Optimization Letters

On Weierstrass extreme value theorem

--Manuscript Draft--

Manuscript Number:	OPTL-D-12-00332R2
Full Title:	On Weierstrass extreme value theorem
Article Type:	Short Communication
Keywords:	global optima, existence of optimal solutions
Corresponding Author:	Juan Enrique Martínez-Legaz Universitat Autònoma de Barcelona Bellaterra, Barcelona SPAIN
Corresponding Author Secondary Information:	
Corresponding Author's Institution:	Universitat Autònoma de Barcelona
Corresponding Author's Secondary Institution:	
First Author:	Juan Enrique Martínez-Legaz
First Author Secondary Information:	
Order of Authors:	Juan Enrique Martínez-Legaz
Order of Authors Secondary Information:	
Abstract:	We show that suitable restatements of the classical Weierstrass extreme value theorem give necessary and sufficient conditions for the existence of a global minimum and of both a global minimum and a global maximum.

On Weierstrass extreme value theorem*

Juan Enrique Martínez-Legaz†‡
Departament d'Economia i d'Història Econòmica
Universitat Autònoma de Barcelona
08193 Bellaterra
Spain
Phone: (+34)935811366
Fax: (+34)935812012

Abstract

We show that suitable restatements of the classical Weierstrass extreme value theorem give necessary and sufficient conditions for the existence of a global minimum and of both a global minimum and a global maximum.

1 Introduction

The classical Weierstrass extreme value theorem asserts that a real-valued continuous function f on a compact topological space attains a global minimum and a global maximum. In fact a stronger statement says that if f is lower semicontinuous (but not necessarily continuous) then f attains a global minimum (though not necessarily a global maximum). The classical result easily follows from the latter statement by applying it to f and to $-f$, since lower semicontinuity of these two functions is equivalent to continuity of f. An immediate but generally neglected observation is that the conclusion of this theorem, that is, the existence of a global minimum, does not refer to any specific topology; consequently, one is free to consider the most convenient topology (other than any "natural" topology in the problem under consideration) to investigate the existence of a global minimum (see, e.g., [1, Thm. 1.1]). The aim of this note is to point out that, under this formulation, the Weierstrass extreme value theorem provides a necessary and sufficient condition for the existence of a global minimum. Similarly, we will show that another suitable restatement provides a necessary and sufficient condition for the existence of both a global minimum

*MSC 2010 Subject Classification. Primary 49J27; secondary 90C48

†The author has been supported by the MICINN of Spain, Grant MTM2011-29064-C03-01. He is affiliated to MOVE (Markets, Organizations and Votes in Economics).

‡*E-mail address:* JuanEnrique.Martinez.Legaz@uab.cat

and a global maximum. All this will be done in the next section, in which with no extra effort we will prove the results in the more general setting of functions taking values in a totally ordered set C.

We will endow C with its order topology (see, e.g., [2]), namely, the one having as a subbase all the sets of the types $(c,+\infty) := \{y \in C : c \prec y\}$ and $(-\infty, c) := \{y \in C : y \prec c\}$, with $c \in C$. If X is a topological space, a function $f : X \longrightarrow C$ is said to be lower semicontinuous if the set $\{f \succ c\} := f^{-1}((c,+\infty))$ is open for every $c \in C$.

2 Results

Theorem 1 *Let X be a nonempty set, C be a totally ordered set, endowed with the order topology, and $f : X \longrightarrow C$. The following statements are equivalent:*

(1) X is compact in the coarsest topology that makes f lower semicontinuous.

(2) There exists a topology on X which makes X compact and f lower semicontinuous.

(3) f has a global minimum.

Proof. Implication (1) $\Longrightarrow$ (2) is obvious. Implication (2) $\Longrightarrow$ (3) is Weierstrass theorem, which in our general setting still admits its standard proof: If f has no global minimum then the open sets $\{f \succ f(x)\}$ $(x \in X)$ make an open cover of X, which, by compactness, admits a finite subcover $\{f \succ f(x_1)\}, ..., \{f \succ f(x_n)\}$, but then the x_i for which $f(x_i)$ is the smallest among $\{f(x_1), ..., f(x_n)\}$ is a global minimum, thus yielding a contradiction.

It only remains to prove implication (3) $\Longrightarrow$ (1). Assume (3) and consider the topology having as base X and all sets of the type $\{f \succ c\}$ $(c \in C)$, that is, the open sets are X and all sets of the form $\bigcup_{i \in I} \{f \succ c_i\}$. Obviously, the topology thus defined makes f lower semicontinuous and is the coarsest with this property. We will now prove that X is compact in this topology. Let $\mathcal{O}$ be an open cover of X, and take a global minimum $x \in X$ of f. One has $x \in U$ for some $U \in \mathcal{O}$. Then either $U = X$ or $U = \bigcup_{i \in I} \{f \succ c_i\}$ for some family $\{c_i\}_{i \in I}$ of elements of C. In the latter case, since $x \in U = \bigcup_{i \in I} \{f \succ c_i\}$, it turns out that $U = X$ too. Thus $\mathcal{O}$ admits the open cover $\{X\}$, which shows that X is compact. ■

