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From Spectral Methods to the Geometrical Approximation for PDEs in Finance

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Introduction

The most popular market model in continuos time is the Black-Scholes model. It assumes for the underlying process, a geometric Brownian motion with constant volatility, that is

$$dS_t = rS_t dt + \sigma S_t d\tilde{W}_t,$$

$$dB_t = rB_t dt,$$

where r is the constant risk-free rate, S_t is the stock and σ is the constant volatility of the stock. Under these assumptions, closed form solutions for the values of European call and put options, are derived by use of the PDE method. We want to discuss by present work, the PDE approach in most complicated cases of market models. Our objective is to use two different techniques that are respectively Spectral Methods and Geometrical Approximation (the latter introduced by us), in order to compute the price of the derivatives for the following kinds of contracts: Double Barrier options in Black-Scholes model; Vanilla options and Barrier options in Heston's model; Vanilla options in SABR model. We have structured the work in five chapters:

Chapters 1 and 2, respectively show the theoretical foundations of the parabolic PDE, and the Black-Scholes market model.

In chapter 3, we are going to consider Double Barrier Options, in Black-Scholes model, that belongs to the kind of exotic options, in which case we have a deterministic volatility function $\sigma(t)$. For example we consider the value of a Knock-out, down-and-out Call option, that is given by the solution of the Black-Scholes equation with appropriate boundary conditions, but we are able to discuss also the cases in which we have Knock-in options, or we have a Put option and do not a Call. To grant the existence and uniqueness of the solution, it is necessary to define the boundary condition and the initial condition. Also we require that when the value of the underlying asset hits the two barriers, lower (L) and upper (H), the option is cancelled in our case, but it could be activated for knock-in options . The best method to solve the above problem, is the using of the Spectral Theory, which allows to write the price of the Knock-out or Knock-in options, as series expansion. We are going to compare the spectral method with others, studying also the computational complexity.

In chapter 4, we propose a new technique, that we have called the **Geometrical Approximation** method. We are going to consider well known stochastic volatility market models. The assumption of constant volatility isn't reasonable in a real market, since we require different values for the volatility parameter for different strikes and different expiries to match market prices. The volatility parameter that is required in the Black-Scholes formula to reproduce market prices is called the implied volatility. To obtain market prices of options maturing at a certain date, volatility needs to be a function of the strike. This function is the so called volatility skew or smile. Furthermore for a fixed strike we also need different volatility parameters to match the market prices of options maturing on different dates written on the same underlying, hence volatility is a function of both the strike and the expiry date of the derivative security. This bivariate function is called the volatility surface. There are two prominent ways of working around this problem, namely, local volatility models and stochastic volatility models. For local volatility models the assumption of constant volatility made in Black and Scholes (1973) is relaxed. The underlying risk-neutral stochastic process becomes

$$dS_t = r(t)S_t dt + \sigma(t, S_t)S_t d\tilde{W}_t,$$

where $r(t)$ is the instantaneous forward rate of maturity t implied by the yield curve and the function $\sigma(S_t, t)$ is chosen (calibrated) such that the model is consistent with market data, see Dupire (1994), Derman and Kani (1994) and (Wilmott, 2000). It is claimed in Hagan et al. (2002) that local volatility models predict that the smile shifts to higher prices (resp. lower prices) when the price of the underlying decreases (resp. increases). This is in contrast with the market behaviour where the smile shifts to higher prices (resp. lower prices) when the price of the underlying increases (resp. decreases). Another way of working around the inconsistency introduced by constant volatility is by introducing a stochastic process for the volatility itself; such models are called stochastic volatility models. The major advances in stochastic volatility models are Hull and White (1987), Heston (1993) and Hagan et al. (2002). Such models have the following general form

$$\begin{aligned}
dS_t &= \mu_t S_t dt + \sigma_t^{\delta} a_2(S_t) dW_t^{(1)}, \\
d\sigma_t^j &= b_1(\sigma, t)dt + \alpha\sigma_t^{\delta} dW_t^{(2)}, \\
dW_t^{(1)} dW_t^{(2)} &= \rho dt, \\
dB_r &= rB_t dt,
\end{aligned}$$

and varying its parameters we can obtain them:

• for $\delta = 1$, $j = 1$, $\alpha \neq 0$, $a_2(S) = S^{\beta}$, $\beta \in (0, 1]$ and $b_1 = 0$, we get the SABR model, by Hagan;

• for $\delta = 1$, $j = 2$, $\alpha \neq 0$, $a_2(S) = S$ and $b_1 = k(\theta - \sigma_t^j)$, we get Heston model, by Heston;

• for $\delta = 1$, $\alpha = 0$ and $b_1 = 0$ we get Black-Scholes model with constant volatility, by Black-Scholes-Merton;

where the tradable security S_t and its volatility σ_t are correlated, i.e.,

$$< dW_t^{(1)}, dW_t^{(2)} >= \rho dt.$$

Using the above indicated general market model, from Itô's lemma, it is possible to derive, under mild additional assumptions, the partial differential equation satisfied by the value function of a European contingent claim. For this purpose, one needs first to specify the market price of volatility risk $\lambda(\sigma, t)$. The market price for the risk is associated with the Girsanov transformation of the underlying probability measure leading to a particular martingale measure. Let us observe that pricing of contingent claims using the market price of volatility risk is not preferences-free. The price function $f = f(t, S, \sigma)$ of a European contingent claim has to satisfy a specific PDE of the form:

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \rho\sigma S b(\sigma, t)\frac{\partial^2 f}{\partial S \partial t} + \frac{1}{2}b(\sigma, t)^2 \frac{\partial^2 f}{\partial \sigma^2} + rS\frac{\partial f}{\partial S} + \left[a(\sigma, t) + \lambda(\sigma, t)b(\sigma, t)\right]\frac{\partial f}{\partial \sigma} - rf = 0,$$

with the terminal condition $\Phi(S_T) = f(T, S, \sigma)$ for every $S \in \mathbb{R}_+$ and $\sigma \in \mathbb{R}_+$.

We are going to use some geometrical transformations in order to simplify the above pricing PDE. Our idea is to determine, by Ito's lemma, the exact PDE for derivative pricing in Heston and SABR market model and instead to use the exact pay-off function, for example, for a Vanilla Call option $(S_T - E)^+$, we consider this $(S_T e^{\varepsilon_T} - E)^+$, where ε_T is a stochastic process linked to volatility (or variance). What mean ε_T will be clear later. Hence, we are able to solve the exact PDE, but with different Cauchy's condition (with respect to original problem).

In other words it is possible to approximate our closed form solution obtained by considerations on property of continuity of Feynman-Kač formula, with the solutions computed using the numerical techniques known in literature.

Finally, in Chapter 5 we are going to present another approximation technique, again for the Heston model, based on different idea, respect to Geometrical Approximation method. In fact in this case we are going to choose a particular volatility risk price, so that, the drift term of the variance processes is equal to zero. Also by the latter procedure, that we name Perturbative Method, we are able to evaluate the Vanilla Options, and not only, through an approximate solution in closed form, that can be used also for pricing several kinds of derivatives contracts, and we have used here also for computing the price of the knock-out Barrier Options.

Chapter 1

Theoretical Foundations

In this chapter we are going to introduce the basic notions of the PDEs of parabolic kind with particular attention on the properties of the density function. Besides, we are going to show some results of Functional Analysis that we are going to use in Chapter 3. We also introduce some geometrical considerations on PDEs associated to stochastic volatility market models, that we will be discussed in more detail in Chapter 4 and Chapter 5.

1.1 Canonical Diffusion Process

The determination of the prices of financial derivative securities can be reduced to solving partial differential equations; in Chapter 2, we are going to introduce the Black-Scholes model as a relevant example. The PDEs used in Finance are typically of parabolic kind, thus it is important to introduce some notions about this topic.

Let $\mathbb{L}^2(I, m)$ be a Hilbert space of real-valued functions with domain an interval $I \subset \mathbb{R}^n$, and with m random measure: in probability theory, a random measure is a measure of random events. A random measure of the form:

$$m = \sum_{n=1}^{N} \delta(X_n),$$

where δ is the Dirac measure, and X_n are random variables, is called a point process or random counting measure. This random measure describes the set of N particles, whose locations are given by the (generally vector valued) random variables X_n. Random measures are useful in the description and analysis of Monte Carlo methods, such as Monte Carlo numerical quadrature and particle filters.

The functions that belong to $\mathbb{L}^2(I, m)$ are square-integrable, and the inner product in $\mathbb{L}^2(I, m)$ is defined as follows:

$$\langle f, g \rangle = \int_I f(x)g(x)m(x)dx. \tag{1.1}$$

We define the infinitesimal operator K_0 in its general form as:

$$K_0 = -\frac{1}{2} \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j}^{n} b_j(x) \frac{\partial}{\partial x_j} + c(x), \qquad x \in \mathbb{R}^n. \tag{1.2}$$

The domain of K_0 in $\mathbb{L}^2(I, m)$ is (McKean 1956, p. 526 and Langer and Schenk 1990, pp. 15):

$$D(K_0) := \{f \in \mathbb{L}^2(I, m) : f, f' \in AC_{loc}(I), K_0 f \in \mathbb{L}^2(I, m) + \text{boundary conditions} \},$$

where $AC_{loc}(I)$ is the space of functions absolutely continuous over each compact subinterval of I. The boundary conditions give a value at the solutions of PDEs on the border.

If K_0 is the generator of a diffusion process with transition density $p_0(x, y, t)$, then the matrix of its diffusion coefficients $(a_{ij}(x))_{i,j}^n$ is given by

$$a_{ij}(x) := lim_{t \to 0} \frac{1}{t} \int_{(\xi : |\xi - x| < \epsilon)} d\xi (\xi_i - x_i)(\xi_j - x_j) p_0(x, \xi, t), \tag{1.3}$$

for any $\epsilon > 0$. Such a process is said to be *canonical* if its transition density can be approximated by the Wiener density, i.e. if

$$p_0(x, \xi, t) \leq c_1 \frac{1}{(2)^{n/2}} exp\left(-c_2 \frac{|x - \xi|^2}{2t}\right), \tag{1.4}$$

where c_1 and c_2 are arbitrary constants. This assumption implies that the coefficients $a_{ij}(x)$, $1 \leq (i,j) \leq n$, of the diffusion matrix are bounded. The processes which are generated by operators of the K_0 form , see eq. (1.2), are the following kind:

$$dx_t = b_j(x_t)dt + a_{i,j}(x_t)dW_t \qquad x_t \in \mathbb{R}^n, \quad \forall t \in [0,T],$$

in which W_t is a Wiener process in $\mathbb{R}^n$; it is an old topic studied in the literature for many situations. Dynkin [29] assumes that the coefficients a_{ij} and b_j are bounded and the diffusion coefficient $a_{i,j}$ satisfies the following ellipticity condition:

$$\sum_{i,j=1}^{n} \lambda_i \lambda_j a_{ij}(x) \geq \gamma \sum_{j=1}^{n} |\lambda|^2, \tag{1.5}$$

for all $(\lambda_1, ..., \lambda_n) \in \mathbb{R}^n$, where γ is an arbitrary constant. It then follows the corresponding transition density can be estimated by the Wiener density: see Dynkin (Theorem 0.5, pp.229 [29]). Kochubei [69] gives rather general conditions on the coefficients of the operator in (1.2) which guarantee that it generates a Feller semigroup (Markov process) with continuous density. In fact the assumption on the coefficients are the following:

(a) The functions a_{ij} are twice differentiable and the partial derivatives $\frac{\partial^2 a_{ij}}{\partial x_k \partial x_l}$, $1 \leq k, l \leq n$ belong to $L^1_{loc}(\mathbb{R}^n)$, space of locally summable functions on $\mathbb{R}$. The functions $\sum_{j=1}^n \frac{\partial a_{ij}}{\partial x_j}$, $1 \leq i \leq n$, and $\sum_{ij}^n \frac{\partial^2 a_{ij}}{\partial x_i \partial x_j}$ are locally Hölder continuos.

(b) The functions b_j belong to $\mathbb{L}^n_{loc}(\mathbb{R}^n)$ and the function $\sum_{j=1}^n \frac{\partial b_j}{\partial x_j}$ belongs to $\mathbb{L}^{max(1,n/2)}_{loc}(\mathbb{R}^n) = \{f \text{ Borel-measurable: } \|1_I f\|_{max(1,n/2)} < \infty, I \subseteq \mathbb{R}^n \quad \text{compact}\}$.

(c) The inequality

$$\sum_{ij}^{n} \frac{\partial^2 a_{ij}(x)}{\partial x_i \partial x_j} \leq 0, \tag{1.6}$$

holds for all $x \in \mathbb{R}^n$. In addition,

$$c(x) \geq max \left(\sum_{j=1}^{n} \frac{\partial b_j(x)}{\partial x_j}, 0 \right) \geq 0, \tag{1.7}$$

for all $x \in \mathbb{R}^n$.
Under the assumption (a), (b) and (c) the operator K_0 generates a Feller semigroup and hence a Markov process

$$\{(\Omega, \mathbb{F}, \mathbb{P}), (X(t) : t \geq 0), (\theta_t : t \geq 0), (E, \varepsilon)\}, \tag{1.8}$$

with state space $E = \mathbb{R}^n$ and with a probability density function $p_0(x,y,t)$ for which all the conditions hereafter $A_1 \to A_4$ are satisfied:

A1. Markov property:
$p_0(x, y, t)$ is non-negative and it verifies the Chapman-Kolmogorov identity, i.e.,

$$\int dz p_0(s, x, z) p_0(t, z, \xi) = p_0(t + s, x, \xi) \qquad t > s, \quad x, \xi \in E \tag{1.9}$$

A2. Feller property:
For every $f \in C_\infty(E)$ the function $x \to \int dm(\xi) f(\xi) p_0(x, \xi, t)$ belongs to $C_\infty(E)$, where $m(\xi)$ is a random-measure.

A3. Continuity:
For every $f \in C_\infty(E)$ and for every $x \in E$ the following identity is true:

$$lim_{t \to 0} \int dm(y) f(y) p_0(t, x, y) = f(x). \tag{1.10}$$

A4. Symmetry:
The function $p_0(t, x, y)$ is symmetric: $p_0(t, x, y) = p_0(t, y, x)$ for all $t > 0$, and for all x and y in E.

The PDE, whose generator is the K_0 operator, is of parabolic kind if and only if the determinant function of $a_{i,j}(x)$ is equal to zero $\forall x \in I \subset \mathbb{R}^n$.

1.1.1 Nonnegative Solutions of Cauchy's Problem

Let be given the parabolic operators K_0 (1.2). in which we separate the spacial variables x_j from the temporal variables t as follows:

$$K_0 f := \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}(t, x) \frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_{j}^{n} b_j(t, x) \frac{\partial f}{\partial x_j} + c(t, x) f - \frac{\partial f}{\partial t} = 0, \qquad x \in \mathbb{R}^n, \qquad t \in \mathbb{R}^+.$$

We make the following hypothesis:

(1) Let the coefficients $a_{i,j}(t, x), b_j(t, x), c(t, x)$ be real functions.
Let the matrix $a_{i,j}$ be symmetric and positive semi-defined.
Let the coefficient c be inferiorly limited:

$$c_0 := inf\{c\} \in R.$$

(2) Suppose there exists a constant M, such that:

$$|a_{i,j}(t, x)| < M, \qquad |b_j(t, x)| < M(1 + |x|), \qquad |c(t, x)| < M(1 + |x^2|),$$

$$\forall (t, x) \in \mathbb{R} \times \mathbb{R}^+, \qquad i, j = 1 \cdots\cdots N.$$

(3) Suppose that the operator K_0 has a fundamental solution $G(t,x|s,y)$, namely, for the Cauchy's problem:

$$\begin{cases} K_0 f = 0 & (t,x) \in]s,+\infty[\times\mathbb{R}^N, \\ \\ f(0,x) = \phi(x) & x \in \mathbb{R}^N, \end{cases}$$

we have:

$$f(t,x) = \int_{\mathbb{R}^N} G(t,x|s,y)\phi(y)dy.$$

Suppose there exists $\lambda > 0$, so that, $\forall T$, $i = 1,\cdots N$, $t \in]s,s+T[$, $x,y \in R^N$ and apply the following relations:

$$\frac{1}{M}G_{\frac{1}{\lambda}}(t,x|s,y) \leq G(t,x|s,y) \leq MG_\lambda(t,x|s,y), \qquad \left|\frac{\partial G(t,x|s,y)}{\partial y_i}\right| \leq \frac{M}{\sqrt{t-s}}G_\lambda(t,x|s,y),$$

where M is positive constant linked to T, and $G_\lambda(t,x|s,y)$ is the fundamental solution of the heat operator $\frac{\lambda}{2}\nabla_x^2 - \partial_t$, in $\mathbb{R}^{N+1}$:

$$G_\lambda(t,x|,s,y) = \frac{1}{(2\pi\lambda(t-s))^{\frac{N}{2}}}\exp\left(\frac{|x-y|^2}{2\lambda(t-s)}\right).$$

(4) Suppose there exists the adjoint operator K_0^* of K_0, so that:

$$K_0^* = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(t,x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j}^{n} b_j^*(t,x)\frac{\partial}{\partial x_j} - c^*(t,x) + \frac{\partial}{\partial t} \qquad x \in \mathbb{R}^n \qquad t \in \mathbb{R}^+.$$

where

$$b_i^* = -b_i + \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}a_{i,j}, \qquad c^* = c + \frac{1}{2}\sum_{i,j}^{N}\frac{\partial}{\partial x_i x_j}a_{i,j} - \sum_{i,j=1}^{N}\frac{\partial}{\partial x_j}b_j.$$

and verify the same growing hypothesis of the coefficients b_j and c seen before in (2).

Theorem (1.1)
Let Cauchy's problem be given:

$$\begin{cases} K_0 f = 0 & (t,x) \in]0,+\infty[\times\mathbb{R}^N, \\ \\ f(0,x) = \phi(x) & x \in \mathbb{R}^N, \end{cases}$$

so that, let be verified the previous hypothesis (1),(2),(3),(4).
Thus, there exists at most one solution $f \in C^{1,2}$ inferiorly limited.

1.2 The General Form of the Spectral Representation

In Functional Analysis, a very important result is the Spectral Theorem for semigroups of self-adjoint operators. By the latter theorem, we are able to write the solution of PDEs of parabolic kind as series expansion of eigenfunctions of the associated Sturm-Liouville problem. This is a relevant topic in Mathematical Finance. In order to use the Spectral Theorem in Finance, we introduce some theoretical notions to produce a spectral representation. Define the value of a derivative security as $V(X_t, t)$, where X_t is a random variable in $\mathbb{R}$ and t is the actual time. Now write $V(x,t) = [P_t f]$ where $x \in \mathbb{R}$ and K_0, see eq.(1.2) for $n = 1$, is the generator operator of P_t for $f \in \mathbb{L}^2(I, m)$ and $t \geq 0$; that is

$$K_0 = -\frac{1}{2}a(x)\frac{\partial^2}{\partial x^2} + b(x)\frac{\partial}{\partial x} + c(x), \qquad x \in I \subset \mathbb{R},$$

$$[P_t f] = exp(-tK_0)f = \int_I p_0(x, \xi, t) f(\xi) dm(\xi), \tag{1.11}$$

and

$$p_0(x, \xi, t) = m(\xi) \int_{(-\infty,0]} d\lambda \left(e^{\lambda t} \sum_{i,j=1}^{\infty} \phi_j(x, \lambda)\phi_i(\xi, \lambda) \right) \qquad t > 0, \quad x, \xi \in I, \tag{1.12}$$

where $\phi_{i=1,\cdots,N}(x, \lambda)$ are the eigenfunctions, of Sturm-Liouville problem associated with K_0 (see Borodin and Salmin 1996, Chapter II for details on one-dimensional diffusion and for a quick review of the Spectral Theorem, or see also Ito and McKean (1974), in which it is possible to find a general spectral representation for the semigroup of a one-dimensional diffusion with killing in $\mathbb{L}^2(I, m)$). When the operator P_t is self-adjoint on its domain, and this is bounded too, by Hilbert-Schmidt Theorem, we know that its spectrum is simple and purely discrete (Elliot 1954 and McKean 1956, Theorem (3.1). Let $(\lambda_i)_{i=1}^N$, $0 > \lambda_1 > \lambda_2 > ...$, $lim_{N\to\infty}\lambda_N = -\infty$, be the eigenvalues of ODE associated to K_0 infinitesimal generator, obtained from the latter by separation variable method, and let $(\phi_i)_{i=1}^N$ be the corresponding eigenfunctions orthonormalized so that $||\phi_N||^2 = 1$ and

$$\langle \phi_i, \phi_j \rangle = \int_I \phi_i(x)\phi_j(x)m(x)dx = \delta_{i,j} = \begin{cases} 0 & \text{for i} \neq j, \\ 1 & \text{for i=j.} \end{cases}$$

McKean (1956), proves a number of smoothness properties for $p_0(x, \xi, t)$. For $f \in \mathbb{L}^2(I, m)$, the equation (1.11) can be rewritten as a spectral expansion:

$$[P_t f](x) = \int_{(-\infty,0]} d\lambda e^{\lambda t} \sum_{i,j=1}^{\infty} \phi_i(x, \lambda) c_j(\lambda) \qquad x \in I, t > 0, \tag{1.13}$$

where the expansion coefficients are

$$c_i(\lambda) = \int_I d\xi m(\xi) f(\xi) \phi_i(\xi, \lambda) \qquad i = 1, \cdots, N \tag{1.14}$$

and the Parseval equality holds

$$||f||^2 = \int_{(-\infty,0]} \sum_{i,j=1}^{\infty} c_i(\lambda)c_j(\lambda)d\lambda. \tag{1.15}$$

Thus we have at the time zero, by Feynman-Kač formula, that the present value or price of derivative security can then be represented as risk-neutral expectation of discounted payoff:

$$V(T,x) = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_0^T r(s)ds} f(X_T)|X_0 = \overline{x}\right],$$

where f is the solution of the parabolic PDE: $K_0 f(x) = 0$, and $f(X_T)$ is the pay-off of a derivative contract. Then the spectral representation for the density and the spectral expansion for the value function simplify to the following series converges uniformly on the compact squares in $(I \times I)$; summarising, we can write:

$$p_0(x,\xi,t) = m(\xi)\sum_{i=1}^{\infty} e^{\lambda_i t}\phi_i(x)\phi_i(\xi), \qquad x,\xi \in I, \quad t > 0, \tag{1.16}$$

and the derivative security price is given by:

$$\begin{aligned} &V(T,x) = [P_T f](x) = \sum_{i=1}^{\infty} c_i e^{\lambda_n T}\phi_i(x), \\ &f \in \mathbb{L}^2(I,m), \qquad x \in I, \quad T > t > 0, \\ &c_i = \langle f, \phi_i \rangle, \quad ||f||^2 = \sum_{i=1}^{\infty} c_i^2. \end{aligned} \tag{1.17}$$

In chapter 3 we will use the Spectral Theory in order to price Double Barrier options, following the approach of Pelsser (2000) [98] and Linetsky (2003) [85].

1.3 Geometrical point of view of Stochastic Problems

The study of PDEs on stochastic volatility market models, can benefit from some geometrical considerations. Here we borrow from Bourgade and Croissant (2005) the theoretical foundations on which we are going to build our approximation method.

We consider a stochastic market model, with stochastic coefficients a_1, b_1 for drift and a_2, b_2 for diffusion, with constant continuously compounded risk free interest rate r, in which the underlying asset S_t follows a geometric Brownian motion and there is a derivative security whose value is given by $f(t,S,\sigma)$ with payoff $f(T,S,\sigma) = \Phi(S_T)$, thus we can write respect to natural probability measure $\mathbb{P}$:

$$\begin{aligned}
&dS_t = a_1(\sigma_t, S_t)dt + a_2(\sigma_t, S_t)dW_t^{(1)},\\
&d\sigma_t = b_1(\sigma_t)dt + b_2(\sigma_t)dW_t^{(2)},\\
&dB_t = rB_t dt,\\
&f(T, S, \sigma) = \Phi(S_T).
\end{aligned}$$

Note that the drift and the diffusion coefficients do not depend on time, with $< dW^{(1)}, dW^{(2)} >= \rho dt$, where ρ is a constant correlation coefficient. Choosing to rewrite the above SDEs with respect to martingale measure $\mathbb{Q}$ we have :

$$\begin{aligned}
&dS_t = rS_t dt + a_2(\sigma_t, S_t)d\tilde{W}_t^{(1)},\\
&d\sigma_t = \tilde{b}_1(\sigma_t)dt + b_2(\sigma_t)d\tilde{W}_t^{(2)},\\
&dB_t = rB_t dt,\\
&f(T, S, \sigma) = \Phi(S_T).
\end{aligned} \tag{1.18}$$

From (5.2), through the Ito's lemma, we have the backward Fokker Planck equation written as follows:

$$\frac{\partial f}{\partial t} + \frac{1}{2}\left(a_2^2\frac{\partial^2 f}{\partial S^2} + 2\rho a_2 b_2 \frac{\partial^2 f}{\partial\sigma\partial S} + b_2^2\frac{\partial^2 f}{\partial\sigma^2}\right) + rS\frac{\partial f}{\partial S} + \tilde{b}_1\frac{\partial f}{\partial\sigma} = rf, \tag{1.19}$$

in which we have imposed $S_t = S$ following the PDE notation, and $\tilde{b}_1(\sigma_t) = b_1(\sigma_t) + \lambda(\sigma_t)$, where $\lambda(\sigma_t)$ is an arbitrary price of volatility risk. We now focus our attention on the squared term; and define the metric $g_{i,j}$ as follows:

$$\begin{aligned}
&g_{ij} = \frac{1}{(1-\rho^2)a_2^2 b_2^2}\begin{pmatrix} b_2^2 & -\rho a_2 b_2 \\ -\rho a_2 b_2 & a_2^2 \end{pmatrix} \qquad i, j = 1, 2\\
&g_{ij}^{-1} = \begin{pmatrix} a_2^2 & \rho a_2 b_2 \\ \rho a_2 b_2 & b_2^2 \end{pmatrix}\\
&g = det\,(g_{ij}) = \frac{1}{(1-\rho^2)a_2^2 b_2^2}
\end{aligned} \tag{1.20}$$

thus we can write the equation (1.19) in more compact and elegant way:

$$\Delta = g^{-1/2}\partial_i\left(g^{1/2}g_{ij}\partial_j\right), \tag{1.21}$$

$$\frac{\partial f}{\partial t} + \left(\frac{1}{2}\Delta + V\right)f = rf,$$

where V is a first-order operator. Therefore, since the matrix $g_{i,j}$ is symmetric exists its diagonal form, that we call $h_{i,j}$, and we can rewrite the equation (1.19) as follows:

$$\Delta = g^{-1/2}\partial_i \left(g^{1/2} h_{ij} \partial_j \right).$$

At this point we can transform the second-order term of the PDE (1.19), in its diagonal form:

$$\frac{\partial f_1}{\partial t} + \left(\frac{1}{2} g^{-1/2}\partial_i \left(g^{1/2} h_{ij} \partial_j \right) + V \right) f_1 = r f_1,$$

where f_1 is the new function written as function of the new variables, variables with respect to which $g_{i,j}$ becomes the diagonal matrix $h_{i,j}$.

We are going to use this approach in Chapter 4 and in Chapter 5, in order to compute the fair price of options in the Heston model and in the SABR model.

Chapter 2

The Black-Scholes Model

In this chapter we briefly introduce the basic principles on which it is possible to build a model in continuous time under the assumption of perfect markets. The Black-Scholes model has been one of the first models of this kind, and it has both a historical and conceptual importance. In fact, it represents the benchmark for PDE pricing methods in quantitative finance. Therefore, in order to derive the Black-Scholes PDE and its transformation in canonical form, we introduce the notions of arbitrage, of perfect markets, of completeness, of replicant portfolios and some useful variable transformations.

2.1 Arbitrage Arguments

An *arbitrage* is the simultaneous purchase and sale of assets, by means of which one can obtain a profit with no risk. The classic example of arbitrage is the following. Assume that we have two assets A and B with the same price, and that the asset B will be certainly worth more than asset A at a given maturity T. In this case, the arbitrageur will buy B and sell A at the same time, thus earning a sure profit equal to difference at maturity T between the values of B and A. The concept of arbitrage is fundamental to build a market model which makes sense. In an efficient market, indeed, there should not be arbitrage opportunities. Then, we are going to assume absence of arbitrage throughout the whole Thesis. We also make the following assumptions: both the borrowing interest rate and the lending interest rate are equal, short selling is allowed with no costs, the assets and options are infinitively divisible, and there is are transaction costs of any kind. When all these assumptions are verified, we say that we are in a *perfect market*.

2.1.1 Black-Scholes

We assume that in the market there are two assets, a risk-free bond B_t with constant interest rate r and a stock S_t. Define a portfolio as a couple of coefficients (α, β) representing the quantities invested in each. We indicate by V_t the value of the portfolio:

$$V_t = \alpha S_t + \beta B_t. \tag{2.1}$$

The absence of arbitrage opportunities is equivalent to assume that all strategies in which we have risk-free portfolios have the same rate of return r.

Let us elaborate on this last point. Suppose that V is the value of portfolio and that during a time step dt the return of the portfolio dV is risk-free.
If we have:

$$dV > rV dt,$$

then an arbitrageur could make a risk-free profit $dV - rV dt$ during the time step dt by borrowing an amount V from a bank and investing it in the portfolio. Conversely, if

$$dV < rV dt,$$

then the arbitrageur can assume a short position and invest V in a bank and get a net income $rV dt - dV$ during the time step dt without taking any risk. Thus only when

$$dV = rV dt,$$

the arbitrage condition is verified and it is impossible to realise a profit without taking any risk.

We now introduce the concept of *financial derivative*. In financial terms, a derivative is a financial instrument or, more simply, an agreement between two people or two counterparts that has a value determined by the price of some other asset which is called the underlying. It is then a financial contract whose value is linked to the future price movements of the underlying asset, which can be a share, a currency or any other financial asset (including another derivative). There are many kinds of derivatives, with the most notable being swaps, futures, and options. However, since a derivative can be placed on any sort of security, the scope of all derivatives is potentially endless. In this Thesis, we define a derivative as an agreement between two parties that is contingent on a future outcome of the underlying S_t. Thus, we will define a derivative via its pay-off function $\Phi(S_T)$ at maturity T.

Let f denote the value of an option that depends on the value of the underlying asset S_t and on time t, i.e., $f = f(S,t)$. A market model is said to be a *complete market* if, for every derivative security, there exists at least a portfolio that reproduces its value at any time (for more details on this concept and asset pricing theorems, see Ingersoll, pp. 45-46). One can prove, that the Black-Scholes model we are dealing with in this Chapter is complete (see Ingersoll, pp. 304). Assume that in an infinitesimal time step dt, the underlying asset pays out a dividend $Sqdt$, where q is a constant known as the *dividend yield*. Further, assume that S_t follows a geometric Brownian motion with constants μ and σ. We then have the following market model:

$$\begin{aligned} &dS_t = \mu S_t dt + \sigma S_t dW_t, \\ &dB_t = rB_t dt, \\ &f(S_T, T) = \Phi(S_T), \end{aligned} \tag{2.2}$$

where W_t is a standard Brownian motion. According to Ito's lemma, the random walk followed by f is given by

$$df = \frac{\partial f}{\partial S} dS + \left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} \right) dt. \tag{2.3}$$

Hence we require f to have at least one t derivative and two S derivatives.

