it can be shown [14] that any generalized complex structure of type $t = n$ is the B-field transform of a complex structure.

5.2.3 Almost structures and integrability condition

Similarly to what we have seen in Section 4.1.1, if we want to transport linear generalized complex structures on a manifold, firstly we have to define an almost generalized complex structure and then we have to specify an integrability condition for it.

We can introduce the generalization of an almost complex structure in several ways, in fact

Definition 5.2.3. Let M be a smooth manifold such that $dim(M) = 2n$. A **generalized almost complex structure** is given by the following equivalent data

1. An almost complex structure $\mathfrak{J}$ on E, which is orthogonal with respect to the metric η, namely

$$\eta(\mathfrak{J}u, \mathfrak{J}v) = \eta(u,v) \qquad \forall\, u, v \in T \oplus T^* \tag{5.92}$$

   or in other words

$$\mathfrak{J}^*\mathfrak{J} = \mathfrak{J}\mathfrak{J}^* = 1 \tag{5.93}$$

2. A maximal isotropic subbundle $L_{\mathfrak{J}} \subset (T \oplus T^*)^{\mathbb{C}}$ of real index $r(L_{\mathfrak{J}}) = 0$, namely such that $L_{\mathfrak{J}} \cap \overline{L_{\mathfrak{J}}} = 0$

Moreover, as we will see in Section 5.3.3, we can define it also by using the pure spinors. From 1. in Definition 5.2.3 we understand why $\mathfrak{J}$ is called *generalized complex structure*, by comparing with Definitions 4.1.3 and 4.1.4.

Next we can focus on the definition of a condition of integrability for a generalized almost complex structure. As we will see it interpolates between the two integrability conditions which we already know, that is $d\omega = 0$ for the symplectic case (where ω is the fundamental form), and $[T^{1,0}, T^{1,0}] \subset T^{1,0}$ for the complex case.

Definition 5.2.4. The generalized complex structure $\mathfrak{J}$ is **integrable** if its $+i$-eigenbundle $L_{\mathfrak{J}} \subset E^{\mathbb{C}}$ is Courant involutive. Alternatively, a generalized complex structure is an involutive maximal isotropic with real index $r(L_{\mathfrak{J}}) = 0$.

Next we can mention one of the most remarkable results of the GCG [14, 15].

Theorem 9. ***Generalized Darboux Theorem***
Let M be a smooth manifold and let E be the generalized tangent bundle induced by the closed three-form $H \in H^3(M, \mathbb{R})$. Let M be endowed with a generalized complex structure $\mathfrak{J}$ over the generalized tangent bundle E. Then for each $p \in M$ which is a regular point, there exists a neighborhood U of p which is equivalent to the product of an open set in $\mathbb{C}^k$ and an open set in the symplectic space $\mathbb{R}^{n-2k}$, defined by the standard symplectic two-form, where k is the type of the generalized complex structure.

Since the portion of the symplectic and complex component of the local product is fixed by the type of the generalized complex structure, which is constant in a neighborhood of a regular point, but in general can change on the manifold, a particular phenomenon can arise, called *type jumping* [14].

5.3 Generalized Kähler geometry

Finally we can generalize the concepts of Kähler and Calabi-Yau manifolds to the generalized complex case.

5.3.1 Generalized metric

The Definition 4.2 of a Kähler manifold provides for the presence of a Riemannian metric on the manifold M. Then reasonably we have to define a Riemannian metric on the generalized tangent bundle E, before to generalize the concept of Kähler manifold to the generalized complex case. The $O(n, n)$ structure of the generalized tangent bundle E is fundamental, in fact we use the indefinite metric η on E to give the following

Definition 5.3.1. The **generalized metric** is a subbundle $C_+ \subset E$ such that $dim(C_+) = n$ on which the metric induced by restriction of η is positive definite.

After denoting the orthogonal complement of C_+ by C_- (on which the induced metric by η is negative definite) we obtain that G is a bilinear form on the tangent bundle E defined as

$$\begin{aligned} G : E \times E &\rightarrow \mathbb{R} \\ \mathsf{G} \equiv \eta|_{C_+} &- \eta|_{C_-} \end{aligned} \tag{5.94}$$

is positive definite and symmetric, since η is. It obeys to the constraints

$$\mathsf{G}^2 = 1 \qquad \mathsf{G}^* = \mathsf{G} \tag{5.95}$$

which is clearly diagonalizable with eigenvalues ± 1, and $C_\pm$ are just its ± 1-eigenspaces. We can define the projectors

$$P_+ = \frac{1}{2}(1 - \mathsf{G}) \qquad P_- = \frac{1}{2}(1 + \mathsf{G}) \tag{5.96}$$

which project respectively on the $+1$-eigenspace and on the -1-eigenspace.

The definition of the generalized metric is equivalent from a topological point of view to the reduction of the structure group from $O(n, n)$ to $O(n) \times O(n)$ [14].

After having complexified the generalized tangent bundle E to obtain

$$E^{\mathbb{C}} \tag{5.97}$$

we can make a further reduction to $U(n)\times U(n)$ by introducing an almost complex structure which is compatible with the almost generalized metric G. This requirement translates into the condition

$$\mathfrak{J}\mathsf{G} = \mathsf{G}\mathfrak{J} \tag{5.98}$$

We can observe that

$$(\mathsf{G}\mathfrak{J})^2 = -1 \tag{5.99}$$

in fact $(\mathsf{G}\mathfrak{J})^2 = \mathsf{G}\mathfrak{J}\mathsf{G}\mathfrak{J} = \mathsf{G}(\mathfrak{J}\mathfrak{J})\mathsf{G} = -\mathsf{G}^2 = -1$, so that it defines a further generalized complex structure.

The last arguments lead us to the intuition that, as well as for the Kähler geometry a pair structures are needed to fix the theory - like the complex structure and the fundamental form then also in the generalized complex case a pair of structures is necessary. These are obviously a pair of generalized complex structures. This discussion finds a formalization in the following ultimate [14]

Definition 5.3.2. A **generalized Kähler structure** is a pair $(\mathfrak{J}_1, \mathfrak{J}_2)$ of generalized complex structures such that

1. $[\mathfrak{J}_1, \mathfrak{J}_2] = 0$.
2. $\mathsf{G} = -\mathfrak{J}_1\mathfrak{J}_2$ is a positive definite metric on the generalized tangent bundle E.

Here is evident that 1. is a the most obvious generalization of the invariance of the fundamental form under the action of the complex structure, in Equation (4.57). On the other hand 2. is the natural generalization of the fact that, from Definition 4.2.7, we get

$$\omega = Jg \quad \Rightarrow \quad J\omega = J^2 g = -g \tag{5.100}$$

We can recognize the classical Kähler and symplectic structures by studying the following [40, 14]

Example 5.3.1. Let (M, g, J, ω) be a Kähler manifold and consider the trivial generalized tangent bundle $E = T \oplus T^*$. The definition of the generalized complex structures $\mathfrak{J}_J$ and $\mathfrak{J}_\omega$ are given in Equations (5.80) and (5.81). It is immediate to see that $[\mathfrak{J}_J, \mathfrak{J}_\omega] = 0$ using the fact that $J^T\omega = \omega J^{-1} = -\omega J$. Besides

$$\mathsf{G} = -\mathfrak{J}_J\mathfrak{J}_\omega = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} \tag{5.101}$$

is a positive definite metric on $T \oplus T^*$. Hence the pair $(\mathfrak{J}_J, \mathfrak{J}_\omega)$ defines a generalized Kähler structure.

It's straightforward to see that we can obtain new generalized Kähler structures from a pair $(\mathfrak{J}_1, \mathfrak{J}_2)$ by applying B-transforms, for any B-closed forms. In fact the pair $(\mathfrak{J}_1^B, \mathfrak{J}_2^B) = (\mathcal{B}\mathfrak{J}_1\mathcal{B}^{-1}, \mathcal{B}\mathfrak{J}_2\mathcal{B}^{-1})$ defines a new generalized Kähler structure, since the condition 1. in Definition 5.3.2 is not modified

$$[\mathfrak{J}_1^B, \mathfrak{J}_2^B] = [\mathcal{B}\mathfrak{J}_1\mathcal{B}^{-1}, \mathcal{B}\mathfrak{J}_2\mathcal{B}^{-1}] = \mathcal{B}\mathfrak{J}_1\mathcal{B}^{-1}\mathcal{B}\mathfrak{J}_2\mathcal{B}^{-1} - \mathcal{B}\mathfrak{J}_2\mathcal{B}^{-1}\mathcal{B}\mathfrak{J}_1\mathcal{B}^{-1} = \mathcal{B}[\mathfrak{J}_1, \mathfrak{J}_2]\mathcal{B}^{-1} = 0 \tag{5.102}$$

and also condition 2. is preserved, since

$$\mathfrak{J}_1^B\mathfrak{J}_2^B = \mathcal{B}\mathfrak{J}_1\mathcal{B}^{-1}\mathcal{B}\mathfrak{J}_2\mathcal{B}^{-1} = \mathcal{B}\mathfrak{J}_1\mathfrak{J}_2\mathcal{B}^{-1} = -\mathcal{B}\mathsf{G}\mathcal{B}^{-1} = -\mathsf{G}^B \tag{5.103}$$

and G^B is positive definite too, since $\mathcal{B}$ is orthogonal, as we have seen in Section 5.1.1.

By applying a B-transformation to $(\mathfrak{J}_J, \mathfrak{J}_\omega)$ we obtain

$$\mathfrak{J}_J^B = \begin{pmatrix} J & 0 \\ BJ + J^T B & -J^T \end{pmatrix} \qquad \mathfrak{J}_\omega^B = \begin{pmatrix} \omega^{-1}B & -\omega^{-1} \\ \omega + B\omega^{-1}B & -B\omega^{-1} \end{pmatrix} \tag{5.104}$$

and

$$\mathsf{G}^B = \begin{pmatrix} -g^{-1}B & g^{-1} \\ g - Bg^{-1}B & Bg^{-1} \end{pmatrix} \tag{5.105}$$

where $g - Bg^{-1}B$ is a Riemannian metric for any two form B, restricted to the tangent bundle.

It seems that each metric can be obtained from a Kähler metric by B-action. However this is not always the case, since when B is not a closed form, the structure obtained are well defined only on the generalized tangent bundle E, whose transition functions encode the non-closed B field. In conclusion, a generalized Kähler structure is not the B-transform of a generalized Kahler structure defined on the trivial generalized tangent bundle, but it can be a more general structure, which encodes highly non-trivial patchings such as those of a generalized tangent bundle.

More generally it can be shown [14] that, given any generalized Kähler structure, the generalized metric takes the form

$$\mathsf{G} = \begin{pmatrix} -g^{-1}B & g^{-1} \\ g - Bg^{-1}B & Bg^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix} \tag{5.106}$$

Namely a generalized Kähler metric is completely determined by a Riemannian metric g together with a two-form $B \in \Lambda^2 T^*$. Finally it's useful to know that

Proposition 5.3.1. $C_\pm$ is locally the graph of $B \pm g : T \to T^*$.

Proposition 5.3.1 means that if we write the conditions which determine $C_\pm$, namely

$$\mathsf{G}v = \begin{pmatrix} -g^{-1}B & g^{-1} \\ g - Bg^{-1}B & Bg^{-1} \end{pmatrix} \begin{pmatrix} X \\ \xi \end{pmatrix} = \begin{pmatrix} X \\ \xi \end{pmatrix} = \pm v \qquad \forall\, v \in \mathfrak{X}(E) \tag{5.107}$$

we obtain for example from the first Equation $g^{-1}BX + g^{-1}\xi = \pm X$, or in other words $\xi = (B \pm g)X$. The second Equation is automatically satisfied.

5.3.2 Vielbein formalism

Despite of the name, the generalized metric is conveniently seen as an automorphism of the generalized tangent bundle. We know that G is a symmetric tensor

$$\mathsf{G} = \mathsf{G}^* \tag{5.108}$$

However, due to the the definition of the transpose map *, the matrix associated to G in Equation (5.106) doesn't look actually as a symmetric matrix. This is due to the fact that each block of the matrix represents a different map, namely

$$-g^{-1}B : \quad V^* \quad \to \quad V \qquad g^{-1} : \quad V^* \quad \mapsto \quad V \tag{5.109}$$

$$g - Bg^{-1}B : \quad V \quad \mapsto \quad V^* \qquad Bg^{-1} : \quad V^* \quad \mapsto \quad V^* \tag{5.110}$$

and then their transposes are

$$\left(-g^{-1}B\right)^* : \quad V \quad \to \quad V^* \qquad (g^{-1})^* : \quad V \quad \mapsto \quad V^* \tag{5.111}$$

$$(g - Bg^{-1}B)^* : \quad V^* \quad \mapsto \quad V \qquad (Bg^{-1})^* : \quad V \quad \mapsto \quad V \tag{5.112}$$

In this framework is immediate to recognize that the transposition amounts to transpose the matrix with respect to the secondary diagonal. This can be easily achieved by noticing that, written in components

$$\mathsf{G} = \begin{pmatrix} -(g^{-1}B)^i{}_j & g^{ij} \\ g_{ij} - (Bg^{-1}B)_{ij} & (Bg^{-1})_i{}^j \end{pmatrix} \tag{5.113}$$

where the top indices act on components of a form, while the bottom indices act on the components of a vector. For example g^{ij} acts on the components of a form ξ_i and returns the component of a vector. In this framework it is simple to understand that for example an object such as

$$\mathcal{O}^i{}_j : \quad X^j \quad \mapsto \quad \mathcal{O}^i{}_j X^j \tag{5.114}$$

acts on the components of a vector and returns the components of a vector. The same happens for forms

$$\mathcal{O}_j{}^i : \quad \xi_i \quad \mapsto \quad \mathcal{O}_j{}^i \xi_i \tag{5.115}$$

While an object which has the indices on the same line, has a transpose in the usual meaning of the term. The use of index notation which in this case help us to understand why the notion of transposition is not the usual one, in general is a very useful way to perfom calculus in the generalized complex framework.

In order to achieve the usual meaning of transposition with respect to the primary diagonal of the matrix, we can simply multiplicate the generalized metric by the indefinite metric η, which has matrix

$$\eta = \begin{pmatrix} 0 & \delta^i{}_j \\ \delta_i{}^j & 0 \end{pmatrix} \tag{5.116}$$

and then one obtains

$$\mathcal{H} = \eta \mathsf{G} = \begin{pmatrix} g - Bg^{-1}B & Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix} \tag{5.117}$$

or written in terms of components

$$\mathcal{H} = \begin{pmatrix} (g - Bg^{-1}B)_{ij} & (Bg^{-1})_i{}^j \\ -(g^{-1}B)^i{}_j & (g^{-1})^{ij} \end{pmatrix} \tag{5.118}$$

which is symmetric in the usual sense.

We can introduce two sets of n ordinary vielbein which form two basis respectively for $C_\pm$

$$\{e_\pm\} \qquad \{\hat{e}_\pm\} \tag{5.119}$$

such that $\hat{e}_\pm$ are the inverses of $e_\pm$, namely

$$e^a_{\pm i}\hat{e}^i_{\pm b} = \delta^a_b \qquad \hat{e}^i_{\pm b}e^b_{\pm j} = \delta^i_j \tag{5.120}$$

They obey the obvious relations

$$g_{ij} = \delta_{ab}e^a_{\pm i}e^b_{\pm j} \qquad g^{ij} = \hat{e}^i_{\pm a}\hat{e}^j_{\pm b}\delta^{ab} \tag{5.121}$$

We take $e_\pm$ to be a basis for $C_\pm$. With this conventions we can build a set of $2n$ generalized vielbeins $\{E\}$ which parametrize the coset

$$O(n,n)\big/(O(n) \times O(n)) \tag{5.122}$$

In particular if one explicitly writes

$$E = \frac{1}{\sqrt{2}}\begin{pmatrix} e_+ - \hat{e}^T_+ B & \hat{e}^T_+ \\ -e_- - \hat{e}^T_- B & \hat{e}^T_- \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \hat{e}^T_+(g - B) & \hat{e}^T_+ \\ -\hat{e}^T_-(g + B) & \hat{e}^T_- \end{pmatrix} \tag{5.123}$$

the metrics η and $\mathcal{H}$ take the form

$$\eta = E^T\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}E \qquad \mathcal{H} = E^T\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}E \tag{5.124}$$

Let us notice that the $O(n) \times O(n)$ acts by the left and simply rotates the set of vielbeins with a matrix of the form

$$E \mapsto KE \qquad K = \begin{pmatrix} O_+ & 0 \\ 0 & O_- \end{pmatrix} \tag{5.125}$$

where $O_\pm \in O(n)$.

The action of $O(n,n)$ is much more interesting, since as it can be easily seen by using the indices formalism, it acts on the generalized metric $\mathcal{H}$ as

$$\mathcal{H} \quad \mapsto \quad O^T\mathcal{H}O \tag{5.126}$$

where

$$O = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{5.127}$$

By using Equation (5.124) and the fact that $O \in O(n,n)$ acts on generalized vielbeins by the right

$$E \quad \mapsto \quad EO \tag{5.128}$$

we immediately find the transformation rules for the ordinary vielbeins

$$\hat{e}_+ \mapsto [d^T + b^T(B+g)]\hat{e}_+ \equiv \tilde{\hat{e}}_+ \qquad \hat{e}_- \mapsto [d^T + b^T(B-g)]\hat{e}_- \equiv \tilde{\hat{e}}_- \tag{5.129}$$

5.3.3 Spinor bundle in GCG

The complete machinery of the GCG can be equivalently described in terms of spinors. This alternative description is very useful for many purposes, since it reduces to work with forms, which are particularly easy to handle.

We can also transport the machinery developed in Section 5.1.2 on a generalized complex manifold. In particular the Clifford action of a generalized vector is now defined locally on forms $\Phi \in \Lambda T|^*_{U_\alpha}$. Its action is given by

$$v_\alpha \cdot \Phi = i_{X_\alpha}\Phi_\alpha + \xi_\alpha \wedge \Phi_\alpha \tag{5.130}$$

where $v = X + \xi$. As usual we will drop the subscript α which refers to the open set U_α, where it is unnecessary. After introducing the $Spin(n,n)$ gamma matrices $\{\check{\Gamma}_i, \hat{\Gamma}^i\}$ we can rewrite the last Equation

$$X \cdot \Phi = (X^i \check{\Gamma}_i + \xi_i \hat{\Gamma}^i)\Phi \tag{5.131}$$

Since the forms are twisted in the overlaps $U_{\alpha\beta}$ by a two-form $dA_{\alpha\beta}$ as described in Equation (5.67), the requirement for the Clifford action in Equation (5.130) to be globally defined, $\Phi^\pm_\alpha \in S^\pm(E)$ have to obey the following patching condition

$$\Phi_\alpha = e^{dA_{\alpha\beta}}\Phi_\beta \tag{5.132}$$

which means that

$$\psi = e^{B_\alpha}\Phi^\pm_\alpha = e^{B_\beta}\Phi^\pm_\beta \tag{5.133}$$

is globally defined on $S^\pm(E)$.

A crucial point is that the exterior derivative is well defined on $S^\pm(E)$, since it is identifiable with $\Lambda^\pm T^*$, and it maps

$$d: \quad S^\pm(E) \quad \longmapsto \quad S^\mp(E) \tag{5.134}$$

As we mentioned at the beginning of the Section, every geometric properties of a generalized complex manifold can be rewritten in terms of pure spinors. Their Definition is identical to the linear case, Definition 5.1.5. Then we can immediately observe that a pure spinor Φ can be associated to each generalized complex structure $\mathfrak{J}$ by the relation

$$L_\Phi = L_{\mathfrak{J}} \tag{5.135}$$

where $L\phi$ is the maximal isotropic subbundle which define the pure spinor, while $L_{\mathfrak{J}}$ is the $+i$-eigenbundle associated to the generalized complex structure $\mathfrak{J}$.