It is worth observing that the coarsest topology $\mathcal{T}$ that makes f lower semicontinuous is very peculiar. If $f(X)$ is finite then the cardinality of $\mathcal{T}$ is that of $f(X)$ plus 1, since in this case the open sets are X and $\{f \succ f(x)\}$ $(x \in X)$. On the other hand, in the general case the boundary of every nonempty open set coincides with its complement. Consequently, every closed set different from X has an empty interior. Therefore, every nonempty set different from X has a nonempty boundary; in other words, X is connected. In fact, it is easy to see that every set is connected.

As an easy illustrative example, consider the case when $f : \mathbb{R} \longrightarrow \mathbb{R}$ is the characteristic function of the set of irrational numbers, that is, $f(x) = 0$ if x is rational and $f(x) = 1$ otherwise. The coarsest topology that makes f lower semicontinuous has exactly three open sets: $\emptyset$, $\mathbb{R}$ and the set of irrational numbers. This is in sharp contrast with the standard topology of the real line, with respect to which f has no lower semicontinuity property and has a very strange topological structure.

We also want to point out that the standard textbook version of Weierstrass extreme value theorem, namely, the one stating that a real-valued continuous function f on a compact topological space attains both a global minimum and a global maximum, does not admit a restatement of the type of Theorem 1. Consider, for instance, the set $X := \left\{\frac{1}{n} : n = 1, 2, 3, ...\right\} \cup \{-1\}$ and the function $f : X \longrightarrow \mathbb{R}$ defined by $f(x) := x$, which attains a global minimum at $x := -1$ and a global maximum at $x := 1$. Clearly, the only topology on X under which f is continuous is the discrete one, but X is not compact under this topology.

In spite of the preceding example, a suitable reformulation of the classical version of Weierstrass theorem allows for a restatement similar to Theorem 1:

Theorem 2 *Let X be a nonempty set, C be a totally ordered set, endowed with the order topology, and $f : X \longrightarrow C$. The following statements are equivalent:*

(1) There exist $c_1, c_2 \in C$ such that the sets $\{f \preceq c_1\} := \{x \in X : f(x) \preceq c_1\}$ and $\{f \succeq c_2\} := \{x \in X : f(x) \succeq c_2\}$ are nonempty and compact in the coarsest topology that makes f continuous.

(2) There exist $c_1, c_2 \in C$ such that the sets $\{f \preceq c_1\} := \{x \in X : f(x) \preceq c_1\}$ and $\{f \succeq c_2\} := \{x \in X : f(x) \succeq c_2\}$ are nonempty and compact in some topology which makes f continuous.

(3) f has a global minimum and a global maximum.

Proof. Implication (1) $\Longrightarrow$ (2) is obvious. Implication (2) $\Longrightarrow$ (3) follows from Weierstrass theorem, as a global minimum (maximum) of f over $\{f \preceq c_1\}$ ($\{f \succeq c_2\}$, respectively) is a global minimum (maximum, respectively) of f over the whole set X. To prove implication (3) $\Longrightarrow$ (1), take a global minimum x_1 and a global maximum x_2 of f and set $c_i := f(x_i)$ $(i = 1, 2)$. We will now prove that the nonempty set $\{f \preceq c_1\}$ is compact. Let $\mathcal{O}$ be an open cover of $\{f \preceq c_1\}$. Since $x_1 \in \{f \preceq c_1\}$, one has $x_1 \in U$ for some $U \in \mathcal{O}$. As U is the preimage under f of some open set in C and the sets $(c, +\infty)$, $(-\infty, c')$ and $(c, +\infty) \cap (-\infty, c')$ make a base of the topology of C when c and c' run over the whole of C, x_1 must belong to some subset of U of one of the following three types: $\{f \succ c\}$, $\{f \prec c'\} := f^{-1}((-\infty, c'))$ and $\{f \succ c\} \cap \{f \prec c'\}$. Clearly, such a subset contains $f^{-1}(c_1) = \{f \preceq c_1\}$; hence $\mathcal{O}$ admits the open cover $\{U\}$, which shows that $\{f \preceq c_1\}$ is compact. One can similarly prove the compactness of the nonempty set $\{f \succeq c_2\}$. ■

Acknowledgment. I am grateful to Maxim I. Todorov for his helpful and stimulating comments. Thanks are also due to an anonymous referee for his/her careful report, pointing out some important corrections and observing that the conditional completeness assumption on C made in the initial version was superfluous.

References

[1] A. Ioffe, R.E. Lucchetti, Generic well-posedness in minimization problems, Abstr. Appl. Anal. 2005 (4) 343–360.

[2] S. Willard, General topology, Addison-Wesley Publishing Co., Reading, Massachussets - London-Don Mills, Ontario, 1970.