Now we can construct a portfolio (or strategy) that hedges the value of derivative security f at any time. We consider the equation (2.1) and suppose that our portfolio receives $Sqdt$ for every asset held, the earnings for the owner of the portfolio during the time step dt is:

$$dV = \alpha dS + \alpha Sqdt + r\beta B_t dt, \tag{2.4}$$

and by completeness we have $dV = df$. Using (2.3), we find the Black-Scholes partial differential equation:

$$\begin{aligned} &\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + (r-q)S\frac{\partial f}{\partial S} = rf, \quad t \in [0,T], \quad S \in [0,+\infty) \\ &f(S,T) = \phi(S), \end{aligned} \tag{2.5}$$

in which we have chosen $\alpha = \frac{\partial f}{\partial S}$ in order to eliminate the random component. The key idea to derive this equation is then to eliminate the uncertainty. The linear differential operator is given by

$$\frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r-q)S\frac{\partial}{\partial S} - r$$

and it has a financial interpretation as a measure of the difference between the return on hedged option portfolio

$$\frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} - qS\frac{\partial}{\partial S},$$

and the return on a bank deposit

$$r\left(1 - S\frac{\partial}{\partial S}\right).$$

Although the difference between the two returns is identically zero for options in European style, it is different from zero for options in American style. In this Thesis, we are going to consider only European options, or options that can be reduced to the European case, such as double-barrier options.

Remarks From the Black-Scholes equation (2.5), we know that the parameter μ in (2.2) does not affect the option price, i.e., the option price determined by this equation is independent of the average return rate of an asset price per unit time. Besides, from the derivation the Black-Scholes equation (2.5), we know that this partial differential equation holds for any option (or portfolio of options) whose value depends only on S and t.

In order to determine a unique solution of the Black-Scholes equation, the solution at the expiry, $t = T$, needs to be given. These conditions are called the final conditions for the partial differential equation.

2.1.2 Black-Scholes Equation in its Canonical form

We introduce some transformations by which we reduce the Black-Scholes equations to the heat equation, since Green's function for the heat equation has a known analytical expression.

The transformations of variables that turn the equation of Black-Scholes (2.5) into a heat equation, for constant volatility and constant interest rate, read as follows

$$\begin{aligned}
&Y = \ln S + \left(r - q - \frac{1}{2}\sigma^2\right)(T-t),\\
&\tau = \frac{1}{2}\sigma^2 (T-t),\\
&f(S,t) = e^{-r(T-t)}F(Y,\tau).
\end{aligned} \tag{2.6}$$

Substituting the relations (2.6) in the equation of Black-Scholes (2.5), we get the canonical form of a parabolic kind PDE:

$$\frac{\partial F}{\partial \tau} = \frac{\partial^2 F}{\partial Y^2}, \qquad Y \in (-\infty, \infty), \qquad t \in [0, T]. \tag{2.7}$$

In more general cases, we can have (r, q, σ) as time dependent parameters. In these cases the transformation that changes the Black-Scholes equation into a heat equation (Canonical form of parabolic PDE) is given by:

$$\begin{aligned} Y &= \ln S + \int_t^T \left(r(s) - q(s) - \frac{1}{2}\sigma^2(s) \right) ds, \\ \tau &= \frac{1}{2} \int_t^T \sigma^2(s) ds, \\ f(S, t) &= e^{-\int_t^T r(s)ds} F(Y, \tau). \end{aligned} \tag{2.8}$$

In Chapter 4, we will illustrate suitable coordinate transformations for market models with stochastic volatility, which will start from geometric considerations devised to simplify our problem. This will come at the cost of an approximation of the pay-off function $\phi(S_T)$ that will reduce the PDE to one easier to solve.

Chapter 3

Spectral methods in Finance

In this Chapter we are going to discuss techniques for pricing Double Barrier option. We are going to compare the computational efficiency of the following numerical methods, which are routinely used in finance:
1) Laplace Transform method,
2) expansion of normal distribution,
3) spectral expansion by Fourier series,
4) Monte-Carlo simulation.

3.1 Pricing Double Barrier Options

Barrier Options belong to the class of Exotic Options. These are usually traded between companies and banks and are not quoted on an exchange. In this case, we usually say that they are traded in the over-the-counter market (OTC). Most Exotic Options are quite complicated, and their final values depend not only on the asset price at expiry but also on the asset price at previous times. They are determined by part or the whole of the path of the asset price during the life of the option. These options are called path-dependent Exotic Options. Over the time, several papers have studied the issue of evaluating the price of Barrier Options and Double-Barrier Options in the Black-Scholes market model.

The Barrier options or the Double-Barrier Options can be of two kinds: **knock-out**, or **knock-in**. They are options that either become worthless or exercised if the underlying asset value reaches the level, so-called Barrier level. Thus a **down-and-out** Call is identical to a European Call with the additional provision that the contract is cancelled if the underlying asset price hits a pre-specified lower barrier level. An **up-and-out** Call is the same, except that contract is cancelled when the underlying asset price first reaches a pre-specified upper barrier level. One can repeat the same reasoning for the **down-and-out** and **up-and-out** Put options. **Knock-in** options are complementary to the **knock-out** options. **Knock-out** Double-Barrier Options are cancelled when the underlying asset first reaches either the upper or lower barrier, contrariwise in the case of **knock-in** Double-Barrier.

In (1992) Kunitomo and Ikeda obtain a pricing formulas expressing the prices of Double-Barrier knock-out Calls and Puts, by infinite series of normal probabilities. Later in (1996) Geman and Yor using the probabilistic methods, derive an expressions for the Laplace transform of the Double-Barrier Option price. Taleb (1997) discusses practical issues with trading and hedging double-barrier options. In (2000) Schroder inverts the Laplace transform analytically using the Cauchy Residue Theorem, expresses the resulting trigonometric series in terms of Theta functions, and studies its convergence and numerical properties. Again, in (2000) Pelsser considers several variations on the basic Double-Barrier knock-out options, including binary Double-Barrier options and expresses their pricing formulae in terms of trigonometric series. In last, Linetsky (2003) by using the spectral method obtains the fair price of a Double-Barrier options, studying the spectrum properties of the Black-Scholes infinitesimal operator.

In the present work, we are going to show the Linetsky idea comparing numerically his method with those obtained by above indicated authors. Our contribution is that to study the computational complexity of the problem, and to argue that the spectral method is the "best" numerical method.

We assume to be in a Black-Scholes market model, with constant continuously compounded risk free interest rate r, in which the underlying asset S_t follows a geometric Brownian motion with constant volatility σ and dividend yield q, and there is a derivative security with payoff $f(S,T)$:

$$\begin{aligned} &dS_t = \mu S_t dt + \sigma S_t dW_t \qquad S \in [0,+\infty), \\ &dB_t = rB_t dt, \\ &f(S,T) = \Phi(S_T). \end{aligned}$$

Then, according to the standard option pricing theory (see, eg. Duffie, 1996), the **knock-out** Double Barrier Call value at the contract inception $t = 0$ is given by the discounted risk-neutral expectation of its pay-off at expiration $t = T$:

$$C(T,S,K,H,L) = e^{-rT}\mathbb{E}_{\mathbb{Q}}\left[(S_T - K)^+ \times \mathbf{1}_{\{\forall t \in [0,T]: L < S_t < H}|S_0\right].$$

In other words, if the value of the underlying asset S hits the two barriers, the lower L or the upper H, at any point in time, the pay-off of the option is zero.

In what follows we use the PDE formalism, thus we replace S_t with S. The PDE for pricing **knock-out** Double Barrier Call is given by the following Cauchy-Dirichlet's problem:

$$\begin{aligned} &\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + (r-q)S\frac{\partial f}{\partial S} - rf = 0, \\ &S \in [L,H], \qquad t \in [0,T], \\ &f(T,S) = (S-K)^+ \times \mathbf{1}_{\{\forall t \in [0,T]: L < S < H\}}, \\ &f(t,L) = 0, \\ &f(t,H) = 0, \end{aligned} \tag{3.1}$$

in which we can restrict our domain to the interval $[L,H]$, since our pay-off is zero for $S < L, S > H$. The PDE (3.1) verifies the Hilbert-Schmidt Theorem, and by this one we are able to compute its solution as series expansion of its eigenfunctions (see Chapter 1 for theoretical details).

Through the transformation of variables described in Chapter 2, the Black-Scholes equation (3.1) assumes the canonical form:

$$\frac{\partial F}{\partial \tau} = \frac{\partial^2 F}{\partial Y^2}, \tag{3.2}$$

$$\begin{aligned} &Y \in [A,B], \qquad \tau \in \left[0, \frac{1}{2}\sigma^2(T-t)\right], \\ &F(A,\tau) = 0, \qquad F(B,\tau) = 0, \\ &F(Y,0) = (e^Y L - k)^+ \times \mathbf{1}_{\{\forall t \in [0,T]: A < Y < B\}}, \end{aligned}$$

where $A(t) = (\ln L + \theta(T-t))$, $B(t) = (\ln H + \theta(T-t))$ and $\theta = (r - q - \frac{1}{2}\sigma^2)$.
It is clear that in an analogous way we can write the price of a Knock-in Option.

Pricing Theorem Under the Black-Scholes framework the arbitrage-price of a Knock-out Call Double Barrier Option is given by:

$$
\begin{aligned}
f(S,t) &= \int_L^H dS' e^{-r(T-t)} (S'-K)^+ \times \mathbf{1}_{\{\forall t \in [0,T]: L \le S \le H\}} G(S', S, t) \\
&= \int_L^H \frac{dS'}{S'} e^{-r(T-t)} (S'-K)^+ \times \mathbf{1}_{\{\forall t \in [0,T]: L \le S \le H\}} \quad \times \\
&\frac{2}{\ln(H/L)} \sum_{n=-\infty}^{\infty} e^{-\frac{(n\pi)^2}{(\ln(H/L))^2}\left(\frac{1}{2}\sigma^2(T-t)\right)} \sin n\pi \left(\frac{\ln(S'/L)}{\ln(H/L)}\right) \sin n\pi \left(\frac{\ln(S/L)}{\ln(H/L)}\right)
\end{aligned}
\tag{3.3}
$$

where $S' = e^{\xi} L$ is equal to S at the time $t = T$, for every underlying asset value $S \in [L, H]$;

Proof:
let the PDE in canonical form of parabolic kind of second order be given, with the following boundary conditions:

$$
\frac{\partial F}{\partial \tau} = \frac{\partial^2 F}{\partial Y^2}, \tag{3.4}
$$

$$
\begin{aligned}
& Y \in [A, B], \qquad \tau \in \left[0, \overline{T}\right], \qquad \overline{T} = \frac{1}{2}\sigma^2 (T-t), \\
& F(A, \tau) = 0, \qquad F(B, \tau) = 0, \\
& F(Y, 0) = (e^Y - K)^+,
\end{aligned}
$$

where we have also chosen the initial condition as $(e^Y - K)^+$. Now we can use the separation method of variables and rewrite the function $F(Y, \tau) = U(Y) \times W(\tau)$. In this way the PDE (3.1) becomes a system of two ODEs in which one is a linear differential equation of first order with respect to t, and the other is the Sturm-Liouville problem of the second order:

$$
\frac{\partial U(Y) W(\tau)}{\partial \tau} = \frac{\partial^2 U(Y) W(\tau)}{\partial Y^2}. \tag{3.5}
$$

Thus we have

$$
U(Y) \frac{\partial W(\tau)}{\partial \tau} = W(\tau) \frac{\partial^2 U(Y)}{\partial Y^2}, \tag{3.6}
$$

$$
\frac{1}{W(\tau)} \frac{\partial W(\tau)}{\partial \tau} = \frac{1}{U(Y)} \frac{\partial^2 U(Y)}{\partial Y^2}. \tag{3.7}
$$

Therefore the left hand side depends only on the variable τ and the right hand side depends only on the variable Y; then we can set the left hand side and the right hand side equal to a negative constant [M. A. Al-Gwaiz, Sturm-Liouville Theory and its Applications, 2008]:

$$
\frac{1}{W(\tau)} \frac{dW(\tau)}{d\tau} = -\lambda^2 \tag{3.8}
$$

$$\frac{1}{U(Y)}\frac{d^2U(Y)}{dY^2} = -\lambda^2. \tag{3.9}$$

Note that we have chosen a negative constant $-\lambda^2$, because it makes the function $F(Y, \tau)$ bounded. Solving the above system of ODEs, we have:

$$W(\tau) = W(0)e^{-\lambda^2\tau}, \tag{3.10}$$

$$\frac{d^2U(Y)}{dY^2} + \lambda^2 U(Y) = 0. \tag{3.11}$$

Equation (3.8) is solved and its solution is obtained from equation (3.10). The equation (3.11) plus the boundary conditions is a Sturm-Liouville problem:

$$\frac{d^2U(Y)}{dY^2} + \lambda^2 U(Y) = 0, \qquad Y \in [A, B], \tag{3.12}$$

$$U(A) = 0, \qquad U(B) = 0.$$

In order to change the interval of the definition and to simplify the computation, we introduce the following variable:

$$Y = \eta + A \qquad \Longrightarrow \qquad \eta = Y - A$$

Hence, we have $U(Y) = U(\eta + L) = \aleph(\eta)$, where $\eta \in [0, l]$ and $l = B - A$,

$$\frac{dU(Y)}{dy} = \frac{d\aleph}{d\eta}, \qquad \frac{d^2U(Y)}{dY^2} = \frac{d^2\aleph}{d\eta^2}.$$

Equation (3.9) is now defined over the interval $[0, l]$

$$\frac{d^2\aleph(\eta)}{d\eta^2} + \lambda^2\aleph(\eta) = 0, \qquad \eta \in [0, l],$$

$$\aleph(0) = 0, \qquad \aleph(l) = 0.$$

The solution of the equation is given by the following relation:

$$\aleph(\eta) = \sum_{n=1}^{+\infty}\left[sin\frac{n\pi\eta}{l}\right], \tag{3.13}$$

where α_n is equal to zero for the boundary condition $\aleph(0) = 0$. At this point, after we have substituted the variable Y with η, thus defining $F(Y, \tau) = \overline{F}(\eta, \tau)$ we can write the solution of the heat equation (5.7) as follows:

$$\frac{\partial\overline{F}}{\partial\tau} = \frac{\partial^2\overline{F}}{\partial\eta^2}, \tag{3.14}$$

$$\eta \in [0, l], \qquad \tau \in \left[0, \overline{T}\right], \qquad \overline{T} = \frac{1}{2}\sigma^2(T - t),$$

$$\overline{F}(0, \tau) = 0, \qquad F(l, \tau) = 0,$$

$$\overline{F}(\eta, 0) = (e^{\eta+A} - K)^+.$$

Remembering that $\overline{F}(\eta, \tau) = \aleph(\eta)W(\tau)$, we have:

$$\overline{F}(\eta, \tau) = \sum_{n=1}^{+\infty} e^{-\left(\frac{n\pi}{l}\right)^2 \tau} \left[c_n \sin\left(\frac{n\pi\eta}{l}\right)\right], \tag{3.15}$$

and this is true if and only if:

$$c_n = \frac{2}{l} \int_0^l d\xi (e^{\xi+A} - K)^+ \sin\left(\frac{n\pi\xi}{l}\right), \tag{3.16}$$

$$\overline{F}(\eta, \tau) = \sum_{n=1}^{+\infty} e^{-\left(\frac{n\pi}{l}\right)^2 \tau} \left[\frac{2}{l} \int_0^l d\xi (e^{\xi+A} - K)^+ sin\left(\frac{n\pi\xi}{l}\right) \sin\left(\frac{n\pi\eta}{l}\right)\right], \tag{3.17}$$

$$\overline{F}(\eta, \tau) = \int_0^l d\xi (e^{\xi+A} - K)^+ \left[\frac{2}{l} \sum_{n=1}^{+\infty} e^{-\left(\frac{n\pi}{l}\right)^2 \tau} \sin\left(\frac{n\pi\xi}{l}\right) \sin\left(\frac{n\pi\eta}{l}\right)\right]. \tag{3.18}$$

In order to simplify the above relation we introduce the Green's function:

$$G(\tau, \eta, \xi) = \left[\frac{2}{l} \sum_{n=1}^{+\infty} e^{-\left(\frac{n\pi}{l}\right)^2 \tau} \sin\left(\frac{n\pi\xi}{l}\right) \sin\left(\frac{n\pi\eta}{l}\right)\right], \qquad \eta, \xi \in [0, l],$$

so that we may write, in a very elegant way, the solution of the parabolic PDE in canonical form of the second order, as follows:

$$f(S, t) = \overline{F}(\eta, \tau) = \int_0^l d\xi (e^{\xi+A} - K)^+ G(\tau, \eta, \xi), \tag{3.19}$$

and using the Poisson transform, we can write the Green's function in the form of a difference between two normal distributions:

$$\begin{aligned} G(\tau, \eta, \xi) &= \left[\frac{2}{l} \sum_{n=1}^{+\infty} e^{-\left(\frac{n\pi}{l}\right)^2 \tau} \sin\left(\frac{n\pi\xi}{l}\right) \sin\left(\frac{n\pi\eta}{l}\right)\right] \\ &= \frac{1}{2\sqrt{\pi\tau}} \sum_{n=-\infty}^{+\infty} \left[e^{-\frac{(\eta-\xi+2nl)^2}{4\tau}} - e^{-\frac{(\eta+\xi+2nl)^2}{4\tau}}\right]. \end{aligned} \tag{3.20}$$

3.2 Numerical Implementation and Computational Complexity

The problem of computing the transition density $G(\tau, \eta, \xi)$ is a classic one (see Feller, 1971, pp.341-3 and pp. 478, or Cox and Miller, 1965) and there exist at least three different ways to write it explicitly.

The first way is an inverse Laplace transform of the resolvent $K(s,\eta,\xi)$:

$$K(s,\eta,\xi)=\frac{e^{-\sqrt{2s}|\eta-\xi|}}{\sqrt{2s}}+\frac{e^{\sqrt{2s}(\eta-\xi)}+e^{-\sqrt{2s}(\eta-\xi)}-e^{\sqrt{2s}(\eta+\xi)}-e^{-\sqrt{2s}(\eta+\xi-2l)}}{\sqrt{2s}(e^{\sqrt{2s}l}-1)} \tag{3.21}$$

where

$$K(s,\eta,\xi)=\int_0^{+\infty}e^{-s\tau}G(\tau,\eta,\xi)d\tau. \tag{3.22}$$

The function K solves the ordinary differential equation

$$\frac{1}{2}\frac{d^2K}{d\eta^2}-sK=-\delta(\eta-\xi),$$

with boundary conditions:

$$K(s,0,\xi)=K(s,l,\xi)=0.$$

The second way, as we have seen before in Eq. (3.20), is a series of normal densities:

$$G(\tau,\eta,\xi)=\frac{1}{2\sqrt{\pi\tau}}\sum_{j=-\infty}^{+\infty}\left[e^{-\frac{(\eta-\xi+2jl)^2}{4\tau}}-e^{-\frac{(\eta+\xi+2jl)^2}{4\tau}}\right]. \tag{3.23}$$

Finally, we write the Green function as a Fourier series:

$$G(\tau,\eta,\xi)=\left[\frac{2}{l}\sum_{j=1}^{+\infty}e^{-\left(\frac{\omega_j^2\tau}{2}\right)}\sin\left(\omega_j\eta\right)\sin\left(\omega_j\xi\right)\right],\qquad \omega_j=\frac{j\pi}{l}. \tag{3.24}$$

Geman and Yor (1996), following the equation (3.22), compute the Laplace transform of the resolvent (4.30) (for any complex number s with $Re(s)>\alpha^2/2$):

$$\int_0^{+\infty}e^{\alpha\xi}K(s,\eta,\xi)d\xi=g_1(s)+g_2(s), \tag{3.25}$$

where

$$g_1(s)=\frac{\left[e^{\alpha u+(l-\eta)\sqrt{2s}}-e^{\alpha\kappa+(\kappa-\eta)\sqrt{2s}}-e^{\alpha l+(l+\eta)\sqrt{2s}}+e^{\alpha\kappa+(\kappa+\eta)\sqrt{2s}}\right]}{\sqrt{2s}\left(2l\sqrt{2s}-1\right)\left(\sqrt{2s}+\alpha\right)}$$
$$+\frac{\left[e^{\alpha\eta+(\eta-\kappa)\sqrt{2s}}+e^{\alpha l+(l-\eta)\sqrt{2s}}-e^{\alpha l+(\kappa-l)\sqrt{2s}}-e^{\alpha\kappa+(2l-\kappa-\eta)\sqrt{2s}}\right]}{\sqrt{2s}\left(2u\sqrt{2s}-1\right)\left(\sqrt{2s}-\alpha\right)},$$

$$g_2(s) = \begin{cases} \frac{1}{\sqrt{2s}(\sqrt{2s}+\alpha)}\left(e^{\alpha\eta} - e^{\alpha\kappa+(\kappa-\eta)\sqrt{2s}}\right) \\ \qquad +\frac{1}{\sqrt{2s}(\sqrt{2s}-\alpha)}\left(e^{\alpha\eta} - e^{\alpha l+(l-\eta)\sqrt{2s}}\right), & \kappa \le \eta \le l \\ \frac{1}{\sqrt{2s}(\sqrt{2s}-\alpha)}\left(e^{\alpha\eta+(\eta-\kappa)\sqrt{2s}} - e^{\alpha l+(l-\eta)\sqrt{2s}}\right), & 0 \le \eta \le \kappa \end{cases}$$

(for details see Geman and Yor, 1996), but this method, from the computational point of view, is not very efficient, because it needs to compute the inverse Laplace transform numerically.

Alternatively, substituting the expansion (3.23) in (3.19) and performing the integration term-by-term leads to the representation as an infinite sum of normal probabilities (see Kunitomo and Ikeda, 1992).

Finally, an alternative representation is obtained by integrating the Fourier series (5.18) term-by-term (see Pelsser, 2000; Linetsky, 2003). where $\omega_n = \frac{n\pi}{u}$. The identity between (3.23) and (5.18), is a classic example of the Poisson summation formula (Feller, 1971). However, series (3.23) and (5.18) have very different numerical convergence properties, in fact the latter converges quicker. See Schroder (2000) for details. As we are going to show below, the technique proposed by Pelsser (2000) and Linetsky (2003) is faster than those proposed by Geman Yor (1996) and Kunitomo Ikeda (1992). The method of computing the arbitrage price of double barrier options through "Fourier expansion" is also simple to implement. In fact it is possible to write down a simple algorithm, in order to get the correct value for Double-Barrier Options just by summing eigenfunctions.

Anther popular method in Quantitative Finance is Monte-Carlo simulations; to evaluate the price of Double Barrier options by Monte-Carlo, we compute approximately five thousand integrals (see Geman Yor, 1996) and the standard deviation of the Monte-Carlo price is computed on a sample of 200 evaluation. Therefore, Monte-Carlo method is more expansive than others, and from computational complexity point of view it is not very efficient method.

The computational complexity of spectral expansion depends to the coefficients c_n in (3.16) that are the weights in the sum (3.27), and these integrals are computed numerically:

$$c_n = \frac{2}{\ln(H/L)} \int_0^{\ln(H/L)} d\xi (e^{\xi} L - K)^+ \sin n\pi \left(\frac{\xi}{\ln \frac{H}{L}} \right). \tag{3.26}$$

The price is given by following relation:

$$f(S,t) = e^{-\int_t^T r(s)ds} \sum_{n=-\infty}^{+\infty} e^{-\left(\frac{n\pi}{\ln \frac{H}{L}}\right)^2 \frac{1}{2}\sigma^2 (T-t)} \left[c_n \sin n\pi \left(\frac{\ln(S/L)}{\ln(H/L)} \right) \right]. \tag{3.27}$$

Note that $e^{-\left(\frac{n\pi}{\ln \frac{H}{L}}\right)^2 \frac{1}{2}\sigma^2 (T-t)}$ decreases quickly. Thus, choosing a small number, ϵ, and defining $n(\epsilon)$ by means of:

$$exp\left[-\left(\frac{n\pi}{\ln \frac{H}{L}} \right)^2 \frac{1}{2}\sigma^2 (T-t) \right] = \epsilon, \tag{3.28}$$

we have:

$$\left(\frac{n\pi}{\ln\left(\frac{H}{L}\right)}\right)^2 = \frac{2}{\sigma^2(T-t)}\ln\left(\frac{1}{\epsilon}\right), \tag{3.29}$$

$$n(\epsilon) = \frac{1}{\pi}\ln\left(\frac{H}{L}\right)\sqrt{\frac{2}{\sigma^2(T-t)}\ln\left(\frac{1}{\epsilon}\right)}. \tag{3.30}$$

Now, in order to study the computational complexity of our problem, we define

$$a_n = e^{-\left(\frac{n\pi}{\ln(H/L)}\right)^2 \frac{1}{2}\sigma^2(T-t)} c_n, \tag{3.31}$$

so that we have

$$a_n = e^{-\left(\frac{n\pi}{\ln(H/L)}\right)^2 \frac{1}{2}\sigma^2(T-t)} \frac{2}{\ln(H/L)} \int_0^{\ln(H/L)} d\xi (e^{\xi}L - K)^+ \sin n\pi\left(\frac{\xi}{\ln(H/L)}\right), \tag{3.32}$$

the value of the integral is a function of n and it results to be

$$\int_0^{\ln(H/L)} d\xi (e^{\xi}L - K)^+ \sin n\pi\left(\frac{\xi}{\ln(H/L)}\right) = \int_{\ln(K/L)}^{\ln(H/L)} d\xi (e^{\xi}L - K) \sin n\pi\left(\frac{\xi}{\ln(H/L)}\right),$$

$$= (-1)^{n+1}K\left(\frac{n^2\pi^2(\ln(H/L) - n\pi)}{L\ln(H/L)\left[(\ln H/L)^2 + (n\pi)^2\right]}\right).$$

Thus we have:

$$c_n = (-1)^{n+1}K\left(\frac{2n^2\pi^2(\ln(H/L) - n\pi)}{L(\ln H/L)^2\left[(\ln H/L)^2 + (n\pi)^2\right]}\right),$$

and

$$a_n = (-1)^{n+1}K\left(\frac{2n^2\pi^2(\ln(H/L) - n\pi)}{L(\ln H/L)^2\left[(\ln H/L)^2 + (n\pi)^2\right]}\right) e^{-\left(\frac{n\pi}{\ln(H/L)}\right)^2 \frac{1}{2}\sigma^2(T-t)}.$$

Let b_n be:

$$b_n = (-1)^{n}K\left(\frac{2n^2\pi^2}{L(\ln H/L)^2\left[(\ln H/L)^2 + (n\pi)^2\right]}\right) e^{-\left(\frac{n\pi}{\ln(H/L)}\right)^2 \frac{1}{2}\sigma^2(T-t)}.$$

We can bound a_n with b_n, in other words $a_n < b_n$ and since $lim_{n\to+\infty} b_n = 0$, this proves that also a_n goes to zero for $n \to +\infty$. Therefore if the following condition: $\left(\frac{n\pi}{\ln(H/L)}\right)^2 \frac{1}{2}\sigma^2(T-t) \geq 1$ is satisfied, the coefficients a_n goes to zero very quickly. In practice, it is enough to compute only the terms up to $n = 3$ to get an accurate solution. If the above condition is not satisfied, more coefficients are needed. Figures 3.1, 3.2, 3.3, which are for three different choices of the parameters, show that a_n converges to zero in very few steps. In all figures, we have chosen $\epsilon = 10^{-4}$. This shows that the spectral method is very efficient. Let us observe that, the larger the lifetime of the option, the smaller is the value of the number $n(\epsilon)$. Hence for a fixed ϵ, we can compute approximately the value of $f(S,t)$ using the partial sum of $n(\epsilon)$ eigenfunctions. By the analytical formula, it is possible to manage the accuracy by choosing the number of eigenfunctions. The obtained results are shown in the Table provided below. Our results are compatible with those obtained by Geman Yor(1996), Kunitomo Ikeda(1992) and with those obtained by using the Monte-Carlo method. The advantage of the Spectral method, is that it decreases the computational complexity. Hence, when it is possible to write the price in terms of a series expansion, it is always worth to do so. The Fourier expansion introduced by Pelsser requires a number of operations which is two orders of magnitude less than the Monte-Carlo method and one order of magnitude less than the Laplace transform method used by Geman and Yor (1996).

Table 3.1: $S(0) = 2$, $(T-t) = 1 - year$

Numerical Techniques	σ	r	k	H	L	f(X, T-t)
Monte Carlo price (st, dev. 0.003)	0.2	0.02	2	2.5	1.5	0.0425
	0.5	0.05	2	3	1.5	0.0191
	0.5	0.05	1.75	3	1	0.0772
Geman-Yor price	0.2	0.02	2	2.5	1.5	0.0411
	0.5	0.05	2	3	1.5	0.0178
	0.5	0.05	1.75	3	1	0.0762
Kunitomo-Ikeda	0.2	0.02	2	2.5	1.5	0.0411
	0.5	0.05	2	3	1.5	0.0179
	0.5	0.05	1.75	3	1	0.0762
Fourier	0.2	0.02	2	2.5	1.5	0.0412
	0.5	0.05	2	3	1.5	0.0178
	0.5	0.05	1.75	3	1	0.0754

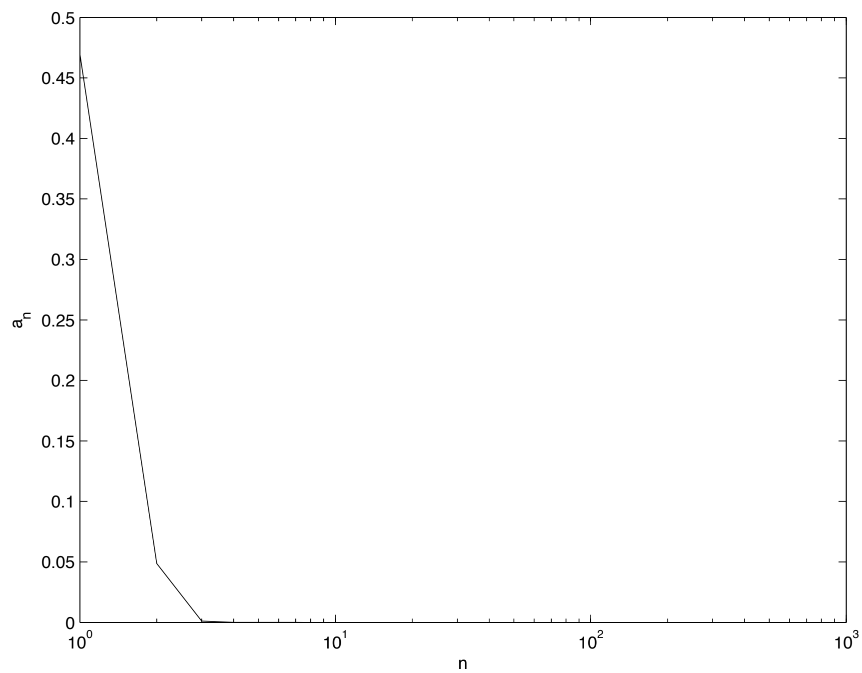

Figure 3.1: Shows a_n as a function of n in the case $\sigma = 0.2, r = 0.02, K = 2, H = 2.5, L = 1.5$.