Using the vielbeins defined in Equation (5.123) one can introduce a basis to diagonalize the $O(n) \times O(n)$ structure induced by a generalized metric. In fact

$$\begin{pmatrix} \Gamma^+ \\ \Gamma^- \end{pmatrix} = E^{-T}\begin{pmatrix} \check{\Gamma} \\ \hat{\Gamma} \end{pmatrix} = \begin{pmatrix} \hat{e}^T_+ \left(\check{\Gamma} + (g-B)\hat{\Gamma}\right) \\ \hat{e}^T_- \left(\check{\Gamma} - (g+B)\hat{\Gamma}\right) \end{pmatrix} \tag{5.136}$$

such that

$$\{\Gamma^+_a, \Gamma^-_b\} = 0 \qquad \{\Gamma^+_a, \Gamma^+_b\} = 2\delta_{ab} \qquad \{\Gamma^-_a, \Gamma^-_b\} = -2\delta_{ab} \tag{5.137}$$

The main point is that one can decompose the $Spin(n,n)$ spinors in representations of $Spin(n,0) \times Spin(n,0)$. In fact, if γ_a are the $Spin(n,0)$ matrices, then one can write

$$\Gamma^+_a = \gamma_a \otimes 1 \qquad \Gamma^-_a = \gamma_7 \otimes \gamma_a \tag{5.138}$$

where $\gamma_7 = \gamma_1 \cdots \gamma_n$ is the volume form of the Clifford algebra, as defined in Equation (3.1.6). Equation (5.138) is true only in the case in which n is even and $\frac{n}{2}$ is odd, which is the relevant one to study $SU(3) \times SU(3)$ structures in the next Section.

The corresponding decomposition of $Spin(n,n)$ pure spinors $\Phi^\pm$ is written

$$\Phi^+ = \eta^1_+ \otimes \overline{\eta}^2_+ + \eta^1_- \otimes \overline{\eta}^2_- \qquad \Phi^- = \eta^1_+ \otimes \overline{\eta}^2_- + \eta^1_- \otimes \overline{\eta}^2_+ \tag{5.139}$$

where η_+ is a chiral $Spin(6,0)$ pure spinor, while η_- is a chiral $Spin(0,6)$ pure spinor. They obey the relations

$$-i\gamma_7\eta_\pm = \eta_\pm \tag{5.140}$$

namely they have the same chirality.

5.3.4 SU(3) × SU(3) structures

The obvious generalization in CGC of the $SU(3)$ structures we mentioned in Section 4.3 is given by a pair of them. The result is a $SU(3) \times SU(3)$ structure.

As we have seen the introduction of both a generalized complex structure $\mathfrak{J}$ and a compatible generalized metric G on a smooth manifold M reduces the structure group to $U(n) \times U(n)$. One further reduction can be achieved if one can define a pair of indipendent $SU(3)$ structures

$$(\omega_1, \Omega_1) \qquad (\omega_2, \Omega_2) \tag{5.141}$$

as in Section 4.3.

The $SU(3) \times SU(3)$ structure can also be determined by a pair of globally defined, non vanishing pure spinors (η^1_+, η^2_+). In fact $SU(3) \times SU(3)$ is the subgroup of $O(6,6)$ under which they remain invariant.

The link between the two formulations is given by the relation

$$J^+_{ij} = -i\overline{\eta}^1_+ \gamma_{ij} \eta^1_+ \qquad \Omega^+_{ijk} = -i\overline{\eta}^1_- \gamma_{ijk} \eta^1_+ \tag{5.142}$$

$$J^-_{ij} = -i\overline{\eta}^2_+ \gamma_{ij} \eta^2_+ \qquad \Omega^-_{ijk} = -i\overline{\eta}^2_- \gamma_{ijk} \eta^2_+ \tag{5.143}$$

where η^i_- is the charge conjugation of the spinor η^i_+, namely

$$\eta_- = C\eta^* \tag{5.144}$$

where C is the charge conjugation matrix such that $\gamma^*_a = -C^{-1}\gamma_a C$.

A third way to define a $SU(3) \times SU(3)$ structure is to encode the information given by the two invariant and nowhere vanishing spinors (η^1_+, η^2_+) in a pair of $Spin(6,6)$ pure spinors as in Equation (5.139)

$$\Phi^+ = \eta^1_+ \otimes \overline{\eta}^2_+ + \eta^1_- \otimes \overline{\eta}^2_- \qquad \Phi^- = \eta^1_+ \otimes \overline{\eta}^2_- + \eta^1_- \otimes \overline{\eta}^2_+ \tag{5.145}$$

The two $SU(3)$ structures are defined respectively on C_+ and on C_-. We define two sets of vielbeins $\{e^a_+\}$ and $\{e^a_-\}$ respectively on C_+ and on C_-, so that the two $SU(3)$ structures take the nice standard form

$$\omega_\pm = e^1_\pm \wedge e^4_\pm + e^2_\pm \wedge e^5_\pm + e^3_\pm \wedge e^6_\pm \tag{5.146}$$

$$\Omega_\pm = (e^1_\pm + ie^4_\pm) \wedge (e^2_\pm + ie^5_\pm) \wedge (e^3_\pm + ie^6_\pm) \tag{5.147}$$

Even if it is not necessary, usually one assumes that $e_+ = e_-$ for simplicity. Moreover we can introduce the curved gamma matrices, which are simply defined by the relation

$$\gamma_i = e^a{}_i \gamma_a \tag{5.148}$$

The main idea to proceed is that the $Spin(6)$ spinors (η^1_+, η^2_+) form an angle which is not necessarily constant throughout the manifold M. Let us denote by η_- the charge conjugated of η_+. Then we can write in general the decomposition

$$\eta^2_+ = e^{i\theta}\cos(\varphi) + \frac{1}{2} z^i \sin(\varphi) \gamma_i \eta^1_- \tag{5.149}$$

where the angle φ denotes the angle between η^1_+ and η^2_+, and varies in the interval $0 \leq \varphi \leq \frac{\pi}{2}$. z^i is such that $|z|^2 = 2$. Now consider the mutually orthogonal spinors

$$\eta_+ \qquad \chi_+ = \frac{1}{2} z^i \gamma_i \eta_- \tag{5.150}$$

such that

$$\overline{\eta}_+ \eta_+ = 1 \qquad \overline{\chi}_+ \chi_+ = 1 \qquad \overline{\eta}_+ \chi_+ = 0 \tag{5.151}$$

We can rewrite

$$\eta^1_+ = e^{i\frac{\theta}{2}} \eta_+ \qquad \eta^2_+ = e^{-i\frac{\theta}{2}} \left(\cos(\varphi)\eta_+ + \sin(\varphi)\chi_+\right) \tag{5.152}$$

Note that at points where $\sin(\varphi) = 0$, the spinor χ doesn't need to be defined. In the other points on the manifold the orthogonal spinors η_+ and χ_+ define a **local SU(2) structure**, which is simply described by

$$z^i = \overline{\eta}_- \gamma^i \chi_+ \tag{5.153}$$

$$\omega_{ij} = \frac{i}{2}\overline{\eta}_{+}\gamma_{ij}\eta_{+} - \frac{i}{2}\overline{\chi}_{+}\gamma_{ij}\chi_{+} \tag{5.154}$$

$$\Omega_{ij} = \overline{\chi}_{+}\gamma_{ij}\eta_{+} \tag{5.155}$$

Then in general the pure spinors defining a $SU(2)$ local structure take the form

$$\Phi^{+} = e^{-\phi-B+\frac{1}{2}z\wedge\bar{z}}\left(\overline{k}_{||}e^{-ij} - i\overline{k}_{\perp}\omega\right) \tag{5.156}$$

$$\Phi^{-} = e^{-\phi-B}z\left(k_{\perp}e^{-ij} + ik_{||}\omega\right) \tag{5.157}$$

6

T-duality

6.1 T-duality for the NLSM

We define the NLSM as the theory of maps

$$\phi : \Sigma \to M \tag{6.1}$$

where Σ is a compact Riemann surface called the *worldsheet*, while M is a Riemann manifold called the *target space*. Let us assume that $dim(M) = D \geq 2$. In order to completely define a NLSM we need also some geometric data about the target space, that is

- A Riemannian metric g.
- A closed three-form $H \in \Lambda^3 T^*$.

The generic action for a NLSM can be written as a sum

$$S_0 = S_P + S_{WZ} \tag{6.2}$$

where S_P is the *Polyakov action*

$$\begin{aligned} S_P = & \quad \frac{1}{2}\int_\Sigma d^2\sigma \quad \sqrt{h}h^{\mu\nu} g_{ij}(X)\partial_\mu X^i \partial_\nu X^j \\ = & \quad \frac{1}{2}\int_\Sigma \quad g_{ij} dX^i \wedge *dX^j \end{aligned} \tag{6.3}$$

and $h_{\mu\nu}$ is a pseudo-Riemannian metric on the worldsheet Σ ($h = det(h_{\mu\nu})$), while $\{\sigma^\mu\}_{\mu\in I_2}$ are the local coordinates on the worldsheet Σ. $d^2\sigma \equiv d\sigma^1 \wedge d\sigma^2$ and in the second expression we note that dX^i denotes the pullback to a worlsheet one-form $dX^i \equiv \phi^*(dX^i) = \partial_\mu X^i d\sigma^\mu$. The subscript 0 in Equation (6.2) means that the action in ungauged.

Often we will take $h_{\mu\nu}$ to be the flat pseudo-Riemannian metric in two dimensions $h_{\mu\nu} = \eta_{\mu\nu}$ and $\eta_{00} = -\eta_{11} = 1$, so that with this choice of gauge the action becomes

$$S_P = \frac{1}{2}\int_\Sigma d^2\sigma \quad g_{ij}\partial_\mu X^i \partial^\mu X^j \tag{6.4}$$

X^i are the local coordinates on the target space, which locally describe the map ϕ. S_{WZ} is the *Wess-Zumino term* associated to the three-form H defining the NLSM. If H is exact, so that we can write $H = db$, then S_{WZ} takes the form

$$\begin{aligned} S_{WZ} = & \quad \frac{1}{2}\int_\Sigma d^2\sigma \quad \epsilon^{\mu\nu} b_{ij}(X)\partial_\mu X^i \partial_\nu X^j \\ = & \quad \int_\Sigma \quad \phi^* b \end{aligned} \tag{6.5}$$

where $b = \frac{1}{2} dX^i \wedge dX^j$. We can rewrite in terms of H

$$\begin{aligned} S_{WZ} = & \quad \frac{1}{3}\int_\Omega d^3\sigma \quad \epsilon^{\mu\nu\rho} H_{ijk}(X)\partial_\mu X^i \partial_\nu X^j \partial_\rho X^k \\ = & \quad \int_\Omega \quad \phi^* H \end{aligned} \tag{6.6}$$

where Ω is any three-manifold such that $\partial\Omega = \Sigma$, and $H = \frac{1}{3}dX^i \wedge dX^j \wedge dX^k$.

If H is not exact, then the action depends on the choice of Ω, but the difference between two different choices takes the form

$$S_{WZ}(\Omega') - S_{WZ}(\Omega) = \int_{\Omega'-\Omega} \phi^* H = \int_{\phi(\Omega'-\Omega)} H \tag{6.7}$$

where $\Omega' - \Omega$ is the three-manifold obtained by glueing Ω' to Ω along their common boundary with opposite orientations. The result is a topological number which depends only on the cohomology class of H and on the homology class of $\phi(\Omega' - \Omega)$. Since it is only a number, it doesn't affect the classical equations of motion. However it could lead to an ambiguity in the quantum theory, since the Euclidean functional integral

$$\int [dX] e^{-S} \tag{6.8}$$

should be modified by a phase $exp\left(i \int_{\phi(\Omega'-\Omega)} H\right)$. The ambiguity is eliminated and the functional integral well defined if

$$\frac{1}{2\pi}[H] \in H^3(M, \mathbb{Z}) \tag{6.9}$$

i.e. if $\frac{1}{2\pi}[H]$ is an integral cohomology class.

Let us notice that by introducing the light-cone coordinates $\sigma^{\pm} = \frac{1}{\sqrt{2}}\left(\sigma^0 \pm \sigma^1\right)$ we can rewrite

$$S = \int_\Sigma d^2\sigma \quad \mathcal{E}_{ij} \partial_+ X^i \partial_- X^j \tag{6.10}$$

where $\mathcal{E}_{ij} = g_{ij} + b_{ij}$. We assumed that $\epsilon^{01} = 1$.

Let us study how a transformation of the fields of the form

$$\delta X^i = \alpha^l K_l^i \tag{6.11}$$

affects the NLSM. For the moment let us consider only global transformations, that is transformation such that α is a constant. Firstly, let us notice that

$$\delta S = \int d^2\sigma \quad \alpha^l \left[K_l^k \partial_k g_{ij} + g_{kj}\partial_i K_l^k + g_{ik}\partial_j K_l^k\right] \partial^\mu X^i \partial_\mu X^j \tag{6.12}$$

so that $\delta S = 0$ if and only if $K_l^k \partial_k g_{ij} + g_{kj}\partial_i K_l^k + g_{ik}\partial_j K_l^k = 0 \quad \forall\, l \in I_d$ $(d \leq D)$, namely if and only if K_l is a Killing vector for each l.

Since we are studying the context of T-duality, we will deal only with abelian isometry groups. We also notice that

$$\delta S_{WZ} = \int_\Sigma d^2\sigma \quad \alpha^l K_l^i H_{ijk} \partial_\mu X^j \partial_\nu X^k \epsilon^{\mu\nu} \tag{6.13}$$

which is a surface term only if $i_{K_l} H = K_l^i H_{ijk}$ is an exact two-form. This means that there must be a set of globally defined one-forms v_l such that

$$i_{K_l} H = dv_l \qquad \forall\, l \in I_d \tag{6.14}$$

The compactness of Σ assures that the transformation in Equation (6.11) leaves invariant both S_P and S_{WZ}.

6.1.1 Gauging the NLSM

The gauging of an abelian isometry, as it is well known, is extremely simple. It consists in promoting the symmetry in Equation (6.11) to a local one, by simply replacing the constant α with a parameter which depends on x and by introducing a set of connections C^l. As we Know from Chapter 2 the connection transforms as

$$\delta C^l = d\alpha^l \tag{6.15}$$

The gauged version of the action S_P is trivially provided by the minimal coupling

$$S_P^G = \frac{1}{2}\int_\Sigma d^2\sigma \quad g_{ij} D_\mu X^i D_\nu X^j \tag{6.16}$$

where the covariant derivative is

$$D_\mu X^i = \partial_\mu X^i - C_i^l K_l^i \tag{6.17}$$

while the field-strenght is given by

$$\mathscr{G}^l = dC^l \tag{6.18}$$

In order to gauge the Wess-Zumino term we have to proceed to successive additions of terms, each of which cancels the variation of the previous one. In particular one obtains that the complete gauged Wess-Zumino term is given by

$$S_{WZ}^G = S_{WZ} + S_1 + S_2 \tag{6.19}$$

where

$$S_1 = \int_\Sigma d^2\sigma \quad \epsilon^{\mu\nu} A_\mu^l v_{li} \partial_\nu X^i \tag{6.20}$$

serves to cancel δS_{WZ}. Moreover

$$S_2 = -\frac{1}{2}\int_\Sigma d^2\sigma \quad \epsilon^{\mu\nu} v_{[l|i|} K_{m]}^i A_\mu^l A_\nu^m \tag{6.21}$$

serves to cancel δS_1. Fortunately, if the (sub-)group of the isometries which are gauged is anomaly-free (that is the case of an abelian isometry group in two dimensions) we have that $\delta S_2 = 0$.

We can rewrite S_{WZ}^G in the good-looking way

$$S_{WZ}^G = \int_\Omega d^3\sigma \quad \left\{\frac{1}{3} H_{ijk} D_\mu X^i D_\nu X^j D_\rho X^k + \mathscr{G}_{\mu\nu}^l v_{li} D_\rho X^i\right\} \tag{6.22}$$

so that the whole action takes the nice form

$$S^G = \frac{1}{2}\int_\Sigma \quad g_{ij} DX^i \wedge *DX^j + \int_\Omega \quad \left\{\frac{1}{3} H_{ijk} DX^i \wedge DX^j \wedge DX^k + \mathscr{G}^l \wedge v_{li} DX^i\right\} \tag{6.23}$$

In [46, 47, 11] it is shown that the costraints which are needed for gauging the NLSM are

1. $i_{K_l} H = dv_l$ for some globally defined one-forms v_a.
2. $\mathfrak{L}_{K_l} H = 0$.
3. $\mathfrak{L}_{K_l} v_m = 0$.
4. $i_{K_l} i_{K_m} H = -dB_{lm}$ for some antysimmetric, globally defined functions $B_{lm} = i_{K_l} v_m$.
5. $i_{K_l} i_{K_m} i_{K_n} H = 0$.

We will refer to these contraints as the *Gauging Conditions (GC).*

The procedure of integrating out the gauge field is quite general. It consists in rewriting S^G in the following form

$$S^G = S_0 + \int_\Sigma d^2\sigma \left(-C_\mu^l J_l^\mu + \frac{1}{2} C_\mu^l C_\nu^m \left[G_{lm}\eta^{\mu\nu} + B_{lm}\epsilon^{\mu\nu}\right]\right) \tag{6.24}$$

where we have chosen the flat Minkowskian metric $h_{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ for the worldsheet. G_{lm} is defined to be

$$G_{lm} = g_{ij} K_l^i K_m^j \tag{6.25}$$

In the light-cone coordinates $\sigma^\pm = \frac{1}{\sqrt{2}}\left(\sigma^0 \pm \sigma^1\right)$ the indices are raised and lowered by the metric $\eta^{+-} = \epsilon^{+-} = 1$ we get

$$S^G = S_0 + \int_\Sigma d^2\sigma \left(-C_+^l J_l^+ - C_-^l J_l^- + C_+^l E_{lm} C_-^m\right) \tag{6.26}$$

where

$$E_{lm} = G_{lm} + B_{lm} \qquad J_{l\pm} = (K_{li} \pm v_{li})\partial_\pm X^i \tag{6.27}$$

If E_{ab} is everywhere invertible it's easy to find the Equations of motions for the gauge fields

$$\begin{aligned} \tfrac{\partial S^G}{\partial C^l_+} &= -J^+_l + E_{lm}C^m_- = 0 \\ \tfrac{\partial S^G}{\partial C^l_-} &= -J^-_l + C^m_+ E_{ml} = 0 \end{aligned} \tag{6.28}$$

from which we get

$$C^l_- = (E^{-1})^{lm} J_{-m} \tag{6.29}$$

$$C^l_+ = J_{+m}(E^{-1})^{ml} \tag{6.30}$$

By inserting the expression for $C^l_\pm$ into Equation (6.26) we obtain

$$S' = S_0 - \int_\Sigma d^2\sigma \quad J^-_l (E^{-1})^{lm} J^+_m \tag{6.31}$$

This can be rewritten

$$S' = \int_\Sigma d^2\sigma \quad \mathcal{E}'_{ij}\partial_+ X^i \partial_- X^j \tag{6.32}$$

where

$$\mathcal{E}'_{ij} = \mathcal{E}_{ij} - (K_{li} + v_{li})(E^{-1})^{lm}(K_{mj} - v_{mj}) \tag{6.33}$$

It turns out that E is invertible if and only if the isometry group acts without fixed points. Since $\mathcal{E}$ is the object which contains the geometrical informations about the sigma model, gauging the isometries and then integrating the gauge fields out amounts to change the geometry of the sigma model from $\mathcal{E}_{ij}$ to $\mathcal{E}'_{ij}$.

6.1.2 The geometry of the gauged NLSM

Let us study in some detail the geometry of a gauged NLSM. We consider NLSMs whose isometries are generated by a set of globally defined Killing vector fields $\{K_l\}_{l\in I_d}$. Moreover, we will consider only NLSM whose isometry group $\mathcal{G}$ is an abelian group, which acts freely on M. Then the d Killing vector fields are commuting vectors

$$[K_l, K_m] = 0 \qquad \forall\, l, m \in I_d \tag{6.34}$$

We will always denote the indices which refer to the set of Killing vectors as $l, m, n, \ldots$.