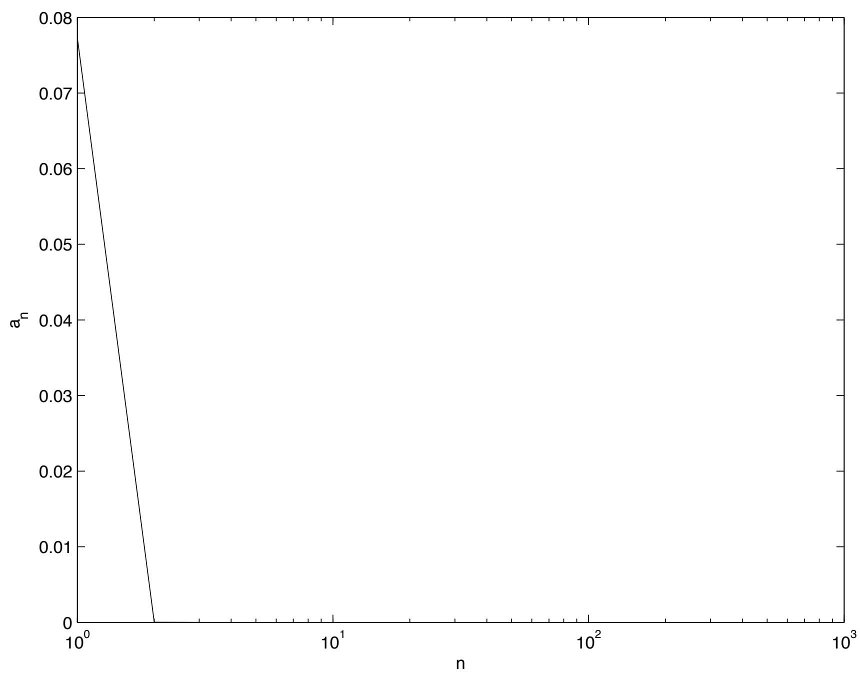

Figure 3.2: Shows a_n as a function of n in the case $\sigma = 0.5, r = 0.05, K = 2, H = 3, L = 1.5$.

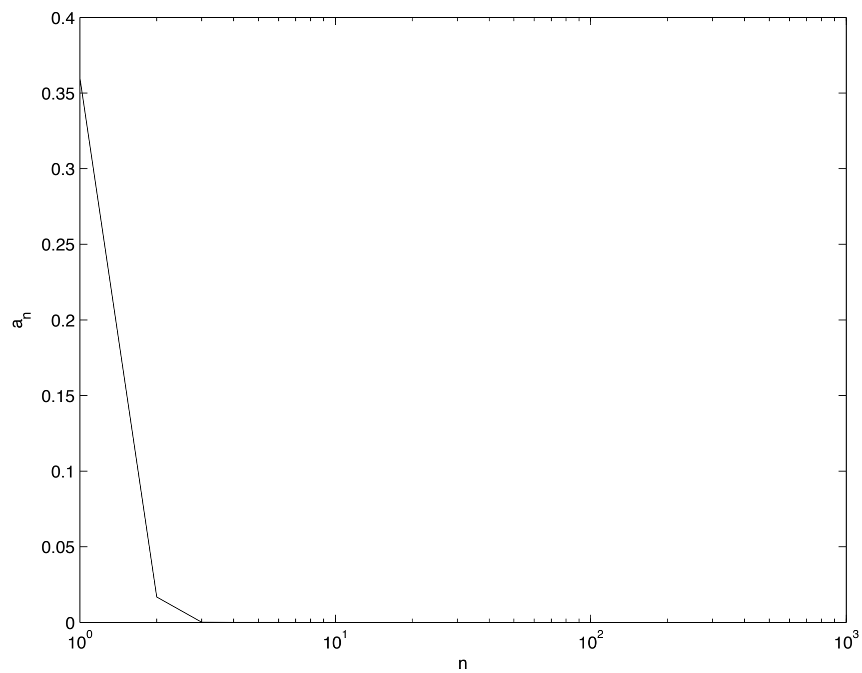

Figure 3.3: Shows a_n as a function of n in the case $\sigma = 0.5, r = 0.05, K = 1.75, H = 3, L = 1$.

Chapter 4

Stochastic Volatility and Geometrical Approximation

In this chapter, we introduce a method that we name Geometrical Approximation (G. A.), by which it is possible to study stochastic volatility market models (as Heston and SABR). The G. A. intends to be an alternative method useful to obtain the approximated price of Vanilla Options.

4.1 Introduction

There are many economic, empirical, and mathematical reasons for choosing a model as Heston (see Cont 2001 for a detailed statistical/empirical analysis). Empirical studies have shown that an asset log-return distribution is non-Gaussian. The Heston model is characterised by heavy tails and high peaks at zero (leptokurtic). There is also empirical evidence and economic arguments that suggest that equity returns and implied volatility are negatively correlated (also termed the leverage effect). This departure from normality is a plague of the Black-Scholes-Merton model, in contrast, of Heston model in which this phenomenon is expected.

The assumption of constant volatility is not reasonable, since we require different values for the volatility parameter for different strikes and different expiries to match market prices. The volatility parameter that is required in the Black-Scholes formula to reproduce market prices is called the implied volatility. This is a critical internal inconsistency, since the implied volatility of the underlying should not be dependent on the specifications of the contract.

There are two prominent ways of working around this problem, namely, local volatility models and stochastic volatility models. Earlier examples of stochastic volatility models are: Hull and White (1987). Scott (1987, 1991) Wiggins (1987) Stein and Stein (1991), Heston (1993). Typically, the stock price dynamics and the volatility dynamics are governed by two (possibly correlated) Brownian motions, and thus the price risk and the volatility risk are partially separated (they may even be "orthogonal" if the two driving Brownian motions are independent as, for instance, in the case of Hull White 1987 model). It is an empirical truth that the value of the stock price and the volatility risk are closely tied to each other. The latter property is an undesirable features if we wish to use the Black-Scholes model to hedge volatility risk.

4.2 Stochastic volatility models

In a continuous-time framework, the generic form of stochastic volatility market models is the following:

$$\begin{aligned} dS_t &= \mu(S_t,t)dt + \sigma_t S_t dW_t^{(1)} \\ d\sigma_t &= a(\sigma_t,t)dt + b(\sigma_t,t)dW_t^{(2)}, \end{aligned} \tag{4.1}$$

with the stochastic volatility σ_t (also known as the instantaneous volatility or the spot volatility). $W^{(1)}$ and $W^{(2)}$are standard one-dimensional Brownian motions defined on some filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$, with the cross-variation satisfying $dW_t^{(1)} dW_t^{(2)} = \rho dt$ for some constant $\rho \in [-1, 1]$. Recall that the Brownian motions $W^{(1)}$ and $W^{(2)}$ are mutually independent if and only if they are uncorrelated, that is, when $\rho = 0$.

For the **Asset Pricing Theorems** we have to that the drift of S_t processes is rS_t under the risk-neutral probability measure, and we obtain this by using Girsanov and Radom Nykodim Theorems. We indicate in what follows the probability measure equivalent to $\mathbb{P}$ with $\mathbb{Q}$, and respects at the latter, our model becomes:

$$dS_t = rS_t dt + \sigma_t S_t d\tilde{W}_t^{(1)}, \tag{4.2}$$

with the spot volatility σ satisfying

$$d\sigma_t = \tilde{a}(\sigma_t, t)dt + b(\sigma_t, t)d\tilde{W}_t^{(2)}, \tag{4.3}$$

for some drift coefficient $\tilde{a}_t$. We shall adopt a commonly standard convention

$$\tilde{a}_t(\sigma_t, t) = a(\sigma_t, t) + \lambda(\sigma_t, t)b(\sigma_t, t), \tag{4.4}$$

for some (sufficiently regular) functions $\lambda(\sigma_t, t)$. The presence of the additional term in the drift of the stochastic spot volatility σ under an equivalent martingale measure is an immediate consequence of Girsanov's theorem. The particular form of this term and its selection is an important question. It is usually motivated by practical considerations.

The stochastic volatility models are not complete, and thus a typical contingent claim (such as a European option) cannot be priced by arbitrage arguments only. In other words, the standard replication arguments cannot be any longer applied to most contingent claims. For this reason, the issue of valuation of derivative securities under market incompleteness has attracted considerable attention in recent years, and various alternative approaches to this problem were subsequently developed.

4.2.1 PDE Approach

Under (4.2), (4.3) from Itô's lemma, it is possible to derive under mild additional assumptions, the partial differential equation satisfied by the value function of a European contingent claim, that is a PDE representing a two-dimensional process. For this purpose, one needs first to specify the market price of volatility risk $\lambda(\sigma, t)$. The market price for the risk is associated with the Girsanov transformation of the underlying probability measure leading to a particular martingale measure. Let us observe that pricing of contingent claims using the market price of volatility risk is not preferences-free. Usually, one assumes that the representative investor is risk-averse and has a constant relative risk-aversion utility function.

To illustrate the PDE approach mentioned above, assume that the dynamic of two dimensional diffusion process (S, σ), under a martingale measure, is given by (4.2), (4.3), with Brownian motions $\tilde{W}_t^{(1)}$ and $\tilde{W}_t^{(2)}$ such that $d\tilde{W}_t^{(1)} d\tilde{W}_t^{(2)} = \rho dt$ for some values of $\rho \in [-1, 1]$. Suppose also that both processes S and σ, are nonnegative. Then the price function $f = f(S, \sigma, t)$ of a European contingent claim (see for instance, Garman (1976) or Hull and White (1976)) solves, for a generic stochastic volatility market models, the following PDE:

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \rho\sigma S b(\sigma, t)\frac{\partial^2 f}{\partial S \partial t} + \frac{1}{2}b(\sigma, t)^2 \frac{\partial^2 f}{\partial \sigma^2} + rS\frac{\partial f}{\partial S} + [a(\sigma, t) + \lambda(\sigma, t)b(\sigma, t)]\frac{\partial f}{\partial \sigma} - rf = 0, \tag{4.5}$$

with the terminal condition $\phi(S_T) = f(T, S, \sigma)$ for every $S \in \mathbb{R}_+$ and $\sigma \in \mathbb{R}_+$.
Let us remark once again that we do not claim here that $\mathbb{Q}$ is a unique martingale measure for a given model. Hence, unless volatility-based derivatives are assumed to be among primary assets, the market price of volatility risk needs to be exogenously specified.

4.3 Heston model

The stochastic volatility model, proposed by Heston (1993), assumes that the asset price S_t satisfies the following SDE

$$dS_t = \mu S_t dt + \sqrt{\nu_t} S_t dW_t^{(1)}, \qquad S \in [0, \infty), \tag{4.6}$$

with the instantaneous variance ν_t is also a stochastic process

$$d\nu_t = k(\theta - \nu_t)dt + \alpha\sqrt{\nu_t} dW_t^{(2)}, \qquad \nu \in (0, \infty), \quad k, \theta, \alpha \in \mathbb{R}, \tag{4.7}$$

where $W^{(1)}$ and $W^{(2)}$ are standard one-dimensional Brownian motions defined on filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$, which the cross-variation $\langle W^{(1)}, W^{(2)} \rangle = \rho t$ for some values of $\rho \in [-1, 1]$. The pricing function f and the market price of volatility risk λ are both functions of variables (S, ν, t). The stochastic volatility models have the drawback of needing an exogenous hypothesis, that is the risk price of volatility. Usually in literature this is given by the following process:

$$\lambda(\nu_t, t) = \lambda\sqrt{\nu_t},$$

for some constant λ such that $\lambda\alpha \neq k$; the choice of $\lambda(t, \nu)$ is an important theoretical problem that we discuss later in the next sections. Hence, under a martingale measure $\mathbb{Q}$, equations (4.6), (4.7) become

$$dS_t = rS_t dt + \sqrt{\nu_t} S_t d\tilde{W}_t^{(1)}, \tag{4.8}$$

and

$$d\nu_t = \kappa(\Theta - \nu_t)dt + \alpha\sqrt{\nu_t} d\tilde{W}_t^{(2)}, \tag{4.9}$$

where we set

$$\kappa = (k - \lambda\alpha), \qquad \Theta = \theta k (k - \lambda\alpha)^{-1}, \tag{4.10}$$

and where $\tilde{W}_t^{(1)}$ and $\tilde{W}_t^{(2)}$ are standard one-dimensional Brownian motions respect to $\mathbb{Q}$ such that $\langle d\tilde{W}_t^{(1)}, d\tilde{W}_t^{(2)} \rangle = \rho dt$. It is now easy to see that the pricing PDE for European derivatives in Heston model, by Itô's lemma, has the following form:

$$\frac{\partial f}{\partial t} + \frac{1}{2}\nu S^2 \frac{\partial^2 f}{\partial S^2} + \rho\nu\alpha S \frac{\partial^2 f}{\partial S \partial \nu} + \frac{1}{2}\nu\alpha^2 \frac{\partial^2 f}{\partial \nu^2} + \kappa(\Theta - \nu)\frac{\partial f}{\partial \nu} + rS\frac{\partial f}{\partial S} - rf = 0, \tag{4.11}$$

with the terminal condition $f(S,\nu,T) = \phi(S)$ for every $S \in \mathbb{R}_+$, $\nu \in \mathbb{R}_+$ and $t \in [0,T]$, and with the boundary conditions:

$$
\begin{aligned}
&f(t,S,\nu) = \phi(S), \\
&f(t,0,\nu) = 0, \\
&\frac{\partial f}{\partial S}(t,S,\infty) = 1, \\
&rS\frac{\partial f}{\partial S}(t,S,0) + \kappa\Theta\frac{\partial f}{\partial \nu}(t,S,0) - rf(t,S,0) + \frac{\partial f}{\partial t}(t,S,0) = 0, \\
&f(t,S,\infty) = S.
\end{aligned}
\tag{4.12}
$$

We take here for granted the existence and uniqueness of (nonnegative) solutions S and ν to Heston's SDE. It is common to assume $2K\Theta/\alpha^2 > 1$ (Feller's condition) , so that, if $\nu_0 > 0$ the solution of the SDE (4.9) is strictly positive .

4.4 Pricing methods for the Heston model

In the literature, the alternative methods used to price the European options in the Heston model, can be divided into two categories:

(1) Numerical methods;

(2) Approximation methods.

In the first case, the objective of the analysis is to find the exact price, and the approximation comes from the numerical analysis. In the second case, one starts from on approximated problem in order to find an analytical solution. The method proposed in section 4.4 belongs to this second line of research.

4.4.1 Numerical methods

(a) **Fourier transform method**
The historic solution, the first solution of the Heston's PDE, was in 1993 [57]; where the author computes the solution written in semi-closed form. The solution for PDE (4.11) pricing a European Call options, is the following:

$$C(t,S,\nu) = S\mathbf{P}_1 - Ee^{-r(T-t)}\mathbf{P}_2,$$

where $\mathbf{P}_1$ and $\mathbf{P}_2$ are:

$$\mathbf{P}_j(T, x, \nu; \ln E) = \frac{1}{2} + \frac{1}{\pi}\int_0^{+\infty} Re\left[\frac{e^{-\imath\phi\ln E} f_j(t, x, \nu; \phi)}{\imath\phi}\right] d\phi, \qquad j = 1, 2 \tag{4.13}$$

and f_j is the characteristic function: $f_j(t, x, \nu; \phi) = e^{C_j(T-t;\phi)+D_j(T-t;\phi)\nu+\imath\phi x}$, where $x = \ln S$ (see the Appendix for more details).

$$C_j(T-t;\phi) = r\phi\imath(T-t) + \frac{a}{\alpha^2}\left[(b_j - \alpha\rho\phi\imath + d_j)(T-t) - 2\ln\left(\frac{1 - g_j e^{d_j(T-t)}}{1 - g_j}\right)\right],$$

$$D_j(T-t;\phi) = \frac{b_j - \alpha\rho\phi\imath + d_j}{\alpha^2}\left[\frac{1 - e^{d_j(T-t)}}{1 - g_j e^{d_j(T-t)}}\right],$$

and

$$g_j = \frac{b_j - \alpha\rho\phi\imath + d_j}{b_j - \alpha\rho\phi\imath - d_j},$$

$$d_j = \sqrt{(\alpha\rho\phi\imath - b_j)^2 - \alpha^2(2u_j\phi\imath - \phi^2)},$$

$$a = \kappa\Theta, \quad b_1 = \kappa + \lambda - \alpha\rho, \quad b_2 = \kappa + \lambda, \quad u_1 = \frac{1}{2} \quad u_2 = -\frac{1}{2};$$

where λ is the price of the volatility risk and it is usually chosen equal to zero. The formula (4.13) is not straightforward to implement. The inverse Fourier transform method is a numerical method based on numerical integration techniques; so that using it we have some problems to define the correct domain of integration. When the volatility of volatility α becomes small, integration of the Fourier transform might lose some accuracy [80]. Several solutions have been proposed in the literature, which suggest a robust implementation. The principal integration techniques are: Gauss-Legendre, Gauss-Lobatto or FFT. In the recent years, this techniques have been improved by several works; to name a few: Jaeckel Kahl [62], and Albrecher et al. [2]. Referring to (4.13), Jaekel and Kahl prove that the $Re\left[e^{-\imath\phi\ln E} f_j(t, x, \nu; \phi)/\imath\phi\right]$ have limits on both ends. They suggest to use an adaptive integration algorithm to correct the oscillatory behaviour [58]. Albrecher et al. suggest to transform the characteristic function f_j of the log forward asset price ($F = e^{r(T-t)}S$) as follows:

$$\tilde{f}_j = \mathbb{E}_{\mathbb{Q}}\left[e^{\imath\phi\ln F}\right] = \begin{cases} \tilde{f}_1 = e^{\tilde{C}_1(T-t;\phi)+\tilde{D}_1(T-t;\phi)\nu+\imath\phi\ln F}, \\ \tilde{f}_2 = e^{\tilde{C}_2(T-t;\phi)+\tilde{D}_2(T-t;\phi)\nu+\imath\phi\ln F}, \end{cases}$$

where

$$\begin{cases} \tilde{C}_1(T-t;\phi) = \frac{a}{\alpha^2}\left[(\kappa - \alpha\rho\phi\imath + d_\phi)(T-t) - 2\ln\left(\frac{1-g_\phi e^{d_\phi(T-t)}}{1-g_\phi}\right)\right], \\ \tilde{D}_1(T-t;\phi) = \frac{\kappa-\alpha\rho\phi\imath+d_\phi}{\alpha^2}\left[\frac{1-e^{d_\phi(T-t)}}{1-g_\phi e^{d_\phi(T-t)}}\right], \\ g_\phi = \frac{\kappa-\alpha\rho\phi\imath+d_\phi}{\kappa-\alpha\rho\phi\imath-d_\phi}, \\ d_\phi = \sqrt{(\alpha\rho\phi\imath-\kappa)^2+\alpha^2(\phi\imath+\phi^2)}, \end{cases}$$

and

$$\begin{cases} \tilde{C}_2(T-t;\phi) = \frac{a}{\alpha^2}\left[\kappa - \alpha\rho\phi\imath + d_\phi)(T-t) - 2\ln\left(\frac{1-\tilde{g}_\phi e^{d_\phi(T-t)}}{1-\tilde{g}_\phi}\right)\right], \\ \tilde{D}_2(T-t;\phi) = \frac{\kappa-\alpha\rho\phi\imath+d_\phi}{\alpha^2}\left[\frac{1-e^{d_\phi(T-t)}}{1-\tilde{g}_\phi e^{d_\phi(T-t)}}\right], \\ \tilde{g}_\phi = \frac{\kappa-\alpha\rho\phi\imath-d_\phi}{\kappa-\alpha\rho\phi\imath+d_\phi} = \frac{1}{g_\phi}, \\ d_\phi = \sqrt{(\alpha\rho\phi\imath-\kappa)^2+\alpha^2(\phi\imath+\phi^2)}. \end{cases}$$

In order to insure that the principal branch of the complex root is always identified. Therefore, following the Abrecher approach one has an algorithmic advantage. See the article in [2] for the explanation.

The Fourier transform method [2], [57], [62] works well for a given range of the model parameters, and it shows robustness properties when the model parameters move, thus one can conclude that it is useful to calibration procedure.

(b) **Finite difference method**
The finite difference method (Crank-Nikolson) is a flexible method which can be used for many pay-offs: European options or certain path dependent derivatives. In this case, the drawback is that we have to approximate the option prices on a grid. Accurate pricing requires a substantial amount of grid points. The theoretical aspects of this methods can be found in the Appendix and, more extensively in [3], [67] .

(c) **Monte Carlo method**
The derivatives price obtained by Monte Carlo method is very sensitive to the effective value of the volatility which is too clumsily approximated by the Euler scheme (see e.g. Sauer, 2005, pp. 460 or the Appendix). From a more theoretical point of view, the question of the true martingality of an asset following Heston dynamics is not clear up to now as studied for instance

by Jourdain in [63]. The Monte Carlo method is the most expensive numerical method, but its accuracy is not better than that given by the Fourier transform or the finite difference method. We are going to use the above numerical techniques in the next section: "Numerical experiments", in order to determine the fair price of a European Call option, comparing the latter methods to each other and also with the Geometrical Approximation.

4.4.2 Approximation method

(a) **Implied Volatility method**
An important contribute to approximation methods for the Heston model is given by Lewis [80]. In order to present the theoretical method proposed by Lewis, we denote $\sigma_T^2(d)$ the implied variance for a $\log$ forward moneyness $d = \frac{\log(E/F_t)}{T}$ and following the author, we write the variance associated to the Heston model as follows:

$$\sigma_T^2(d) = I_0 + I_1(d) + I_2(d),$$

where I_j represents the j-th order approximation for small d, and the parameters set are risk-neutral:

$$\begin{cases} I_0 = \Theta + \frac{\nu_0 - \Theta}{\kappa T}\left(1 - e^{-\kappa T}\right), \\ I_1(d) = \frac{\sigma M_1}{T}\left(\frac{1}{2} + \frac{d}{I_0 T}\right), \\ I_2(d) = \frac{\sigma^2}{T}\left[M_3\left(\frac{d^2}{2 I_0^2 T^2} - \frac{1}{I_0 T} - \frac{1}{8}\right) + M_4\left(\frac{d^2}{I_0^2 T^2} - \frac{d}{I_0 T} - \frac{4 - I_0 T}{4 I_0 T}\right)\right] + \frac{M_1}{2}\left(-\frac{5d^2}{2 I_0^3 T^3} - \frac{d}{I_0 T} + \frac{12 + I_0 T}{8 I_0 T^2}\right), \end{cases}$$

where

$$\begin{cases} M_1 = \frac{\kappa\rho T\left[\Theta - (\nu_0 - \Theta)e^{-\kappa T}\right] + (\nu_0 - 2\Theta)\left(1 - e^{-\kappa T}\right)}{\kappa^2}, \\ M_2 = \frac{\left[\Theta - 2(\nu_0 - \Theta)e^{-\kappa T}\right]T + \frac{1 - e^{-kT}}{\kappa}\left[\left(\nu_0 - \frac{\Theta}{2}\right)(1 + e^{-\kappa T}) - 2\Theta\right]}{2\kappa^2}, \\ M_4 = \frac{\rho^2}{\kappa^3}(\nu_0 - 3\Theta) + \frac{\rho^2 \Theta t}{\kappa^2} + \frac{\rho^2}{\kappa^3}(3\Theta - \nu_0)e^{-\kappa t} + \frac{\rho^2 t}{\kappa^2} e^{-\kappa t}(2\Theta - \nu_0) + \frac{\rho^2 t^2}{2\kappa}(\Theta - \nu_0)e^{-\kappa t}. \end{cases}$$

Lewis's idea is to obtain the implied volatility from the implied variance $\sigma_T^2(d)$, and by this substituting in the Black-Scholes solution, compute an approximate price of the derivatives in the Heston model. The method proposed by Lewis [80], is very useful to study the asymptotic properties of the Heston model, for small and large maturities.

Although the Lewis's work is very useful it is not mathematically rigorous. In this sense, Forde and Jacquier improve, from the theoretical point of view, Lewis's results on small-time asymptotic behaviour and long-time asymptotic behaviour.

In the **small-time** case, using the Gärtener-Ellis theorem from large deviations theory, Forde and Jacquier [36] show that the small-time asymptotic behaviour in the Heston model satisfies a small-time large deviation principle (see Theorem 1.1 [36]). In Theorem 2.5 [36] the authors show how to compute the level, the slope and the curvature of the implied volatility in the small-time, with limit at-the-money. The results obtained by Forde and Jacquier are consistent with those obtained probabilistic methods by Durrleman [31].

Here we present the main result of interest in practice as proved in [36]:

(1) **Theorem (Forde-Jacquier [36], Theorem 2.4 and 2.5)**
Let F_t be a process which follows the Heston dynamic. Then, as $T \to 0$, we have:

$$I(d) = lim_{T\to 0}\sigma_T(d) = \begin{cases} \frac{d}{\sqrt{2\Lambda^*(d)}} & \text{if} \quad d \neq 0 \\ \sqrt{\nu_0}\left[1 + \frac{\rho z}{4} + \left(\frac{1}{24} - \frac{5}{48}\rho^2\right) z^2 + O\left(z^3\right)\right] & \text{if} \quad d \neq 0, \end{cases}$$

where $z = \frac{\rho d}{\nu_0}$ and $\Lambda^*(\cdot)$ is the Legendre transform of the Laplace transform $\Lambda(p) = lim_{T\to 0}\left[t \times \mathbb{E}\left(e^{\frac{p}{T}(X_T - X_0)}\right)\right]$. Where $\sigma_T(d)$ is the implied volatility written on the forward asset price F_t, with strike $E = F_0 e^{d*T}$ maturing at time T (see [32] for details).

Concerning the **large-maturity** behaviour of the Heston model, in [80] Lewis obtains the large-time behaviour of the implied volatility as a second-order polynomial in the forward log-moneyness $X_t = log(F_t)$. Forde and Jacquier, by using of the Large Deviations Theory, extend the Lewiss 0-th order asymptotic result to the case in which the strike is $F_0 e^{d*T}$. In this case when the maturity gets large, the range of possible strikes gets large too. We present here as made before, the main result of interest in practice as proved in [37]:

(2) **Theorem (Forde-Jacquier [37] Corollary 1.7 of Theorem 1.1)**
For $d \in \mathbb{R} - \left\{-\frac{\Theta}{2}, +\frac{\overline{\Theta}}{2}\right\}$ and if $\kappa > \rho\alpha$, then we have the following large-time asymptotic behaviour for $\sigma_T(d)$

$$\sigma^2_\infty(d) = lim_{T\to+\infty} = \begin{cases} 2\left(2V_S^*(-d) + d - 2\sqrt{V_S^*(-d)^2 + V_S^*(-d)d}\right) & \left(d < -\frac{\Theta}{2}, \quad d > \frac{\overline{\Theta}}{2}\right), \\ 2\left(2V_S^*(-d) + d + 2\sqrt{V_S^*(-d)^2 + V_S^*(-d)d}\right) & \left(-\frac{\Theta}{2} < d < \frac{\overline{\Theta}}{2}\right), \end{cases}$$

and

$$\begin{aligned} &lim_{d\to\frac{\Theta}{2}}\overline{\sigma^2}_\infty(d) = \overline{\Theta}, \\ &lim_{d\to-\frac{\Theta}{2}}\overline{\sigma^2}_\infty(d) = \Theta, \end{aligned} \tag{4.14}$$

where $\overline{\Theta} = \frac{\kappa\Theta}{\kappa - \rho\alpha}$ and $V^*(\cdot)$ is the Legendre transform of the Laplace transform of

$$V(p) = lim_{T\to+\infty}\left[\frac{1}{T}\mathbb{E}_{\mathbb{Q}}\left(e^{p(X_T - X_0)}\right)\right],$$

and $V_S^*(d) = V^*(d) - d$.
The symmetric properties of the Heston model are given by (4.14) (see [37] for a precise statement and its proof).

Corollary 1.17 in [37] is the main result, and it is useful to give an approximation value for long-maturity Call options:

$$\mathbb{E}_{\mathbb{Q}}\left(S_T - Ee^{d\times T}\right) \simeq C_{BS}(S_0, S_0e^{d\times T}, T, \sigma_\infty(d)) \qquad T \to +\infty,$$

where $C_{BS}(S_0, E, T, \sigma)$ is the Black-Scholes Call option formula with initial stock price S_0, strike price E, maturity T and volatility σ. Quoting the authors' work: "It can also be used for implied volatility smile extrapolations at large maturities, where Monte Carlo and PDE methods break down". Nevertheless, the large deviation principle is unmannerly; therefore the authors compute the implied variance as $\sigma_T^2(d) = \sigma_\infty^2(d) + \frac{a(d)}{T} + O\left(\frac{1}{T^2}\right)$, where the correlation term $a(d)$ is computed using Laplace's method for Countor integrals. The practical relation obtained from Forde and Jacquier is the following:

$$\mathbb{E}_{\mathbb{Q}}\left(S_T - Ee^{d\times T}\right) \to C_{BS}\left(S_0, S_0e^{d\times T}, T, \sqrt{\sigma_\infty^2(d) + \frac{a(d)}{T} + O\left(\frac{1}{T^2}\right)}\right)$$
$$T \to +\infty.$$

Their results appear to be valuable and easy to use, but their proof need sophisticated mathematical instruments.

(b) **Analytic and Geometric Methods for Heat Kernel**
The Avramidi's technique is an approximation method for PDEs for stochastic volatility models. The author bases his work on the geometrical considerations of Bourgade and Croissant (2005); he shows its application at two cases: SABR and Heston.

On the SABR model he writes the PDE operator L (see pp. 232, "Analytic and Geometric Methods for Heat Kernel Applications in Finance", Preprint 2007) in the form:

$$L = L_0 + L_1,$$

where L_0 is the scalar Laplacian,

$$L_0 = -g^{-1/2}\partial_i g^{1/2} g^{ij} \partial_j,$$

and L_1 is the first order operator:

$$L_1 = \frac{\nu^2}{2} y^2 C(x) \frac{dC(x)}{dx} \partial_x.$$

The heat kernel of the operator L_0 is given by:

$$U_0(t,x,x') = \frac{1}{4\pi t}\sqrt{\frac{\rho r}{\sinh(\rho r)}} exp\left(-\frac{r^2}{4t}\right) \times \left[1 - \frac{t}{4r^2}(\rho^2 r^2 + \rho r \coth(\rho r) - 1) + O(t^2)\right],$$

where $\rho = \frac{\nu}{\sqrt{2}}$ and considering the operator L_1 as a perturbation, the author gets as follows:

$$U(t,x,x') = \left[1 - tL_1 + \frac{t^2}{2}(L_1^2 + [L_0, L_1]) + O(t^3)\right] U_0(t,x,x').$$

That reduces itself to Hagan formula if we restrict to the linear order in L_1 (see "Analytic and Geometric Methods for Heat Kernel Applications in Finance", Preprint 2007, for theoretical details, pp.231).

On the Heston model, Avramidi [6] shows as solve the Heston's PDE, through Fourier transform, some coordinate transformations, and a perturbative expansion of the heat kernel.

In other words, starting from the known Heston's PDE, written using the Mellin's coordinates ($x = \ln S, u = \frac{\nu}{\eta}, \tau = \frac{\eta}{2}(T-t)$):

$$\left[\frac{\partial}{\partial \tau} - u(\frac{\partial^2}{\partial x^2} + 2\rho\frac{\partial^2}{\partial x \partial u} + \frac{\partial^2}{\partial u^2} + (u - 2\frac{r}{\eta})\frac{\partial}{\partial x} + 2\frac{\lambda}{\eta}\left(u - \frac{\overline{\nu}}{\eta}\right)\frac{\partial}{\partial u} + 2\frac{r}{\eta}\right] U(\tau,x,u,x',u') = 0,$$

and using the Fourier transform one has:

$$U(\tau,x,u,x',u') = \int_{-\infty}^{\imath+\infty} \frac{dp}{2\pi} e^{\imath p(x-x')}\tilde{U}(\tau,p,u,u'),$$

where a is a real constant that must be chosen in such a way that the integral converges; then the function $\tilde{U}$ satisfies the following PDE:

$$\left[\partial_\tau - u\partial_u^2 + (2\beta_1 u + \beta_0)\partial_u + \gamma_1 u + \gamma_0\right]\tilde{U} = 0,$$

where

$$\beta_1 = \frac{\lambda}{\eta} - \imath\rho p, \qquad \beta_0 = -2\frac{\lambda}{\eta^2}\overline{\nu},$$

$$\gamma_1 = p^2 + \imath p, \qquad \gamma_0 = 2\frac{r}{\eta}(1 - \imath p).$$

After many calculi (see "Analytic and Geometric Methods for Heat Kernel Applications in Finance", Preprint 2007, for theoretical details, pp.239) it is possible to find the following solution:

$$U(\tau,x,u,x',u') = \int_{\imath a-\infty}^{\imath a+\infty} \frac{dp}{2\pi}\int_{b-\imath\infty}^{b+\imath\infty} \frac{dq}{2\pi\imath} e^{\imath p(x-x') + qu - q_0 u'} \left(\frac{q_0 - q_1}{q - q_1}\right)^{a_1} \times \left(\frac{q_0 - q_2}{q - q_2}\right)^{a_2},$$

that is rather complicated since it involves two complex integrals. In order to simplify this problem, Avramidi choses a particular heat kernel:

$$P_n(\tau, x, u) = \int_0^{+\infty} dx' \int_0^{+\infty} du' U(\tau, x, u, x', u') e^{nx'},$$

which can be rewrite as series expansion of complex integral that can be solved by residue theory:

$$P_n(\tau, x, u) = \int_{\imath-\infty}^{\imath+\infty} \frac{dp}{2\pi} \frac{1}{(\imath p - n)} e^{\imath p x - (\gamma_0 + \beta_0 \beta_1)\tau} \times exp\left(-\frac{u\gamma_1 \sinh(D\tau)}{\beta_1 \sinh(D\tau) + D\cosh(D\tau)}\right) \times \left(\cosh(D\tau) + \beta_1 \frac{\sinh(D\tau)}{D}\right)^{\beta_0}.$$

Thus, by Avramidi method is possible to compute the kernel of the Heston's PDE as series expansion of complex integral, and it is an alternative method to evaluate derivatives in the Heston model.

4.5 Geometrical Approximation method: Heston model

In this section, we propose an approximation technique for pricing European options through the solution of PDE (4.11). We call it the **Geometrical Approximation Method** for derivatives pricing, since the solution method for the PDE is based on geometrical considerations. We will apply this methodology to the Heston [24] and SABR [27] model, but the proposed technique potentially applies for more general specifications of the diffusion processes.

For ease of exposition, we now focus on the Heston model. The proposed technique is based on a stochastic approximation of the Cauchy condition of the PDE (4.11): it consists in using the final condition $\Phi\left(S_T e^{\varepsilon_T}\right)$ where $\varepsilon_T = \rho(\overline{\nu} - \nu_T)/\alpha$, instead of the standard pay-off function $\Phi(S_T)$. Notice that ε_T is a stochastic quantity and $\overline{\nu}$ is the risk-neutral expected value of ν_T, see equation (4.15) below.

The advantage of the approximation is of providing closed formulas for European options. The geometrical interpretation of such an approximation will be more apparent when computing the price of a European call option, where we impose that $\rho = \sin\theta_\rho$ and $\sqrt{1-\rho^2} = \cos\theta_\rho$, $\theta_\rho \in (-\pi/2, \pi/2)$. Notice that if $\rho = 0$ there is no approximation and the formulas are exact. In all other cases, we will have to estimate the approximation error.

By the form of the approximation, we can see that it will make not a big difference with the actual option price if $\varepsilon_T = \rho(\overline{\nu} - \nu_T)/\alpha$ is small. To compute the mean and the variance of ε_T, we need the mean and the variance of ν_T which are given, in the Heston model, by:

$$\overline{\nu} = \mathbb{E}_Q[\nu_T] = \left[(\nu_0 - \Theta)e^{-\kappa T} + \Theta\right]. \tag{4.15}$$

Where ϵ_T is given by:

$$\varepsilon_T = \frac{\rho(\overline{\nu} - \nu_T)}{\alpha} = \frac{\rho}{\alpha}\left[(\nu_0 - \Theta)e^{-\kappa T} + \Theta - \nu_T\right], \tag{4.16}$$

and thus

$$\mathbb{E}_Q(\varepsilon_T) = 0.$$

It is worth noting that for $\alpha = 0$ we have a singularity, but this one is not interesting case, since for $\alpha = 0$ we have a deterministic variance (4.9), thus one can consider $\alpha \in \mathbb{R}^+ - \{0\}$. Regarding the variance we have,

$$\begin{aligned} Var(\nu_T) = \nu_0 \frac{\alpha^2}{\kappa}\left(e^{-\kappa T} - e^{-2\kappa T}\right) + \frac{\theta\alpha^2}{2\kappa}\left(1 - e^{-\kappa T}\right)^2 \\ = \frac{\alpha^2}{\kappa}\left(1 - e^{-\kappa T}\right)\left[\left(\nu_0 - \frac{\Theta}{2}\right)e^{-\kappa T} + \frac{\Theta}{2}\right], \end{aligned} \tag{4.17}$$

so that

$$Var(\varepsilon_T) = Var\left(\frac{\rho}{\alpha}(\overline{\nu} - \nu_T)\right) = \mathbb{E}_{\mathbb{Q}}\left[\varepsilon_T^2\right] - \mathbb{E}_{\mathbb{Q}}\left[\varepsilon_T\right]^2, \tag{4.18}$$

where $\mathbb{E}_{\mathbb{Q}}[\varepsilon_T] = 0$, thus we have:

$$\begin{aligned} Var(\varepsilon_T) = \frac{\rho^2}{\alpha^2}\left[\mathbb{E}_Q\left(\nu_T^2\right) - \overline{\nu}^2\right] \\ = \frac{\rho^2}{\alpha^2} Var(\nu_T) = \frac{\rho^2}{\kappa}\left(1 - e^{-\kappa T}\right)\left[\left(\nu_0 - \frac{\Theta}{2}\right)e^{-\kappa T} + \frac{\Theta}{2}\right]. \end{aligned}$$

Both for the variance of the variance processes ν_T and the variance for ε_T processes we a have a singularity for $\kappa = 0$, i.e., for $k = \lambda\alpha$, see (4.10), but for hypothesis $k \neq \lambda\alpha$, as we have just said in the previous section, and thus we consider $\kappa \neq 0$. Let us compute the limits of these quantities:

$$\begin{aligned} &lim_{T\to 0} Var(\nu_T) = 0, \\ &lim_{T\to +\infty} Var(\nu_T) = \frac{\alpha^2\Theta}{2\kappa}, \end{aligned} \tag{4.19}$$

therefore we have for ε_T:

$$\begin{aligned} &lim_{T\to 0} Var(\varepsilon_T) = 0, \\ &lim_{T\to +\infty} Var(\varepsilon_T) = \frac{\rho^2\Theta}{2\kappa}. \end{aligned} \tag{4.20}$$

Thus the variance ν_T goes to zero for T small and when T is large the variance approaches to $\alpha^2\Theta/2\kappa$. Equivalently our stochastic error ε_T goes to 0 for T small and when T is large it approaches to $(\rho^2\Theta/2\kappa)$, that is great or equal to zero.

In what follows we use the PDE formalism, thus we replace S_t with S and ν_t with ν. In order to evaluate a European Call option in the Heston model, using the G. A. method, we have to modify the pay-off as follows: $\left(Se^{\frac{\rho(\overline{\nu}-\nu)}{\alpha}} - E\right)^+$, that reduces itself at maturity to a pay-off near at the $(S-E)^+$ giving us an approximate Call option price. It is worth noting that $\left(Se^{\frac{\rho(\overline{\nu}-\nu)}{\alpha}} - E\right)^+$ can be also written as follows $e^{\frac{\rho\overline{\nu}}{\alpha}}\left(Se^{\frac{-\rho\nu}{\alpha}} - Ee^{\frac{-\rho\overline{\nu}}{\alpha}}\right)^+$, in which $Ee^{\frac{-\rho\overline{\nu}}{\alpha}}$ is a constant. For simplicity we put $Ee^{\frac{-\rho\overline{\nu}}{\alpha}} = \overline{E}$, in order to rewrite our modified pay-off as follows: $e^{\frac{\rho\overline{\nu}}{\alpha}}\left(Se^{\frac{-\rho\nu}{\alpha}} - \overline{E}\right)^+$.
Our formulation of Heston problem for derivatives pricing, is a Cauchy's problem:

$$\frac{\partial f}{\partial t} + \frac{1}{2}\nu S^2\frac{\partial^2 f}{\partial S^2} + \rho\nu\alpha S\frac{\partial^2 f}{\partial S\partial\nu} + \frac{1}{2}\nu\alpha^2\frac{\partial^2 f}{\partial\nu^2} + \kappa(\Theta-\nu)\frac{\partial f}{\partial\nu} + rS\frac{\partial f}{\partial S} - rf = 0,$$

$$S\in[0,+\infty), \qquad \nu\in[0,+\infty),$$

$$f(T,S,\nu) = e^{\frac{\rho\overline{\nu}}{\alpha}}\left(Se^{\frac{-\rho\nu}{\alpha}} - \overline{E}\right)^+, \qquad t\in[0,T],$$

$f(t,S,\nu)\in C^{2,1}\left([0,+\infty)\times[0,+\infty)\times[0,T]\right)$. For the Theorem (1.1) (see Chapter 1 for theoretical aspects on parabolic operators), we start saying that we are able to define our problem as a case study well-posed.

We now simplify the PDE (4.11) at hand. To this end, let us introduce new variables $x,\tilde{\nu}$ and a new function f_1:

$$S = e^x, \quad \nu = \tilde{\nu}\alpha, \qquad x\in(-\infty,+\infty), \qquad \nu\in[0,+\infty), \qquad t\in[0,T],$$
$$f(t,S,\nu) = e^{-r(T-t)}f_1(t,x,\tilde{\nu}).$$

The determinant of the jacobian matrix has not singularities within the domain, but only for $S=0$ which is a boundary point, and this one guarantees us that the new PDE is:

$$\frac{\partial f_1}{\partial t} + \frac{1}{2}\tilde{\nu}\alpha\left(\frac{\partial^2 f_1}{\partial x^2} + 2\rho\frac{\partial^2 f_1}{\partial x\partial\tilde{\nu}} + \frac{\partial^2 f_1}{\partial\tilde{\nu}^2}\right) + \frac{\kappa}{\alpha}\left(\Theta-\tilde{\nu}\alpha\right)\frac{\partial f_1}{\partial\tilde{\nu}} + \left(r - \frac{1}{2}\tilde{\nu}\alpha\right)\frac{\partial f_1}{\partial x} = 0,$$
$$f_1(T,x,\tilde{\nu}) = e^{\frac{\rho\overline{\nu}}{\alpha}}(e^{x-\rho\tilde{\nu}} - \overline{E})^+, \tag{4.21}$$

in which we have considered the modified pay-off: $e^{\frac{\rho\overline{\nu}}{\alpha}}(e^{x-\rho\tilde{\nu}} - \overline{E})^+$, instead of $(e^x - E)^+$ with respect to the new variables $x,\tilde{\nu}$. Now we consider only the terms that have derivatives of the second order and after that, we try a new set of coordinates that transform the PDE in a simpler form. As a first step, we write the characteristic equation associated to the second order terms of our PDE (4.21) , then we compute its roots:

$$\frac{\partial^2 f_1}{\partial x^2} + 2\rho \frac{\partial^2 f_1}{\partial x \partial \tilde{\nu}} + \frac{\partial^2 f_1}{\partial \tilde{\nu}^2} = 0.$$

The characteristic equation results to be

$$\left(\frac{dx}{d\tilde{\nu}}\right)^2 - 2\rho\left(\frac{dx}{d\tilde{\nu}}\right) + 1 = 0,$$

$$\Delta = 4^2(1-\rho^2) \leq 0, \qquad \rho \in (-1,1),$$

so that the squared term is of elliptic kind, and the roots belong at the set of complex numbers

$$\left(\frac{dx}{d\tilde{\nu}}\right)_{1/2} = \rho \pm \imath\sqrt{1-\rho^2}.$$

At this point we can define the characteristic lines (remembering what was said in chapter 1, these are also defined as geodesics) as follows

$$x - (\rho + \imath\sqrt{1-\rho^2})\tilde{\nu} = z,$$
$$x - (\rho - \imath\sqrt{1-\rho^2})\tilde{\nu} = w.$$

Through another change of variables, we obtain a linear system easy to solve:

$$z = \xi + \imath\eta, \qquad w = \xi - \imath\eta,$$

so that results $w = \overline{z}$

$$\begin{aligned} \tilde{\nu} = -\frac{\eta}{\sqrt{1-\rho^2}}, \qquad & x = \frac{\xi\sqrt{1-\rho^2} - \rho\eta}{\sqrt{1-\rho^2}}, \\ \eta = -\tilde{\nu}\sqrt{1-\rho^2}, \qquad & \xi = x - \rho\tilde{\nu}, \end{aligned} \tag{4.22}$$

where $\eta \in (-\infty, 0)$ and $\xi \in (-\infty, +\infty)$, in which we have a singularity for $\rho = \pm 1$, while the determinant of the jacobian matrix has not singularities, in fact this one is equal to $\left(-\sqrt{1-\rho^2}\right)$. Thus we have to transform f_1 in another function f_2 dependents of the new variables (t, ξ, η).

At this point, it is fundamental to make the following geometrical consideration, in order to understand our method. We have defined a new system of coordinates, where $\vec{e}_\eta, \vec{e}_\xi, \vec{e}_t$, are orthogonal directions; we can think of $x, \tilde{\nu}$ as vectors, whose projections on the axes are respectively given by

$$\vec{x} = (0)\vec{e}_\eta + (x)\vec{e}_\xi, \qquad \vec{\tilde{\nu}} = (\tilde{\nu}\cos\theta_\rho)\vec{e}_\eta + (\tilde{\nu}\sin\theta_\rho)\vec{e}_\xi,$$

where, we have supposed $\rho = \sin\theta_\rho$ and $\sqrt{1-\rho^2} = \cos\theta_\rho$, $\theta_\rho \in (-\pi/2, \pi/2)$. Now we can define a new vector, that we call $\vec{V}$, whose projections are

$$\vec{V} \equiv (V_\eta, V_\xi), \qquad V_\eta = -\tilde{\nu}\cos\theta_\rho, \qquad V_\xi = x - \tilde{\nu}\sin\theta_\rho,$$

by which, we can show the vectorial relation that exists between the variables $(x, \tilde{\nu})$.

Now, from the Cauchy's condition, we are able to write the new function f_2, as function of variables t and $V_\xi(x, \tilde{\nu})$, because, the function f depends, at the time T, only on the projection terms upon the axis ξ, therefore, because of the continuity properties of the Feynman-Kač formula, we can suppose that is true at any time t:

$$f_1(t, x, \tilde{\nu}) = f_2(t, V_\xi(x, \tilde{\nu})), \qquad t \in [0, T],$$

now we may substitute them in the old squared term

$$\frac{\partial^2 f_1}{\partial x^2} + 2\rho\frac{\partial^2 f_1}{\partial x \partial\tilde{\nu}} + \frac{\partial^2 f_1}{\partial\tilde{\nu}^2} = (1-\rho^2)\nabla^2_{V_\xi} f_2(t, V_\xi(x, \tilde{\nu})),$$

where $\nabla^2_{V_\xi}(\cdot)$ is the Laplace's operator with respect to the variable V_ξ . Thus, the new PDE of Heston's model has become:

$$\frac{\partial f_2}{\partial t} - \frac{\alpha V_\eta}{\sqrt{1-\rho^2}}\left[\frac{(1-\rho^2)}{2}\frac{\partial^2 f_2}{\partial V_\xi^2} - \left(\frac{1}{2} - \frac{\kappa}{\alpha}\rho\theta\right)\frac{\partial f_2}{\partial V_\xi}\right] + \left(r - \frac{\kappa}{\alpha}\rho\theta\right)\frac{\partial f_2}{\partial V_\xi} = 0, \tag{4.23}$$

with a singularity for $\rho = \pm 1$, and where the following final condition is given by

$$f_2(T, V_\xi) = e^{\frac{\rho\overline{\nu}}{\alpha}}\left(e^{V_\xi} - \overline{E}\right)^+,$$

where

$$V_\xi = x - \rho\tilde{\nu} = \ln S - \frac{\rho}{\alpha}\nu, \qquad V_\xi \in (-\infty, +\infty).$$

Now, we can compute the solution of PDE (4.23) in closed form.

Another change of coordinates is sufficient to simplify last PDE. We may define a new transformation of coordinate and the new function f_3, as follows

$$\gamma = V_\xi + \left(r - \frac{k}{\alpha}\rho\theta\right)(T-t), \qquad \gamma \in (-\infty, +\infty),$$
$$\tau = \int_t^T \nu_s ds, \qquad \tau \in \left[0, \int_0^T \nu_s ds\right],$$
$$f_2(t, V_\xi) = f_3(\tau(t), \gamma(t, V_\xi)).$$

The determinant of the jacobian matrix, that is equal to ν, has not singularities within the domain, but only for $\nu = +\infty$ which is a boundary point.

Substituting what we have just found, in the previous equation, we finally have a simpler Cauchy's problem:

$$\begin{aligned}
&\frac{\partial f_3}{\partial \tau} = \frac{(1-\rho^2)}{2}\nabla^2_\gamma f_3 - \left(\frac{1}{2} - \frac{\kappa\rho}{\alpha}\right)\frac{\partial f_3}{\partial \gamma}, \\
&f_3(0,\gamma) = e^{\frac{\rho\overline{\nu}}{\alpha}}\left(e^\gamma - \overline{E}\right)^+, \\
&\gamma \in (-\infty, \infty), \quad \tau \in \left[0, \int_0^T \nu_s ds\right],
\end{aligned} \tag{4.24}$$

where $\nabla^2_\gamma(\cdot)$ is the Laplace's operator with respect to the γ-variable. Now we can rewrite the function f_3 as follows, in order to obtain the one-dimensional heat equation:

$$f_3(\tau,\gamma) = e^{a\tau + b\gamma} f_4(\tau,\gamma),$$

where

$$a = -\frac{(1/2 - \kappa\rho/\alpha)^2}{2(1-\rho^2)}, \qquad b = \frac{(1/2 - \kappa\rho/\alpha)}{(1-\rho^2)},$$

so we have

$$\frac{\partial f_4}{\partial \tau} = \frac{(1-\rho^2)}{2}\nabla^2_\gamma f_4.$$

Also in this case we have singularities for $\rho = \pm 1$ on a and b, as well as for the variable τ.

At this point we have another problem that has a simpler solution (see A. D. Polyanin, Handbook of Linear PDE, pp.45):

$$\begin{aligned}
&\frac{\partial f_4}{\partial \tau} = \frac{(1-\rho^2)}{2}\nabla^2_\gamma f_4, \qquad \gamma \in (-\infty, +\infty), \quad \tau \in \left[0, \int_0^T \nu_s ds\right], \\
&f_4(0,\gamma) = e^{\frac{\rho\overline{\nu}}{\alpha}}\left(e^\gamma - \overline{E}\right)^+.
\end{aligned}$$

We now are able to write the solution, that is

$$\begin{aligned}
f_4(\tau,\gamma) &= \frac{1}{\sqrt{2\pi(1-\rho^2)\tau}}\int_{-\infty}^{+\infty} d\gamma' f_4(0,\gamma') \exp\left[-\frac{(\gamma'-\gamma)^2}{2(1-\rho^2)\tau}\right] \\
&= \int_{-\infty}^{\infty} d\gamma' f_4(0,\gamma') G(\gamma', 0|\gamma, \tau),
\end{aligned} \tag{4.25}$$

where

$$G(\gamma',0|\gamma,\tau)=\frac{1}{\sqrt{2\pi(1-\rho^2)\tau}}\exp\left[-\frac{(\gamma'-\gamma)^2}{2(1-\rho^2)\tau}\right],$$

and

$$f(t,S,\nu)=e^{-r(T-t)+a\tau+b\gamma}f_4(\tau,\gamma),$$

$$f(T,S,\nu)=e^{b\gamma}f_4(0,\gamma),$$

$$f_4(0,\gamma)=e^{-b\gamma}e^{\frac{\rho\overline{\nu}}{\alpha}}\left(e^{\gamma}-\overline{E}\right)^+,$$

for which we have

$$f_4(\tau,\gamma)=\frac{1}{\sqrt{2\pi(1-\rho^2)\tau}}\int_{-\infty}^{+\infty}d\gamma' e^{-b\gamma'}e^{\frac{\rho\overline{\nu}}{\alpha}}\left(e^{\gamma'}-\overline{E}\right)^+\exp\left[-\frac{(\gamma'-\gamma)^2}{2(1-\rho^2)\tau}\right]$$

$$=\frac{1}{\sqrt{2\pi(1-\rho^2)\tau}}\int_{\ln\overline{E}}^{+\infty}d\gamma' e^{-b\gamma'}e^{\frac{\rho\overline{\nu}}{\alpha}}\left(e^{\gamma'}-\overline{E}\right)\exp\left[-\frac{(\gamma'-\gamma)^2}{2(1-\rho^2)\tau}\right].$$

Thus follows the solution on the modified payoff function, by which approximate the Heston solution for Call options:

$$f(t,S,\nu)=\frac{e^{-r(T-t)+a\tau+b\gamma}}{\sqrt{2\pi(1-\rho^2)\tau}}\int_{\ln\overline{E}}^{+\infty}d\gamma' e^{-b\gamma'}e^{\frac{\rho\overline{\nu}}{\alpha}}\left(e^{\gamma'}-\overline{E}\right)\exp\left[-\frac{(\gamma'-\gamma)^2}{2(1-\rho^2)\tau}\right]$$

$$=Se^{\frac{\rho}{\alpha}(\overline{\nu}-\nu)}e^{\delta}\mathbf{N}(d_1^\rho)-Ee^{-r(T-t)}\mathbf{N}(d_2^\rho), \quad (4.26)$$

where

$$\delta=-\left[\frac{\kappa}{\alpha}\rho\Theta-\left(a+\frac{(1-b)^2(1-\rho^2)}{2}\right)\frac{1}{T-t}\int_t^T\nu_s ds\right](T-t)$$

$$=-\left[\frac{\kappa}{\alpha}\rho\Theta+\left(\frac{\rho^2}{2}-\frac{\kappa}{\alpha}\rho\right)\frac{1}{T-t}\int_t^T\nu_s ds\right](T-t), \quad (4.27)$$

$$d_1^\rho=\frac{\ln(Se^{\frac{\rho}{\alpha}(\overline{\nu}-\nu)}/E)+\left[\left(r-\frac{\kappa}{\alpha}\rho\Theta\right)+(1-b)(1-\rho^2)\frac{1}{T-t}\int_t^T\nu_s ds\right](T-t)}{\sqrt{(1-\rho^2)\int_t^T\nu_s ds}}$$

$$=\frac{\ln(Se^{\frac{\rho}{\alpha}(\overline{\nu}-\nu)}/E)+\left[\left(r-\frac{\kappa}{\alpha}\rho\Theta\right)+\left(\frac{1}{2}+\frac{\kappa\rho}{\alpha}-\rho^2\right)\frac{1}{T-t}\int_t^T\nu_s ds\right](T-t)}{\sqrt{(1-\rho^2)\int_t^T\nu_s ds}}, \quad (4.28)$$

$$d_2^{\rho} = \frac{\ln(Se^{\frac{\rho}{\alpha}(\overline{\nu}-\nu)}/E) + \left[\left(r - \frac{\kappa}{\alpha}\rho\Theta\right) - b(1-\rho^2)\frac{1}{T-t}\int_t^T \nu_s ds\right](T-t)}{\sqrt{(1-\rho^2)\int_t^T \nu_s ds}}$$
$$= \frac{\ln(Se^{\frac{\rho}{\alpha}(\overline{\nu}-\nu)}/E) + \left[\left(r - \frac{\kappa}{\alpha}\rho\Theta\right) - \left(\frac{1}{2} - \frac{\kappa\rho}{\alpha}\right)\frac{1}{T-t}\int_t^T \nu_s ds\right](T-t)}{\sqrt{(1-\rho^2)\int_t^T \nu_s ds}}. \quad (4.29)$$

From theoretical viewpoint, the G. A. method is compatible with the Hull-White formula (see "Derivatives in Financial Markets with Stochastic Volatility", authors J. P. Fouque, G. Papanicolau, K. R. Sircar, pp.51).

In fact for $\rho = 0$ we have that our solution reduce itself to the Black-Scholes solution with average volatility over the time, instead of the spot volatility.
From 4.26, 4.27, 4.28, 4.29 follows that for $\rho = 0$ we have $\varepsilon = 0$, $\delta = 0$ and

$$d_1^{\rho=0} = \frac{\ln(S/E) + \left(r + \frac{1}{2}\frac{1}{T-t}\int_t^T \nu_s ds\right)(T-t)}{\sqrt{\int_t^T \nu_s ds}},$$

$$d_2^{\rho=0} = \frac{\ln(S/E) + \left(r - \frac{1}{2}\frac{1}{T-t}\int_t^T \nu_s ds\right)(T-t)}{\sqrt{\int_t^T \nu_s ds}},$$

then we find the Hull-White formula:

$$f(t,S,\nu)_{\rho=0} = S\mathbf{N}(d_1^{\rho=0}) - Ee^{-r(T-t)}\mathbf{N}(d_2^{\rho=0}) = C_{BS}(t,S,\sqrt{\tilde{\nu}}),$$

where $\tilde{\nu} = \frac{1}{T-t}\int_t^T \nu_s ds$.

On the boundary of its domain, the function $f(t,S,\nu)$ (4.26), assumes the following values:

$$f(t,0,\nu) = 0,$$

$$f(t,S,0) = \begin{cases} \left(Se^{\frac{\rho}{\alpha}\overline{\nu}}\right)e^{-\frac{\kappa}{\alpha}\rho\Theta(T-t)} - Ee^{-r(T-t)} & d_1^{\rho}, d_2^{\rho} > 0 \\ 0 & d_1^{\rho}, d_2^{\rho} < 0 \end{cases}$$

$$f(t,+\infty,\nu)=+\infty,$$

$$f(t,S,+\infty)=\begin{cases} \quad\quad 0 & b>1,\\ \begin{cases}\rho>0 & 0\\ \rho<0 & +\infty\end{cases} & 0<b<1,\\ \begin{cases}\rho>0 & \left(Se^{\frac{\rho}{\alpha}\overline{\nu}}-Ee^{-r(T-t)}\right)\\ \rho<0 & +\infty\end{cases} & b<0,\end{cases}$$

$$\frac{\partial f}{\partial S}(t,+\infty,\nu)=e^{\varepsilon+\delta},$$

$$\frac{\partial^2 f}{\partial S^2}(t,+\infty,\nu)=0.$$

Concluding for a European Call option we have:

$$C_{\rho,\alpha,\Theta,\kappa}(t,S,\nu)=(Se^{\varepsilon})\,e^{\delta}\mathbf{N}(d_1^{\rho})-Ee^{-r(T-t)}\mathbf{N}(d_2^{\rho}), \tag{4.30}$$

where $\varepsilon=\frac{\rho}{\alpha}(\overline{\nu}-\nu)$. For a Put we have:

$$P_{\rho,\alpha,\Theta,\kappa}(t,S,\nu)=Ee^{-r(T-t)}\mathbf{N}(-d_2^{\rho})-(Se^{\varepsilon})\,e^{\delta}\mathbf{N}(-d_1^{\rho}). \tag{4.31}$$

As one can see from the solution (4.26), the singularities due to the new variables vanish when we use the old variables. The terms in which the singularities for $\rho=\pm1$ don't simplify, are the arguments d_1^{ρ} and d_2^{ρ} of the normal distributions $\mathbf{N}$.

The proceeding of the solution (4.26) for $\rho\to\pm1$ is given by the limit:

$$\lim_{\rho\to\pm1}\left[Se^{\frac{\rho}{\alpha}(\overline{\nu}-\nu)}e^{\delta}\mathbf{N}(d_1^{\rho})-Ee^{-r(T-t)}\mathbf{N}(d_2^{\rho})\right].$$

It is known by (4.30) that $d_2^{\rho}=d_1^{\rho}-\sqrt{(1-\rho^2)\tilde{\nu}(T-t)}$ and for $\rho\to\pm1$ we have $d_2^{\rho}=d_1^{\rho}$. Thus, there are two possible cases, $d_1^{\rho},d_2^{\rho}>0$ and $d_1^{\rho},d_2^{\rho}<0$.

For $d_1^{\rho},d_2^{\rho}<0$, the limit will be:

$$\lim_{\rho\to\pm1}\left[Se^{\frac{\rho}{\alpha}(\overline{\nu}-\nu)}e^{\delta}\mathbf{N}(d_1^{\rho})-Ee^{-r(T-t)}\mathbf{N}(d_2^{\rho})\right]=0.$$

For $d_1^{\rho},d_2^{\rho}>0$, the limit will be:

$$\lim_{\rho\to\pm1}\left[Se^{\frac{\rho}{\alpha}(\overline{\nu}-\nu)}e^{\delta}\mathbf{N}(d_1^{\rho})-Ee^{-r(T-t)}\mathbf{N}(d_2^{\rho})\right]$$
$$=Se^{\frac{\pm1}{\alpha}(\overline{\nu}-\nu)}e^{-\left[\frac{\pm\kappa}{\alpha}\Theta+\left(\frac{1}{2}-\frac{\pm\kappa}{\alpha}\right)\tilde{\nu}\right](T-t)}-Ee^{-r(T-t)}.$$

Since the G. A. method is an approximation technique, we need to compute, if it is possible, how is the error. Defining an Error function as follows:

$$Err(\rho) = \frac{\|C^{GA}(t,S,\nu) - C^{H}(t,S,\nu)\|}{E}, \tag{4.32}$$

where $\|\cdot\|$ is the norm of $\mathbb{L}^1$ space. We cannot compute (4.32) analytically, even if we could do it with a Fourier transform, which has to be computed numerically. But it is worth noting that by the triangle inequality one can say at least, from theoretical point of view, that the Error function (4.32) is bounded.