The geometry of a NLSM is completely determined by the data (Σ, M, g, H)

1. The worldsheet Σ.
2. The target space (M, g).
3. The closed three-form $H \in B^3(M)$.

Consequently, once specified the worldsheet Σ, we will denote each NLSM by (M, g, H).

The action of the isometry group $\mathcal{G}$ on M defines the space of orbits

$$N \equiv {}^{M}\!\big/_{\mathcal{G}} \tag{6.35}$$

In the case of $U(1)$ actions it turns out that N is a manifold and the natural projection on the space of orbits

$$\pi : M \to N \tag{6.36}$$

defines a principal bundle whose fiber is $\mathcal{G}$.

In this setup, a form ω is *horizontal* if

$$i_{K_l}\omega = 0 \tag{6.37}$$

while it is *invariant* if

$$\mathfrak{L}_{K_l}\omega = 0 \tag{6.38}$$

A form which is both horizontal and invariant is *basic*. A basic form can be thought as a form on the base space N (hence the name *basic*), as it can be shown that it is the pullback with respect to the projection $\pi : M \to N$ of a form on N.

A metric g is *horizontal* if

$$g(K_l, X) = 0 \qquad \forall\, X \in TM \tag{6.39}$$

g is *invariant* if

$$\mathfrak{L}_{K_l} g = 0 \tag{6.40}$$

and it is *basic* if it is both horizontal and invariant. As before, a basic metric can be thought as a metric on the base space.

We can define a set of d one-forms $\{\xi^l\}$, which are dual to the Killing vectors, namely

$$i_{K_l}\xi^m = \delta_l^m \tag{6.41}$$

by writing

$$\xi_i^l = G^{lm} g_{ij} K_m^j \tag{6.42}$$

where G^{lm} is the inverse of G_{lm} defined in Equation (6.25). The two-forms

$$F^l = d\xi^l \tag{6.43}$$

are horizontal. The metric on M can be written as

$$g = \overline{g} + G_{lm}\xi^l \otimes \xi^m \tag{6.44}$$

where $\overline{g}$ is basic, while the term $G_{lm}\xi^l \otimes \xi^m$ encodes the restriction of the metric on the fibers as well as the mixed matrix elements which connect the base with the fibers.

We can redefine the coordinates in each patch to obtain "adapted" coordinates $X^i = (X^a, Y^\mu)$ in which the Killing vectors take the nice form

$$K_l^i \frac{\partial}{\partial X^i} = \frac{\partial}{\partial X^l} \tag{6.45}$$

The Y^μ coordinates parametrize the base space N. The set of adapted coordinates induce also the splitting of the one-forms

$$\xi^l = dX^l + A^l \tag{6.46}$$

where A^l are local connections on N, and are horizontal. By looking at Equation (6.44) we immediately understand the meaning of the local connections A^l. They encode the metric informations in the directions which connect the base with the fibers. Since $F^l = dA^l$, the two-forms F^l are the local curvatures.

From 1. and 4. of the GC one can obtain the splitting

$$v_l = \overline{\xi}_l - B_{lm}\xi^m \tag{6.47}$$

where $\overline{\xi}_l$ is the basic component of v_l. We can define the basic two-form

$$\widetilde{F}_l = d\overline{\xi}_l \tag{6.48}$$

Then the closed three-form H can be expanded as

$$H = \overline{H} + (i_{K_l}H) \wedge \xi^l + \frac{1}{2}(i_{K_l}i_{K_m}H) \wedge \xi^l \wedge \xi^m - \frac{1}{6}(i_{K_l}i_{K_m}i_{K_n}H) \wedge \xi^l \wedge \xi^m \wedge \xi^n \tag{6.49}$$

where $\overline{H}$ is horizontal. After some simple algebra it can be rewritten

$$H = \overline{H} + \widetilde{F}_l \wedge \xi^l + dB \tag{6.50}$$

where

$$B = \frac{1}{2}B_{lm}\xi^l \wedge \xi^m \tag{6.51}$$

is a globally defined two-form. Since $dH = 0$, then

$$\overline{H} = -\widetilde{F}_l \wedge F^l \tag{6.52}$$

and $\overline{H}$ is basic. Moreover locally we can write $H = db$. It turns out that

$$b = \overline{b} + \xi^l \wedge \overline{\xi}_l + B \tag{6.53}$$

where as it is intuitive $\overline{b}$ is a basic two-form.

As we have seen, the new geometry obtained by integrating out the gauge fields is encoded into $\mathcal{E}'_{ij}$. It turns out to be

$$\mathcal{E}'_{ij} = \mathcal{E}_{ij} - \xi^l_i E_{lm} \xi^m_j + \overline{\xi}_{li} (E^{-1})^{lm} \overline{\xi}_{mj} - \overline{\xi}_{li} \xi^l_j + \xi^l_i \overline{\xi}_{lj} \tag{6.54}$$

We extract from $\mathcal{E}'_{ij}$ the symmetric and antysimmetric parts, respectively

$$\widetilde{G}^{lm} = (E^{-1})^{(lm)} \qquad \widetilde{B}^{lm} = (E^{-1})^{[lm]} \tag{6.55}$$

We can write for the new b-field

$$b' = b - \overline{\xi}_l \wedge \xi^l - \xi^l_i B_{lm} \xi^m_j + \overline{\xi}_{li} \widetilde{B}^{lm} \overline{\xi}_{mj} \tag{6.56}$$

The new geometry (M, g', H') is given by

$$g' = g - G_{lm} \xi^l \otimes \xi^m + \widetilde{G}^{lm} \overline{\xi}_l \otimes \overline{\xi}_m = \overline{g} + \widetilde{G}^{lm} \overline{\xi}_l \otimes \overline{\xi}_m \tag{6.57}$$

and by

$$H' = \overline{H} + \overline{\xi}_l \wedge F^l + d\widetilde{B} \tag{6.58}$$

where

$$\widetilde{B} = \frac{1}{2} \widetilde{B}^{lm} \overline{\xi}_l \wedge \overline{\xi}_m \tag{6.59}$$

It's fundamental to notice that both g' and H' are basic with respect to all of the Killing vectors. This implies that the new NLSM is invariant under the local symmetries as in Equation (6.11).

6.1.3 The NLSM on the trivial T^3

In this Section we want to see at work the formalism studied in Sections 6.1.1, 6.1.2. So we consider the simplest non-trivial example. It turns out to be the three-torus T^3.

The three-torus T^3

The three-torus T^3 is a flat smooth manifold, which can be defined as

$$\mathsf{T}^3 = S^1 \times S^1 \times S^1 \tag{6.60}$$

In Section 2.1.1 we have seen that the smooth structure on T^3 is simply defined by taking the triple product of the atlas of the S^1 atlas in Example 2.1.1.

It is often convenient to encode the whole smooth structure by the identifications

$$(x, y, z) \sim (x + 1, y, z) \tag{6.61}$$
$$(x, y, z) \sim (x, y + 1, z) \qquad (x, y, z) \in \mathbb{R}^3 \tag{6.62}$$
$$(x, y, z) \sim (x, y, z + 1) \tag{6.63}$$

so that T^3 can be thought as the quotient of $\mathbb{R}^3$ with respect to the above identifications. Equations (6.61), (6.62) and (6.63) tell us that we are considering circles of length 1 or equivalently of radius $R = \frac{1}{2\pi}$.

This means that in each change of charts from U_α to U_β the coordinates on each of the circles of the torus T^3 are shifted by a combination of the three transformations

$$x^\alpha \quad \mapsto \quad x^\beta = x^\alpha + 1 \tag{6.64}$$
$$y^\alpha \quad \mapsto \quad y^\beta = y^\alpha + 1 \tag{6.65}$$
$$z^\alpha \quad \mapsto \quad z^\beta = z^\alpha + 1 \tag{6.66}$$

Since these are the transformations defining the change of the charts the Jacobian of a combination of the transformations in Equations (6.64), (6.65), (6.66) tells us that the three torus is an oriented manifold. Then it is possible to define a global three-form H, which is proportional to the volume form.

Since the above Jacobian is always equal to 1, H is globally defined and we can write

$$H = h dx^\alpha \wedge dy^\alpha \wedge dz^\alpha \tag{6.67}$$

in each chart U_α which induces coordinates $\{x^\alpha, y^\alpha, z^\alpha\}$. We take $h \in \mathbb{Z}$ for later convenience.

If we take the trivial metric

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{6.68}$$

then each of the vectors which generate the translations along the fibers

$$\frac{\partial}{\partial x^\alpha} \qquad \frac{\partial}{\partial y^\alpha} \qquad \frac{\partial}{\partial z^\alpha} \tag{6.69}$$

represents a Killing vector since Equation (2.128) is trivially satisfied. We will consider the action generated by the Killing vector

$$K^\alpha = \frac{\partial}{\partial z^\alpha} \tag{6.70}$$

that is the translation along the circle parametrized by z^α. From Equations (6.64), (6.65), (6.66) we see that the Killing vector is globally defined since

$$\frac{\partial}{\partial z^\alpha} = \frac{\partial}{\partial z^\beta} \tag{6.71}$$

and then the solution of the Killing Equation (2.128) can be easily glued in the intersections of the charts. Moreover the last Equation allows us to denote the Killing vector associated to the third circle simply by K.

The action generated by K allows to reach each point on the third circle by starting from anyone of them. In fact the flow associated to it as in Equation 2.126 is

$$z^\alpha \quad \mapsto \quad z^\alpha + \epsilon \tag{6.72}$$

Since a circle S^1 is diffeomorphic to $U(1)$ we can think to the fiber as a $U(1)$ group and to the Killing action as the multiplication by a complex number of modulus 1. For this reason we will call the Killing action $U(1)$-action.

Since it acts freely, we can consider the quotient as in Equation (6.35)

$$\mathsf{T}^2 = \mathsf{T}^3 / U(1) \tag{6.73}$$

where we have to remember that the $U(1)$-action is associated to the Killing K. Then T^3 can be thought as the trivial principal bundle

$$\pi : \mathsf{T}^3 \to \mathsf{T}^2 \tag{6.74}$$

with structure group and fiber diffeomorphic to $U(1)$. This is the reason why this example is often called the *trivial* three-torus T^3.

The geometry of a NLSM on T^3

The main point is now to understand if a NLSM on the three torus is a gaugeable one. In particular we ahve to check if

$$i_K H \tag{6.75}$$

is an exact form dv as required by the 1. of the GC.

In the chart U_α we can write

$$i_K H = h dx^\alpha \wedge dy^\alpha \tag{6.76}$$

We can immediately say that this two-form is not exact, since it is a generator of $H^2(\mathsf{T}^2,\mathbb{R})$ as we have seen in Example **??**. We can also see that it is not exact because the Equations

$$i_K H = dv^\alpha \tag{6.77}$$
$$i_K H = dv^\beta \tag{6.78}$$

have solutions respectively on U_α and on U_β

$$v^\alpha = h\left(ax^\alpha dy^\alpha - by^\alpha dx^\alpha\right) \tag{6.79}$$
$$v^\beta = h\left(a'x^\beta dy^\beta - b'y^\beta dx^\beta\right) \tag{6.80}$$

where $a, b, a', b' \in \mathbb{R}$ such that $a + b = 1$ and $a' + b' = 1$.

Unfortunately there is no value of the constants $a, a', b, b' \in \mathbb{R}$ such that the one-form v is globally defined. In other words

$$v^\alpha - v^\beta = d\lambda^{\alpha\beta} \tag{6.81}$$

Now the point is to find the right expressions for $\lambda^{\alpha\beta}$. Obviously they strictly depend on the choice of U_α and U_β, since as we have seen the transition functions are related with this choice. Let us consider for example a change of chart from U_α to U_β which is encoded by the transition functions

$$\begin{array}{lll} x^\alpha & \mapsto & x^\beta = x^\alpha + 1 \\ y^\alpha & \mapsto & y^\beta = y^\alpha \\ z^\alpha & \mapsto & z^\beta \end{array} \tag{6.82}$$

Then we can write

$$\begin{aligned} &\tfrac{1}{h}\left(v^\alpha - v^\beta\right) = ax^\alpha dy^\alpha - by^\alpha dx^\alpha - a'x^\beta dy^\beta - b'y^\beta dx^\beta = \\ &= ax^\alpha dy^\alpha - by^\alpha dx^\alpha - a'x^\alpha dy^\alpha - a'dy^\alpha + b'y^\alpha dx^\alpha = (a-a')x^\alpha dy^\alpha - (b-b')y^\alpha dx^\alpha - a'dy^\alpha \end{aligned} \tag{6.83}$$

Since the difference between Equations (6.77) and (6.78) gives

$$d(v^\alpha - v^\beta) = 0 \tag{6.84}$$

we have that $v^\alpha - v^\beta$ is exact, so that the terms proportional to $a - a'$ and $b - b'$ must vanish. This fact implie that

$$a = a' \qquad b = b' \tag{6.85}$$

and then for the choice of charts which are related by the transition functions in Equations (6.91) we get

$$\lambda^{\alpha\beta} = -ahy^\alpha \tag{6.86}$$

By computing $v^\beta - v^\alpha$ we obtain

$$v^\beta - v^\alpha = ahy^\beta = ahy^\alpha \tag{6.87}$$

The interesting point is that

$$\lambda^{\alpha\beta} + \lambda^{\beta\alpha} = 0 \tag{6.88}$$

It easy to see that if the transition functions realted to the choice of α and β is given by

$$\begin{array}{lll} x^\alpha & \mapsto & x^\beta = x^\alpha \\ y^\alpha & \mapsto & y^\beta = y^\alpha + 1 \\ z^\alpha & \mapsto & z^\beta \end{array} \tag{6.89}$$

then the $\lambda^{\alpha\beta}$ are given by

$$\lambda^{\alpha\beta} = ahx^\alpha \qquad \lambda^{\beta\alpha} = -ahx^\beta = -ahx^\alpha \qquad \lambda^{\alpha\beta} + \lambda^{\beta\alpha} = 0 \tag{6.90}$$

So finally let us compute the $\lambda^{\alpha\beta}$ in the last case in which the transition functions between U_α and U_β are given by

$$\begin{array}{lll} x^\alpha & \mapsto & x^\beta = x^\alpha + 1 \\ y^\alpha & \mapsto & y^\beta = y^\alpha + 1 \\ z^\alpha & \mapsto & z^\beta \end{array} \tag{6.91}$$

Proceeding as before we get that

$$v^\alpha - v^\beta = -ahdy^\alpha + bhdx^\alpha = d\lambda^{\alpha\beta}$$
$$v^\beta - v^\alpha = ahdy^\beta - bhdx^\beta = d\lambda^{\beta\alpha} \tag{6.92}$$

and then

$$\lambda^{\alpha\beta} = -ahy^\alpha + bhx^\alpha$$
$$\lambda^{\beta\alpha} = ahy^\beta - bhx^\beta = ahy^\alpha + ah - bhx^\alpha - bh \tag{6.93}$$

so that

$$\lambda^{\alpha\beta} + \lambda^{\beta\alpha} = h(a-b) \tag{6.94}$$

If we want the $\lambda^{\alpha\beta}$ to respect the condition in Equation (6.90) we have to require

$$a = b = \frac{1}{2} \tag{6.95}$$

With this choice is also easy to prove that for each choiche of U_α and U_β we get

$$\lambda^{\alpha\beta} + \lambda^{\beta\gamma} + \lambda^{\gamma\alpha} = 0 \tag{6.96}$$

so that $\lambda^{\alpha\beta}$ form a cocycle.

From Example 2.3.1 we obtain that v is a local connection for a circle bundle $\hat{M}$ over M, so that we can add a $U(1)$ fiber to the $U(1)$ principal bundle.

Let $\hat{\pi} : \hat{M} \to M$ be the projection of the new circle bundle over M be and let $\hat{X}^\alpha$ be the coordinate which parametrizes the new circle in the chart U_α. $\hat{M}$ is locally described in U_α by the set of coordinates $X^I \equiv (x^\alpha, y^\alpha, z^\alpha, \hat{X}^\alpha)$ and it is called the **doubled space** because the dimension of the fiber is doubled by inserting the new fiber related to $\hat{X}$.

Since X is the fiber coordinate of a circle bundle it satisfies

$$\hat{X}^\alpha - \hat{X}^\beta = -\lambda^{\alpha\beta} \tag{6.97}$$

The main point here is that by using the transition functions of $\hat{X}^\alpha$ we can lift the one-form v to a globally defined one

$$\hat{v} = d\hat{X}^\alpha + v^\alpha \tag{6.98}$$

$\hat{v}$ is obviously globally defined since in the overlap $U_{\alpha\beta}$

$$\hat{v}^\alpha = d\hat{X}^\alpha + v^\alpha = d\hat{X}^\beta - d\lambda^{\alpha\beta} + v^\beta + d\lambda^{\alpha\beta} = d\hat{X}^\beta + v^\beta = \hat{v}^\beta \tag{6.99}$$

We can also lift the metric g and the three form H in the simplest way by pull-back

$$\hat{g} = \hat{\pi}^* g \qquad\qquad \hat{H} = \hat{\pi}^* H \tag{6.100}$$

where $\hat{g}$ and $\hat{H}$ have vanishing components in the new direction

$$\hat{g}\left(\tfrac{\partial}{\partial \hat{X}}, \cdot\right) = 0 \tag{6.101}$$
$$i_{\frac{\partial}{\partial \hat{X}}} \hat{H} = 0 \tag{6.102}$$

and they remain indipendent from $\hat{X}$. Then the lifted K, $\hat{K}$ remains a Killing vector for the lifted metric $\hat{g}$. Moreover $\hat{H}$ remains invariant with respect to the Killing vector $\hat{K}$. The lifted Killing vector $\hat{K}$ is the same vector as before, except that now it is thought as a vector of the doubled bundle $\hat{M}$.

It's immediate to notice that, because of Equations (6.101) and (6.102) we obtain a new Killing vector for free

$$\widetilde{K} = \frac{\partial}{\partial \hat{X}} \tag{6.103}$$

with respect to which H is invariant. $\widetilde{K}$ commutes with $\hat{K}$ since the two $U(1)$ actions are indipendent of each other

$$[\hat{K},\widetilde{K}] = \frac{\partial}{\partial z}\frac{\partial}{\partial X} - \frac{\partial}{\partial X}\frac{\partial}{\partial z} = 0 \tag{6.104}$$

Then we can define the lift of the one-form in Equation (6.42), which is dual to $\hat{K}$

$$\hat{\xi} = G\hat{g}_{IJ}\hat{K}^I dX^J = G\hat{g}_{ij}\hat{K}^i dX^j = \xi \tag{6.105}$$

In other words $\hat{\xi}$ remains the same one-form dual to K which can be defined on M, but it can be thought as a one-form on $\hat{M}$ with vanishing components on the new direction X. In our particular case, since the metric is the diagonal one in Equation (6.68), then the local connection A vanishes and the one-form $\hat{\xi}$ takes in U_α the form

$$\hat{\xi} = dz^\alpha \tag{6.106}$$

Now it's important to find a one-form which is dual to $\widetilde{K}$. We already have an object which behaves in the right way, which is $\hat{v}$. In fact

$$\hat{v}(\widetilde{K}) = d\hat{X}\left(\frac{\partial}{\partial\hat{X}}\right) + \hat{v}\left(\frac{\partial}{\partial\hat{X}}\right) = d\hat{X}\left(\frac{\partial}{\partial\hat{X}}\right) = 1 \tag{6.107}$$

Then we can redefine

$$\widetilde{\xi} = \hat{v} \tag{6.108}$$

for writing convenience. As for $\hat{\xi}$, in general there exists a local connection $\widetilde{A}$ such that

$$\widetilde{\xi} = d\hat{X} + \widetilde{A} \tag{6.109}$$

and the curvature

$$\widetilde{F} = d\widetilde{A} \tag{6.110}$$

Gauging the NLSM on T^3

Once we have found a global one-form like $\hat{v}$ it's possible to gauge the sigma model $(\hat{M},\hat{H},\hat{g})$. The new action is easily built since the new direction parametrized by $\hat{X}$ is null both for $\hat{g}$ and for $\hat{H}$. Then there is only a slight modification to the gauged action in Equation (6.23)

$$\hat{S}^G = \frac{1}{2}\int_\Sigma g_{ij}DX^i\wedge *DX^j + \int_\Omega\left\{\frac{1}{3}H_{ijk}DX^i\wedge DX^j\wedge DX^k + \mathscr{G}\wedge\hat{v}_I DX^I\right\} \tag{6.111}$$

where

$$D_\mu x = \partial_\mu x \quad D_\mu y = \partial_\mu y \quad D_\mu z = \partial_\mu z - C_\mu \quad D_\mu X = \partial_\mu X \tag{6.112}$$

and we denote

- $x^l = (x,y)$
- $x^i = (x,y,z)$
- $\hat{x}^i = (x,y,\hat{X})$
- $X^I = (x,y,z,\hat{X})$

We can easily rewrite

$$\hat{S}^G = \int_\Sigma d^2\sigma \quad \left\{\frac{1}{2}(g_{ij}+b_{ij})\partial_\mu x^i\partial_\nu x^j - C_\mu J^\mu + C_\mu\epsilon^{\mu\nu}\partial_\nu\hat{X} + \frac{G}{2}\eta^{\mu\nu}C_\mu C_\nu\right\} \tag{6.113}$$

where

$$J^\mu = (K_i\eta^{\mu\nu} - v_i\epsilon^{\mu\nu})\partial_\nu x^i \tag{6.114}$$

In the T^3 example it turns out that $K_i = \delta_{iz}$. Moreover v is horizontal so that $v_z = 0$.