Proposition: Theoretical estimation of Errors

Let be given the Call Option prices by G. A. method and by Fourier Transform method:

$$C^{GA}(t,S,\nu) = (Se^{\varepsilon})\, e^{\delta}\mathbf{N}_1 - Ee^{-r(T-t)}\mathbf{N}_2$$

$$C^{H}(t,S,\nu) = SP_1 - Ee^{-r(T-t)}P_2$$

where $\mathbf{P}_1$ and $\mathbf{P}_2$ are the distributions of the Heston's solution (4.13), and we have replaced $\mathbf{N}(d_1^{\rho})$ with $\mathbf{N}_1$ and $\mathbf{N}(d_2^{\rho})$ with $\mathbf{N}_2$ for computing simplicity.
Thus we have:

$$\|C^{GA}(t,S,\nu) - C^{H}(t,S,\nu)\| = \|S\left(e^{\varepsilon+\delta}\times\mathbf{N}_1 - P_1\right) - Ee^{-r(T-t)}\left(\mathbf{N}_2 - P_2\right)\|$$

$$= \|S\left(e^{\varepsilon+\delta}\times\mathbf{N}_1 - \mathbf{P}_1\right) + Ee^{-r(T-t)}\left(\mathbf{P}_2 - \mathbf{N}_2\right)\|$$

$$\leq \|S\left(e^{\varepsilon+\delta}\times\mathbf{N}_1 - \mathbf{P}_1\right)\| + \|Ee^{-r(T-t)}\left(\mathbf{P}_2 - \mathbf{N}_2\right)\|.$$

For $S = E$, we have:

$$Err(\rho) = \|\left(e^{\varepsilon+\delta}\times\mathbf{N}_1 - \mathbf{P}_1\right) + e^{-r(T-t)}\left(\mathbf{P}_2 - \mathbf{N}_2\right)\|$$

$$\leq \|\left(e^{\varepsilon+\delta}\times\mathbf{N}_1 - \mathbf{P}_1\right)\| + e^{-r(T-t)}\|\left(\mathbf{P}_2 - \mathbf{N}_2\right)\|.$$

For $S = E(1 \pm 10\%\sqrt{\Theta T})$, we have:

$$Err(\rho) = (1 \pm 10\%\sqrt{\Theta T})\|\left(e^{\varepsilon+\delta}\times\mathbf{N}_1 - \mathbf{P}_1\right) + e^{-r(T-t)}\left(\mathbf{P}_2 - \mathbf{N}_2\right)\|$$

$$\leq (1 \pm 10\%\sqrt{\Theta T})\|\left(e^{\varepsilon+\delta}\times\mathbf{N}_1 - \mathbf{P}_1\right)\| + e^{-r(T-t)}\|\left(\mathbf{P}_2 - \mathbf{N}_2\right)\|.$$

In other words we can say that the Error between the G.A method and Fourier method, is given by the difference among the normal distributions $\mathbf{N}_{j=1,2}$ and the distributions $\mathbf{P}_{j=1,2}$, or again, how the distribution of the Heston model is different to a Normal distribution.

4.5.1 Greeks and Put-Call-Parity

In order to find the best hedging strategy, we use a replicant portfolio. So, we need to know the value of the first and second, derivative of the price, with respect to S, that we respectively call $\Delta_{(\cdot)}$ and $\Gamma_{(\cdot)}$ strategies for a European call option and European put option.

In the Heston model the best way in literature to compute the Greeks by Fourier method is to use Shaw's approach [111]. This one is an alternative Fourier method, in which one has the advantage that the computation of the Greeks is straightforward. Δ and Γ can be got just by multiplying the Shaw's integral V with $-\imath\frac{\phi}{S}$ and $-\frac{\phi^2}{S^2}$ respectively.

$$V = \frac{1}{2\pi}e^{-r\tau}\int_{\imath c-\infty}^{\imath c+\infty} e^{\imath\phi x}\tilde{W}(\phi,\nu,0)G(\phi,\nu,\tau)d\phi,$$

where the Green's function has the form:

$$G(\phi,\nu,\tau) = e^{C(\tau,\phi)+\nu D(\tau,\phi)},$$

and

$$W(x,\nu,\tau) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-\imath\phi x}\tilde{W}(\phi,\nu,\tau)d\phi,$$

$$\tilde{W}(\phi,\nu,\tau) = \int_{-\infty}^{+\infty} e^{\imath\phi x}W(x,\nu,\tau)dx.$$

The initial condition of $\tilde{W}(\phi,\nu,\tau)$ for European Call is

$$\tilde{W}(\phi,\nu,0) = \int_{-\infty}^{+\infty} e^{\imath}W(x,\nu,0)dx = \int_{\log E}^{+\infty}\left(e^{(1+\imath\phi)x} - Ee^{\imath x}\right)dx = \frac{E^{1+\imath\phi}}{\imath\phi-\phi^2},$$

where the integral has to evaluated at $Im(\phi) > 1$; $x = \log S + r(T-t)$, $\tau = T-t$ and $W = Ve^{r(T-t)}$.

Using the Geometrical Approximation method we are able to provide in closed form the Greeks, proving the **Put-Call-Parity** relation. The proposed solution is simpler than Shaw's solution as one can see comparing them each other. In other words, by G .A. method, we obtain approximated solutions with formulas simple as well as in the B. S. case, but working with stochastic volatility market model.

$$\Delta_{call} = \frac{\partial C_{\rho,\alpha,\Theta,\kappa}}{\partial S} = (e^{\varepsilon})e^{\delta}N(d_1^{\rho}),$$

$$\Gamma_{call} = \frac{Ee^{\delta-(d_1^{\rho})^2/2}}{S\sqrt{2\pi(1-\rho^2)\nu(T-t)}},$$

and

$$\Delta_{put} = \frac{\partial P_{\rho,\alpha,\Theta,\kappa}}{\partial S} = -(e^{\varepsilon})e^{\delta}N(-d_1^{\rho}),$$

$$\Gamma_{put} = \frac{Ee^{\delta-(d_1^{\rho})^2/2}}{S\sqrt{2\pi(1-\rho^2)\nu(T-t)}},$$

thus we have

$$\Gamma_{put} = \Gamma_{call}.$$

The **Put-Call-Parity** condition is verified, and this is consistent with the assumption that we are in a free arbitrage market (see figures 4.1, 4.2).

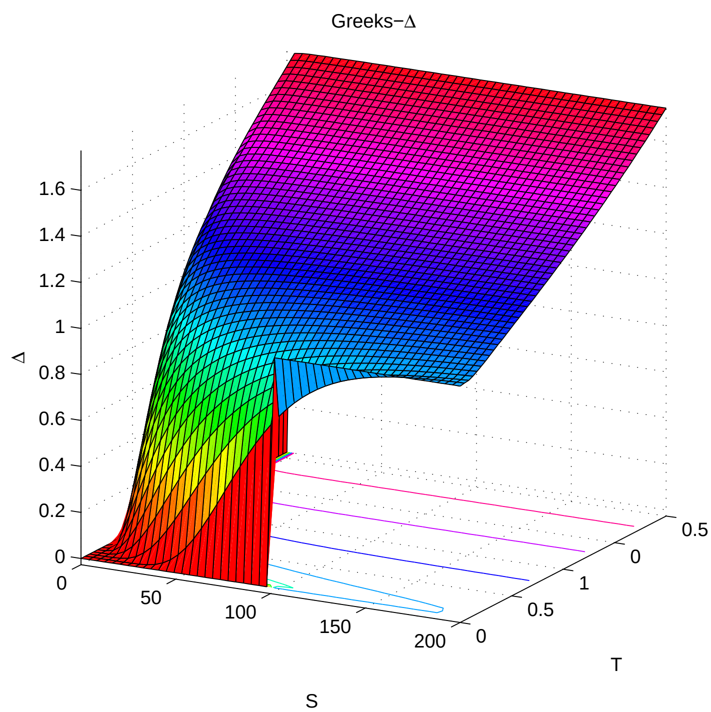

Figure 4.1: Greeks-Δ, for Call option computed by G. A. method in the Heston model, with the following parameter set: $S_0 = E = 100$, $\alpha = 0.39$, $\nu_0 = 0.03$, $\Theta = 0.04$, $\rho = -0.1$, $T = 1 - year$

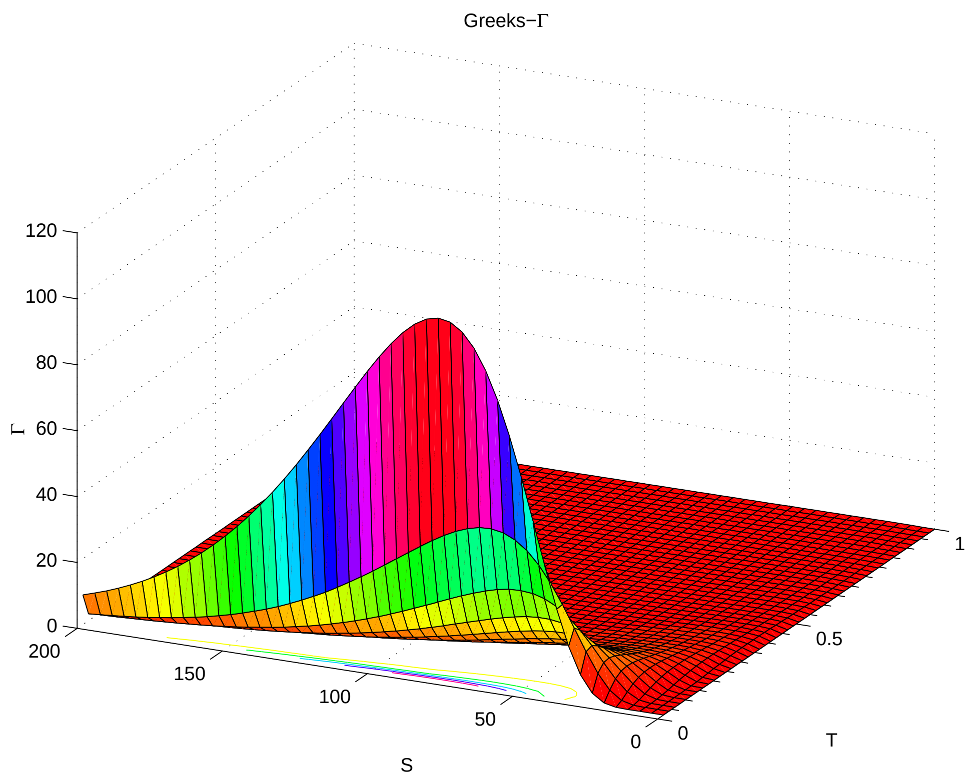

Figure 4.2: Greeks-Γ, for Call option computed by G. A. method in the Heston model, with the following parameter set: $S_0 = E = 100$, $\alpha = 0.39$, $\nu_0 = 0.03$, $\Theta = 0.04$, $\rho = -0.1$, $T = 1 - year$

4.6 Numerical comparison of G. A. with alternative methods

In this section we compare the G. A. method proposed here with the aforementioned numerical procedures proposed in the literature. The purpose of this section is then to evaluate the impact of the approximation needed by the G. A. method to get a closed-form solutions for Call prices, in the case of sensible parameter values.

Before we compare the Call option prices, we compute three volatility surfaces using the Fourier transform method. The Matlab code for implementing Fourier transform is transcript in the Appendix. We use the parameter set $\kappa = 2$, $\Theta = 0.04$, $\alpha = 0.4$, $\nu_0 = 0.038$,r = 0.01,$S_0 = 1$, $E \in [0.8, 1.2]$, $t \in [0.5, 3]$, which are the values proposed in [95]. To illustrate the impact of the correlation coefficient, we use $\rho = 0, +0.5, -0.5$ in Figures 4.3, 4.4, 4.5 respectively. The figures show that the impact of ρ is on the skewness of the volatility surface. With $\rho = 0$ (Figure 4.3) we observe the classical volatility smile whose magnitude decreases with maturity. With $\rho \neq 0$, the smile is instead skewed, see Figures 4.4 and 4.5.

We now compare the G. A. method with the three numerical methods (Fourier transform, Finite Difference method and Monte Carlo simulations). For values of $\rho = -1, +1$ we have two degenerate cases, but we are not going to consider these cases in our numerical experiments. The parameter values are those estimated in Bakshi, Cao and Chen (1997) and reported in table IV of their paper. Bakshi, Cao and Chen estimate the correlation under three different measure [6], and this results to be around -0.64 for derivatives on an underlying asset, and around -0.28 for derivatives on an index; the latter value has been obtained by the sample time-series correlation

Table 4.1: Call option prices computed with the G. A. method and three alternative numerical methods: Fourier transform method, Finite Difference Method (F. D. M.) and Monte Carlo method (M. C.). Parameter values are those in Bakshi, Cao and Chen (1997) namely $\kappa = 1.15$, $\Theta = 0.04$, $\alpha = 0.39$ and $\rho = -0.64$. We have chosen $r = 3\%$ $E = 100$, $\nu_0 = 0.03$ and three different maturities T. At-the-money options (ATM) have $S_0 = E$; in-the-money options (INM) have $S_0 = E(1 + 10\%\sqrt{\Theta T})$ and out-of-the money options (OTM) have $S_0 = E(1 - 10\%\sqrt{\Theta T})$.

$(T = 6/12)$				
	G. A.	Fourier	F. D. M.	M.C
ATM	5.4265	5.5707	5.5461	5.5574
INM	6.3111	6.5265	6.5030	6.5123
OTM	4.6244	4.6766	4.6491	4.6643
$(T = 9/12)$				
	G. A.	Fourier	F. D. M.	M.C
ATM	7.0120	7.0500	7.112	7.0300
INM	8.1344	8.2561	8.2115	8.2347
OTM	5.9894	5.9177	5.8770	5.8991
$(T = 1)$				
	G. A.	Fourier	F. D. M.	M.C
ATM	8.4300	8.3816	8.3293	8.3550
INM	9.7652	9.8020	9.7503	9.7734
OTM	7.2086	7.0445	6.9907	7.0196

Table 4.2: Same as table (4.2) with $\rho = -0.28$.

($T = 6/12$)				
	G. A.	Fourier	F. D. M.	M. C.
ATM	5.5022	5.5548	5.5479	5.5524
INM	6.3531	6.4566	66.4434	6.4431
OTM	4.7178	4.7229	4.7077	4.7117
($T = 9/12$)				
	G. A.	Fourier	F. D. M.	M. C.
ATM	6.9829	7.0383	7.0297	7.0196
INM	8.0571	8.1761	8.1692	8.1558
OTM	5.9895	5.9851	5.9757	5.9680
($T = 1$)				
	G. A.	Fourier	F. D. M.	M. C.
ATM	8.3093	8.3751	8.3905	8.3501
INM	9.5816	9.7171	9.7352	9.6901
OTM	7.1294	7.1286	7.1425	7.1058

between daily S&P 500 index returns and daily changes in the implied volatility of a given option model. The G. A. method needs low correlation in order to reduce the stochastic error e^{ε_T}, so that for $\rho = -0.28$ we have that the G. A. prices, see table 4.2, are precise with an error around 3%.

Before to discuss a case of high correlation, it is worth noting that, the G. A method gets the price of the modified payoff, which for negative correlation is always out the money. In other word, if we supposed to be in the case at money, in which we have $S_0 = E = 100$, the G. A method gets the price for $S_0 e^{\varepsilon_0} < E$. Thus for $\rho < 0$ we get the price for an option out the money. For $|\rho|$ smaller than 0.3, this problem is neglectable for maturity T within $1 - year$, but for $|\rho|$ bigger than 0.3, we have to consider this one, adding the initial error capitalised at rate r, i.e., we supposed to buy a bond at time zero so that at maturity its value is $|S_0 e^{\varepsilon_0} - E| e^{r(T-t)}$. This procedure is reasonable for high level of correlation in which the difference at time zero between $S_0 e^{\varepsilon_0} - E$ is sensible. In fact by this reasoning, for $\rho = -0.64$, the G. A. technique is precise with an error around 5%, that otherwise would have been around 10%, see table 4.1 (see the code in Appendix).

Using a correlation lower than before, for example $\rho = -0.1$, we obtain that the G. A. prices are close to the Fourier prices, as we are going to show later by Figures 4.10, 4.11, 4.12. In fact, in this case we make an error less than 2%, see table 4.3.
The integration formula (4.13) we use is based on Gauss-Legendre approximation with 20 points; the routine is implemented in Matlab and reported in the Appendix. To implement the Monte Carlo method we have written a code in $C++$, in which we use a time step of $1/250$ (in yearly units) and $N = 10^6$ trajectories (see the Appendix for details); either it is possible to use the "Fairmat" software (there exists an academic version) based on the Monte Carlo method, by which obtain the prices of every kind of derivative contracts. For Finite Difference method we use the results in the website $(http://Kluge.in-chemnitz.de)$, [3], [68] but in the Appendix one can find the code of Crank-Nicolson algorithm written in MatLab.

Table 4.3: Same as table (4.1) with $\rho = -0.1$.

	G. A.	Fourier	F. D. M.	M. C.
($T = 6/12$)				
ATM	5.6247	5.5419	5.5348	5.5300
INM	6.4689	6.4152	6.4089	6.4022
OTM	4.8449	4.7413	4.7338	4.7305
($T = 9/12$)				
	G. A.	Fourier	F. D. M.	M. C.
ATM	7.1051	7.0212	7.0368	7.0032
INM	8.1681	8.1231	8.1407	8.1035
OTM	6.1202	6.0078	6.0231	5.9915
($T = 1$)				
	G. A.	Fourier	F. D. M.	M. C.
ATM	8.4165	8.3547	8.4334	8.3307
INM	9.6725	9.6556	9.7383	9.6295
OTM	7.2499	7.1542	7.2308	7.1324

It is worth spending some words to describe the distribution of the stochastic error term e^{ε_T} which, in the G. A. method, is assumed to be close to 1. Let us remark that the stochastic error is given by:

$$e^{\varepsilon_T} = e^{\frac{\rho\left\{\left[(\nu_0-\Theta)e^{-\kappa T}+\Theta\right]-\nu_T\right\}}{\alpha}}, \tag{4.33}$$

thus its stochasticity fully depends on the random variable ν_T. We simulate the Heston model with the parameter values in [89], with different values of the correlation coefficient ρ. The distribution of e^{ε_T} is reported in Figures 4.6, 4.7, 4.8, 4.9 with $(\rho = -0.1, T = 1/12)$, $(\rho = -0.1, T = 1)$, $(\rho = -0.9, T = 1/12)$, $(\rho = -0.9, T = 1)$. As one can see the stochastic error distributes approximatively as a gaussian. In the case of low correlation we have a distribution close to 1 and small variance, either for short maturities than for long maturities. Unlike for the high correlation in which case we have again a distribution for the error, but with higher variance.

Finally, we evaluate the difference between the option price value computed with the Fourier method and the G. A. method as a function of the maturity of the option. Figures 4.10, 4.11, 4.12 show the results, see the captions for parameter values. We can see by those Figures 4.10, 4.11, 4.12 that the distance of the G. A. value from the Fourier transform value is reduced when we move in the money and at the money and not out the money. We can observe that our solution (4.30) verifies the law of monotonicity with respect to maturity.

Concluding, the numerical experiments highlight that the G. A. method can be used for evaluating option contracts when the correlation is low. In our experiments with sensible data, we find that a reasonable estimate of the approximation error is around 1% (clearly, the precise value depends on the parameter set adopted). This error increases with maturity and correlation coefficient (as well as inversely with the mean-reversion speed). It appears clear that a crucial parameter is the correlation, with the G. A. being more reliable for low values of the correlation. In other words, the accuracy of he G. A. method is mostly determined by the magnitude of the correlation. Markets in which the price/volatility correlation is low, and thus the G. A. method seems more promising, are the Electricity Markets, see [28], [99].

$Volatility \quad surface \quad in \quad the \quad Heston \quad model$

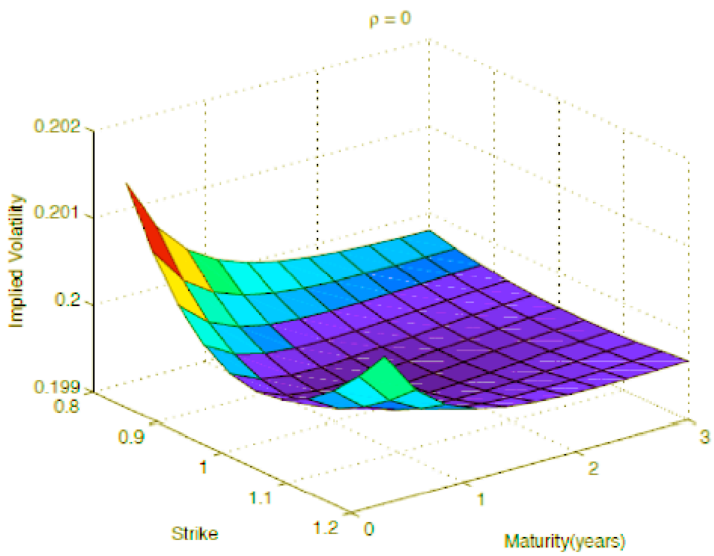

Figure 4.3: Implied volatility surface in the Heston model, $\rho = 0$, $k = 2$, $\Theta = 0.04$, $\alpha = 0.4$, $\nu_0 = 0.04$,r = 0.01,$S_0 = 1$, $E \in [0.8, 1.2]$, $t \in [0.5, 3]$

$Volatility \quad surface \quad in \quad the \quad Heston \quad model$

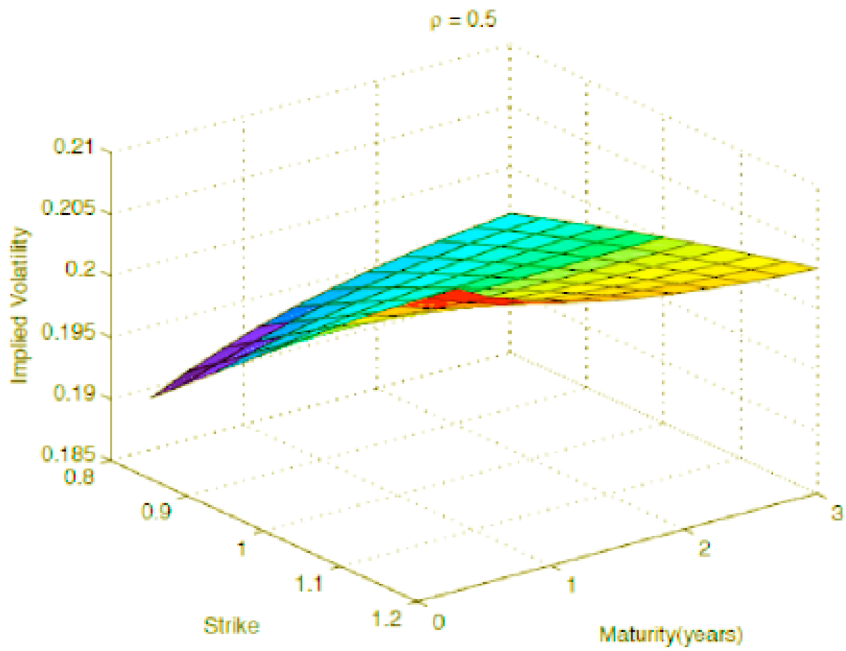

Figure 4.4: Implied volatility surface in the Heston model, $\rho = +0.5$, $k = 2$, $\Theta = 0.04$, $\alpha = 0.4$, $\nu_0 = 0.04$,r = 0.01,$S_0 = 1$, $E \in [0.8, 1.2]$, $t \in [0.5, 3]$

$Volatility \quad surface \quad in \quad the \quad Heston \quad model$

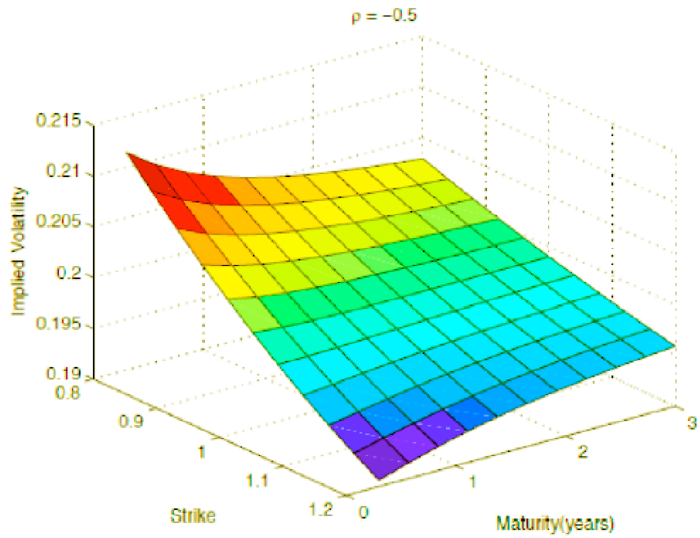

Figure 4.5: Implied volatility surface in the Heston model, $\rho = -0.5$, $k = 2$, $\Theta = 0.04$, $\alpha = 0.4$, $\nu_0 = 0.04$,r = 0.01,$S_0 = 1$, $E \in [0.8, 1.2]$, $t \in [0.5, 3]$

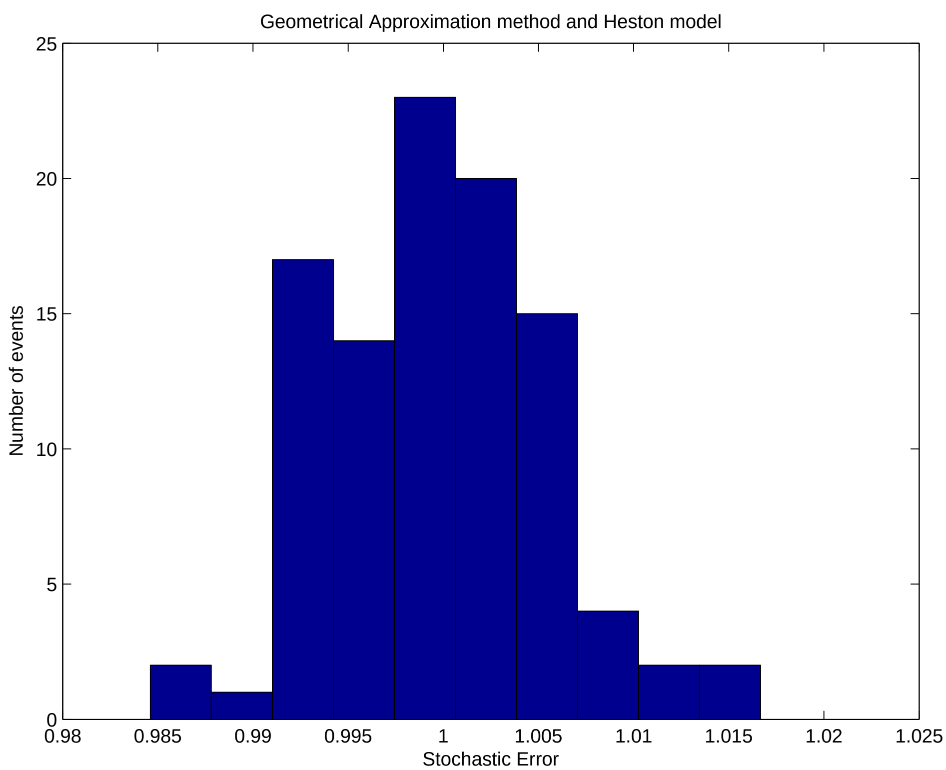

Figure 4.6: Distribution of the stochastic error e^{ε_T}, obtained via simulation with parameter values in [6] and $\rho = -0.1$, $T = 1$-month

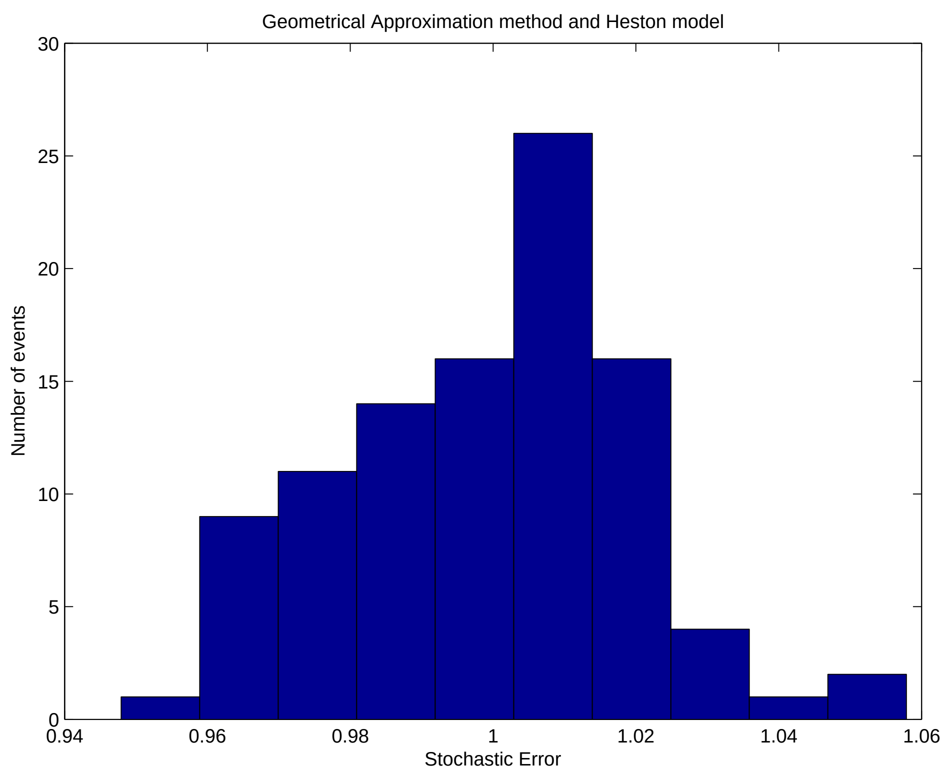

Figure 4.7: Distribution of the stochastic error e^{ε_T}, obtained via simulation with parameter values in [6] and $\rho = -0.1$, $T = 1$-year

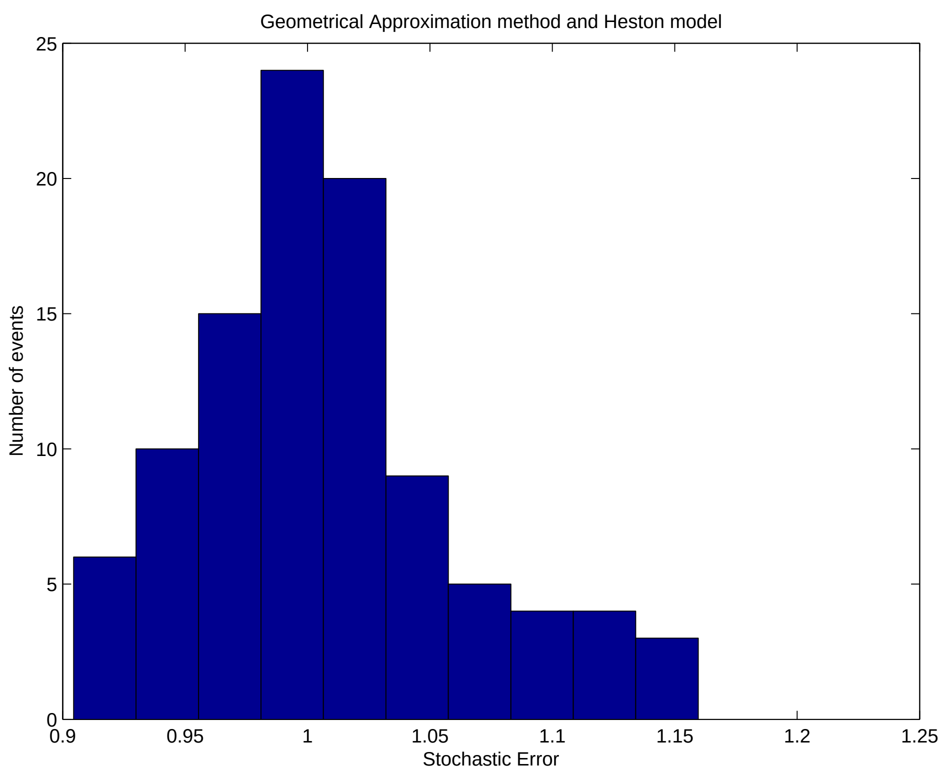

Figure 4.8: Distribution of the stochastic error e^{ε_T}, obtained via simulation with parameter values in [6] and $\rho = -0.9$, $T = 1$-month

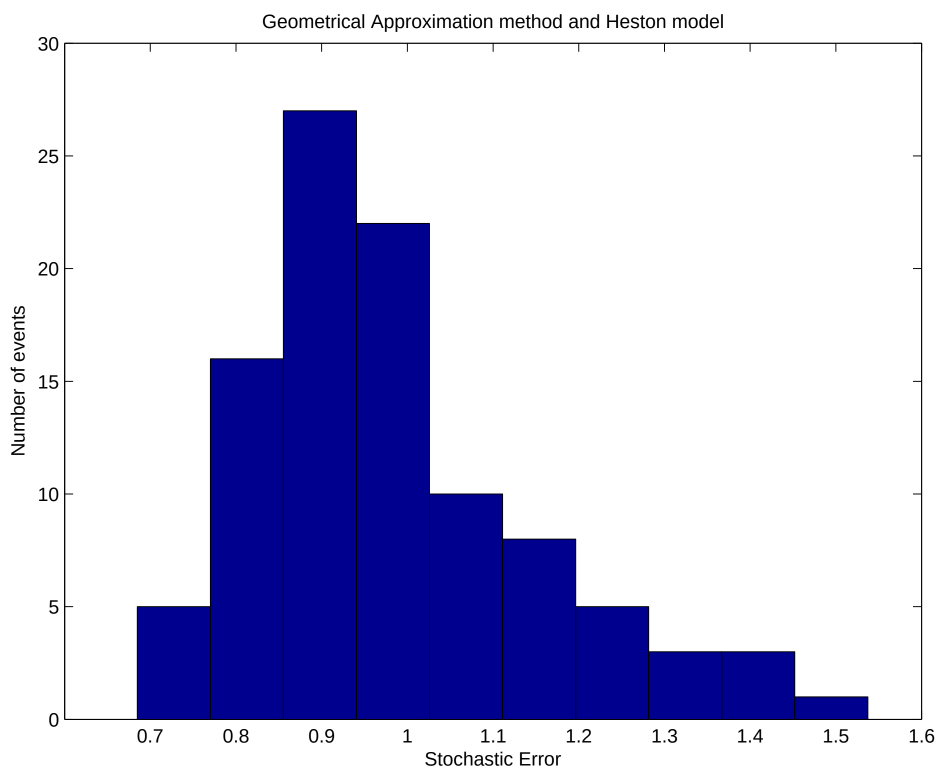

Figure 4.9: Distribution of the stochastic error e^{ε_T}, obtained via simulation with parameter values in [6] and $\rho = -0.9$, $T = 1$-year

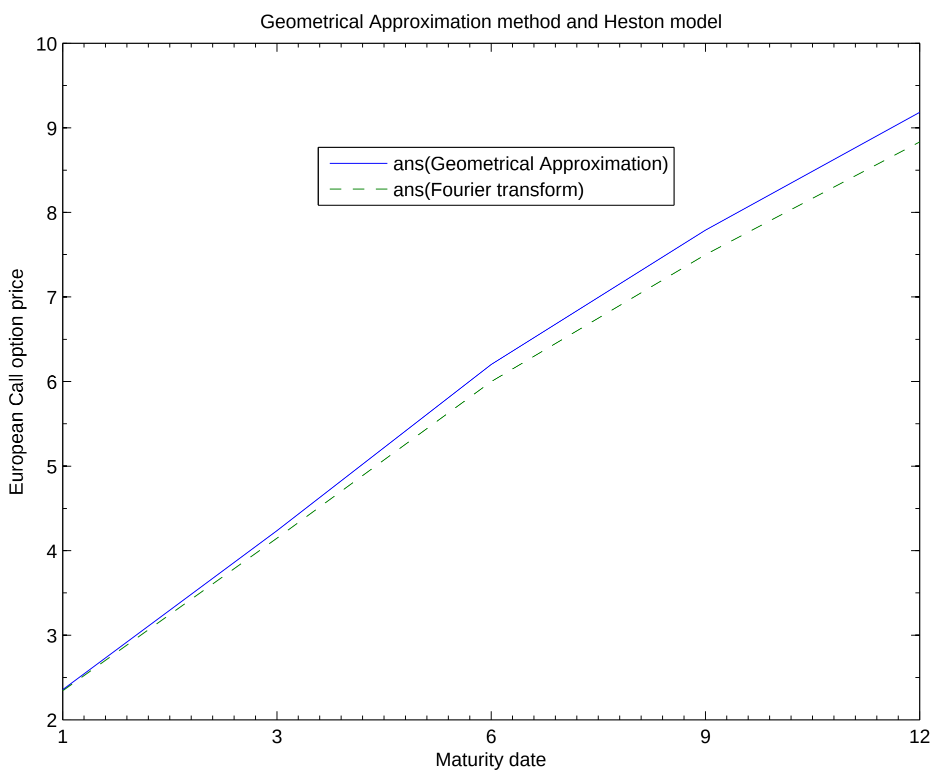

Figure 4.10: **At the money** - Call prices computed with the Fourier integral and with the Geometrical Approximation as a function of the maturity, for $\rho = -0.1$, $k = 1.15$, $\Theta = 0.04$, $\nu_0 = 0.038$, $\alpha = 0.39$, [6]. The points of the abscissas are the maturities in months.

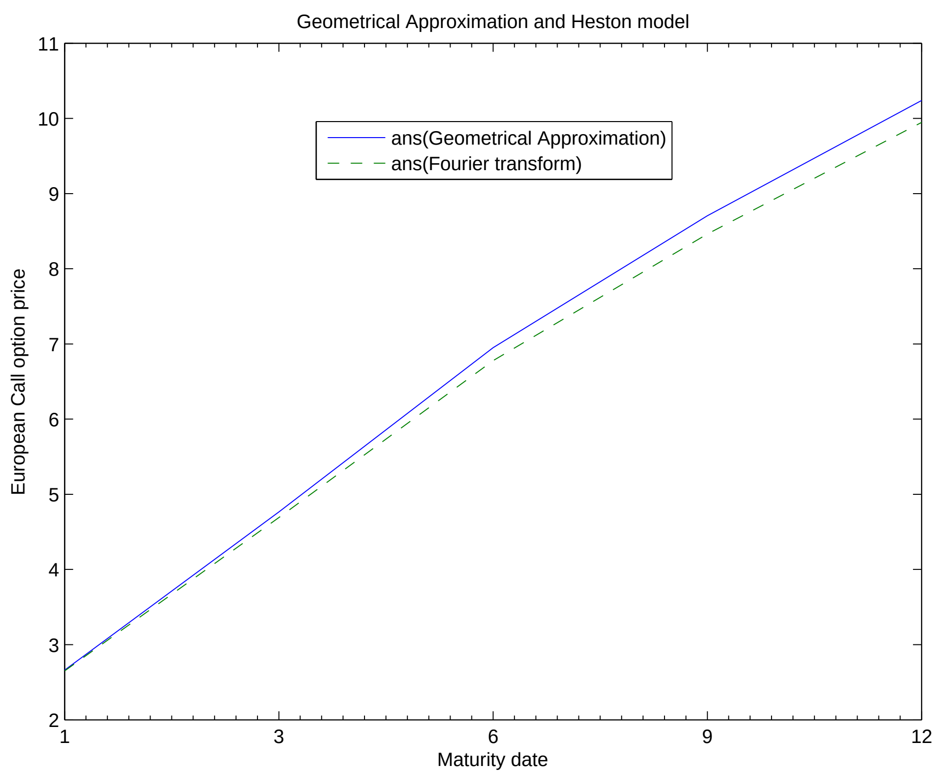

Figure 4.11: **In the money** - Call prices computed with the Fourier integral and with the Geometrical Approximation as a function of the maturity, for $\rho = -0.1$, $k = 1.15$, $\Theta = 0.04$, $\nu_0 = 0.038$ $\alpha = 0.39$, [6]. The points of the abscissas are the maturities in months.

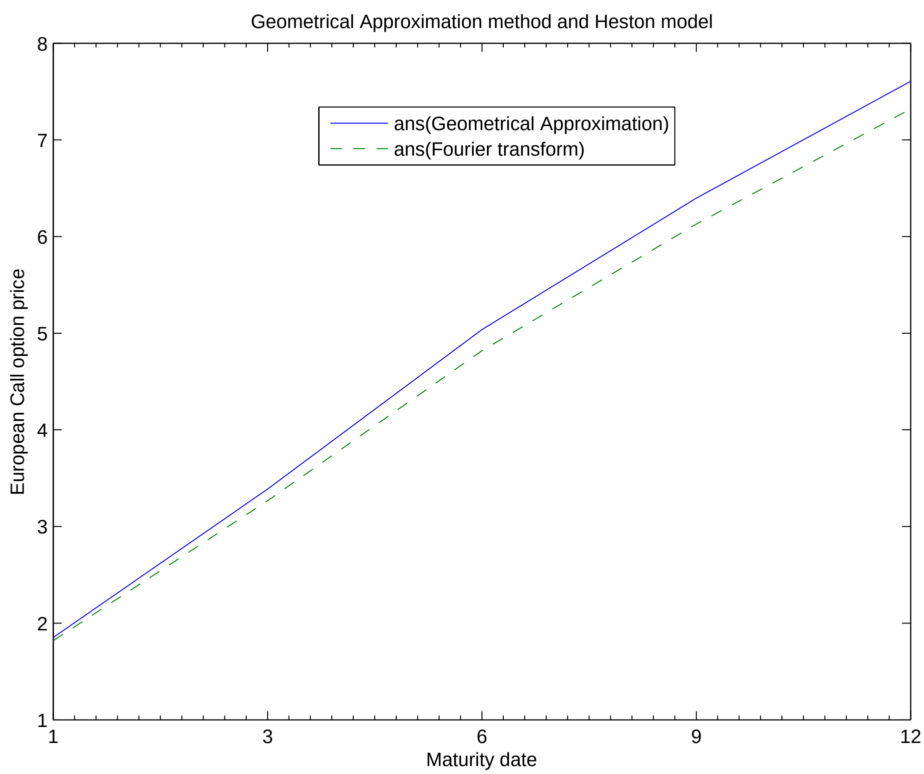

Figure 4.12: **Out the money**- Call prices computed with the Fourier integral and with the Geometrical Approximation as a function of the maturity, for $\rho = -0.1$, $k = 1.15$, $\Theta = 0.04$, $\nu_0 = 0.038$ $\alpha = 0.39$, [6]. The points of the abscissas are the maturities in months.

4.7 SABR model

In a quite recent paper by Hagan et al. (2002-2003), the authors examine the issue of dynamics of the implied volatility smile. They argue that any model based on the local volatility function incorrectly predicts the future behaviour of the smile, i.e., when the price of the underlying asset decreases, local volatility models predict that the smile shifts to higher prices. Similarly an increase of the price results in a shift of the smile to lower prices. It was observed that the market behaviour of the smile is precisely the opposite. Thus, the local volatility model has the inherent flaw of predicting the wrong dynamics of the Black-Scholes implied volatility. Consequently, hedging strategies based on such a model may be worse than the hedging strategies evaluated for the naive model with constant volatility as those of Black-Scholes models. A particular model proposed and analysed by Hagan et al. (2002-2003) is specified as follows: under a martingale measure $\mathbb{Q}$ the forward price is assumed to obey the SDE

$$dF_t = \sigma_t^F F_t^{\beta} d\tilde{W}_t^{(1)} \qquad \beta \in (0,1], \tag{4.34}$$

and

$$d\sigma_t^F = \alpha \sigma_t^F d\tilde{W}_t^{(2)} \qquad \alpha \in \mathbb{R}, \tag{4.35}$$

where $\tilde{W}_t^{(1)}$ and $\tilde{W}_t^{(2)}$ are Brownian motions with respect to a common filtration $\mathbb{F}^W$, with a constant correlation coefficient $\rho \in [-1,1]$. The model given by (4.34)-(4.35) is known as the SABR model. It can be seen as a natural extension of the classical CEV model, proposed by Cox(1975). The model can be accurately fitted to the observed implied volatility curve for a single maturity T. A more complicated version of the model is needed if we wish to fit volatility smiles at several different maturities. More importantly, the model seems to predict the correct dynamics of the implied volatility skews (as opposed to the CEV model or any model based on the concept of a local volatility function). To support this claim, Hagan et al. (2002) derive and study the approximate formulas for the implied Black and Bachelier volatilities in the SABR model. It appears that the Black implied volatility $\hat{\sigma}(K,T)$, in this model can be represented as follows:

$$\hat{\sigma}(E,T) = \frac{\sigma_0}{(F_0/E)^{(1-\beta)/2}\left(1 + \frac{(1-\beta)^2}{24}\ln^2(F_0/E) + \frac{(1-\beta)^4}{1920}\ln^4(F_0/E) + ...\right)} \times \frac{z}{\chi(z)}\left\{1 + \left[\frac{(1-\beta)^2\sigma_0^2}{24(F_0E)^{(1-\beta)}} + \frac{\rho\beta\sigma_0\alpha}{4(F_0E)^{(1-\beta)/2}} + \frac{(2-3\rho^2)\alpha^2}{24}\right]T + ...\right\}, \tag{4.36}$$

where E is the strike price, F_0 is the underlying asset value at the time $t=0$ and σ_0 is the value of the volatility at time $t=0$,

$$z = \frac{\alpha}{\sigma_0}(F_0/E)^{(1-\beta)/2}\ln(F_0/E),$$

and

$$\chi(z) = \ln\left\{\frac{\sqrt{1-2\rho z + z^2} + z - \rho}{1-\rho}\right\}.$$

In the case of at-the-money option, the formula above reduces to

$$\hat{\sigma}(F_0,T) = \frac{\sigma_0}{F_0^{(1-\beta)}}\left\{1+\left[\frac{(1-\beta)^2\sigma_0^2}{24(F_0)^{2(1-\beta)}}+\frac{\rho\beta\sigma_0\alpha}{4(F_0)^{(1-\beta)}}+\frac{(2-3\rho^2)\alpha^2}{24}\right]T+.......\right\}.$$

It is worth noting that the SABR model has an accurate asymptotic solution. This solution, as well as its implications for pricing the interest derivatives, has been described in [54]. In the paper written by P. Hagan , A. Lesniewski and D. Woodward, has been obtained the improving of the results presented in [55]. First the authors present a more systematic framework for generating an accurate, asymptotic form of the probability distribution in the SABR model; and second they address the issue of low strikes, or the behaviour of the model as the forward rate approaches zero. The study is based on a WKB type expansion for the heat kernel of a perturbed Laplace-Beltrami operator on a suitable hyperbolic Riemannian manifold [55].

4.7.1 Geometrical Approximation method: SABR model

It is our intention use the Geometrical Approximation method also for SABR market model. We can accomplish this only in the case in which the parameter β is equal to 1 [27].

Let the following market, under natural measure $\mathbb{P}$, be given:

$$\begin{aligned}
&dS_t = \mu_t^{(S)} S_t dt + \sigma_t S_t dW_t^{(1)},\\
&d\sigma_t = \mu_t^{(\sigma)} \sigma_t dt + \alpha\sigma_t dW_t^{(2)},\\
&dB_t = rB_t dt,\\
&dW_{t,(\mathbb{P})}^{(1)} dW_t^{(2)} = \rho dt,\\
&f(T,S_T,\sigma_T) = \phi(S_T),
\end{aligned} \tag{4.37}$$

in which S_t is the underlying asset value at time t, σ_t is the stochastic volatility, ρ is the correlation factor between $W^{(1)}$, $W^{(2)}$, that are Brownian motions, and finally B_t is a zero coupon bond with borrowing interest rate r, and $f(T,S_T,\sigma_T) = \phi(S_T)$ is a derivative contract. The market price risk for S_t is given by

$$\lambda_t^{(S)}(S_t,\sigma_t,t) = \frac{r-\mu_t^{(S)}}{\sigma_t}. \tag{4.38}$$

Now we choose the market price of volatility risk, in order to have the SABR model with $\beta = 1$, as follows:

$$\lambda_t^{(\sigma)}(\sigma_t,t) = \frac{-\mu_t^{(\sigma)}}{\alpha}. \tag{4.39}$$

Under the martingale measure $\mathbb{Q}$, the forward price is assumed to obey the SDE:

$$\begin{aligned}
&dF_t = \sigma_t^{(F)} F_t d\tilde{W}_t^{(1)}, \qquad F_t \in [0,\infty), t \in [0,T], \beta \in (0,1),\\
&d\sigma_t^{(F)} = \alpha \sigma_t^{(F)} d\tilde{W}_t^{(2)}, \qquad \alpha \in \mathbb{R},\\
&d\tilde{W}_t^{(1)} d\tilde{W}_t^{(2)} = \rho dt, \qquad \rho \in [-1,1]\\
&dB_t = rB_t dt,\\
&f(T, F_T, \sigma_T^F) = \Phi(F_T),
\end{aligned}$$

where $F_t = e^{r(T-t)} S_t$ is the forward price, and $\Phi(F_T)$ is a generic derivative contract. As well as in the Heston case, we compute the expected value of volatility process also for SABR model:

$$\overline{\sigma} = \mathbb{E}_{\mathbb{Q}}[\sigma_T] = \sigma_0 e^{\frac{\alpha^2}{2}T},$$

and we define $\varepsilon_T = \frac{\rho}{\alpha}(\overline{\sigma} - \sigma_T)$. Also in this case we have: $\mathbb{E}_{\mathbb{Q}}[\varepsilon_T] = 0$. We now are going to repeat the same procedure that we have used in Heston model in order to compute the price of the Vanilla Options.

As seen for Heston case, we use a modified pay-off instead of $(F_T - E)^+$. For a Call we have: $\left(F_T e^{\frac{\rho}{\alpha}\varepsilon_T} - E\right)^+ = \left(F_T e^{\frac{\rho}{\alpha}(\overline{\sigma}-\sigma_T)} - E\right)^+ = e^{\frac{\rho\overline{\sigma}}{\alpha}}\left(F_T e^{\frac{-\rho\sigma_T}{\alpha}} - E e^{\frac{-\rho\overline{\sigma}}{\alpha}}\right)^+$, finally considering $\overline{E} = E e^{\frac{-\rho\overline{\sigma}}{\alpha}}$, we have the following modified pay-off $e^{\frac{\rho\overline{\sigma}}{\alpha}}\left(F_T e^{-\frac{\rho\sigma_T}{\alpha}} - \overline{E}\right)^+$, as well as for the Heston case.
We use the PDE formalism, thus we replace F_t with F and σ_t with σ. The Cauchy's problem for the SABR case is given by:

$$\begin{aligned}
&\frac{\partial f}{\partial t} + \frac{1}{2}(\sigma)^2\left(F^2\frac{\partial^2 f}{\partial F^2} + 2\rho F\alpha\frac{\partial^2 f}{\partial F\partial\sigma} + \alpha^2\frac{\partial^2 f}{\partial \sigma^2}\right) - rf = 0 \qquad F \in [0,+\infty), \quad \sigma \in [0,+\infty),\\
&f(0,F,\sigma) = e^{\frac{\rho\overline{\sigma}}{\alpha}}\left(Fe^{-\frac{\rho\sigma}{\alpha}} - \overline{E}\right)^+, \qquad t \in [0,T],
\end{aligned} \tag{4.40}$$

for which $f \in C^{2,1}\left([0,+\infty) \times [0,+\infty) \times [0,T]\right)$.

In order to simplify eq. (4.40), we change some variables:

$$
\begin{aligned}
&x = \ln F, \qquad x \in (-\infty, \infty) \quad t \in [0, T] \\
&\tilde{\sigma} = \frac{\sigma}{\alpha}, \qquad \alpha \in \mathbb{R}, \quad \tilde{\sigma} \in [0, \infty); \\
&f(t, F, \sigma) = e^{-r(T-t)} f_1(t, x, \tilde{\sigma}),
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial f_1}{\partial t} + \frac{1}{2}(\tilde{\sigma})^2\alpha^2 \left(\frac{\partial^2 f_1}{\partial x^2} + 2\rho \frac{\partial^2 f_1}{\partial x \partial \tilde{\sigma}} + \frac{\partial^2 f_1}{\partial \tilde{\sigma}^2} \right), -\frac{1}{2}(\tilde{\sigma})^2\alpha^2 \frac{\partial f_1}{\partial x} = 0, \\
&f_1(T, x, \tilde{\sigma}) = e^{-\frac{\rho\tilde{\sigma}}{\alpha}} \left(e^{x-\rho\tilde{\sigma}} - \overline{E} \right)^+,
\end{aligned}
$$

in which we have considered the modified payoff: $e^{-\frac{\rho\tilde{\sigma}}{\alpha}} \left(e^{x-\rho\tilde{\sigma}} - \overline{E} \right)^+$, instead of $(e^x - E)^+$ with respect to the new variables $x, \tilde{\sigma}$. Using the same method that we have used in the previous sections, we have:

$$
\begin{aligned}
&V_\xi = x - \rho\tilde{\sigma}, \qquad V_\xi \in (-\infty, +\infty), \\
&V_\eta = -\tilde{\sigma}\sqrt{1-\rho^2}, \qquad V_\eta \in (-\infty, 0], \\
&\tau = \frac{1}{2}\int_t^T \sigma_s^2 ds, \qquad \tau \in \left[0, \frac{1}{2}\int_0^T \sigma_s^2 ds \right], \\
&f_1(t, x, \tilde{\sigma}, t) = f_2(\tau(t, V_\eta), V_\xi(x, \tilde{\sigma}))
\end{aligned}
$$

The PDE to solve is:

$$
\begin{aligned}
&\frac{\partial f_2}{\partial \tau} = (1-\rho^2)\frac{\partial^2 f_2}{\partial V_\xi^2} - \frac{\partial f_2}{\partial V_\xi}, \\
&f_2(0, V_\xi) = e^{\frac{\rho\tilde{\sigma}}{\alpha}} \left(e^{V_\xi} - \overline{E} \right)^+.
\end{aligned}
$$

Now in order to eliminate the linear term, we make the following transformation

$$
f_2(\tau, V_\xi) = e^{\frac{V_\xi}{2(1-\rho^2)}} f_3(\tau, V_\xi),
$$

and we obtain

$$
\begin{aligned}
&\frac{\partial f_3}{\partial \tau} = (1-\rho^2)\frac{\partial^2 f_3}{\partial V_\xi^2}, \qquad V_\xi \in (-\infty, +\infty), \quad \tau \in \left[0, \frac{1}{2}\int_0^T \sigma_s^2 ds \right], \\
&f_3(0, V_\xi) = e^{-\frac{V_\xi}{2(1-\rho^2)}} e^{\frac{\rho\tilde{\sigma}}{\alpha}} \left(e^{V_\xi} - \overline{E} \right)^+.
\end{aligned}
\tag{4.41}
$$

Thus, the solution of the PDE (4.41) is given by:

$$f(t,F,\sigma) = \frac{e^{-r(T-t)+\frac{V_\xi}{2(1-\rho^2)}}}{2\sqrt{\pi(1-\rho^2)\tau}} \int_{-\infty}^{+\infty} dV'_\xi e^{-\frac{V'_\xi}{2(1-\rho^2)}} e^{\frac{\rho\overline{\sigma}}{\alpha}} \left(e^{V_\xi} - \overline{E}\right)^+ exp\left[-\frac{(V'_\xi - V_\xi)^2}{4(1-\rho^2)\tau}\right]$$
$$= \left(Fe^{-r(T-t)}\right) e^{\frac{\rho}{\alpha}(\overline{\sigma}-\sigma)} e^{\left(\frac{(1-2\rho^2)}{4(1-\rho^2)}\int_t^T \sigma_s^2 ds\right)} \mathbf{N}(d_1^\rho) - Ee^{-r(T-t)+\frac{\int_t^T \sigma_s^2 ds}{8(1-\rho^2)}} \mathbf{N}(d_2^\rho).$$

Thus we have

$$f(t,F,\sigma) = \left(Fe^{\frac{\rho}{\alpha}(\overline{\sigma}-\sigma)}\right) e^{-\left(r - \frac{(1-2\rho^2)}{8(1-\rho^2)}\frac{1}{T-t}\int_t^T \sigma_s^2 ds\right)(T-t)} \mathbf{N}(d_1^\rho) - Ee^{-\left(r - \frac{\frac{1}{T-t}\int_t^T \sigma_s^2 ds}{8(1-\rho^2)}\right)(T-t)} \mathbf{N}(d_2^\rho), \tag{4.42}$$

where

$$\delta_1^\rho = -\left(r - \frac{(1-2\rho^2)}{8(1-\rho^2)}\frac{1}{T-t}\int_t^T \sigma_s^2 ds\right)(T-t),$$
$$\delta_2^\rho = -\left(r - \frac{\frac{1}{T-t}\int_t^T \sigma_s^2 ds}{8(1-\rho^2)}\right)(T-t), \tag{4.43}$$

and

$$d_1^\rho = \frac{\ln\left(Fe^{\frac{\rho}{\alpha}(\overline{\sigma}-\sigma)}/E\right) + \frac{1}{2}(1-2\rho^2)\int_t^T \sigma_s^2 ds}{\sqrt{(1-\rho^2)\int_t^T \sigma_s^2 ds}},$$
$$d_2^\rho = \frac{\ln\left(Fe^{\frac{\rho}{\alpha}(\overline{\sigma}-\sigma)}/E\right) - \frac{1}{2}\int_t^T \sigma_s^2 ds}{\sqrt{(1-\rho^2)\int_t^T \sigma_s^2 ds}}.$$

It is worth noting that also for SABR model, we are able to obtain the Hull-White formula for $\rho = 0$. The price for a Call option, written on forward value F of underlying asset in a SABR market model for $\beta = 1$, is given by:

$$C(t,F,\sigma) = (Fe^\varepsilon)\, e^{\delta_1^\rho}\mathbf{N}(d_1^\rho) - Ee^{\delta_2^\rho}\mathbf{N}(d_2^\rho), \tag{4.44}$$

where $\varepsilon = \frac{\rho}{\alpha}(\overline{\sigma} - \sigma)$. For a Put option we have:

$$P(t,F,\sigma) = Ee^{\delta_2^\rho}\mathbf{N}(-d_1^\rho) - (Fe^\varepsilon)\, e^{\delta_1^\rho}\mathbf{N}(-d_1^\rho). \tag{4.45}$$

Contrariwise that in Heston model, in the SABR case, we may define a theoretical error function given by the difference between the Hagan (2002) formula and G. A. formula. For a Call we have:

$$Err = \left| C^{Hagan}(t,F,\sigma) - C^{(G.A.)}(t,F,\sigma) \right|$$
$$= \left| F \left[e^{-r(T-t)}\mathbf{N}(d_1) - e^{\varepsilon+\delta_1^\rho}\mathbf{N}(d_1^\rho) \right] - E \left[e^{-r(T-t)}\mathbf{N}(d_2) - e^{\delta_2^\rho}\mathbf{N}(d_2^\rho) \right] \right|,$$

where $d_{1,2} = \frac{log(F/E) \pm \frac{1}{2}\hat{\sigma}^2(E,T)(T-t)}{\sqrt{\hat{\sigma}^2(E,T)(T-t)}}$ and $\hat{\sigma}(E,T) = \sigma_0 \left(\frac{z}{\chi(z)} \right) \left\{ 1 + \left[\frac{\rho\alpha\sigma_0}{4} + \frac{2-3\rho^2}{24}\alpha^2 \right] \times T \right\}$, $z = \frac{\alpha}{\sigma_0} \ln(F_0/E)$.

4.7.2 Greeks and Put-Call-parity

Exactly like in the Heston's model, also in the SABR model, in order to find the best hedging strategy, we use a replicant portfolio. So we need to know the value of the first and second derivative of the price, with respect to F, that we respectively name $\Delta_{(\cdot)}$ and $\Gamma_{(\cdot)}$ strategies:

$$\Delta_{call} = \frac{\partial C(t,F,\sigma)}{\partial F} = (e^\varepsilon)\, e^{\delta_1^\rho}\mathbf{N}(d_1^\rho),$$

$$\Gamma_{call} = \frac{\partial^2 C(t,F,\sigma)}{\partial F^2} = \frac{E e^{\frac{(1-2\rho^2)}{8(1-\rho^2)}\sigma^2(T-t) - \frac{(d_1^\rho)^2}{2}}}{F\sqrt{2\pi\sigma^2(T-t)}},$$

and

$$\Delta_{put} = \frac{\partial P(t,F,\sigma)}{\partial F} = -(e^\varepsilon)\, e^{\delta_1^\rho}\mathbf{N}(-d_1^\rho),$$

$$\Gamma_{put} = \frac{\partial^2 P(t,F,\sigma)}{\partial F^2} = \frac{E e^{\frac{(1-2\rho^2)}{8(1-\rho^2)}\sigma^2(T-t) - \frac{(d_1^\rho)^2}{2}}}{F\sqrt{2\pi\sigma^2(T-t)}},$$

thus we have

$$\Gamma_{put} = \Gamma_{call}.$$

In the figures hereafter 4.13, 4.14, we can see the drawing of the Greeks, Δ and Γ over the time, and these one have a trend compatible with what is known in literature.

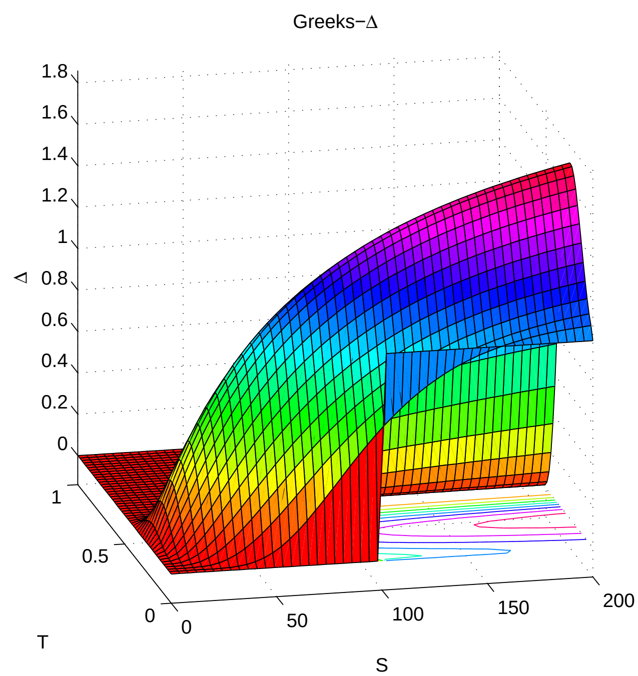

Figure 4.13: Greeks-Δ, for Call option computed by G. A. method in the SABR model, with the following parameter set: $S_0 = E = 100$, $\alpha = 0.29$, $\sigma_0 = 20\%$, $\rho = -0.1$, $T = 1 - year$

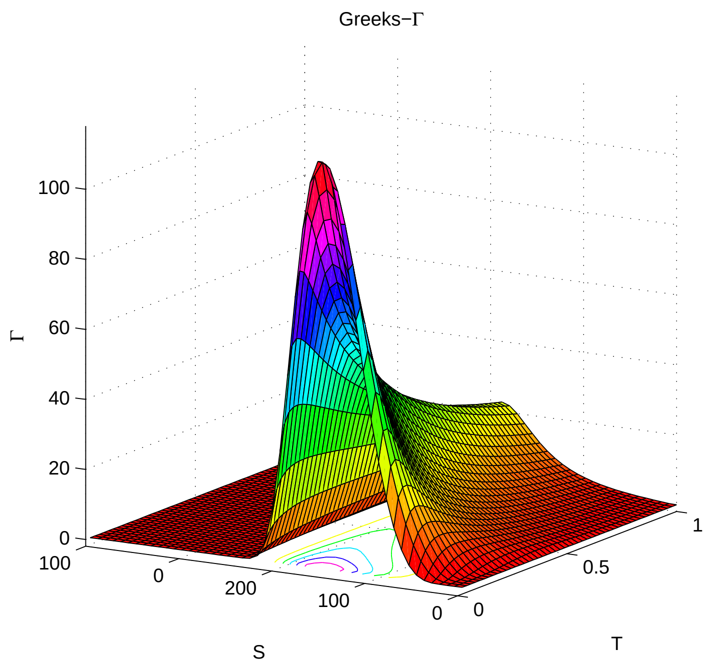

Figure 4.14: Greeks-Γ, for Call option computed by G. A. method in the SABR model, with the following parameter set: $S_0 = E = 100$, $\alpha = 0.29$, $\sigma_0 = 20\%$, $\rho = -0.1$, $T = 1 - year$

4.7.3 Numerical comparison

In this section we compute and compare the Call option prices (see Table 4.4, 4.5) calculated by our approximation method with those obtained by Hagan formula (2002), in the case of $\beta = 1$ (log-normal case):

$$\hat{\sigma}(E,T) = \sigma_0 \left(\frac{z}{\chi(z)}\right)\left\{1 + \left[\frac{\rho\alpha\sigma_0}{4} + \frac{2-3\rho^2}{24}\alpha^2\right] \times T\right\},$$

$$z = \frac{\alpha}{\sigma_0}\ln(F_0/E),$$

and

$$\chi(z) = \ln\left\{\frac{\sqrt{1-2\rho z + z^2} + z - \rho}{1-\rho}\right\}.$$

The algorithms are written in the MatLab code reported in the Appendix. The parameters set used is that obtained in "The SABR LIBOR Market Model" by R.Rebonato, K. MCkay, R.White, 2009, pp.29 [104] and G. West [119].