We can integrate out the gauge fields C_μ by finding its equations of motion

$$\frac{\delta\hat{S}^G}{\delta C_\mu} = -J^\mu + \epsilon^{\mu\nu}\partial_\nu\hat{X} + GC^\mu = 0 \tag{6.115}$$

from which we get

$$C^\mu = \frac{1}{G}\left[J^\mu - \epsilon^{\mu\nu}\partial_\nu \hat{X}\right] \tag{6.116}$$

By inserting Equation (6.116) into the action we can find

$$\begin{aligned}\hat{S}^G = \int_\Sigma \Big\{ \tfrac{1}{2}\left(g_{ij}+b_{ij}\right)\eta^{\mu\nu}\partial_\mu x^i \partial_\nu x^j - \\ -\tfrac{1}{G}\left[J_\mu - \eta_{\mu\rho}\epsilon^{\rho\nu}\partial_\nu \hat{X}\right]J^\mu + \tfrac{1}{G}\left[J_\mu - \eta_{\mu\rho}\epsilon^{\rho\nu}\partial_\nu \hat{X}\right]\epsilon^{\mu\lambda}\partial_\lambda \hat{X} + \\ +\tfrac{1}{2G}\eta^{\mu\nu}\left[J_\mu - \eta_{\mu\rho}\epsilon^{\rho\tau}\partial_\tau \hat{X}\right]\left[J_\nu - \eta_{\nu\sigma}\epsilon^{\sigma\lambda}\partial_\lambda \hat{X}\right]\Big\} =\end{aligned} \tag{6.117}$$

$$\begin{aligned}= \int_\Sigma \Big\{ \tfrac{1}{2}\left(g_{ij}+b_{ij}\right)\eta^{\mu\nu}\partial_\mu x^i \partial_\nu x^j - \\ -\tfrac{1}{G}J_\mu J^\mu + \tfrac{1}{G}J_\mu \epsilon^{\mu\nu}\partial_\nu \hat{X} + \tfrac{1}{G}J_\mu \epsilon^{\mu\lambda}\partial_\lambda \hat{X} + \tfrac{1}{G}\epsilon^{\mu\lambda}\eta_{\lambda\rho}\epsilon^{\rho\nu}\partial_\nu \hat{X}\partial_\mu \hat{X} + \\ +\tfrac{1}{2G}J_\mu J^\mu - \tfrac{1}{2G}J_\mu \epsilon^{\mu\nu}\partial_\nu \hat{X} - \tfrac{1}{2G}J_\mu \epsilon^{\mu\nu}\partial_\nu \hat{X} - \tfrac{1}{2G}\epsilon^{\mu\lambda}\eta_{\lambda\rho}\epsilon^{\rho\nu}\partial_\mu \hat{X}\partial_\nu \hat{X}\Big\} =\end{aligned} \tag{6.118}$$

$$\begin{aligned}= \int_\Sigma \Big\{ \tfrac{1}{2}\left(g_{ij}+b_{ij}\right)\eta^{\mu\nu}\partial_\mu x^i \partial_\nu x^j - \\ -\tfrac{1}{2G}J_\mu J^\mu + \tfrac{1}{G}J_\mu \epsilon^{\mu\nu}\partial_\nu \hat{X} + \tfrac{1}{2G}\eta^{\mu\nu}\partial_\mu \hat{X}\partial_\nu \hat{X}\Big\} =\end{aligned} \tag{6.119}$$

By substituting the expression for J_μ in Equation (6.114), after some simple algebra we get the expressions for the new geometry. In fact

$$\begin{aligned}= \int_\Sigma \Big\{ \tfrac{1}{2}(g_{ij}+b_{ij})\partial_\mu x^i \partial_\nu x^j - \tfrac{1}{2G}\left[k_i \eta^{\mu\nu} - v_i \epsilon^{\mu\nu}\right]\left[k_j \delta_\mu{}^\lambda - v_j \eta_{\mu\rho}\epsilon^{\rho\lambda}\right]\partial_\nu x^i \partial_\lambda x^j + \\ +\tfrac{1}{G}\left[k_i \delta_\mu{}^\nu - v_i \eta_{\mu\rho}\epsilon^{\rho\nu}\right]\epsilon^{\mu\lambda}\partial_\nu x^i \partial_\lambda \hat{X} + \tfrac{1}{2G}\eta^{\mu\nu}\partial_\mu \hat{X}\partial_\nu \hat{X}\Big\} =\end{aligned} \tag{6.120}$$

$$\begin{aligned}= \int_\Sigma \Big\{ \tfrac{1}{2}(g_{ij}+b_{ij})\partial_\mu x^i \partial_\nu x^j - \tfrac{1}{2G}k_i k_j \eta^{\mu\nu}\partial_\mu x^i \partial_\nu x^j + \tfrac{1}{2G}k_i v_j \epsilon^{\mu\nu}\partial_\mu x^i \partial_\nu x^j + \\ +\tfrac{1}{2G}v_i k_j \epsilon^{\mu\nu}\partial_\nu x^i \partial_\mu x^j + \tfrac{1}{2G}v_i v_j \epsilon^{\mu\lambda}\eta_{\lambda\rho}\epsilon^{\rho\nu}\partial_\mu x^i \partial_\nu x^j + \tfrac{1}{G}k_i \epsilon^{\mu\nu}\partial_\mu x^i \partial_\nu \hat{X} + + \\ +\tfrac{1}{G}v_i \epsilon^{\mu\lambda}\eta_{\lambda\rho}\epsilon^{\rho\nu}\partial_\mu x^i \partial_\nu \hat{X} + \tfrac{1}{2G}\eta^{\mu\nu}\partial_\mu \hat{X}\partial_\nu \hat{X}\Big\} =\end{aligned} \tag{6.121}$$

$$\begin{aligned}= \int_\Sigma \Big\{ \tfrac{1}{2}(g_{ij}+b_{ij})\partial_\mu x^i \partial_\nu x^j - \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu z \partial_\nu z + \tfrac{1}{2}v_j \epsilon^{\mu\nu}\partial_\mu z \partial_\nu x^j + \\ +\tfrac{1}{2}v_i \epsilon^{\mu\nu}\partial_\nu x^i \partial_\mu z + \tfrac{1}{2G}v_i v_j \eta^{\mu\nu}\partial_\mu x^i \partial_\nu x^j + \epsilon^{\mu\nu}\partial_\mu z \partial_\nu \hat{X} + \\ \tfrac{1}{G}v_i \eta^{\mu\nu}\partial_\mu x^i \partial_\nu \hat{X} + \tfrac{1}{2G}\eta^{\mu\nu}\partial_\mu \hat{X}\partial_\nu \hat{X}\Big\} =\end{aligned} \tag{6.122}$$

$$\begin{aligned}= \int_\Sigma \Big\{ \tfrac{1}{2}g_{lm}\partial_\mu x^l \partial_\nu x^m + \tfrac{1}{2}\epsilon^{\mu\nu}b_{ij}\partial_\mu x^i \partial_\nu x^j + \tfrac{1}{G}v_i \eta^{\mu\nu}\partial_\mu x^i \partial_\nu \hat{X} + v_j \epsilon^{\mu\nu}\partial_\mu z \partial_\nu x^j + \\ +\epsilon^{\mu\nu}\partial_\mu z \partial_\nu \hat{X} + \tfrac{1}{2G}v_i v_j \eta^{\mu\nu}\partial_\mu x^i \partial_\nu x^j + \tfrac{1}{2G}\eta^{\mu\nu}\partial_\mu \hat{X}\partial_\nu \hat{X}\Big\} =\end{aligned} \tag{6.123}$$

from which we can read the new geometry encoded by

$$g' = g - G dz \otimes dz + \tfrac{1}{G}\hat{v}\otimes \hat{v} \tag{6.124}$$

$$b' = b - \hat{v}\wedge dz \tag{6.125}$$

which locally coincides with the result expected by the *Buscher rules* that is

$$\hat{g}_{\hat{X}\hat{X}} = \frac{1}{G} \qquad \hat{g}_{\hat{X}l} = \hat{g}_{l\hat{X}} = \frac{v_l}{G} \qquad \hat{g}_{lm} = g_{lm} + \frac{v_l v_m}{G} \tag{6.126}$$

and

$$\hat{b}_{\hat{X}l} = -\hat{b}_{l\hat{X}} = 0 \qquad \hat{b}_{lm} = b_{lm} \tag{6.127}$$

The term

$$-d\hat{X}\wedge dz \tag{6.128}$$

which comes out by writing explicitly $\hat{v} = d\hat{X} + v$ in Equation (6.125) is the price we have to pay for having adopted a globally well defined procedure for the gauging of the NLSM, which involves a doubled space. In particular it doesn't appear in the Buscher rules, since they exchange one circle with the related dual, without doubling the fiber degrees of freedom.

6.1.4 Gauging the ungaugeable

The results reported in the previous Section can be easily extended to the general case with several globally defined Killing vectors $\{K_l\}_{l\in I_d}$.

The starting point is the assumption of the violation of 1. of the GC, supposing that $i_{K_l}H$ is not exact. Then in each U_α we can find a one-form v_l^α such that

$$i_{K_l}H = dv^\alpha \tag{6.129}$$

In each overlap $U_{\alpha\beta}$ we obtain a set of functions $\lambda_l^{\alpha\beta}$ such that

$$v^\alpha - v^\beta = d\lambda_l^{\alpha\beta} \tag{6.130}$$

When each $\lambda_l^{\alpha\beta}$ is such that in each triple overlap $U_{\alpha\beta\gamma}$ the cocycle condition

$$\lambda_l^{\alpha\beta} + \lambda_l^{\beta\gamma} + \lambda_l^{\gamma\alpha} = 0 \tag{6.131}$$

is satisfied, then each v_l^α defines the local connection for a $U(1)$ principal bundle over M, $\hat{\pi} : \hat{M} \to M$. We obtain a torus principal bundle T^d over M.

We can choose fiber coordinates X_l^α in each U_α such that

$$X_l^\alpha - X_l^\beta = -\lambda_l^{\alpha\beta} \tag{6.132}$$

and the lifted one-forms

$$\hat{v}_l^\alpha = dX_l^\alpha + v_l^\alpha \tag{6.133}$$

are globally defined.

M can be locally described by a set of doubled coordinates

$$X^I = (Y^\mu, X^l, \hat{X}_l) \tag{6.134}$$

and both the metric g and the three-form H can be pull-back on $T^*\hat{M}$ using the projection map $\hat{\pi}$. They are transported in the trivial way, so that the only non-vanishing components of the metric $\hat{g}$ are those of the original one, and the same is true for $\hat{H}$.

The first point which deserves special attention because it is substantially different from the T^3 example is the definition of the lift of the Killing vectors. The most general lift provides for a twist described by a set of functions θ_{lm} as follows

$$\hat{K}_l = K_l + \theta_{lm}\frac{\partial}{\partial \hat{X}_m} \tag{6.135}$$

The requirement that $\hat{K}_l$ is a globally defined vector implies that

$$\theta_{lm}^\alpha - \theta_{lm}^\beta = -i_{K_l} d\lambda_m^{\alpha\beta} \tag{6.136}$$

The vectors $\hat{K}_l$ are trivially Killing vectors on $\hat{M}$

$$\hat{\mathfrak{L}}_{K_l}\hat{g} = 0 \qquad\qquad \hat{\mathfrak{L}}_{\hat{K}_l}\hat{H} = 0 \tag{6.137}$$

Finally we have a new NLSM $(\hat{M}, \hat{g}, \hat{H})$. We will immediately see that it is gaugeable.

In fact from Equation (6.135) it turns out that

$$i_{\hat{K}_l}\hat{v}_m = i_{K_l}v_m + \theta_{lm} \tag{6.138}$$

and we can define

$$\theta_{lm} = B_{lm} - i_{K_l}v_m \tag{6.139}$$

with $B_{lm} = -B_{lm}$, so that

$$i_{\hat{K}_l}\hat{v}_m + i_{\hat{K}_m}\hat{v}_l = 0 \tag{6.140}$$

and

$$i_{\hat{K}_l}\hat{H} = d\hat{v}_l \tag{6.141}$$

since $dv = d\hat{v}$, so that 1. of the GC is obeyed by the new NLSM. The definition in Equation (6.139) explains why in the T^3 case the θ function vanish. In fact if the fiber has dimension one, the antysimmetric functions B_{lm} vanish trivially, while v is horizontal with respect to the unique Killing vector $i_K v = 0$.

It's amazing to notice that in this context, imposing 3. of the GC implies immediately 4. In fact by computing the action of the Lie derivative

$$\hat{\mathfrak{L}}_{\hat{K}_l}\hat{v}_m = i_{\hat{K}_l}i_{\hat{K}_m}\hat{H} + di_{\hat{K}_l}\hat{v}_m = i_{K_l}i_{K_m}H + dB_{lm} \tag{6.142}$$

we obtain that $\hat{\mathfrak{L}}_{\hat{K}_l}\hat{v}_m = 0$ is equivalent to

$$i_{K_l}i_{K_m}H = -dB_{lm} \tag{6.143}$$

Moreover it can be shown that [11]

$$[\hat{K}_l, \hat{K}_m] = -(i_{K_l}i_{K_m}i_{K_n}H)\frac{\partial}{\partial\hat{X}_n} \tag{6.144}$$

so that the whole algebra generated by the Killing vectors is abelian if and only if

$$i_{K_l}i_{K_m}i_{K_n}H = 0 \tag{6.145}$$

It is intuitive to see that also the vectors

$$\widetilde{K}^l = \frac{\partial}{\partial\hat{X}_l} \tag{6.146}$$

are Killing vectors preserving the three-form H. This means that the new NLSM $(\hat{M}, \hat{g}, \hat{H})$ has $2d$ commuting Killing vectors. If G_{lm} is everywhere invertible, then the one-forms

$$\hat{\xi}^l = G^{lm}\hat{g}_{IJ}\hat{K}^I_m dX^J = \xi^l \tag{6.147}$$

are lifted trivially. Finally the one-forms $\widetilde{\xi}_l$ such that

$$\hat{v}_l = \widetilde{\xi}_l - B_{lm}\xi^m \tag{6.148}$$

are horizontal with respect to the lifted Killing vectors $\hat{K}_l$. They are the analogous of the forms $\overline{\xi}_l$ defined in Equation (6.47).

We can choose local adapted coordinates on the fibers

$$X^I = (Y^\mu, \widetilde{X}^l, \widetilde{X}_l) \tag{6.149}$$

such that

$$\hat{K}_l = \frac{\partial}{\partial\widetilde{X}^l} \qquad\qquad \widetilde{K}^l = \frac{\partial}{\partial\widetilde{X}_l} \tag{6.150}$$

where $\widetilde{X}^l = X^l$, while

$$\widetilde{X}_l = \hat{X}_l + f_l \tag{6.151}$$

and $f_l \equiv f_l(X^l, Y^\mu)$ are such that

$$\frac{\partial f_l}{\partial X^m} = -\theta_{ml} \tag{6.152}$$

The gauge action is slightly modified with respect to Equation (6.23), as we have shown in Equation (6.111). In this case

$$D_\mu\hat{X}^I = \partial_\mu X^I - C^l_\mu\hat{K}^I_l \tag{6.153}$$

so that

$$D_\mu\hat{X}_l = \partial_\mu\hat{X}_l + \theta_{lm}C^m_\mu \tag{6.154}$$

As in the case of the three torus T^3, the elimination of the gauge fields C^l_μ brings by means of the equations of motion brings to the new geometry. This procedure is in general encoded in a few simple rules. We denote by a $\widetilde{}$ the dual objects relative to the original geometry (M, g, H). The T-duality map is

$$g = \overline{g} + G_{lm}\xi^l \otimes \xi^m \quad \longmapsto \quad \widetilde{g} = \overline{g} + \widetilde{G}^{lm}\widetilde{\xi}_l \otimes \widetilde{\xi}_m \tag{6.155}$$

$$H = \overline{H} + \widetilde{F}_l \wedge \xi^l + dB \quad \longmapsto \quad \widetilde{H} = \widetilde{\xi}_l \wedge F^l + d\widetilde{B} \tag{6.156}$$

where $E = G + B$ and

$$\widetilde{G}^{lm} = (E^{-1})^{(lm)} \qquad \widetilde{B}^{lm} = (E^{-1})^{[lm]} \tag{6.157}$$

and

$$B = \frac{1}{2}B_{lm}\xi^l \wedge \xi^m \qquad \frac{1}{2}\widetilde{B}^{lm}\widetilde{\xi}_l \wedge \widetilde{\xi}_m \tag{6.158}$$

Moreover

$$F^l = d\xi^l \qquad \widetilde{F}_l = d\widetilde{\xi}_l \tag{6.159}$$

while $\overline{H}$ is such that

$$d\overline{H} = -\widetilde{F}_l \wedge F^l \tag{6.160}$$

There are n Killing vectors on M dual to ξ^l and n Killing vectors $\widetilde{K}^l$ on $\widetilde{M}$ dual to $\widetilde{\xi}$. In adapted coordinates

$$K_l = \frac{\partial}{\partial X^l} \qquad \widetilde{K}^l = \frac{\partial}{\partial \widetilde{X}_l} \tag{6.161}$$

and

$$\xi^l = dX^l + A^l \qquad \widetilde{\xi}_l = d\widetilde{X}_l + \widetilde{A}_l \tag{6.162}$$

The form $\overline{H}$ is basic. It represents the component of the H form which doesn't have legs on the fibered directions. The two-forms F^l and $\widetilde{F}_l$ are also basic. They respectively define the Chern class associated to the l-th circle on the torus and the H-class associated to the l-th dual circle on the dual torus.

The T-duality exchanges the one-forms

$$\xi \quad \longmapsto \quad \widetilde{\xi}_l \tag{6.163}$$

and the torus moduli

$$E \quad \longmapsto \quad \widetilde{E} \equiv E^{-1} \tag{6.164}$$

Also first Chern classes and H classes are exchanged

$$[F_l] \quad \longmapsto \quad [\widetilde{F}^l] \tag{6.165}$$

6.1.5 Global symmetries

There are a pair of global transformations which act naturally on the set of Killing vectors. These are particularly important, since they preserve the physics of the NLSM.

The first one is a $GL(n, \mathbb{Z})$ transformation which acts on the set of Killing vectors $\{K_l\}_{l \in I_d}$ as follows

$$K_l \mapsto K'_l = \sum_m L_l{}^m K_m \tag{6.166}$$

where $L_l{}^m \in O(n, n)$, so that K_l transforms in the covector representation. This is because the generic Killing vector on the smooth manifold M is of the form $\sum_l N^l K_l$. It's necessary to require that $N^l \in \mathbb{Z}$ in order to preserve the periodicity of the orbits generated by the Killing vectors.