As in the Heston case, we simulate the stochastic error, whose stochasticity fully depends from

Table 4.4: Call prices obtained in the SABR model with the G. A. method and with the Hagan formula for the parameter values in [104] in which one has $\alpha = 0.29$, $\rho = -0.71$. We have chosen $r = 3\%$ $E = 100$, $\sigma_0 = 20\%$ and three different maturities T. At-the-money options (ATM) have $F_0 = E$; in-the-money options (INM) have $F_0 = E(1 + 10\%\sqrt{\sigma_0 T})$ and out-of-the money options (OTM) have $F_0 = E(1 - 10\%\sqrt{\sigma_0 T})$.

($T = 1/12$)		
	G. A.	Hagan
ATM	2.3426	2.2956
INM	3.0008	2.9492
OTM	1.7655	1.6605
($T = 3/12$)		
	G. A.	Hagan
ATM	3.9097	3.9495
INM	5.0110	5.1039
OTM	2.9481	2.8821
($T = 6/12$)		
	G. A.	Hagan
ATM	5.3064	5.5295
INM	6.8070	7.1942
OTM	4.0023	4.0742

Table 4.5: Same as table (4.4) with $\rho = -0.1$.

($T = 1/12$)		
	G. A.	Hagan
ATM	2.2855	2.2983
INM	2.9702	2.9389
OTM	1.7152	1.6764
($T = 3/12$)		
	G. A.	Hagan
ATM	3.9241	3.9654
INM	5.0839	5.0795
OTM	2.9615	2.9351
($T = 6/12$)		
	G. A.	Hagan
ATM	5.4885	5.5684
INM	7.0892	7.1575
OTM	4.1643	4.1901

the random variable σ_T. To describe the distribution of the stochastic error term e^{ε_T} which, in the G. A. method is assumed to be close to 1, its analytical form is:

$$e^{\varepsilon_T} = e^{\frac{\rho\left[\sigma_0 e^{\left(\frac{\alpha^2}{2}T\right)} - \sigma_T\right]}{\alpha}}. \tag{4.46}$$

We simulate the SABR model with the above parameter values, and different correlation ρ. The distribution of e^{ε_T} is reported in Figures 4.15, 4.16, with $(\rho = -0.1, T = 1 - year)$, $(\rho = -0.9, T = 1 - year)$. As one can see the stochastic error distributes as a gaussian. In the case of low correlation we have a chart close to 1 and small variance. Unlike for the high correlation in which case we have again a gaussian distribution for the error, but with higher variance.

Finally, we evaluate the difference between the option price value computed with the Hagan method and the G. A. method as a function of the maturity of the option. Figures 4.17, 4.18, 4.19 show that the distance of the G. A. value from the Hagan value is reduced when we move in the money and at the money unlike out the money. Besides, we can observe that our solution (4.30) verifies the law of monotonicity with respect to maturity.

Concluding, in our experiments with sensible data ($\alpha = 0.29$, $\rho = -0.71$), we find that a reasonable estimate of the approximation error is around 5% , and it becomes around 2% for low correlation $\rho = -0.1$, see Table 4.4, 4.5 (clearly, the precise value depends on the parameter set adopted). This error increases with maturity and correlation coefficient. In other words, the accuracy of the G. A. method is determined by the magnitude of the correlation.

Again from (4.46), we have that for low values of correlation our method gives accurate prices, because our stochastic error is close to 1.

Although in the SABR case the G. A. technique presents some differences due to the form of the expected value. In fact for the SABR model the expected value for $T \to 0$ and $T \to +\infty$ is equal to:

$$\mathbb{E}_{\mathbb{Q}}[\sigma_T] = \sigma_0 e^{\frac{\alpha^2}{2}T},$$

$$lim_{T\to 0}\mathbb{E}_{\mathbb{Q}}[\sigma_T] = \sigma_0, \qquad lim_{T\to +\infty}\mathbb{E}_{\mathbb{Q}}[\sigma_T] = +\infty,$$

and differently from the Heston case, for maturity date that goes to infinity, the expected value is unbounded. Thus we can conclude that our method in the SABR model loses accuracy when the maturity increases.

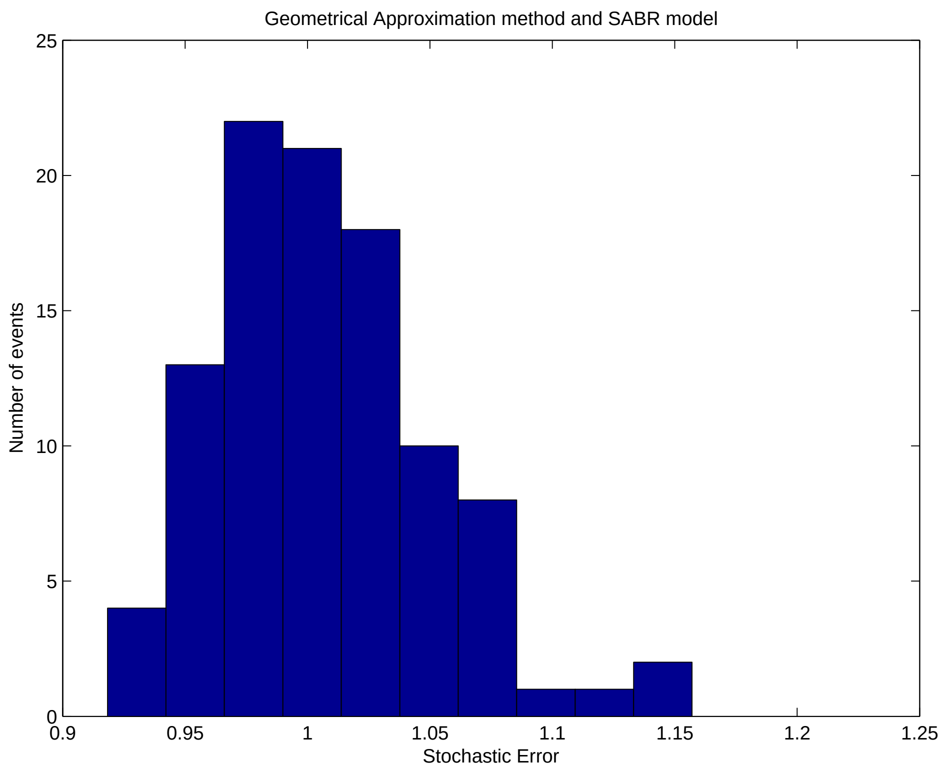

Figure 4.15: Distribution of the stochastic error e^{ε_T}, obtained via simulation, for $\beta = 1$, $\rho = -0.1$, $\sigma_0 = 20\%$ $\alpha = 0.29$, $T = 1$-year

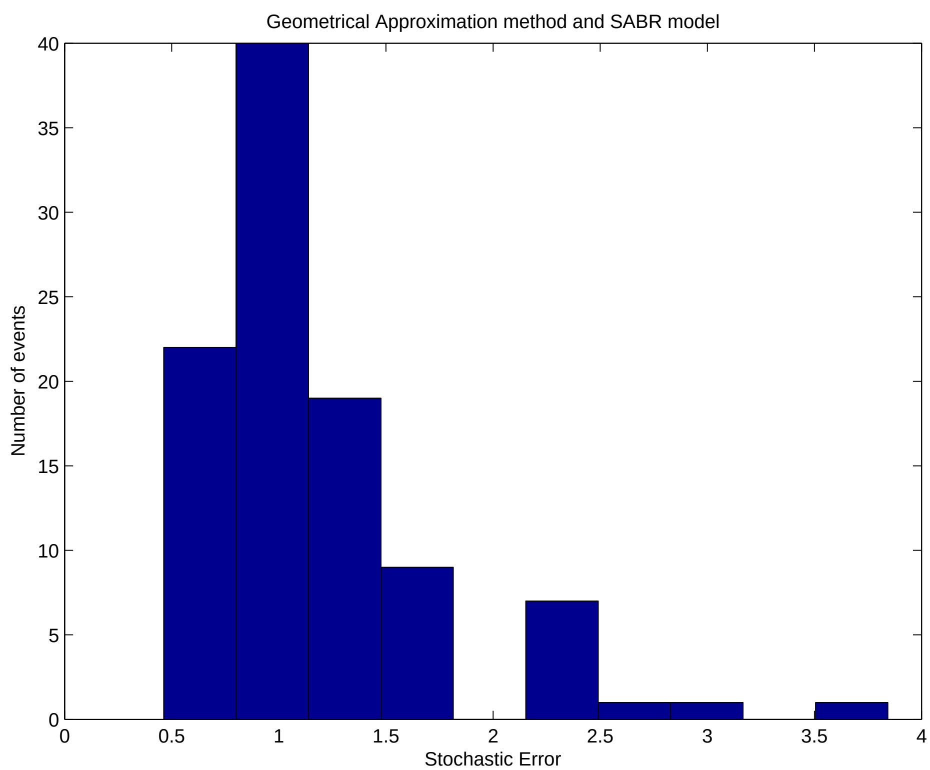

Figure 4.16: Distribution of the stochastic error e^{ε_T}, obtained via simulation, for $\beta = 1$, $\rho = -0.9$, $\sigma_0 = 20\%$ $\alpha = 0.29$, $T = 1$-year

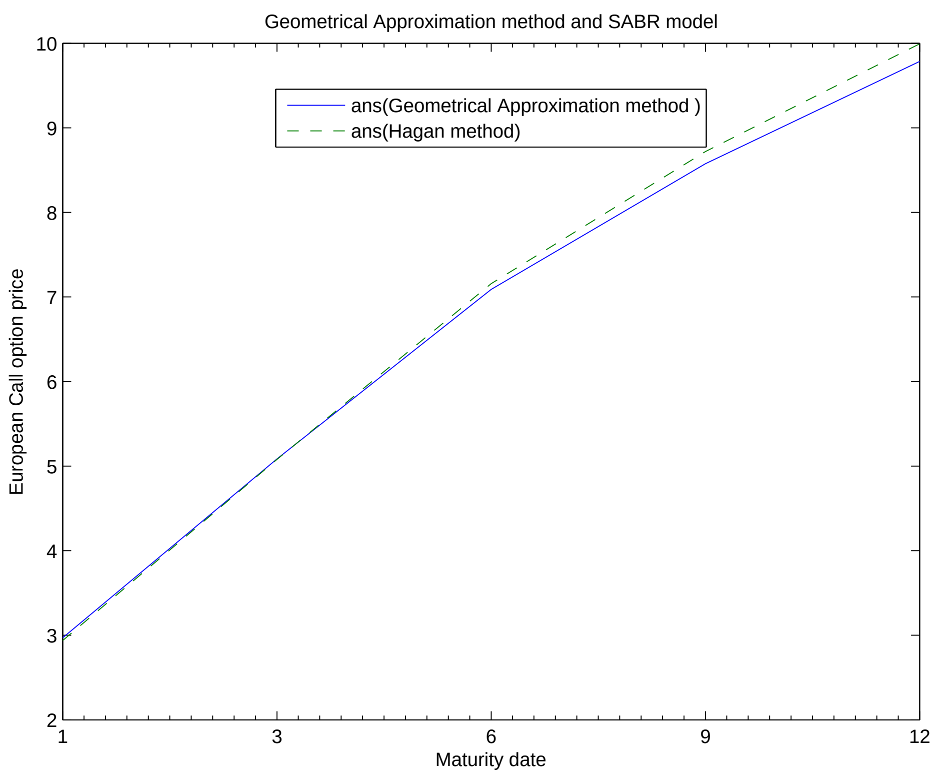

Figure 4.17: **In the money** - Call option prices computed with the Hagan method and with the Geometrical Approximation method as a function of the maturity, for $\rho = -0.1$, $\sigma_0 = 20\%$ $\alpha = 0.29$. The points of the abscissas are the maturities in months.

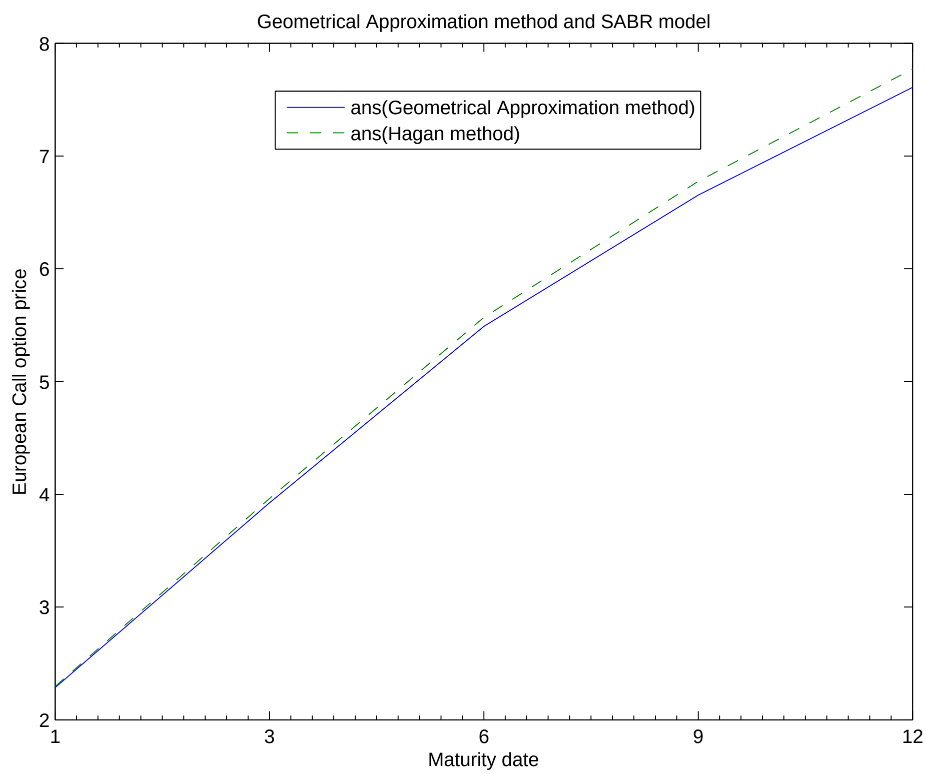

Figure 4.18: **At the money** - Call option prices computed with the Hagan method and with the Geometrical Approximation method as a function of the maturity, for $\rho = -0.1$, $\sigma_0 = 20\%$ $\alpha = 0.29$. The points of the abscissas are the maturities in months.

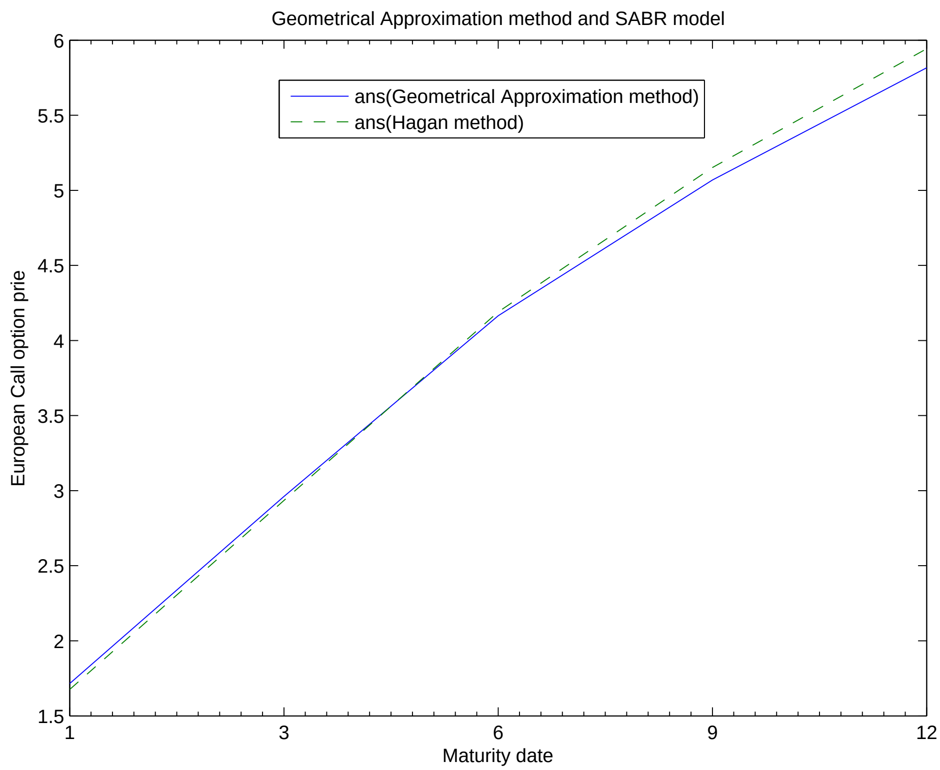

Figure 4.19: **Out the money** - Call option prices computed with the Hagan method and with the Geometrical Approximation method as a function of the maturity, for $\rho = -0.1$, $\sigma_0 = 20\%$ $\alpha = 0.29$. The points of the abscissas are the maturities in months.

Chapter 5

Perturbative method: Heston model with drift zero

In this Chapter, we present a perturbative method by a particular choice of the volatility risk price in the Heston model, namely such that the drift term of the risk-neutral stochastic volatility process is zero. This will allow us to introduce an approximating technique for solving the pricing PDE in the Heston case.

5.1 Perturbative methods: Vanilla Options

In this Chapter we want to present an alternative approach to discuss the Heston model, based on perturbative expansion, by a particular choice of the volatility price of risk, of PDE (4.11), i.e., such that the drift term of the risk-neutral stochastic volatility process is zero [25]. As will be clear later, our formula gives an accurate price close to that obtained by Fourier transform formula, choosing the following risk-neutral parameters set: $\kappa = 0, \Theta = indeterminate$, as we will see in the section Numerical experiments. The advantage is that in this case we are also able to compute an approximate solution of the Heston PDE (4.11) which can be written in closed form. Our solution is easy to implement and by the latter also we are able to compute the Greeks in closed form.

Our methodology consists to choose the following volatility risk price

$$\lambda(t, \nu_t) = \frac{k(\theta - \nu_t)}{\alpha\sqrt{\nu_t}}, \tag{5.1}$$

namely such that the Heston market model becomes the following:

$$\begin{aligned}
&dS_t = rS_t dt + \sqrt{\nu_t} S_t d\tilde{W}_t^{(1)}, \\
&d\nu_t = \alpha\sqrt{\nu_t} d\tilde{W}_t^{(2)}, \qquad \alpha \in \mathbb{R}^+, \\
&d\tilde{W}_t^{(1)} d\tilde{W}_t^{(2)} = \rho dt, \qquad \rho \in (-1, +1), \\
&dB_t = rB_t dt,
\end{aligned} \tag{5.2}$$

in the risk-neutral measure $\mathbb{Q}$. Notice that for the above process the Feller condition is not fulfilled, thus the volatility has an absorbing state at 0. This is not a problem since first, this is the risk-neutral evolution of the volatility, and second because of the analytical tractability of this particular problem which might provide handy formulas for the volatility surface. Notice that also the popular SABR model analysed later in the Thesis has a zero risk-neutral drift. As seen before, when the volatility is a Markov Itô processes, we have a pricing function for European derivatives of the form $f(t, S_t, \nu_t)$ from no-arbitrage arguments, as in the Black-Scholes case, the function $f(t, S_t, \nu_t)$ satisfies a partial differential equation with two space dimensions (S and ν). In what follows we use the PDE formalism, thus we replace S_t with S and ν_t with ν. Thus we can write by Feynman-Kač formula the PDE pricing , that is Cauchy's problem, as follows:

$$\begin{aligned}
&\frac{\partial f}{\partial t} + \frac{1}{2}\nu\left(S^2\frac{\partial^2 f}{\partial S^2} + 2\rho\alpha S\frac{\partial^2 f}{\partial S\partial\nu} + \alpha^2\frac{\partial^2 f}{\partial\nu^2}\right) + rS\frac{\partial f}{\partial S} - rf = 0, \\
&f(T, S, \nu) = \Phi(S) \qquad \rho \in (-1, +1), \qquad \alpha \in \mathbb{R}^+, \\
&S \in [0, +\infty) \qquad \nu \in [0, +\infty) \qquad t \in [0, T],
\end{aligned} \tag{5.3}$$

for which $f(t, S, \nu) \in C^{2,2}\left([0, +\infty) \times [0, +\infty) \times [0, T]\right)$, such that, $\Phi(S)$ is a general pay-off function for a derivative security; in what follows we consider a call option: $\Phi(S) = (S - E)^+$. In order to manage a simpler PDE, we make some coordinate transformations; these ones are equal to those just discussed in Chapter 4, therefore in what follows we are assuming them, without any discussions:

$$1^{st}\text{ transformation}\begin{cases} x = \ln S, & x \in (-\infty, +\infty), \\ \tilde{\nu} = \nu/\alpha, & \tilde{\nu} \in [0, +\infty), \\ f(t, S, \nu) = f_1(t, x, \tilde{\nu}) e^{-r(T-t)}, \end{cases}$$

thus we have:

$$\frac{\partial f_1}{\partial t} + \frac{1}{2}\tilde{\nu}\left(\frac{\partial^2 f_1}{\partial x^2} + 2\rho\frac{\partial^2 f_1}{\partial x \partial \tilde{\nu}} + \frac{\partial^2 f_1}{\partial \tilde{\nu}^2}\right) + \left(r - \frac{1}{2}\tilde{\nu}\right)\frac{\partial f_1}{\partial x} - r f_1 = 0,$$
$$f_1(T, x, \tilde{\nu}) = (e^x - E)^+, \qquad \rho \in (-1, +1), \qquad \alpha \in \mathbb{R}^+,$$
$$x \in (-\infty, +\infty), \qquad \tilde{\nu} \in [0, +\infty), \qquad t \in [0, T].$$

Again, we make another coordinates transformation:

$$2^{nd}\text{transformation}\begin{cases} \xi = x - \rho\tilde{\nu}, & \xi \in (-\infty, +\infty), \\ \eta = -\tilde{\nu}\sqrt{1-\rho^2}, & \eta \in (-\infty, 0], \\ f_1(t, x, \tilde{\nu}) = f_2(t, \xi, \eta), \end{cases}$$

and we have:

$$\frac{\partial f_2}{\partial t} + \frac{\alpha\eta}{2\sqrt{1-\rho^2}}(1-\rho^2)\left(\frac{\partial^2 f_2}{\partial \xi^2} + \frac{\partial^2 f_2}{\partial \eta^2}\right) + \left(r - \frac{\alpha\eta}{2\sqrt{1-\rho^2}}\right)\frac{\partial f_2}{\partial \xi} - r f_2 = 0,$$
$$f_2(T, \xi, \eta) = \left(e^{\xi - \rho\eta/\sqrt{1-\rho^2}} - E\right)^+, \qquad \rho \in (-1, +1), \qquad \alpha \in \mathbb{R}^+,$$
$$\xi \in (-\infty, +\infty), \qquad \eta \in (-\infty, 0], \qquad t \in [0, T].$$

Finally, by the third coordinates transformation:

$$3^{rd}\text{transformation}\begin{cases} \gamma = \xi + r(T-t), & \gamma \in (-\infty, +\infty), \\ \delta = -\eta, & \delta \in [0, +\infty), \\ \tau = -\frac{\alpha\eta}{2\sqrt{1-\rho^2}}(T-t), & \tau \in [0, +\infty), \\ f_2(t, \xi, \eta) = f_3(\tau, \gamma, \delta), \end{cases}$$

we obtain the following PDE:

$$\frac{\partial f_3}{\partial \tau} - (1-\rho^2)\left(\frac{\partial^2 f_3}{\partial \gamma^2} + \frac{\partial^2 f_3}{\partial \delta^2} + 2\phi\frac{\partial^2 f_3}{\partial \delta \partial \tau} + \phi^2\frac{\partial^2 f_3}{\partial \tau^2}\right) + \frac{\partial f_3}{\partial \gamma} = 0,$$
$$f_3(\tau, \gamma, \delta) = \left(e^{\gamma + \rho\delta/\sqrt{1-\rho^2}} - E\right)^+, \qquad \rho \in (-1, +1), \qquad \alpha \in \mathbb{R}^+,$$
$$\gamma \in (-\infty, +\infty), \qquad \delta \in [0, +\infty), \qquad \tau \in [0, +\infty),$$

where $\phi = \frac{\alpha(T-t)}{2\sqrt{1-\rho^2}}$. It is worth noting that $\rho \in (-1, +1)$ and thus ϕ can assume only finite values.

At this point we can make the subsequent considerations: if we suppose $\alpha \sim 10^{-2}$ thus for maturity date lesser than 1-year the term $(T-t) \sim 10^{-1}$, $2\sqrt{1-\rho^2} \sim 1$ and $\phi \sim 10^{-3}$; obviously we have $\phi^2 \sim 10^{-6}$. Thus it is reasonable to approximate $\phi \simeq 0$.

The PDE that comes out from the above indicated considerations is the following:

$$\left[\frac{\partial f_3}{\partial \tau} - (1-\rho^2)\left(\frac{\partial^2 f_3}{\partial \gamma^2} + \frac{\partial^2 f_3}{\partial \delta^2}\right) - \frac{\partial f_3}{\partial \gamma}\right] = 0,$$

$$\begin{aligned} & f_3(0,\gamma,\delta) = (e^{\gamma+\rho\delta/\sqrt{1-\rho^2}} - E)^+, \qquad \rho \in (-1,+1), \qquad \alpha \in \mathbb{R}^+, \\ & \gamma(-\infty,+\infty), \qquad \delta \in [0,+\infty), \qquad \tau \in [0,+\infty). \end{aligned} \tag{5.4}$$

The new PDE (5.4) is simpler than (5.3) and we now are able to find its solution in closed form. In order to obtain the latter, we impose that:

$$f_3(\tau,\gamma,\delta) = e^{\frac{1}{4(1-\rho^2)}\tau + \frac{1}{2(1-\rho^2)}\gamma} f_4(\tau,\gamma,\delta), \tag{5.5}$$

thus we have:

$$\begin{aligned} & \frac{\partial f_4}{\partial \tau} = (1-\rho^2)\left(\frac{\partial^2 f_4}{\partial \gamma^2} + \frac{\partial^2 f_4}{\partial \delta^2}\right), \\ & f_4(0,\gamma,\delta) = e^{-\frac{\gamma}{2(1-\rho^2)}}\left(e^{\gamma+\rho\delta/\sqrt{1-\rho^2}} - E\right)^+, \\ & \tau \in [0,+\infty), \qquad \delta \in [0,+\infty), \qquad \gamma \in (-\infty,+\infty), \end{aligned} \tag{5.6}$$

where $f(t,S,\nu) = e^{-r(T-t) + \frac{\tau}{4(1-\rho^2)} + \frac{\gamma}{2(1-\rho^2)}} f_4(\tau,\gamma,\delta)$.
The solutions of PDE (5.16) is known in the literature (Andrei D. Polyanin, Handbook of Linear Partial Differential Equations, 2002, pp. 188), and it can be written as integral, whose kernel $G(0,\gamma',\delta'|\tau,\gamma,\delta)$ is a bivariate gaussian function:

$$\begin{aligned} f_4(\tau,\gamma,\delta) = & \int_0^{+\infty} d\delta' \int_{-\infty}^{+\infty} d\gamma' f_4(0,\gamma',\delta') G(0,\gamma',\delta'|\tau,\gamma,\delta) \\ & + (1-\rho^2)\int_0^{\tau} d\tau' \int_{-\infty}^{+\infty} d\gamma' f_4(\tau',\gamma',0) \frac{\partial G(0,\gamma',\delta'|\tau,\gamma,\delta)}{\partial \delta'}|_{\delta'=0}, \end{aligned}$$

where the second term is zero, because for $\delta = 0$, also $\tau = 0$ at any time. Thus we can write as follows:

$$\begin{aligned}
f_4(\tau,\gamma,\delta) &= \int_0^{+\infty} d\delta' \int_{-\infty}^{+\infty} d\gamma' f_4(0,\gamma',\delta') G(0,\gamma',\delta'|\tau,\gamma,\delta) \\
&= \int_0^{+\infty} d\delta' \int_{-\infty}^{+\infty} d\gamma' e^{-\frac{\gamma'}{2(1-\rho^2)}} \left(e^{\gamma'+\rho\delta'/\sqrt{1-\rho^2}} - E\right)^+ G(0,\gamma',\delta'|\tau,\gamma,\delta) \\
&= \int_0^{+\infty} d\delta' \int_{\ln E-\rho\delta'/\sqrt{1-\rho^2}}^{+\infty} d\gamma' e^{-\frac{\gamma'}{2(1-\rho^2)}} \left(e^{\gamma'+\rho\delta'/\sqrt{1-\rho^2}} - E\right) G(0,\gamma',\delta'|\tau,\gamma,\delta),
\end{aligned} \tag{5.7}$$

the kernel $G(0,\gamma',\delta'|\tau,\gamma,\delta)$ is given by:

$$G(0,\gamma',\delta'|\tau,\gamma,\delta) = \frac{1}{4\tau(1-\rho^2)}\left[e^{-\frac{(\gamma'-\gamma)^2+(\delta'-\delta)^2}{4\tau(1-\rho^2)}} - e^{-\frac{(\gamma'-\gamma)^2+(\delta'+\delta)^2}{4\tau(1-\rho^2)}}\right].$$