This transformation extend naturally to each tensor with "Killing" indices $l, m, n, \ldots$, such as ξ^l, v_l, G_{lm}, B_{lm}. They transform under $GL(n, \mathbb{Z})$ in the right representation, that is

$$\xi^l \quad \longmapsto \quad (L^{-T})^l{}_m \xi^m \tag{6.167}$$

$$v_l \quad \longmapsto \quad L_l{}^m v_m \tag{6.168}$$

$$G_{lm} \quad \longmapsto \quad L_l{}^n G_{np} (L^T)^p{}_m \tag{6.169}$$

$$B_{lm} \quad \longmapsto \quad L_l{}^n B_{np} (L^T)^p{}_m \tag{6.170}$$

The action of the NLSM is invariant under these global transformations.

There is an additional well-behaving global transformation which consists in the shift of the functions B_{lm} by constants Λ_{lm}

$$B_{lm} \quad \longmapsto \quad B_{lm} + \lambda_{lm} \tag{6.171}$$

The classical physics is trivially unchanged, since the constants λ_{lm} does not change H. In the quantum theory we must pay more attention since the action change by an amount

$$\frac{1}{2}\int_{\Sigma} \lambda_{lm} dX^l \wedge dX^m = \int_{\phi(\Sigma)} \lambda \tag{6.172}$$

where $\phi(\Sigma)$ is the embedding via ϕ of the worldsheet Σ into the target space M. This means that its contribution to the functional integral is

$$e^{i\int_{\phi(\Sigma)} \lambda} \tag{6.173}$$

and that this does not bring to anomalies if and only if

$$\frac{1}{2\pi}\int_{\phi(\Sigma)} \lambda \in \mathbb{Z} \tag{6.174}$$

or, by Stoke's Theorem 2 if and only if $\left[\frac{\lambda}{2\pi}\right]$ represents an integral cohomology class

$$\left[\frac{\lambda}{2\pi}\right] \in H^2(M, \mathbb{Z}) \tag{6.175}$$

Then we can say that $GL(n, Z)$ action togheter with a B-shifts preserve a quantum field theory defined by a NLSM as in Equation (6.2), since they map the geometry of a background in an equivalent one from the quantum theory point of view. We have also seen that a T-duality transformation actually does the same. In fact, when possible, it'sufficient to construct the double space over a principal torus bundle and to apply duality transformations in Equations (6.167)-(6.171), to obtain a new quantum theory equivalent to the original one.

The next amazing step is to observe that B-shift and $GL(n, \mathbb{Z})$ action get togheter to generate a larger group which we already encountered. In fact they form a direct subgroup of the orthogonal group $O(n, n, \mathbb{Z})$, as we observed in Section 5.2.1. Since here we are dealing with group constructed over the field of the integer numbers, we will denote it by $\Gamma(\mathbb{Z})$. Surprisingly, we are going to see that also T-duality transformations lies in the $O(n, n, \mathbb{Z})$ group, so that it is also called the **T-duality group**.

The generic element $h \in G(n, \mathbb{Z})$ such that

$$h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{6.176}$$

acts over tensors with lower or upper "Killing" indices as in Equations (6.166) - (6.170). Obviously each $h \in O(n, n, \mathbb{Z})$ preserves the indefinite metric

$$\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{6.177}$$

A non-trivial fact is that E doensn't transform as a tensor under a transformation $E \in O(n, n, \mathbb{Z})$, but [49]

$$E \quad \longmapsto \quad E' = \widetilde{E} = \frac{aE + b}{cE + d} \tag{6.178}$$

where a, b, c, d are the $n \times n$ matrices defined in Equation (6.176). Moreover, the $GL(n, Z)$ subgroup lies in $O(n, n, \mathbb{Z})$ through the following immersion

$$GL(n, \mathbb{Z}) \ni L^l{}_m \quad \longmapsto \quad \begin{pmatrix} L^l{}_m & 0 \\ 0 & (L^{-1})_l{}^m \end{pmatrix} \in O(n, n, \mathbb{Z}) \tag{6.179}$$

which is the same we have seen in Equation (5.7).

6.2 T-duality on $SU(3) \times SU(3)$ structures

As we have seen in Section **??**, if $i_{\mathfrak{L}_{K_m}} i_{\mathfrak{L}_{K_n}} H$ is not a globally defined form, so that the we can not use the formalism developed in Section 6.1.4 to find a globally defined dual background, T-duality bring to a non-geometric dual background. These kind of backgrounds are simply defined as manifolds on which the transition functions for the metric and the B-field admit T-duality transformations. Moreover in this case the T-duality map can be defined only locally.

This situation seems to fit perfectly the framework of Generalized Geometry. In fact we will see that in that context T-duality reduces to a gauge transformation of the generalized metric $\mathcal{H}$ as in Equation (5.126).

The striking fact is that the presence of T-folds is simply encoded by the dual pure spinors which describe the background, as we will see explicitly in Sections 6.2.2, **??**.

6.2.1 T-duality in the generalized formalism

In order to achieve T-duality in the generalized formalism we need to generalize the notion of Killing vector.

The Killing condition $\mathfrak{L}_K g = 0$ for a K which leaves invariant H allows to make a local gauge choice on the B-field. In fact we can always choose $B' = B + d\chi$ such that

$$\mathfrak{L}_K B' = 0 \tag{6.180}$$

by taking $B' = B + d\chi$. This would imply that $\mathfrak{L}_K B' = \mathfrak{L}_K B + \mathfrak{L}_K d\chi = \mathfrak{L}_K B + di_K d\chi = \mathfrak{L}_K B - d\xi$. Since $\mathfrak{L}_v H = 0$ means that $di_K H = di_K dB = d\mathfrak{L}_K B = 0$ then by the Poincarè lemma $\mathfrak{L}_K B$ is locally exact, and then we can always choose χ, or more precisely

$$\xi = -i_K d\chi + df \tag{6.181}$$

such that $\mathfrak{L}_K B' = 0$.

So the generic conditions for applying local T-duality are then described by the two Equations

$$\mathfrak{L}_K g = 0 \qquad\qquad \mathfrak{L}_K B - d\xi = 0 \tag{6.182}$$

which involves a couple of objects (K, ξ). There is an ambiguity in (K, ξ), since ξ is not actually involved in Equation (6.182), but $d\xi$ is.

It turns out that Equation (6.182) describe the conditions which define a generalization of the Lie derivative, adapted to the generalized framework. In this framework, it's immediate to interpret the couple (K, ξ) as the generator of diffeomorphisms on the generalized bundle.

More precisely we can define the **generalized Lie derivative** on sections $v = X + \eta$ of the generalized tangent bundle E, along the generalized vector $w = K + \xi \in \mathfrak{X}(E)$ by the *Dorfman bracket* already defined in Equation (5.76)

$$\mathsf{L}_w v = [Y, X] + (\mathfrak{L}_Y \eta - i_X d\xi) \tag{6.183}$$

Notice that the action of the Dorfman bracket is the most natural one for the bundle structure defined in Equation (5.67). In fact it locally represents the usual diffeomorphism on the vectors and on the one-forms, supplemented by the term $-i_X d\xi$ which represents the local twisting due to the action of the B-field.

The action of the gereralized Lie derivative on the generalized metric $\mathcal{H}$ can be defined by analogy with the action of the Lie derivative on a Riemannian metric g. In particular we require that $\mathsf{L}_v(\phi) = \mathfrak{L}_v(\phi) = i_v d\phi$, which is the usual requirement for the action of the Lie derivative on a scalar map $f \in C^\infty(M)$.

We get

$$\mathsf{L}_v [\mathcal{H}] (w, t) = \mathsf{L}_v [\mathcal{H}(w, t)] + \mathcal{H} [\mathsf{L}_v w, t] + \mathcal{H} [w, \mathsf{L}_v t] \tag{6.184}$$

where $v = V + \lambda$, $w = X + \xi$, $t = Y + \eta$. Then

$$\mathsf{L}_v [\mathcal{H}(w, t)] = \begin{pmatrix} \begin{array}{c} \mathfrak{L}_v g - (\mathfrak{L}_v B - d\xi) g^{-1} B - \\ -B(\mathfrak{L}_v g^{-1} B) - B g^{-1} (\mathfrak{L}_v B - d\xi) \end{array} & (\mathfrak{L}_v B - d\xi) g^{-1} + B(\mathfrak{L}_v g^{-1}) \\ & \\ -g^{-1}(\mathfrak{L}_v B - d\xi) - (\mathfrak{L}_v g^{-1}) B & \mathfrak{L}_v g^{-1} \end{pmatrix} \tag{6.185}$$

Since it turns out to be

$$\mathsf{L}_v \eta = 0 \tag{6.186}$$

then the conditions in Equation (6.182) which are the necessary conditions for T-dualizing a local background along $v \in \mathfrak{X}(E)$ are equivalent to

$$\mathsf{L}_v \mathsf{G} = 0 \tag{6.187}$$

where G is the generalized metric defined in Equation (5.106).

We can always use the arbitrariness in the choice of ξ to normalize v

$$\eta(v, v) = 1 \tag{6.188}$$

so that in adapted coordinates we can write $V = \frac{\partial}{\partial t}$. Then, as we know from Equation (6.181) $\xi = -i_v d\chi + df$, and by choosing $f = t$ we define v as

$$v = \frac{\partial}{\partial t} + \left(dt - i_{\frac{\partial}{\partial t}} d\chi\right) \tag{6.189}$$

The element of the $O(n,n)$ group which correspond to T-duality transformation in Equation (**??**) is

$$\mathsf{T}_v = 1 - 2v\, v^T \eta \tag{6.190}$$

where

$$(\mathsf{T}_v)^i{}_j = \delta^i{}_j - 2v^i v^k \eta_{kj} = \delta^i{}_j \tag{6.191}$$

$$(\mathsf{T}_v)^{ij} = -2v^i v_k \eta^{kj} \tag{6.192}$$

$$(\mathsf{T}_v)_{ij} = -2v_i v^k \eta_{kj} \tag{6.193}$$

$$(\mathsf{T}_v)_i{}^j = \delta_i{}^j - 2v_i v_k \eta^{kj} \tag{6.194}$$

since as we know from Section 5.3.2 $\delta^{ij} = \delta_{ij} = 0$ and $\eta^i{}_j = \eta_i{}^j = 0$.

By making the choice of gauge $\chi = 0$ and by choosing as basis for $T \oplus T^*$

$$\left\{\frac{\partial}{\partial t}, e_2, \ldots, e_n, dt, e^2, \ldots, e^n\right\} \tag{6.195}$$

we obtain the explicit expression fot the matrix which represents T-duality transformation

$$\mathsf{T}_v = \begin{pmatrix} 1 - M & M \\ M & 1 - M \end{pmatrix} \tag{6.196}$$

where

$$v = \frac{\partial}{\partial t} + dt \tag{6.197}$$

and M is a $n \times n$ matrix

$$M = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \tag{6.198}$$

The T-dual generalized metric is a $O(n,n)$ gauge transformation of the generalized metric $\mathcal{H}$, namely

$$\mathcal{H}_\beta = \mathsf{T}_v^T \mathcal{H}_\alpha \mathsf{T}_v \tag{6.199}$$

whre α, β are used to label the open set of the covering $\{U_\alpha\}$.

From Chapter 5 we immediately know what is the action of T-duality transformation on the $Spin(n,n)$ spinors, which is given by the Clifford action

$$\Phi_\beta = v \cdot \Phi_\alpha = i_{\frac{\partial}{\partial t}} \Phi_\alpha + \xi \wedge \Phi_\alpha \tag{6.200}$$

This is the property we are going to use in order to find the dual backgrounds in the Examples of the following Sections.

6.2.2 T-dualities on T^2 fibrations

In the present Section we will show how the formalism developed in Section (6.2.1) works in a couple of Examples.

The situation considered is that of a T^2 fibration $\pi : \mathsf{T}^6 \to \mathsf{T}^4$ with an $SU(3)$-structure defined by the symplectic and canonical forms

$$\omega = e^1 \wedge e^4 + e^2 \wedge e^5 + e^3 \wedge e^6 \qquad \Omega = (e^1 + ie^4) \wedge (e^2 + ie^5) \wedge (e^3 + ie^6) \tag{6.201}$$

where $\{e^a\}_{a \in I_6}$ is a basis of viebeins on the total space T^6. We will consider a trivial fibration so that in the fibered direction i we can write

$$e^i = r_i dx^i \tag{6.202}$$

where r_i is exactly the radius of the fibered circle in the i direction.

As we know from Section 5.3.4 the pure spinors which describe this structure are given by

$$\Phi^+ = e^{-\phi - B - i\omega} \qquad \Phi^- = e^{-\phi - B}\Omega \tag{6.203}$$

We will distinguish two cases, since the choice of the B-field directions is not equivalent with respect to the $SU(3)$ structures. In particular we will study what happens if the B-field couples or not the symplectic structure directions.

Coupling the symplectic directions

Let us consider the case of a B field whose legs lies in the e^1 and e^4 directions

$$B = \frac{b}{r_1 r_4} e^1 \wedge e^4 = b\, dx^1 \wedge dx^4 \tag{6.204}$$

where b is a function of the base. For example if $b = hx^6$ then the B field generates the H flux

$$H = h dx^1 \wedge dx^4 \wedge dx^6 \tag{6.205}$$

We will perform two T-dualities along the v_1 and v_2 directions, where

$$v_1 = \frac{\partial}{\partial x^1} + dx^1 \qquad v_2 = \frac{\partial}{\partial x_2} + dx^2 \tag{6.206}$$

We have to find the T-dual pure spinors

$$\widetilde{\Phi}^+ \equiv \mathsf{T}(\Phi^+) \qquad \widetilde{\Phi}^- = \mathsf{T}(\Phi^-) \tag{6.207}$$

where

$$\mathsf{T} = \mathsf{T}_{v_1} \mathsf{T}_{v_4} \qquad \mathsf{T}_{v_i}(\Phi^\pm) = v_i \cdot \Phi^\pm \tag{6.208}$$

where as we have seen in Section 6.2.1 T_{v_i} acts on the pure spinor by Clifford action.

In performing calculus we will omit the symbol $\wedge$ for writing convenience.

Let us start by computing

$$e^{-i\omega} = 1 - i\omega + \frac{(-i)^2}{2}\omega^2 + \frac{(-i)^3}{6}\omega^3 \tag{6.209}$$

where

$$\begin{aligned} \tfrac{(-i)^2}{2}\omega^2 &= -\tfrac{1}{2}\left[e^1e^4 + e^2e^5 + e^3e^6\right]\left[e^1e^4 + e^2e^5 + e^3e^6\right] = \\ = -\tfrac{1}{2}&\left[e^1e^4e^2e^5 + e^1e^4e^3e^6 + e^2e^5e^1e^4 + e^2e^5e^3e^6 + e^3e^6e^1e^4 + e^3e^6e^2e^5\right] = \\ &= -\left[e^1e^4e^2e^5 + e^1e^4e^3e^6 + e^2e^5e^3e^6\right] \end{aligned} \tag{6.210}$$

then

$$\begin{aligned} \tfrac{(-i)^3}{6}\omega^3 &= \tfrac{(-i)^3}{6}\omega^2 \wedge \omega = \\ = \tfrac{i}{3}&\left[e^1e^4e^2e^5 + e^1e^4e^3e^6 + e^2e^5e^3e^6\right]\left[e^1e^4 + e^2e^5 + e^3e^6\right] = \\ &= ie^1e^4e^2e^5e^3e^6 \end{aligned} \tag{6.211}$$

Next we can easily find that

$$e^{-B-i\omega} = \left[1 - \tfrac{b}{r_1 r_4} e^1 \wedge e^4\right]\left[1 - ie^1e^4 - ie^2e^5 - ie^3e^6 - e^1e^4e^2e^5 - e^2e^5e^3e^6 + ie^1e^4e^2e^5e^3e^6\right] =$$
$$= 1 - \left(i + \tfrac{b}{r_1r_4}\right) e^1e^4 - ie^2e^5 - ie^3e^6 - \left(i\tfrac{ib}{r_1r_4}\right) e^1e^4e^2e^5 - \left(1 - \tfrac{ib}{r_1r_4}\right) e^1e^4e^3e^6 -$$
$$-e^2e^5e^3e^6 + \left(i + \tfrac{b}{r_1r_4}\right) e^1e^4e^2e^5e^3e^6 =$$
$$= 1 - (b + ir_1r_4)dx^1dx^4 - ie^2e^5 - ie^3e^6 + i(b + ir_1r_4)dx^1dx^4e^2e^5 + i(b + ir_1r_4)dx^1dx^4e^3e^6 -$$
$$-e^2e^5e^3e^6 + (b + ir_1r_4)dx^1dx^4e^2e^5e^3e^6 \tag{6.212}$$

Then we can apply the first T-duality transformation to $e^{-\phi-B-i\omega}$ along v_4

$$\mathsf{T}_{v_4}(e^{-B-i\omega}) = \left(\tfrac{\partial}{\partial x^4} + dx^4\right) \cdot (e^{-B-i\omega}) =$$
$$= (b + ir_1r_4)dx^1 - i(b + ir_1r_4)dx^1e^2e^5 - i(b + ir_1r_4)dx^1e^3e^6 - (b + ir_1r_4)dx^1e^2e^5e^3e^6+$$
$$+dx^4 - idx^4e^2e^5 - idx^4e^3e^6 - dx^4e^2e^5e^3e^6 \tag{6.213}$$

and then the second T-duality map T_{v_1} to obtain

$$\mathsf{T}_{v_1}(\mathsf{T}_{v_4}\left(e^{-\phi-B-i\omega}\right)) = (b + ir_1r_4) - i(b + ir_1r_4)e^2e^5 - i(b + ir_1r_4)e^3e^6 -$$
$$-(b + ir_1r_4)e^2e^5e^3e^6 + dx^1dx^4 - idx^1dx^4e^3e^6 - dx^1dx^4e^2e^5e^3e^6 =$$
$$= (b + ir_1r_4) + \tfrac{1}{r_1r_4}e^1e^4 - i(b + ir_1r_4)e^2e^5 - i(b + ir_1r_4)e^3e^6 -$$
$$-(b + ir_1r_4)e^2e^5e^3e^6 - \tfrac{i}{r_1r_4}e^1e^4e^2e^5 - \tfrac{i}{r_1r_4}e^1e^4e^3e^6 - \tfrac{1}{r_1r_4}e^1e^4e^2e^5e^3e^6 =$$
$$= (b + ir_1r_4)\left[1 + \tfrac{e^1e^4}{r_1r_4(b+ir_1r_4)} - ie^2e^5 - ie^3e^6 - e^2e^5e^3e^6 -\right.$$
$$\left.- \tfrac{ie^1e^4e^2e^5}{r_1r_4(b+ir_1r_4)} - \tfrac{ie^1e^4e^3e^6}{r_1r_4(b+ir_1r_4)} - \tfrac{e^1e^4e^2e^5e^3e^6}{r_1r_4(b+ir_1r_4)}\right] = \tag{6.214}$$
$$= (b + ir_1r_4)e^{-\widetilde{B}-i\widetilde{\omega}} \tag{6.215}$$

where

$$\widetilde{\omega} = \frac{e^1 \wedge e^4}{b^2 + r_1^2r_4^2} + e^2 \wedge e^5 + e^3 \wedge e^6 \tag{6.216}$$

$$\widetilde{B} = -\frac{b}{r_1r_4(b^2 + r_1^2r_4^2)}e^1 \wedge e^4 \tag{6.217}$$

in fact we can write

$$e^{-i\widetilde{\omega}} = 1 - i\widetilde{\omega} + \frac{(-i)^2}{2}\widetilde{\omega}^2 + \frac{(-i)^3}{6}\widetilde{\omega}^3 = 1 - \frac{i}{b^2 + r_1^2r_4^2}e^1e^4 - ie^2e^5 - ie^3e^6 -$$
$$-\frac{1}{b^2 + r_1^2r_4^2}e^1e^4e^2e^5 - \frac{1}{b^2 + r_1^2r_4^2}e^1e^4e^3e^6 - e^2e^5e^3e^6 + \frac{i}{b^2 + r_1^2r_4^2}e^1e^4e^2e^5e^3e^6 \tag{6.218}$$

Thus

$$e^{-\widetilde{B}-i\widetilde{\omega}} = \left[1 + \tfrac{b}{r_1r_4(b^2+r_1^2r_4^2)}e^1e^4\right] e^{-i\widetilde{\omega}} =$$
$$= 1 - \tfrac{ie^1e^4}{b^2+r_1^2r_4^2} - ie^2e^5 - ie^3e^6 - \tfrac{e^1e^4e^2e^5}{b^2+r_1^2r_4^2} - \tfrac{e^1e^4e^3e^6}{b^2+r_1^2r_4^2} - e^2e^5e^3e^6 + i\tfrac{e^1e^4e^2e^5e^3e^6}{b^2+r_1^2r_4^2}+$$
$$+\tfrac{b\ e^1e^4}{r_1r_4(b^2+r_1^2r_4^2)} - \tfrac{ib\ e^1e^4e^2e^5}{r_1r_4(b^2+r_1^2r_4^2)} - \tfrac{ib\ e^1e^4e^3e^6}{r_1r_4(b^2+r_1^2r_4^2)} - \tfrac{b\ e^1e^4e^2e^5e^3e^6}{r_1r_4(b^2+r_1^2r_4^2)} =$$
$$= 1 + \tfrac{e^1e^4}{r_1r_4(b+ir_1r_4)} - ie^2e^5 - ie^3e^6 - e^2e^5e^3e^6 -$$
$$- \tfrac{ie^1e^4e^2e^5}{r_1r_4(b+ir_1r_4)} - \tfrac{ie^1e^4e^3e^6}{r_1r_4(b+ir_1r_4)} - \tfrac{e^1e^4e^2e^5e^3e^6}{r_1r_4(b+ir_1r_4)} \tag{6.219}$$

which proves the Equation (6.215). Moreover, since the effect of the two T-dualities on the dilaton is given by

$$e^{-\phi} \quad \mapsto \quad \frac{e^{-\widetilde{\phi}}}{\sqrt{b^2 + r_1^2r_4^2}} \tag{6.220}$$

Then the total trasformation on the pure spinor is given by

$$\mathsf{T}(e^{-\widetilde{\phi}-\widetilde{B}-i\widetilde{\omega}}) = \frac{b + ir_1r_4}{\sqrt{b^2 + r_1^2r_4^2}}e^{-\widetilde{\phi}-\widetilde{B}-i\widetilde{\omega}} = = (\cos(\theta_+) + i\sin(\theta_+))\, e^{-\widetilde{\phi}-\widetilde{B}-i\widetilde{\omega}} = e^{i\theta_+}e^{-\widetilde{\phi}-\widetilde{B}-i\widetilde{\omega}}$$

where

$$\sin(\theta_+) = \frac{r_1 r_4}{\sqrt{b^2 + r_1^2 r_4^2}} \qquad \cos(\theta_+) = \frac{b}{\sqrt{b^2 + r_1^2 r_4^2}} \tag{6.221}$$

We can turn to the pure spinor Φ^-, and compute

$$\begin{aligned}\Omega = e^1e^2e^3 + ie^1e^2e^6 + ie^1e^5e^3 + ie^4e^2e^3 - e^1e^5e^6 - e^4e^2e^6 - e^4e^5e^3 - ie^4e^5e^6 = \\ = r_1dx^1e^2e^3 + ir_1dx^1e^2e^6 + ir_1dx^1e^5e^3 + ir_4dx^4e^2e^3 - \\ -r_1dx^1e^5e^6 - r_4dx^4e^2e^6 - r_4dx^4e^5e^3 - ir_4dx^4e^5e^6\end{aligned} \tag{6.222}$$

It's obvious that

$$e^{-B}\Omega = \Omega \tag{6.223}$$

since each term in Equation (6.222) contains either a dx^1 or a dx^4 term. Then

$$\begin{aligned}\mathsf{T}_{v_4}(\Phi^-) = \left(\tfrac{\partial}{\partial x^4} + dx^4\right) \cdot e^{-B}\Omega = \\ = ir_4e^2e^3 - r_4e^2e^6 - r_4e^5e^3 - ir_4e^5e^6 + r_1dx^4dx^1e^2e^3 + ir_1dx^4dx^1e^2e^6 + \\ +ir_1dx^4dx^1e^5e^3 - r_1dx^4dx^1e^5e^6\end{aligned} \tag{6.224}$$

And then

$$\begin{aligned}\mathsf{T}(e^{-\phi-\widetilde{B}}\Omega) = \mathsf{T}_{v_1}(\mathsf{T}_{v_4}(\Phi^-)) = \left(\tfrac{\partial}{\partial x^1} + dx^1\right) \cdot (\mathsf{T}_{v_4}(\Phi^-)) = \\ = -r_1dx^4e^2e^3 - ir_1dx^4e^2e^6 - ir_1dx^4e^5e^3 + r_1dx^4e^5e^6 + \\ +ir_4dx^1e^2e^3 - r_4dx^1e^2e^6 - r_4dx^1e^5e^3 - ir_4dx^1e^5e^6 = \\ = -\tfrac{r_1}{r_4}e^4e^2e^3 - i\tfrac{r_1}{r_4}e^4e^2e^6 - i\tfrac{r_1}{r_4}e^4e^5e^3 + \tfrac{r_1}{r_4}e^4e^5e^6 + \\ +i\tfrac{r_4}{r_1}e^1e^2e^3 - \tfrac{r_4}{r_1}e^1e^2e^6 - \tfrac{r_4}{r_1}e^1e^5e^3 - i\tfrac{r_4}{r_1}e^1e^5e^6 =\end{aligned} \tag{6.225}$$

After inserting the part with the dilaton we obtain

$$\begin{aligned}\mathsf{T}(e^{-\phi-\widetilde{B}}\Omega) = \frac{e^{-\widetilde{\phi}}}{\sqrt{b^2+r_1^2r_4^2}}\mathsf{T}(e^{-B}\Omega) = \\ = \frac{e^{-\widetilde{\phi}}}{\sqrt{b^2+r_1^2r_4^2}}\left(\tfrac{b^2+r_1^2r_4^2}{b^2+r_1^2r_4^2}\right)\mathsf{T}(e^{-B}\Omega) = \\ = \left(\tfrac{b^2+r_1^2r_4^2}{b^2+r_1^2r_4^2}\right)\frac{ie^{-\widetilde{\phi}}}{\sqrt{b^2+r_1^2r_4^2}}\Big\{\tfrac{r_4}{r_1}e^1e^2e^3 + i\tfrac{r_4}{r_1}e^1e^2e^6 + i\tfrac{r_4}{r_1}e^1e^5e^3 + \\ +i\tfrac{r_1}{r_4}e^4e^2e^3 - \tfrac{r_4}{r_1}e^1e^5e^6 - \tfrac{r_1}{r_4}e^4e^2e^6 - \tfrac{r_1}{r_4}e^4e^5e^3 - i\tfrac{r_1}{r_4}e^4e^5e^6\Big\} = \\ = \left(i\frac{b+ir_1r_4}{\sqrt{b^2+r_1^2r_4^2}}\right)e^{-\widetilde{\phi}}\left(\tfrac{b-ir_1r_4}{b^2+r_1^2r_4^2}\right)\Big\{\tfrac{r_4}{r_1}e^1e^2e^3 + i\tfrac{r_4}{r_1}e^1e^2e^6 + i\tfrac{r_4}{r_1}e^1e^5e^3 + \\ +i\tfrac{r_1}{r_4}e^4e^2e^3 - \tfrac{r_4}{r_1}e^1e^5e^6 - \tfrac{r_1}{r_4}e^4e^2e^6 - \tfrac{r_1}{r_4}e^4e^5e^3 - i\tfrac{r_1}{r_4}e^4e^5e^6\Big\} = \\ = ie^{i\theta_+}e^{-\widetilde{\phi}-\widetilde{B}}\widetilde{\Omega} = e^{i(\theta_++\frac{\pi}{2})}e^{-\widetilde{\phi}-\widetilde{B}}\widetilde{\Omega} \equiv e^{i\theta_-}e^{-\widetilde{\phi}-\widetilde{B}}\widetilde{\Omega}\end{aligned} \tag{6.226}$$

where

$$\widetilde{\Omega} = \frac{b - ir_1r_4}{b^2 + r_1^2r_4^2}\left(\frac{r_4}{r_1}e^1 + i\frac{r_1}{r_4}e^4\right) \wedge (e^2 + ie^5) \wedge (e^3 + ie^6) \tag{6.227}$$

and

$$\theta_- = \theta_+ + \frac{\pi}{2} \tag{6.228}$$

In fact

$$\begin{aligned}\widetilde{\Omega} = \tfrac{b-ir_1r_4}{b^2+r_1^2r_4^2}\Big(\tfrac{r_4}{r_1}e^1e^2e^3 + i\tfrac{r_4}{r_1}e^1e^2e^6 + i\tfrac{r_4}{r_1}e^1e^5e^3 + \\ +i\tfrac{r_1}{r_4}e^4e^2e^3 - \tfrac{r_4}{r_1}e^1e^5e^6 - \tfrac{r_1}{r_4}e^4e^2e^6 - \tfrac{r_1}{r_4}e^4e^5e^3 - i\tfrac{r_1}{r_4}e^4e^5e^6\Big)\end{aligned} \tag{6.229}$$

One can rewrite these results in the basis of the dual vielbeins obtained from Equation (**??**). In this way it can be checked that the dual geometry is again an $SU(3)$ structure.

Decoupling the symplectic directions

Let us consider the case of a B-field whose legs lies in the e^2 and e^3 directions

$$B=\frac{b}{r_2r_3}e^2\wedge e^3=b\,dx^2\wedge dx^3 \tag{6.230}$$

where again $b=hx^6$. The B field generates the H flux

$$H=hdx^2\wedge dx^3\wedge dx^6 \tag{6.231}$$

We will perform two T-dualities along the v_2 and v_3 directions, where

$$v_2=\frac{\partial}{\partial x^2}+dx^2 \qquad v_3=\frac{\partial}{\partial x_3}+dx^3 \tag{6.232}$$

We have to find the T-dual pure spinors

$$\widetilde{\Phi}^+\equiv \mathsf{T}(\Phi^+) \qquad \widetilde{\Phi}^-=\mathsf{T}(\Phi^-) \tag{6.233}$$

where

$$\mathsf{T}=\mathsf{T}_{v_2}\mathsf{T}_{v_3} \qquad \mathsf{T}_{v_i}(\Phi^\pm)=v_i\cdot\Phi^\pm \tag{6.234}$$

where as we have seen in Section 6.2.1 T_{v_i} acts on the pure spinor by Clifford action.

The calculus is similar to that performed in the case of decoupled simplectic directions, but even longer, so we prefer to skip it and to give only the results [16]. The dual spinors can be written in the form of Equations (5.156) and (5.157), where

$$\tilde{z}=-i(\tilde{e}^1_++i\tilde{e}^4_+) \tag{6.235}$$
$$\tilde{j}=\tilde{e}^2_+\wedge\tilde{e}^5_++\tilde{e}^3_+\wedge\tilde{e}^6_+ \tag{6.236}$$
$$\tilde{\omega}=(\tilde{e}^2_++i\tilde{e}^5_+)\wedge(\tilde{e}^3_++i\tilde{e}^6_+) \tag{6.237}$$
$$\tilde{B}=-\tfrac{b}{r_2r_3}\tilde{e}^2_+\wedge\tilde{e}^3_+ \tag{6.238}$$
$$k_\perp=i\tfrac{r_2r_3}{\sqrt{b^2+r_2^2r_3^2}} \tag{6.239}$$
$$k_{||}=\tfrac{b}{b^2+r_2^2r_3^2} \tag{6.240}$$
$$e^{\tilde{\phi}}=e^{-\phi}\sqrt{b^2+r_2^2r_3^2} \tag{6.241}$$

and the dual vielbeins can be found from Equation (5.129)

$$\tilde{e}^2_\pm=\tfrac{\pm r_3e^2-b\frac{r_2}{r_3}e^3}{b^2+r_2^2r_3^2}=r_2\tfrac{\pm r_3^2dx^2-bdx^3}{b^2+r_2^2r_3^2} \tag{6.242}$$
$$\tilde{e}^3_\pm=\tfrac{\pm r_2^2e^3+b\frac{r_3}{r_2}e^2}{b^2r_2^2r_3^2}=r_3\tfrac{\pm r_2^2dx^3+bdx^2}{b^2r_2^2r_3^2} \tag{6.243}$$
$$\tilde{e}^a_\pm=e^a \quad a\neq 2,3 \tag{6.244}$$

Equations (6.233) - (6.241) tell us that the dual geometry obtained is an $SU(2)$ structure, which has been studied in Section 5.3.4. This fact has deep consequences, as we will see in the next Section.
We can check these results by showing that they concide with those suggested by the Buscher rules.

We will denote by $\alpha,\beta,\gamma,\ldots$ the indices referred to the fiber coordinates x^2,x^3, while we will denote by $l,m,n,\ldots$ the indices referred to the base coordinates x^1,x^4,x^5,x^6. Since in this Section the calculus are always carried out locally, we are sure not to create confusion with the indices $\alpha,\beta,\gamma,\ldots$ which are usually employed to label open sets of a covering.

Since the fibration is trivial the metric takes the nice form in blocks

$$g_{\alpha\beta}=\begin{pmatrix} r_2 & 0\\ 0 & r_3\end{pmatrix} \qquad g_{lm}=\begin{pmatrix} g_{11} & g_{14} & g_{15} & g_{16}\\ g_{41} & g_{44} & g_{45} & g_{46}\\ g_{51} & g_{54} & g_{55} & g_{56}\\ g_{61} & g_{64} & g_{65} & g_{66}\end{pmatrix} \tag{6.245}$$

where the first 2×2 matrix is referred to the directions 2 and 3, while the second 4×4 matrix is referred to the other coordinates. The triviality of the fibration tells us that all the components which mix the fiber indices with the base vanish. For the B-field we obtain

$$B_{\alpha\beta} = \begin{pmatrix} 0 & k \\ -k & 0 \end{pmatrix} \tag{6.246}$$

where $k \equiv hx^6$. All the other components of the B-field vanish.

So we can define the action of the NLSM associated to the background described by Equations (6.201) as

$$S = \int d^2\sigma \quad \eta^{\mu\nu} \left\{ g_{\alpha\beta}\partial_\mu x^\alpha \partial_\nu x^\beta + g_{lm}\partial_\mu x^l \partial_\nu x^m + B_{\alpha\beta}\partial_\mu x^\alpha \partial_\nu x^\beta \right\} \tag{6.247}$$

This time the procedure of gauging proceeds regardless of global issues. We simply introduce abelian gauge fields C^α and replace the fiber coordinates by them.

We obtain the following gauged action

$$S^G = \int d^2\sigma \left\{ (g_{\alpha\beta} + B_{\alpha\beta})C^\alpha C^\beta + g_{lm}\partial_+ x^l \partial_- x^m + \theta_\alpha \left(\partial_+ \overline{C}^\alpha - \partial_- C^\alpha \right) \right\} \tag{6.248}$$

where we are supposing that all the light-cone coordinates indices are contracted in the right way even if we will not write exlpicitly them for writing convenience. As usual the antisymmetry of the term which includes the B-field is assured by an ϵ tensor. It turns out that θ_α are the coordinates of the dual circle.

The equation of motion for C^α and $\overline{C}^\alpha$ are

$$\frac{\delta S^G}{\delta C^\alpha} = (g_{\alpha\beta} + B_{\alpha\beta})\overline{C}^\beta + \partial_- \theta_\alpha = 0 \tag{6.249}$$

$$\frac{\delta S^G}{\delta \overline{C}^\alpha} = (g_{\alpha\beta} - B_{\alpha\beta})C^\beta - \partial_+ \theta_\alpha = 0 \tag{6.250}$$

from which one can find that

$$r_2^2 C_2 = \partial_+\theta_2 + bC_3 \qquad \overline{C}_2 = \frac{b}{b^2+r_2^2r_3^2}\partial_-\theta_3 - \frac{r_3^2}{b^2+r_2^2r_3^2}\partial_-\theta_2 \tag{6.251}$$

$$r_3^2 A_3 = \partial_+\theta_3 - bC_2 \qquad \overline{C}_4 = -\frac{r_2^2}{b^2+r_2^2r_3^2}\partial_-\theta_3 - \frac{b}{b^2+r_2^2r_3^2}\partial_-\theta_2 \tag{6.252}$$

and finally

$$C_2 = \frac{r_3^2}{b^2+r_2^2r_3^2}\partial_+\theta_2 + \frac{b}{b^2+r_2^2r_3^2}\partial_+\theta_3 \tag{6.253}$$

$$C_4 = \frac{r_2^2}{b^2+r_2^2r_3^2}\partial_+\theta_3 - \frac{b}{b^2+r_2^2r_3^2}\partial_+\theta_2 \tag{6.254}$$

By inserting the solutions of the equations of motion into the terms which appear in the action S^G we obtain

$$\begin{aligned} r_2^2 C_2\overline{C}_2 &= r_2^2 \left[\frac{r_3^2}{det}\partial_+\theta_2 + \frac{b}{det}\partial_+\theta_3 \right] \left[-\frac{r_3^2}{det}\partial_-\theta_2 + \frac{b}{det}\partial_-\theta_3 \right] \\ &= -\frac{r_2^2r_3^2}{(det)^2}\partial_+\theta_2\partial_-\theta_2 + \frac{b}{(det)^2}\partial_+\theta_2\partial_-\theta_3 - \frac{-br_2^2r_3^2}{(det)^2}\partial_+\theta_3\partial_-\theta_2 + \frac{b^2r_2^2}{(det)^2}\partial_+\theta_3\partial_-\theta_3 \end{aligned} \tag{6.255}$$

where we put $det = b^2 + r_2^2r_3^2$. Analogously

$$r_3^2 C_3\overline{C}_3 = \frac{b^2r_3^2}{(det)^2}\partial_+\theta_2\partial_-\theta_2 + \frac{br_2^2r_3^2}{(det)^2}\partial_+\theta_2\partial_-\theta_3 - \frac{br_2^2r_3^2}{(det)^2}\partial_+\theta_3\partial_-\theta_2 - \frac{r_2^2r_3^2}{(det)^2}\partial_+\theta_3\partial_-\theta_3 \tag{6.256}$$

$$B_{23}C^2\overline{C}^3 = -\frac{b^2r_3^2}{(det)^2}\partial_+\theta_2\partial_-\theta_2 + \frac{b^3}{(det)^2}\partial_+\theta_2\partial_-\theta_3 + \frac{br_2^2r_3^2}{(det)^2}\partial_+\theta_3\partial_-\theta_2 - \frac{b^2r_2^2}{(det)^2}\partial_+\theta_3\partial_-\theta_3 \tag{6.257}$$

$$B_{32}C^3\overline{C}^1 = -\frac{b^2r_3^2}{(det)^2}\partial_+\theta_2\partial_-\theta_2 + \frac{b^3}{(det)^2}\partial_+\theta_2\partial_-\theta_3 + \frac{br_2^2r_3^2}{(det)^2}\partial_+\theta_3\partial_-\theta_2 - \frac{b^2r_2^2}{(det)^2}\partial_+\theta_3\partial_-\theta_3 \tag{6.258}$$

Moreover we get the terms

$$\begin{gathered} -\partial_+\theta_2\overline{C}_2 - \partial_+\theta_3\overline{C}_3 = \\ \frac{-b}{(det)}\partial_+\theta_2\partial_-\theta_3 + \frac{r_3^2}{(det)}\partial_+\theta_2\partial_-\theta_2 + \frac{r_2^2}{(det)}\partial_+\theta_3\partial_-\theta_3 + \frac{b}{(det)}\partial_+\theta_3\partial_-\theta_3 \end{gathered} \tag{6.259}$$

$$\partial_+\theta_2 C_2 + \partial_-\theta_3 C_3 = \\ \tfrac{b}{(det)}\partial_-\theta_2\partial_+\theta_3 + \tfrac{r_3^2}{(det)}\partial_+\theta_2\partial_-\theta_2 + \tfrac{r_2^2}{(det)}\partial_+\theta_3\partial_-\theta_3 - \tfrac{b}{(det)}\partial_+\theta_2\partial_-\theta_3 \tag{6.260}$$

After summing all these terms the final result is the expected one

$$\int d^2\sigma \quad \left\{\frac{r_3^2}{(det)}\partial_+\theta_1\partial_-\theta_1 + \frac{r_2^2}{(det)}\partial_+\theta_3\partial_-\theta_3 - \frac{b}{(det)}\partial_+\theta_2\partial_-\theta_3 + \frac{b}{(det)}\partial_+\theta_3\partial_-\theta_2\right\} \tag{6.261}$$

from which we get the Buscher rules

$$B_{23} = -\frac{b}{b^2 + r_2^2 r_3^2} = -B_{32} \qquad g_{22} = \frac{r_3^2}{b^2 + r_2^2 r_3^2} \qquad g_{33} = \frac{r_2^2}{b^2 + r_2^2 r_3^2} \tag{6.262}$$

If we turn to the vielbein basis we obtain for the B-field the result

$$B_{23} = -\frac{b}{r_2 r_3 (b^2 + r_2^2 r_3^2)} = -B_{32} \tag{6.263}$$

which is the same of Equation (6.238), in the basis of the original vielbeins.

The principal observation is that since we have korked in a local chart, we can not recognize in Equation (6.262) the footprints of non-gometricity. Buscher rules are a local representation of the T-duality map. Strikingly, as we will se in the next Section, the non-geometricity is encoded by the form of the dual pure spinors in Equations (6.233) - (6.241).

7 Conclusions

This thesis has been focused on the analysis of some geometrical aspects of Superstring compactification with H-flux. This area of String Theory has recently received considerable attention from theorists, since it has been shown that H-fluxes can be used to partially break the $N = 2$ supersymmetry in four dimensions to $N = 1$. A similar result is particularly important from the phenomenologic point of view, since the current paradigm of the particle physics provides a $N = 1$ supersymmetric extension of the Standard model.

Since the present work is focused on geometrical questions arising in performing T-duality, it looked necessary to introduce the whole mathematical apparatus necessary to address the issue. In this perspective, we want to stress on the deep importance of the use of G-structures, which we introduced in Chapter 2.1.1 and we used diffusely throughout the thesis. In particular they furnishes a convenient and immediate way to classify all the compactification backgrounds, as we reviewed in Section 4.3.

It is well known that the local form of the T-duality map is given from the *Buscher rules*. They are simply obtained by gauging the the non-linear sigma model arising form a String background and then integrating out the gauge fields via their equations of motion. In this way the new action obtained encodes the new geometry of the dual String background.

We have seen that under certain conditions on the H-flux, there is a way to perform the gauging of a non-linear sigma model and the subsequent elimination of the gauge fields via equations of motion in a globally well defined way. This procedure involves the so called *double space*. In this context we have analysed an explicit example, the three-torus T^3. Although it is a non-physical case - in fact its dimension is 3, and we need a six-manifold to compactify a Superstring theory in a significant way - it provides an excellent example to highlight the mathematical aspects of the issue. In particular we performed explicitly the T-duality on T^3, and we showed that locally the solutions coincides with the results expected form Buscher rules.

The non-geometric String compactifications and the role played by the Generalized Geometry in such a kind of compactifications are the two fundamental points of this thesis.

In fact in Section **??** we have seen what happens if we relax the constraints on the H flux which are needed to achieve a globally defined procedure for T-dualizing the non-linear sigma model, and then to obtain a globally defined String background. It comes out that the dual background is not longer a well defined manifold, since the geometrical objects which define it do not transform as real tensors. In particular they admit B-transformations as transition functions.

In Chapter 5 we have studied the Generalized Complex Geometry, which was developed in the last decade. It provides a new approach to complex and symplectic geometry, and it was born precisely in the physical context of Mirror symmetry, which is a close relative of T-duality. As we have seen in Section 6.2.1, the same definition of the Generalized Geometry encodes in its structure group the group of transformations of T-duality: $O(n, n)$. Again we stress on the importance of the structure group description of the geometry. We focused on Hitchin's approach to generalized Geometry, and then on its nature of connective structure of a gerbe.

In this context we have analysed the form of the T-duality map in the Generalized Geometry formalism in Section 6.2.1, and we also performed explicitly the calculations for finding the dual backgrounds in a couple of examples which are relevant as type II strings backgrounds in Section 6.2.2.

The striking fact is that the arising of the non-geometric backgrounds opens the doors to new surprising points of view on String and even on field theory. In fact it seems that a new kind of non-local transformations must be put on the same footing as diffeomorphisms and gauge transformations. Different attempts have been moved in this direction.

C. Hull has tempted to build a new formalism for String Theory, which is called the *doubled field theory* [11, 12]. Its peculiarity is to provide an action which contains directly the generalized metric as the metric of a space which is similar to the doubled space that we introduced in Section 6.1.4. The main point of this approach is that the T-duality is manifest as a gauge symmetry of the theory. Moreover, not all non-geometric backgrounds are consistent in quantum theory. Conformal, Lorentz and modular invariance on the worldsheet have to be imposed in order for the theory to be well defined.

On the side of the Generalized Complex Geometry, a huge amount of work is still to do. In fact G. Cavalcanti and M. Gualtieri have recently shown that T-duality can be seen as an isomorphism between Courant algebroids [42], which are the most immediate generalization of a Lie algebroid, which in turn is a generalization of the most common Lie algebra. However their work is valid in a case which is even simpler than those studied in Section 6.1.4, in fact its validity is restricted to the cases in which $i_{K_l} i_{K_m} H = 0$. Finally V. Mathai and J. Rosemberg have shown that if the condition $i_{K_l} i_{K_m} H = 0$ is not satisfied, T-dual manifolds can be interpreted as non-commutative spaces [51]. The relation between Generalized Geometry and non-commutative spaces has yet to be investigated.

A
Multilinear algebra

Let us give a brief recall of some basic concepts in multilinear algebra [17].

Let $\{V_i\}_{i\in I_n}$ be a set vector spaces over the field $\mathbb{K}$ such that $dim(V_i) = n_i$ and let T be a vector space over $\mathbb{K}$ such that $dim(W) = n$.

Definition A.0.1. A map

$$f : V_1 \times \cdots \times V_p \to W \tag{A.1}$$

which is separately linear in all its variables is a **multilinear map**, or a **p-linear map**.

We can enunciate the following

Definition A.0.2. Let $V_1, \ldots, V_p$ be vector spaces over the field $\mathbb{K}$ such that $dim(V_i) = n_i$ and let T be a vector space over $\mathbb{K}$ such that $dim(T) = n$. The **tensor product** of $V_1, \ldots, V_p$ is a pair (T, F) where $F : V_1 \times \cdots \times V_p \to T$ is a p-linear map such that

- $\forall\, W$ vector space over $\mathbb{K}$ and $\forall\, p$-linear map $\Phi : V_1 \times \cdots \times V_p \to W \quad \exists!\ \ \widetilde{\Phi} : T \to W$ such that $\Phi = \widetilde{\Phi} \circ F$, namely such that the diagram in Figure A.1 commutes.

$$\begin{tikzcd}
V_1 \times \cdots \times V_p \arrow[r, "\Phi"] \arrow[d, "F"'] & W \\
T \arrow[ur, "\tilde{\Phi}"'] &
\end{tikzcd}$$

Figure A.1: Tensor product.

If the specification of F is not needed due to the context, usually the space T in Definition (A.0.2) is denoted by

$$V_1 \otimes \cdots \otimes V_p \tag{A.2}$$

and called the **tensor product**. An element $w \in V_1 \otimes \cdots \otimes V_p$ is a **tensor**. An element of the form $F(v_1, \ldots, v_p)$ is a **indecomposable tensor** and is denoted by

$$v_1 \otimes \cdots \otimes v_p \tag{A.3}$$

In practice a tensor product can be determined by an isomorphism existing between $T^p(V)$ and the p-multilinear maps, defined by the relation

$$v_1 \otimes \cdots \otimes v_p(\varphi^1, \ldots, \varphi^p) = \prod_{j=1}^{p} \varphi_j(v_j) \tag{A.4}$$

Now we have the necessary knowledge to develop some further structures starting from a vector space V. For instance $T^\bullet(V) = \bigoplus_{j\geq 0} T^j(V)$, where $T^j(V) = \underbrace{V \otimes \cdots \otimes V}_{\text{j times}}$, is the **contravariant tensor algebra of V**.

$T_\bullet(V) = \bigoplus_{j\geq 0} T_j(V)$, where $T_j(V) = \underbrace{V^* \otimes \cdots \otimes V^*}_{\text{j times}}$, is the **covariant tensor algebra of V**. Then trivially $T(V) = \bigoplus_{j,k\geq 0} T_k^j(V)$ - where $T_k^j(V) = \underbrace{V \otimes \cdots \otimes V}_{\text{j times}} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{\text{k times}}$ - is the **tensor algebra of V**. The product which makes $T(V)$ an algebra is obviously $\otimes$, and the sum $+ : V \times V \to V$ is extended component by component to the whole $T(V)$.

Let us denote $\mathcal{T}^p(V) = \sum_{k=0}^{p} \bigotimes_{j=1}^{k} T^j(V)$. It's intuitive that there exists a natural **filtration** of the tensor algebra, that is

$$\mathcal{T}^0(V) \subset \mathcal{T}^1(V) \subset \mathcal{T}^2(V) \cdots \subset T^\bullet(V) \tag{A.5}$$

such that

$$\mathcal{T}^p \otimes \mathcal{T}^q \subseteq \mathcal{T}^{p+q} \qquad \forall\, p, q \in \mathbb{N} \tag{A.6}$$

This makes the tensor algebra a **filtered algebra**.

Definition A.0.3. Let V, W be two vector spaces on $\mathbb{K}$. Let $\varphi \in M(V, \ldots, V; W)$ be a p-linear map. If

$$\varphi(v_{P(1)}, \ldots, v_{P(p)}) = sgn(P)\varphi(v_1, \ldots, v_p) \tag{A.7}$$

for each p-tuple $(v_1, \ldots, v_p) \in \underbrace{V \times \cdots \times V}_{\text{p times}}$ and for each permutation $P \in \mathcal{P}$, where $\mathcal{P}$ is the permutation group of the elements $\{1, \ldots, p\}$, then φ is a *skew-symmetric p-linear map.*

The vector space

$$\Lambda^p(V) = \{v_1 \otimes \cdots \otimes v_n \in T^p(V) | \quad v_1 \otimes \cdots \otimes v_n \text{ is skew-symmetric}\} \tag{A.8}$$

is the *p-th exterior algebra of* V. Elements in $\Lambda^p(V)$ are called alternating p-multivectors. On $\Lambda^p(V)$ we can define the *wedge product* of n vectors $v_i \in V$ as

$$v_1 \wedge \cdots \wedge v_n = \frac{1}{n!} \sum_{P\in\mathcal{P}} v_{P(1)} \otimes \cdots \otimes v_{P(n)} \tag{A.9}$$

where $P \in \mathcal{P}$ and $\mathcal{P}$ denotes the permutation group of the elements $\{1, \ldots, n\}$. It's obvious that n must be less then or equal to the dimension of V, otherwise the wedge product is equal to 0. For example the wedge product of two independent vectors $v_1, v_2 \in V$ is

$$v_1 \wedge v_2 = \frac{1}{2}(v_1 \otimes v_2 - v_2 \otimes v_1) \tag{A.10}$$

Moreover we can notice that the wedge product induces a bilinear map

$$\wedge : \Lambda^p(V) \times \Lambda^q(V) \to \Lambda^{p+q}(V) \qquad \forall\, p, q \quad \text{s.t} \quad p + q \leq n \tag{A.11}$$

The **exterior algebra** of the vector space V is

$$\Lambda(V) = \bigoplus_{0\leq p\leq n} \Lambda^p(V) \tag{A.12}$$

equipped with the wedge product $\wedge : \Lambda(V) \times \Lambda(V)$ induced by the map in Equation (A.11). In Section 3.1.1 we will see the exterior algebra is a special case of a *Clifford algebra*. More specifically we can define it as a quotient of the tensor algebra $T(V)$ by the bilateral ideal generated by the element

$$v \otimes v - 1 \tag{A.13}$$

As it is immediately evident, whis correspond to eliminate all symmetric tensor product of vectors. In fact, for example we can write

$$[0] = [(v + w) \otimes (v + w)] = [v \otimes v] + [v \otimes w + w \otimes v] + [w \otimes w] = [v \otimes w + w \otimes v] \tag{A.14}$$

The exterior algebra is an associative, non-commutative algebra with unity $1 \in \Lambda^0(V^*) \equiv \mathbb{R}$. It is also a *graded algebra*, where the gradation means that

$$\Lambda^p(V) \wedge \Lambda^q(V) \subseteq \Lambda^{p+q}(V) \qquad \forall\, p+q \leq n \tag{A.15}$$

and $\Lambda^p(V) \wedge \Lambda^q(V) = 0$ if $p+q > n$. Each $\Lambda^p(V)$ represents the degree p subspace.

The exterior algebra $\Lambda(V)$ inherits an inner product from the vector space V, if it is endowed with a scalar product $\eta : V \times V \to \mathbb{R}$. In fact let $v = v_1 \wedge \cdots \wedge v_p \in \Lambda^p(V)$ and $w = w_1 \wedge \cdots \wedge w_p \in \Lambda^p(V)$. Then we can define an inner product on $\Lambda^p(V)$ by

$$(v, w) = \det\left(\eta(v_i, w_j)\right) \tag{A.16}$$

and extend it bilinearly to all of $\Lambda^p(V)$. It is also necessary to put $(v, w) = 0$ if $v \in \Lambda^p(V)$ and $w = \Lambda^q(V)$ where $p \neq q$.

The next step is to transport these structures on a smooth manifold M such that $dim(M) = n$. Let us consider the space

$$\Lambda^p(T^*) \equiv \Lambda^p(T^*) = \coprod_{p \in M} \Lambda^p(T_p^* M) \tag{A.17}$$

where as usual $\coprod_{p \in M}$ denotes the disjoint union. This defines a fiber bundle together with the canonical projection $\pi : \Lambda^p(T^*) \to M$ which maps $\Lambda^p(T_p^* M)$ into $p \in M$. The smooth sections of this bundle $\Lambda^p T^* \equiv \Gamma(M, \Lambda^p(T^*))$ are the **differential p-forms over** M. Clearly $\Lambda^p(T^*) = 0 \quad \forall\, p > n$ and the dimension is given by $dim(\Lambda^p(T^*)) = \binom{n}{p}$.

B

Appendix to the integration of forms

In the present Appendix we will list a pair of concepts needed to define the integration of forms over a smooth manifold as we did in Section 2.1.3.

Manifolds with boundaries

We define the boundary of the set $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \quad \text{such that} \quad x^i \geq 0 \quad \forall\, i \in I_n\}$ as the sets

$$\mathbb{R}^n_0 = \{x \in \mathbb{R}_n \quad \text{such that} \quad x^n = 0\} \tag{B.1}$$

Let $U \in R^n_+$ be an open set. We denote by $\partial U = U \cap \mathbb{R}^n_0$ the *boundary of* U. We also denote by $I(U) = U/\partial U$ the *interior of* U.

Let $U, V \subset R^n_+$ and let $f : U \to V$. f is smooth if there exist open sets $U \subset U_1$, $V \subset V_1$ and a smooth map $f_1 : U_1 \to V_1$ such that $f_1|_U = f$.

If $f : U \to V$ is a diffeomorphism then it induces a diffeomorphism between $I(U)$ and $I(V)$ and between ∂U and ∂V.

Let M be a topological space. The couple (U, φ) where U is an open set of M and $\varphi : U \to \varphi(U) \subset \mathbb{R}^n_+$ is a **chart with boundary for M** if φ is a homeomorphism onto the open set $\varphi(U) \subset \mathbb{R}^n_+$.

The obvious substitutions into Definitions 2.1.2, 2.1.3 give us the notions of **atlas with boundaries** and **manifolds with boundaries**.

The **boundary of a manifold M** is denoted by ∂M and defined as the set of points $p \in M$ such that there exists a chart with boundary (U, φ) and $p \in U$, $\varphi(p) \in R^n_0$. The **interior of M** is defined as $I(M) = M/\partial M$.

A smooth manifold with empty boundary is said to be boundaryless, and in this case we recover the usual Definition 2.1.3 of a smooth manifold.

The differentiable structure of a manifold with boundary M induces a differentaible strucuture both on ∂M and on $I(M)$. They become smooth manifolds without boundary respectively of dimension $n - 1$ and n.

Classical examples of manifold with boundaries are the *disk*, whose boundary is a circle and the *three-dimensional ball*, whose boundary is a 2-sphere.

C

Basics in complex linear algebra

Let V be a real vector space such that $dim(V) = n$, and denote $\sqrt{-1} = i$. A complex vector space is naturally associated to V: it is the *complexification of* V

$$V^{\mathbb{C}} = \{(v, w) \in V \oplus V| \quad v, w \in V\} \tag{C.1}$$

It is convenient to denote the elements of $V^{\mathbb{C}}$ in the following way

$$(v, w) \equiv v + iw \quad \forall\, v, w \in V \tag{C.2}$$

A complex vector space structure is immediately given on $V^{\mathbb{C}}$ if one define the sum

$$(v_1 + iw_1) + (v_2 + iw_2) = (v_1 + v_2) + i(w_1 + w_2) \quad \forall\, v_1, v_2, w_1, w_2 \in I_n \tag{C.3}$$

and the scalar multiplication for $\lambda = a + ib \in \mathbb{C}$ and $a, b \in \mathbb{R}$

$$\lambda(v + iw) \equiv (a + ib)(v + iw) = (av - bw) + i(aw + bv) \qquad \forall\, v, w \in I_n \tag{C.4}$$

Each vector $v \in V^{\mathbb{C}}$ can be uniquely written as a sum of the form $v = v_1 + iv_2$, where $v_1, v_2 \in V$. We will denote by $\Re e(v) = v_1$ and $\Im m(v) = v_2$ the *real* and *immaginary* parts of v.

An important involutive operation is naturally defined in $V^{\mathbb{C}}$: the *conjugation*

$$\begin{array}{rcl} \bar{\cdot} : V^{\mathbb{C}} & \to & V^{\mathbb{C}} \\ v & \mapsto & \overline{v} = \Re e(v) - i\Im m(v) \end{array} \tag{C.5}$$

The conjugation is an involution, since $\overline{\overline{v}} = v$ and it is $\mathbb{R}$-linear but it is not $\mathbb{C}$-linear.

In particular we can identify $\mathbb{C}^n \cong \mathbb{R}^{2n}$ via the map

$$(z_1, \ldots, z_n) \quad \mapsto \quad (\Re e(z_1), \ldots, \Re e(z_n), \Im m(z_1), \ldots, \Im m(z_n)) \tag{C.6}$$

where $z_i = \Re e(z_i) + i\Im m(z_i)$ for each $i \in I_n$. In this framework we can rewrite the conjugation as an endomorphism $j : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such as

$$v \quad \mapsto \quad v' = jv \tag{C.7}$$

where j is the $2n \times 2n$ matrix

$$j = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix} \tag{C.8}$$

Let us now recall some notions of the elementary holomorphic functions. Let $U \subseteq \mathbb{C}^n$ be an open set. Define $x^\mu = \Re e(z^\mu)$ and $y^\mu = \Im m(z^\mu)$, where $\mu \in I_n$, so that $z^\mu = x^\mu + iy^\mu$.

Now consider the set $C^\infty_{\mathbb{C}}(U) = \{f : U \to \mathbb{C}| \quad f \quad \text{is smooth}\}$. Then define the operators on $C^\infty_{\mathbb{C}}(U)$

$$\frac{\partial}{\partial z^\mu} = \frac{1}{2}\left(\frac{\partial}{\partial x^\mu} - i\frac{\partial}{\partial y^\mu}\right) \qquad \frac{\partial}{\partial \overline{z}^\mu} = \frac{1}{2}\left(\frac{\partial}{\partial x^\mu} + i\frac{\partial}{\partial y^\mu}\right) \qquad \mu, \overline{\mu} \in I_n \tag{C.9}$$

where it is evident that

$$\overline{\frac{\partial}{\partial z^\mu}} = \frac{\partial}{\partial \overline{z}^\mu} \tag{C.10}$$

The set

$$\left\{ \frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial \overline{z}^\mu} \right\}_{\mu \in I_n} \tag{C.11}$$

contains $2n$ indipendent vectors.

Let $f \equiv f(z, \overline{z}) \in C^\infty_{\mathbb{C}}(U)$. Then if $u = \Re e(f)$ and $v = \Im m(f)$, we can write the Cauchy-Riemann relations in the simple form

$$\begin{aligned} \frac{\partial f}{\partial \overline{z}^\mu} = 0 \quad &\Leftrightarrow \quad \frac{\partial u}{\partial x^\mu} - \frac{\partial v}{\partial y^\mu} + i \left(\frac{\partial v}{\partial x^\mu} + \frac{\partial u}{\partial y^\mu} \right) = 0 \quad \mu, \overline{\mu} \in I_n \\ &\Leftrightarrow \quad \frac{\partial u}{\partial x^\mu} - \frac{\partial v}{\partial y^\mu} = 0 \quad \text{and} \quad \frac{\partial v}{\partial x^\mu} + \frac{\partial u}{\partial y^\mu} = 0 \quad \mu \in I_n \end{aligned} \tag{C.12}$$

A map $f \in C^\infty_{\mathbb{C}}$ such that $\frac{\partial f}{\partial \overline{z}^\mu} = 0$ is a *holomorphic* map and it doesn't depend on $\overline{z}^\mu$. On the contrary a map $f \in C^\infty_{\mathbb{C}}$ such that $\frac{\partial f}{\partial z^\mu} = 0$ is an *antiholomorphic* map and it doesn't depend on z^μ. Also, the coordinates z^μ are called *holomorphic coordinates* while the coordinates $\overline{z}^\mu$ are called *antiholomorphic coordinates*.

After having used the identification in Equation (C.6) it's immediate to see that the holomorficity of a function $f : \mathbb{R}^{2n} \supset U \to \mathbb{R}^{2n}$ is equivalent to the condition

$$j \circ f_*(z) = f_*(z) \circ j \quad \forall\, z \in U \tag{C.13}$$

D

Chern classes

As we have seen, given two smooth manifolds E, M and a fiber F, we can construct many fiber bundles, depending on the choice of the transition functions. Naturally we can ask if there exists a way to measure how much a generic bundle E is different from the trivial one $M \times F$ constructed with the same base manifold M and the same fiber F of E itself. The needed tool to achieve this purpose are the characteristic classes, namely suitable subsets of the cohomology classes over the base space M, which precisely measure the non-triviality of bundles. In this context it is important to notice that a fiber bundle is a topological object since the projection π which defines it is not a diffeomorphism but only a surjiection.

To understand the need for introducing the Chern classes, we need to recall some concept in elementary geometry. We have already recalled the Euler characteristic in Section **??**, which is defined, for a polyhedron as

$$\chi = V - L + F \tag{D.1}$$

where $V = \sharp$ of vertices, $L = \sharp$ of edges, $F = \sharp$ of faces. This formula can be extended to general compact smooth manifolds, since χ turns out to be a combination of the Betti numbers for real manifolds, or a combination of the Hodge numbers for complex manifolds. One of the main theorems of geometry - the **Gauss-Bonnet Theorem** - tells us that the total curvature of a compact manifold is given, for a compact and boundaryless smooth surface Σ by

$$\int_\Sigma K d\Sigma = 2\pi\chi(\Sigma) \tag{D.2}$$

where K is the *Gaussian curvature*, i.e. the product of the two principal curvatures (namely the maximum and the minimum curvatures). The total curvature is an intrinsic object of the surface.

It can be understood by giving some simple examples. Let us consider a flat rectangular sheet of paper. We expect that its curvature is zero, and in effect it is so. Now try to construct a cylinder from the flat sheet of paper. We can do it simply by identifying the point on two opposite edges of the sheet. The two principal curvatures will be 0 and 1 (let us construct a cylinder of radius 1). This means that the total curvature of the cylinder is $0 \times 1 = 0$. This is surprisingly: the total curvature of the cylinder is zero as well as the total curvature of the flat sheet of paper.

The geometrical meaning of this puzzling is that the distance between two fixed points on the flat sheet of paper remains the same both before to roll it (to become a cylinder) and after. Instead a sphere has total curvature 4π, that means that there is no way to "transform" it into a flat sheet without stretching or twisting it. In general each continuous transformation keep the total curvature constant.

Moreover it's easy to be computed if one remembers that $\chi = 2 - 2g$ for a surface, where g is the genus of the surface ($g = 0$ for the sphere, while $g = 1$ for the torus).

The attempt to generalize the Gauss-Bonnet Theorem lead us to the Chern classes. In fact, as we will see, the higher non-vanishing Chern class (that in the case of complex surfaces is the first one) is always the Euler characteristic.

In particular we can notice a substantial difference between the sphere and the torus, that explains us the reason why the first Chern class of the torus vanishes, while the first Chern class of the sphere is different from 0. The reason is that on a sphere it's not possible to define a non-vanishing smooth vector field, a notorius fact (see Figure D.1) which is known as the *impossibility to comb the hair on a sphere*. For surfaces which present such a kind of singularities, the first Chern class can not vanish. On the other hand, the torus has not such obstruction like that, as it's evident from Figure D.1. Thus its first Chern class is zero.

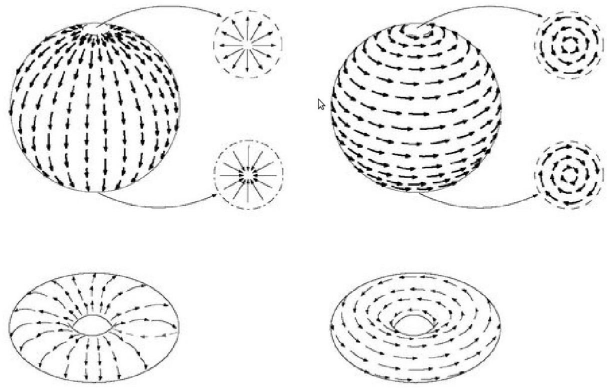


Figure D.1: You can not comb the hair on a sphere, but you can do it on a torus.

The framework in which we will move in the present Section is given by a complex vector bundle E of rank $rk(E) = n$ over the base space M whcih is a smooth manifold of dimension $dim(M) = m$. Its structure group is naturally $GL(n, \mathbb{C})$. Let us start with the fundamental

Definition D.0.4. If $P(Ad_g(y_1), \ldots, Ad_g(y_n)) = P(y_1, \ldots, y_n)$ with $g \in G$ and $y_j \in \mathfrak{g} \quad \forall j \in I_n$, then it is a **symmetric invariant polynomial**. If $y_j = y \quad \forall j \in I_n$, then P is a **invariant polynomial of degree n**

$$P(y, \ldots, y) \equiv P(y^n) \tag{D.3}$$

An example of invariant polynomial is immediately given by the symmetrized trace

$$P(y_1, \ldots, y_n) = str(y_1, \ldots, y_n) = \frac{1}{n!} \sum_{P \in \mathcal{P}} tr(y_{P(1)} \ldots y_{P(n)}) \tag{D.4}$$

where $\mathcal{P}$ denotes the permutation group of the n elements $(1, \ldots, n)$.

Since we are interested in objects as the local connection and the local curvature, we have to extend the definition of invariant polynomial to Lie algebra valued forms. If $x_j = y_j \otimes \omega_j \in \mathfrak{g} \otimes \Lambda^{p_j} T^* U_\alpha$ (see Section 2.2.2), then we simply have

$$P(x_1, \ldots, x_n) = P(y_1 \otimes \omega_1, \ldots, y_n \otimes \omega_n) = \omega_1 \wedge \cdots \wedge \omega_n \, P(y_1, \ldots, y_n) \tag{D.5}$$

For example we have

$$str(x_1, \ldots, x_n) = str(y_1 \otimes \omega_1, \ldots, y_n \otimes \omega_n) = \omega \wedge \cdots \wedge \omega \, str(y_1, \ldots y_n) \tag{D.6}$$

Let $\mathcal{A}$ be a gauge connection over E and $\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ be its related local curvature two-form. Let $\mathcal{A}$ and $\mathcal{F}$ take their values in the Lie algebra $\mathfrak{g}$ of the gauge group G, which is in turn a subgroup of the structure group $GL(n, \mathbb{C})$. The importance of the invariant polynomials resides in the following Proposition

Proposition D.0.1. Let P be an invariant polynomial. The $P(\mathcal{F})$ satisfies

1. $dP(\mathcal{F}) = 0$
2. $P(\mathcal{F}_1) - P(\mathcal{F}_2) = dQ$

where $\mathcal{F}_j$ is the curvature two-form associated to the connection one-form $\mathcal{A}_j$. Finally we can give

Definition D.0.5. The **total Chern class of** E is

$$c(E) = \det \left(1 + \frac{i}{2\pi} \mathcal{F} \right) \tag{D.7}$$

It's evident that $c(E)$ is the direct sum of forms of even degrees. i.e.

$$c(E) = 1 + c_1(E) + c_2(E) + \dots \tag{D.8}$$

Coefficients $c_k(E)$ are the *Chern forms*. One can prove that Equation (D.7) is an invariant polynomial. Then from Proposition D.0.1 we get that each term in the development of Equation (D.7) must vanishes indipendently, so that Chern forms are closed. Consequently they define the *Chern classes*

$$[c_k(E)] \in H^{2k}(M, \mathbb{R}) \tag{D.9}$$

Even if the definition of Chern classes relies to a specific connection one-form $\mathcal{A}$ over E, 2. in Proposition D.0.1 tells us that the difference between Chern forms derived from different connections over E is always an exact form. Then the Chern class isn't modified by a change in the choice of the connection. Obviously different connections lead to different representatives of the cohomology classes $[c_j(E)]$. Moreover, since $\mathcal{F}$ is a two form, if $dim(E) = n$, then $c_j(E) = 0 \quad \forall\, 2j > n$. In any case, independently of $dim(M)$, the last $c_j(E) \neq 0$ is $c_k(E) = \det\left(\frac{i}{2\pi}\mathcal{F}\right)$, thus $c_j(E) = 0 \quad \forall\, j > k$.

Now we will give a method which allow us to find explicitly and easily Chern forms for the general complex vector bundle E. Let $\mathcal{F}$ be the curvature two-form, and let $g \in GL(n, \mathbb{C})$ be the matrix which diagonalizes $\mathcal{F}$, i.e.

$$Ad_g\left(\frac{i}{2\pi}\mathcal{F}\right) = diag(x_1, \dots, x_n) \equiv D \tag{D.10}$$

where $x_1, \dots, x_n$ are suitable two-forms. We can write the total Chern class as

$$c(E) = \det\left(1 + \frac{i}{2\pi}\mathcal{F}\right) = \det(1 + \frac{i}{2\pi}D) = \det\left(diag(1 + x_1, \dots, 1 + x_n)\right) =$$

$$= \prod_{j=1}^{n}(1 + x_j) = 1 + (x_1 + \dots + x_n) + (x_1x_2 + \dots + x_{n-1}x_n) + \dots + (x_1x_2 \dots x_{n-1}x_n) =$$

$$= \mathbf{1} + Tr(\mathcal{A}) + \frac{1}{2}\{(Tr(\mathcal{A})^2 - Tr(\mathcal{A}^2))\} + \dots + \det\mathcal{A} \tag{D.11}$$

From this expansion we immediately understand why the last $c_k(E) \neq 0$ is $\det\left(\frac{i}{2\pi}\mathcal{F}\right)$. Thus, using that $\det(1 + \mathcal{F})$ is an invariant polynomial and then that $\det\left(1 + \frac{i}{2\pi}\mathcal{F}\right) = \det\left(1 + \frac{i}{2\pi}g\mathcal{F}g^{-1}\right) = \det(1 + D)$, we get

$$c_0(E) = 1 \tag{D.12}$$

$$c_1(E) = Tr(D) = Tr\left(\frac{i}{2\pi}g\mathcal{F}g^{-1}\right) = \frac{i}{2\pi}Tr(\mathcal{F}) \tag{D.13}$$

$$c_2(E) = \frac{1}{2}\{(Tr(D))^2 - Tr(D^2)\} = \frac{1}{2}\left(\frac{i}{2\pi}\right)^2\{Tr(\mathcal{F}) \wedge Tr(\mathcal{F}) - Tr(\mathcal{F} \wedge \mathcal{F})\} \tag{D.14}$$

$$\vdots$$

$$c_k(E) = \det D = \left(\frac{i}{2\pi}\right)^2 \det\mathcal{F} \tag{D.15}$$

Furthermore we can define the Euler characteristic of the bundle E as the top Chern class, namely

$$c_k(E) = \chi(E) \qquad k\,\text{is the top index} \tag{D.16}$$

In particular, for Riemann surfaces $c_1(E) = \chi(E)$, as we mentioned at the beginning of the Section.

Moreover we can give some of the most important features of the Chern classes, namely

Proposition D.0.2. Let E, E' be two complex vector bundles over the smooth manifold M, with structure group $GL(n, \mathbb{C})$, and let $f : M \to N$ be a smooth map between two smooth manifolds. Then the following properties hold

1. $c(E) = 1$ if E is a trivial bundle.
2. $c(f^*E) = f^*c(E)$.
3. $c(E \oplus E') = c(E) \wedge c(E')$.

Bibliography

[1] T. Yoneya *Connection of Dual Models to Electrodynamics and Gravidynamics* Prog. Theor. Phys. 51 (1974) 1907-1920

[2] J. Scherk & J. H. Schwarz *Dual Models and the Geometry of Space-Time* Phys.Lett. B52 (1974) 347

[3] P. Ramond *Dual Theory for Free Fermions* Phys.Rev. D3 (1971) 2415-2418

[4] T. Kaluza *Zum Unitätsproblem in der Physik* Sitzungsber. Preuss. Akad. Wiss. Berlin. (Math. Phys.) 96: 69â72

[5] O. Klein *Quantentheorie und fünfdimensionale Relativitätstheorie* Zeitschrift fÃ$\frac{1}{4}$r Physik A 37 (12): 895â906

[6] M. B. Greene, J. H. Schwartz, E. Witten *Superstrings theory: Introduction. Vol. 1* Cambridge University press

[7] M. B. Greene, J. H. Schwartz, E. Witten *Superstrings theory: Loop amplitudes, anomalies and phenomenology. Vol. 2*

[8] T. R. Taylor & C. Vafa *RR flux on Calabi-Yau and partial supersymmetry breaking* Phys. Lett. B474 (2000) 130 arXiv:hep-th/9912152

[9] A. Font & S. Theisen *Introduction to String Compactification* In Geometric and Topological Methods for Quantum Field Theory (pp. 101-181). Berlin / Heidelberg: Springer.

[10] M Graña *Flux compactifications in string theory: a comprehensive review* arXiv:hep-th/0509003v3

[11] C. M. Hull *Global aspects of* T*-duality, gauged sigma models and* T*-folds* arXiv:hep-th/0604178v3

[12] C. M. Hull *A geometry for non-geometric backgrounds* arxiv:hep-th/0406102v3

[13] P. Bouwknegt & J. Evslin & V. Mathai *T-Duality: Topology Change from H-flux* arxiv:hep-th/0306062v2

[14] M. Gualtieri *Generalized complex geometry* University of Oxford arXiv:math/0401221v1

[15] P. Koerber *Lectures on Generalized Complex Geometry for physicists* Instituut voor Theoretische Fysica arXiv:1006.1536v2

[16] M. Graña, R. Minasian, M. Petrini, D. Waldram T*-duality, Generalized Geometry and Non-Geometric Backgrounds* arXiv:hep-th/0807.4527v1

[17] M. Abate & F. Tovena *Geometria differenziale* Universitá di Pisa

[18] L. Fatibene & M. Francaviglia *Natural and gauge natural formalism for classical field theories* Universitá di Torino

[19] M. Nakahara *Geometry and topology in physics* Shizuoka University

[20] R. A. Bertlmann *Anomalies in Quantum Field Theory* University of Vienna

[21] K. B. Marathe & G. Martucci *The mathematical foundations of gauge theories* Elsevier Science publishers B. V.

[22] G. Gentili & F. Podestá & E. Vesentini. *Lezioni di geometria differenziale* 1995 Bollati Boringhieri editore

[23] P. Menotti *Lectures on gravitation theory* Universitá di Pisa

[24] A. Moroianu *Lectures on Kähler geometry* 2008

[25] E. P. van den Ban *Lectures notes on Riemannian geometry* 2008

[26] S. Cacciatori & B. van Geemen *Metodi matematici per la fisica matematica* Universitá di Milano

[27] J. Milnor & J. D. Stasheff *Characteristic classes* American Mathematical Society

[28] R. M. Wald *General Relativity* University of Chicago Press, 1984

[29] P. Candelas *Lectures on complex geometry* University of Texas, 1987

[30] S. Vandoren *Lectures on Riemannian geometry, Part II: Complex manifolds* Utrecht University

[31] J. Figueroa-O'Farrill *PG course on Spin geometry* Edimburgh mathematical physics group Cambridge University press

[32] D. Anselmi *Introduction to renormalization* Milano, 2009

[33] V. Bouchard *Lectures on complex geometry, Calabi-Yau manifolds and toric geometry* Perimeter Institute, Canada arXiv:hep-th/0702063v1

[34] L. Schwartz *Lectures on complex analytic manifolds* Berlin, Heidelberg

[35] K. Hori & S. Katz & A. Klemm & R. Pandharipande & R. Thomas & C. Vafa & R. Vakil & E. Zaslow *Mirror symmetry* American Mathematical Society

[36] C. Chevalley *The algebraic theory of spinors and Clifford algebra* Springer

[37] H. B. Lawson Jr. & M.-L. Michelsohn *Spin geometry* Princeton University

[38] J. Bures *Harmonic spinors on Riemann surfaces* Proceedings of the winter school "Geometry and physics" Palermo, 1994

[39] Sven-S. Porst *Something on Spin structures*

[40] N. Zabzine *Lectures on Generalized Complex Geometry and Supersymmetry* arXiv:hep-th/0605148v4

[41] M. Casati *Geometrie generalizzate e modelli sigma non-lineari* Università degli studi di Milano - Bicocca

[42] G. R. Cavalcanti & Marco Gualtieri *Generalized complex geometry and* T*-duality* arXiv:1106.1747v1 [math.DG]

[43] J. Persson T*-duality and Generalized Complex Geometry* arXiv:hep-th/0612034v3

[44] B. Zumino *Supersymmetry and Kähler manifolds* Phys.Lett. B87 (1979) 203

[45] S. J. Gates Jr. & C. M. Hull *Twisted multiplets and new supersymmetryc non-linear sigma models* Nucl.Phys. B248 (1984) 157P

[46] C. M. Hull & B. Spence *The gauged non-linear sigma model with Wess-Zumino term* Physics Letters B Volume 232 (1989), 204-210

[47] I. Jack & D. R. T. Jones & N. Mohammedi *Gauging the general sigma model with Wess-Zumino term* Nuclear Physics B332 (1990) 359-379

[48] Y. Kosmann-Schwarzbach *Derived brackets* Lett. Math Phys. 69, 61 (2004) arXiv:math/0312524

[49] A. Giveon & M. Porrati & E. Rabinovici *Target Space Duality in String Theory* arXiv:hep-th/9401139v1

[50] N. M. J. Woodhouse *Geometric quantization. Second edition.* Oxford mathematical monographs. Oxford science publications. The Clarendon press, Oxford University press, New York, 1992

[51] V. Mathai & J. Rosemberg T*-duality for torus bundles with* H*-fluxes via non-commutative topology* Comm. Math. Phys., 253(3):705-721, 2005 arXiv:hep-th/0401168