Therefore we can rewrite:

$$\begin{aligned}
f_4(\tau,\gamma,\delta) = \int_0^{+\infty} d\delta' \int_{\ln E-\rho\delta'/\sqrt{1-\rho^2}}^{+\infty} d\gamma' e^{-\frac{\gamma'}{2(1-\rho^2)}} \left(e^{\gamma'+\rho\delta'/\sqrt{1-\rho^2}} - E\right) \times \\
\frac{1}{4\tau(1-\rho^2)}\left[e^{-\frac{(\gamma'-\gamma)^2+(\delta'-\delta)^2}{4\tau(1-\rho^2)}} - e^{-\frac{(\gamma'-\gamma)^2+(\delta'+\delta)^2}{4\tau(1-\rho^2)}}\right],
\end{aligned}$$

and the fair price of a Call option in European style is given by:

$$\begin{aligned}
&f(t,S,\nu) \\
&= e^{-r(T-t)+\frac{\tau}{4(1-\rho^2)}+\frac{\gamma}{2(1-\rho^2)}} \int_0^{+\infty} d\delta' \int_{\ln E-\rho\delta'/\sqrt{1-\rho^2}}^{+\infty} d\gamma' e^{-\frac{\gamma'}{2(1-\rho^2)}} \left(e^{\gamma'+\rho\delta'/\sqrt{1-\rho^2}} - E\right) \times \\
&\qquad \frac{1}{4\tau(1-\rho^2)}\left[e^{-\frac{(\gamma'-\gamma)^2+(\delta'-\delta)^2}{4\tau(1-\rho^2)}} - e^{-\frac{(\gamma'-\gamma)^2+(\delta'+\delta)^2}{4\tau(1-\rho^2)}}\right].
\end{aligned}$$

Thus, if we indicate with $C(t,S,\nu)$ the value at any time t of a Call option, we have:

$$\begin{aligned}
C(t,S,\nu) = e^{\frac{\nu(T-t)}{4(1-\rho^2)}} S\left[N\left(d_1, a_{0,1}\sqrt{1-\rho^2}\right) - e^{\left(-2\frac{\rho}{\alpha}\nu\right)} N\left(d_2, a_{0,2}\sqrt{1-\rho^2}\right)\right] \\
- e^{\frac{\nu(T-t)}{4(1-\rho^2)}} E e^{-r(T-t)}\left[N\left(\tilde{d}_1, \tilde{a}_{0,1}\sqrt{1-\rho^2}\right) - N\left(\tilde{d}_2, \tilde{a}_{0,2}\sqrt{1-\rho^2}\right)\right].
\end{aligned} \tag{5.8}$$

Differently of Black-Scholes market model with deterministic volatility in which we have one dimensional normal distribution functions, in the Black-Scholes with stochastic volatility. We have the bivariate normal distribution functions, in which the arguments are:

$$\begin{cases}
d_1 = \frac{\nu/\alpha+\rho\nu(T-t)}{\sqrt{\nu(T-t)}}, \\
d_2 = \frac{-\nu/\alpha+\rho\nu(T-t)}{\sqrt{\nu(T-t)}}, \\
a_{0,1} = \frac{\ln(S/E)+(r+\nu/2)(T-t)}{\sqrt{(1-\rho^2)\nu(T-t)}}, \\
a_{0,2} = \frac{\ln(S/E)+(r+\nu/2)(T-t)-2\rho\nu/\alpha}{\sqrt{(1-\rho^2)\nu(T-t)}},
\end{cases}
\qquad
\begin{cases}
\tilde{d}_1 = \frac{\nu/\alpha}{\sqrt{\nu(T-t)}}, \\
\tilde{d}_2 = \frac{-\nu/\alpha}{\sqrt{\nu(T-t)}}, \\
\tilde{a}_{0,1} = \frac{\ln(S_t/E)+(r-\nu/2)(T-t)}{\sqrt{(1-\rho^2)\nu(T-t)}}, \\
\tilde{a}_{0,2} = \frac{\ln(S_t/E)+(r-\nu/2)(T-t)-2\rho\nu_t/\alpha}{\sqrt{(1-\rho^2)\nu(T-t)}}.
\end{cases} \tag{5.9}$$

It is worth noting that for ρ equal to zero we have:

$$d_2 = -d_1, \qquad a_{0,1} = a_{0,2},$$
$$\tilde{d}_2 = -\tilde{d}_1, \qquad \tilde{a}_{0,1} = \tilde{a}_{0,2}.$$

Similarly, we may write the fair price of a Put option in European style:

$$P(t,S,\nu) = e^{\frac{\nu(T-t)}{4(1-\rho^2)}} E e^{-r(T-t)} \left[N\left(\tilde{d}_1, -\tilde{a}_{0,1}\sqrt{1-\rho^2}\right) - N\left(\tilde{d}_2, -\tilde{a}_{0,2}\sqrt{1-\rho^2}\right)\right] - S e^{\frac{\nu(T-t)}{4(1-\rho^2)}} \left[N\left(d_1, -a_{0,1}\sqrt{1-\rho^2}\right) - e^{-2\frac{\rho}{\alpha}\nu} N\left(d_2, -a_{0,2}\sqrt{1-\rho^2}\right)\right]. \quad (5.10)$$

Theoretical Error
The theoretical error in Perturbative method can be evaluated by computing the terms that we have before neglected

$$Err = \left(2\phi \frac{\partial^2}{\partial\delta\partial\tau} + \phi^2 \frac{\partial^2}{\partial\tau^2}\right) f(t,S,\nu),$$

where $\phi = \frac{\alpha(T-t)}{2\sqrt{1-\rho^2}}$ and $f(t,S,\nu)$ is the solution of the PDE (5.4).

As one can see in Numerical Experiments section, the error will be around 1% for maturity lesser than 1-year.

5.1.1 Greeks and Put-Call-Parity

The way to reduce the sensitivity of a portfolio to the movement of something by taking opposite positions in different financial instruments is called hedging. Hedging is a basic concept in finance. For the stochastic volatility market models used in literature, is not possible to write the Greeks in closed form. Contrariwise by our model we are able to compute the Greeks in closed form; and in order to verify the **Put-Call-Parity** relation, we compute only Δ and Γ as follows:

$$\begin{aligned}\Delta_{call} &= \frac{\partial C(t,S,\nu)}{\partial S} \\ &= e^{\frac{\nu(T-t)}{2(1-\rho^2)}} \left[N\left(d_1, a_{0,1}\sqrt{1-\rho^2}\right) - e^{-\frac{2\rho}{\alpha}\nu} N\left(d_2, a_{0,2}\sqrt{1-\rho^2}\right)\right] \\ &+ E \frac{e^{\frac{\nu(T-t)}{2(1-\rho^2)}}}{\sqrt{2(1-\rho^2)\nu(T-t)}} \left[\frac{\partial N\left(d_1, a_{0,1}\sqrt{1-\rho^2}\right)}{\partial a_{0,1}} - e^{-2\frac{\rho}{\alpha}\nu} \frac{\partial N\left(d_2, a_{0,2}\sqrt{1-\rho^2}\right)}{\partial a_{0,2}}\right] \\ &- \left(\frac{E^2}{S}\right) \frac{e^{-\left(r-\frac{\nu}{2(1-\rho^2)}\right)(T-t)}}{\sqrt{2(1-\rho^2)\nu(T-t)}} \left[\frac{\partial N\left(\tilde{d}_1, \tilde{a}_{0,1}\sqrt{1-\rho^2}\right)}{\partial \tilde{a}_{0,1}} - \frac{\partial N\left(\tilde{d}_2, \tilde{a}_{0,2}\sqrt{1-\rho^2}\right)}{\partial \tilde{a}_{0,2}}\right].\end{aligned} \quad (5.11)$$

$$
\begin{aligned}
\Delta_{put} &= \frac{\partial P(t,S,\nu)}{\partial S} \\
&= \left(\frac{E^2}{S}\right) \frac{e^{-\left(r-\frac{\nu}{2(1-\rho^2)}\right)(T-t)}}{\sqrt{2(1-\rho^2)\nu(T-t)}} \left[\frac{\partial N\left(\tilde{d}_1, -\tilde{a}_{0,1}\sqrt{1-\rho^2}\right)}{\partial \tilde{a}_{0,1}} - \frac{\partial N\left(\tilde{d}_2, -\tilde{a}_{0,2}\sqrt{1-\rho^2}\right)}{\partial \tilde{a}_{0,2}}\right] \\
&- e^{\frac{\nu(T-t)}{2(1-\rho^2)}} \left[N\left(d_1, -a_{0,1}\sqrt{1-\rho^2}\right) - e^{-\frac{2\rho}{\alpha}\nu} N\left(d_2, -a_{0,2}\sqrt{1-\rho^2}\right)\right] \\
&- E\frac{e^{\frac{\nu(T-t)}{2(1-\rho^2)}}}{\sqrt{2(1-\rho^2)\nu(T-t)}} \left[\frac{\partial N\left(d_1, -a_{0,1}\sqrt{1-\rho^2}\right)}{\partial a_{0,1}} - e^{-2\frac{\rho}{\alpha}\nu} \frac{\partial N\left(d_2, -a_{0,2}\sqrt{1-\rho^2}\right)}{\partial a_{0,2}}\right].
\end{aligned}
\tag{5.12}
$$

Thus we have:

$$
\begin{aligned}
\Gamma_{call} &= \Gamma_{put} \\
&= \left(\frac{E}{S}\right) \frac{e^{\frac{\nu(T-t)}{2(1-\rho^2)}}}{\sqrt{2(1-\rho^2)\nu(T-t)}} \left[\frac{\partial N\left(d_1, a_{0,1}\sqrt{1-\rho^2}\right)}{\partial a_{0,1}} - e^{-\frac{2\rho}{\alpha}\nu} \frac{\partial N\left(d_2, a_{0,2}\sqrt{1-\rho^2}\right)}{\partial a_{0,2}}\right] \\
&+ \left(\frac{E^2}{S}\right) \frac{e^{\frac{\nu(T-t)}{2(1-\rho^2)}}}{2(1-\rho^2)\nu(T-t)} \left[\frac{\partial^2 N\left(d_1, a_{0,1}\sqrt{1-\rho^2}\right)}{\partial a_{0,1}^2} - e^{-2\frac{\rho}{\alpha}\nu} \frac{\partial^2 N\left(d_2, a_{0,2}\sqrt{1-\rho^2}\right)}{\partial a_{0,2}^2}\right] \\
&- \left(\frac{E^2}{S^2}\right) \frac{e^{-\left(r-\frac{\nu}{2(1-\rho^2)}\right)(T-t)}}{2(1-\rho^2)\nu(T-t)} \left[\frac{\partial^2 N\left(\tilde{d}_1, \tilde{a}_{0,1}\sqrt{1-\rho^2}\right)}{\partial \tilde{a}_{0,1}^2} - \frac{\partial^2 N\left(\tilde{d}_2, \tilde{a}_{0,2}\sqrt{1-\rho^2}\right)}{\partial \tilde{a}_{0,2}^2}\right] \\
&+ \left(\frac{E^2}{S^2}\right) \frac{e^{-\left(r-\frac{\nu}{2(1-\rho^2)}\right)(T-t)}}{\sqrt{2(1-\rho^2)\nu(T-t)}} \left[\frac{\partial N\left(\tilde{d}_1, \tilde{a}_{0,1}\sqrt{1-\rho^2}\right)}{\partial \tilde{a}_{0,1}} - \frac{\partial N\left(\tilde{d}_2, \tilde{a}_{0,2}\sqrt{1-\rho^2}\right)}{\partial \tilde{a}_{0,2}}\right],
\end{aligned}
$$

and we can conclude that our model verifies the **Put-Call-Parity** relation.

5.1.2 Numerical Experiments

In this section we compare the approximated option price (5.8) with the Fourier transform method. We use the following risk-neutral parameter set: $\kappa = 0, \Theta = indeterminate$. We could not find estimates of such a model in the literature. Also in this case we use the integration method based on Gauss-Legendre approximation with 20 points as done previously for the Heston's formula (4.13). As one can see in the table (5.1.2) the prices proposed by our approximation method are close to Fourier prices. Besides it is worth noting that for $\phi << 1$ we find the property that the price of a

Table 5.1: Call option prices computed with formula (5.8) (approximation method) and with the Fourier transform method where $\kappa = 0$, $\Theta = indeterminate$, $\alpha = 0.1$ and $\rho = -0.64$. We have chosen $r = 3\%$ $E = 100$, $\nu_0 = 0.04$ and three different maturities T. At-the-money options (ATM) have $S_0 = E$; in-the-money options (INM) have $S_0 = E(1 + 10\%\sqrt{\nu_0 T})$ and out-of-the money options (OTM) have $S_0 = E(1 - 10\%\sqrt{\nu_0 T})$).

($T = 1/12$)		
	Perturbative method	Fourier
ATM	2.4305	2.4261
INM	2.7337	2.7341
OTM	2.1503	2.1410
($T = 3/12$)		
	Perturbative method	Fourier
ATM	4.3755	4.3524
INM	4.9037	4.8942
OTM	3.8871	3.8499
($T = 6/12$)		
	Perturbative method	Fourier
ATM	6.3790	6.3765
INM	7.1214	7.1322
OTM	5.6925	5.6358

European option in the Heston model follows a Bivariate Normal distribution (5.8). Our methodology loses accuracy for $\alpha \sim 10^{-1}$ and maturity date longer than 1 year. Contrariwise for $\alpha \sim 10^{-2}$ (see Figures 5.1, 5.2, 5.3).

Finally, we evaluate the difference between the option price value computed with the Fourier method and the proposed approximation method as a function of the maturity of the option. Figures 5.1, 5.2, 5.3 show the results, see the captions for parameter values. We can see that the distance among the values given from the two methods is reduced when we move in the money and at the money unlike out the money. In fact one has that the difference among the two methods is lesser than 1% in the money and at the money and becomes around 2% out the money for maturity of 1-year. Besides we can observe that our solution (4.30) verifies the law of monotonicity with respect to maturity.

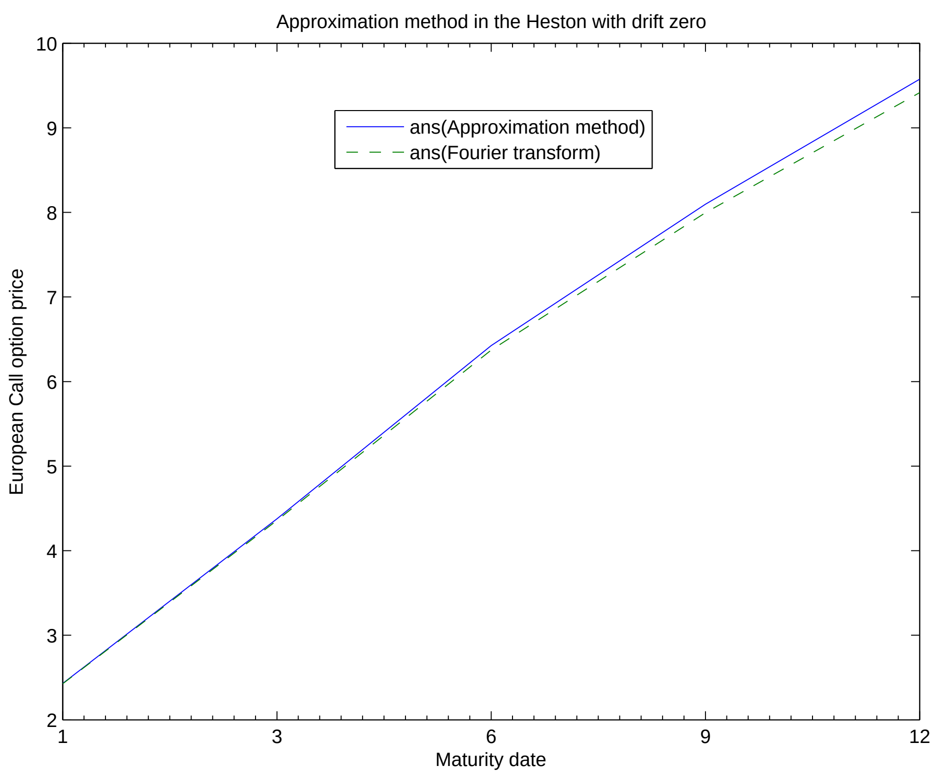

Figure 5.1: **A the money** - Call prices computed with the Fourier integral and with the Approximation method as a function of the maturity, for $\rho = -0.64$, $\kappa = 0$, $\Theta = indeterminate$, $\nu_0 = 0.04$, $\alpha = 0.01$. The points of the abscissas are the maturities in months.

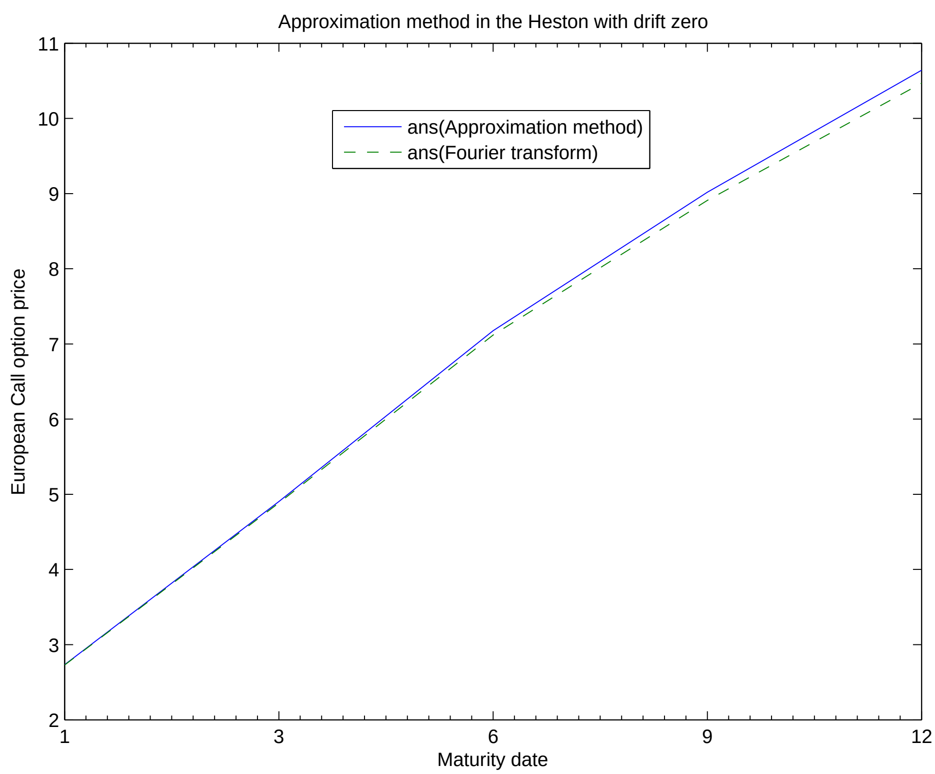

Figure 5.2: **In the money** - Call prices computed with the Fourier integral and with the Approximation method as a function of the maturity, for $\rho = -0.64$, $\kappa = 0$, $\Theta = indeterminate$, $\nu_0 = 0.04$, $\alpha = 0.01$. The points of the abscissas are the maturities in months.

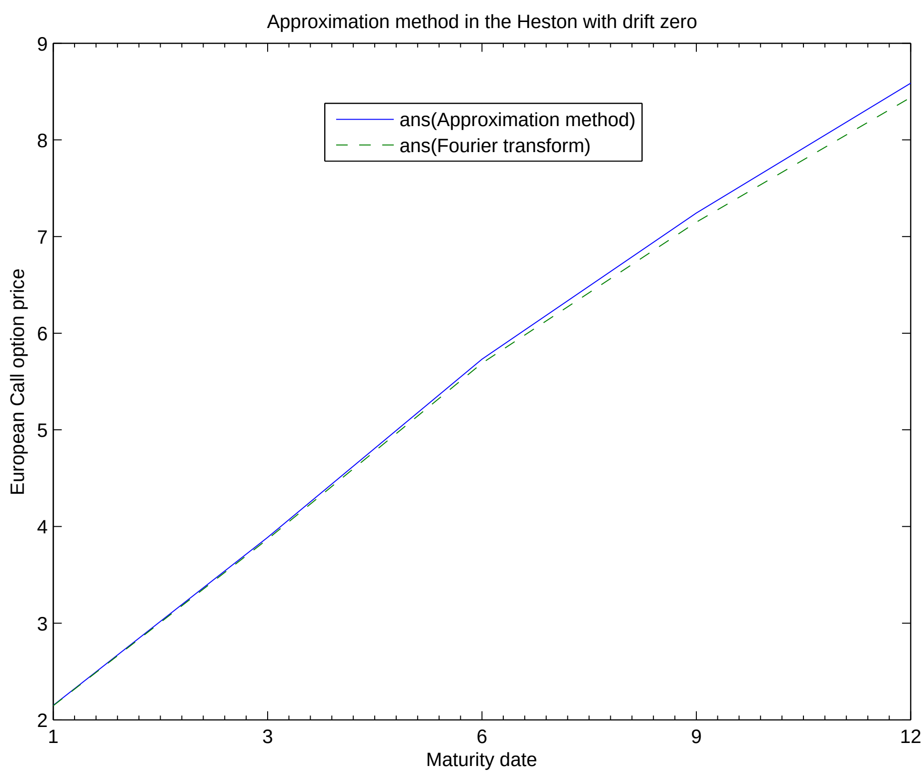

Figure 5.3: **Out the money** - Call prices computed with the Fourier integral and with the Approximation method as a function of the maturity, for $\rho = -0.64$, $\kappa = 0$, $\Theta = indeterminate$, $\nu_0 = 0.04$, $\alpha = 0.01$. The points of the abscissas are the maturities in months.

5.2 Barrier Options

Using the same market model (5.2), and as done in the previous section, using the PDE formalism, for which we have replaced S_t with S and ν_t with ν, we get the fair price of a European **down-and-out** Call option, through the **Perturbative Method** [26].

The Barrier Options can be of two kinds: **knock-out**, or **knock-in**. Barrier Options are options that either become worthless or exercised if the underlying asset value reaches the level, so-called barrier level.
The simplest **knock-out** options are the **down-and-out** Call and the **up-and-out** Put. An option is called a **down-and-out** Call if it is actually a Call when S is always greater than the barrier during the life of the option, and it becomes worthless when S reaches the barrier from above at any time $t < T$ before expiring. We call this barrier a lower barrier L, and in this section we mainly consider the case that such barrier is below the exercise price E. A **down-and-out** Call could be a European style, or an American style option just like a vanilla option. An **up-and-out** Put is similar to a **down-and-out** Call. However, instead of a lower barrier , it has an upper barrier H, which we assume is greater than E. It is a put if S is never above H and becomes worthless when S crosses the barrier H from below at any time $t < T$ prior to expiry.
A **Knock-in** option is a contract that comes into existence if the asset price crosses a barrier. For example, a **down-and-in** Call with a lower barrier L expires worthless unless the asset price reaches the lower barrier from above prior to or at expiry. If it crosses the lower barrier from above at some time before expiry, then the option becomes a vanilla option. An **up-and-in** Put is similar to a **down-and-out** Call, but the barrier is an upper one and the Put option is activated when S crosses the upper barrier from below.

Now let us look at a European **down-and-out** Call option. Let $f = C_L^{out}(t, S, \nu)$ denote the value of this option. If S is always greater than L, then it is a standard Call option in European style.

$f = C_L^{out}(t, S, \nu)$ satisfies the following Cauchy-Dirichlet's problem:

$$
\begin{aligned}
&\frac{\partial f}{\partial t} + \frac{1}{2}\nu S^2 \frac{\partial^2 f}{\partial S^2} + \rho\nu\alpha S \frac{\partial^2 f}{\partial S \partial \nu} + \frac{1}{2}\nu\alpha^2 \frac{\partial^2 f}{\partial \nu^2} + rS\frac{\partial f}{\partial S} - rf = 0,\\
&f(T, S, \nu) = (S - E)^+ \times \mathbf{1}_{\{\forall t \in [0,T] : S > L\}},\\
&f(t, L, \nu) = 0,\\
&S \in [L, \infty), \quad \nu \in [0, +\infty), \quad \text{t} \in [0, T],
\end{aligned}
\tag{5.13}
$$

for which $f(t, S, \nu) \in C^{2,2}\left([0, +\infty) \times [0, +\infty) \times [0, T]\right)$, $E > L$ and $f(T, S, \nu)$ is a **knock-out** Call option payoff.

In order to manage a simpler PDE, we make the same coordinate transformations that we seen in the previous section, and considering worthless the terms $\phi \frac{\partial^2 f_2}{\partial \delta \partial \tau}$ and $\phi^2 \frac{\partial^2 f_2}{\partial \tau^2}$, we have the following PDE:

$$\left[\frac{\partial f_3}{\partial \tau} - (1-\rho^2)\left(\frac{\partial^2 f_3}{\partial \gamma^2} + \frac{\partial^2 f_3}{\partial \delta^2}\right) - \frac{\partial f_3}{\partial \gamma}\right] = 0,$$
$$f_3(0,\gamma,\delta) = \left(e^{\gamma+\rho\delta/\sqrt{1-\rho^2}} - E\right)^+ \times \mathbf{1}_{\{\forall t\in[0,T]:\gamma>A\}}, \qquad \rho \in (-1,+1), \qquad \alpha \in \mathbb{R}^+,$$
$$\gamma(-\infty,+\infty), \qquad \delta \in [0,+\infty), \qquad \tau \in [0,+\infty), \tag{5.14}$$

where $A = \left[\ln L + \frac{r\sqrt{1-\rho^2}}{\alpha}\left(\frac{\tau}{\delta}\right) - \left(\frac{\rho}{\sqrt{1-\rho^2}}\right)\delta\right]$.
The new PDE (5.14) is simpler than (5.13) and we now are able to find its solution in closed-form. In order to obtain the latter, we impose that:

$$f_3(\tau,\gamma,\delta) = e^{\frac{1}{4(1-\rho^2)}\tau + \frac{1}{2(1-\rho^2)}\gamma} f_4(\tau,\gamma,\delta), \tag{5.15}$$

thus we have:

$$\frac{\partial f_4}{\partial \tau} = (1-\rho^2)\left(\frac{\partial^2 f_4}{\partial \gamma^2} + \frac{\partial^2 f_4}{\partial \delta^2}\right)$$
$$f_4(0,\gamma,\delta) = e^{-\frac{\gamma}{2(1-\rho^2)}}\left(e^{\gamma+\rho\delta/\sqrt{1-\rho^2}} - E\right)^+ \times \mathbf{1}_{\{\forall t\in[0,T]:\gamma>A\}},$$
$$\tau \in [0,+\infty), \qquad \delta \in [0,+\infty), \qquad \gamma \in (-\infty,+\infty). \tag{5.16}$$

The solution of the heat equation in the domain $[A,+\infty)\times[0,+\infty)\times[0,+\infty)$, is known in literature (Andrei D. Polyanin, Handbook of Linear Partial Differential Equations, 2002, pp. 189), and the latter is given by :

$$f_4(\tau,\gamma,\delta) = \int_0^{+\infty} d\delta' \int_{A'}^{+\infty} d\gamma' f_4(0,\gamma',\delta') G(0,\gamma',\delta'|\tau,\gamma,\delta)$$
$$+ (1-\rho^2)\int_0^{\tau} d\tau' \int_{A'}^{+\infty} d\gamma' f_4(\tau',\gamma',0)\frac{\partial G(0,\gamma',\delta'|\tau,\gamma,\delta)}{\partial \delta'}|_{\delta'=0},$$
$$+ (1-\rho^2)\int_0^{\tau} d\tau' \int_0^{+\infty} d\delta' f_4(\tau',A',\delta')\frac{\partial G(0,\gamma',\delta'|\tau,\gamma,\delta)}{\partial \gamma'}|_{\gamma'=0}, \tag{5.17}$$

where the Green's function is the following:

$$G(0,\gamma',\delta'|\tau,\gamma,\delta)$$
$$= \frac{1}{4\pi(1-\rho^2)\tau}\left\{e^{-\frac{[\gamma-\gamma'-(A-A')]^2}{4(1-\rho^2)\tau}} - e^{-\frac{[\gamma+\gamma'-(A+A')]^2}{4(1-\rho^2)\tau}}\right\}\left\{e^{-\frac{(\delta-\delta')^2}{4(1-\rho^2)\tau}} - e^{-\frac{(\delta+\delta')^2}{4(1-\rho^2)\tau}}\right\}.$$

The third and second terms of (5.17) are equal to zero; the second for the equations (4.41), in fact if $\delta = 0$ also $\tau = 0$; and the third for hypothesis $f_4(\tau,\gamma = A,\delta) = 0$ at any time.

Then, for $(E \geq L)$, the price of a **down-and-out** Call option is

$$\begin{aligned}
f_4(\tau,\gamma,\delta) &= \int_0^{+\infty} d\delta' \int_{A'}^{+\infty} d\gamma' f_4(0,\gamma',\delta') G(0,\gamma',\delta'|\tau,\gamma,\delta) \\
&= \int_0^{+\infty} d\delta' \int_{A'}^{+\infty} d\gamma' e^{-\frac{\gamma'}{2(1-\rho^2)}} \left(e^{\gamma'+\rho\delta'/\sqrt{1-\rho^2}} - E\right)^+ G(0,\gamma',\delta'|\tau,\gamma,\delta) \\
&= \int_0^{+\infty} d\delta' \int_{A'}^{+\infty} d\gamma' e^{-\frac{\gamma'}{2(1-\rho^2)}} \left(e^{\gamma'+\rho\delta'/\sqrt{1-\rho^2}} - E\right)^+ \times \\
&\frac{1}{4\pi(1-\rho^2)\tau} \left\{ e^{-\frac{[\gamma-\gamma'-(A-A')]^2}{4(1-\rho^2)\tau}} - e^{-\frac{[\gamma+\gamma'-(A+A')]^2}{4(1-\rho^2)\tau}} \right\} \left\{ e^{-\frac{(\delta-\delta')^2}{4(1-\rho^2)\tau}} - e^{-\frac{(\delta+\delta')^2}{4(1-\rho^2)\tau}} \right\} \\
&= \int_0^{+\infty} d\delta' \int_{\ln E - \frac{\rho}{\sqrt{1-\rho^2}}\delta'}^{+\infty} d\gamma' e^{-\frac{\gamma'}{2(1-\rho^2)}} \left(e^{\gamma'+\rho\delta'/\sqrt{1-\rho^2}} - E\right) \times \\
&\qquad \frac{1}{4\pi(1-\rho^2)\tau} \left\{ e^{-\frac{[\gamma-\gamma'-(A-A')]^2}{4(1-\rho^2)\tau}} - e^{-\frac{[\gamma+\gamma'-(A+A')]^2}{4(1-\rho^2)\tau}} \right\} \left\{ e^{-\frac{(\delta-\delta')^2}{4(1-\rho^2)\tau}} - e^{-\frac{(\delta+\delta')^2}{4(1-\rho^2)\tau}} \right\},
\end{aligned}$$

but we know by previous section that:

$$f(t,S,\nu) = e^{-r(T-t)+\frac{\tau}{4(1-\rho^2)}+\frac{\gamma}{2(1-\rho^2)}} f_4(\tau,\gamma,\delta),$$

thus we have

$$\begin{aligned}
&f(t,S,\nu) \\
&= e^{-r(T-t)+\frac{\tau}{4(1-\rho^2)}+\frac{\gamma}{2(1-\rho^2)}} \int_0^{+\infty} d\delta' \int_{\ln E - \frac{\rho}{\sqrt{1-\rho^2}}\delta'}^{+\infty} d\gamma' e^{-\frac{\gamma'}{2(1-\rho^2)}} \left(e^{\gamma'+\rho\delta'/\sqrt{1-\rho^2}} - E\right) \times \\
&\qquad \frac{1}{4\pi(1-\rho^2)\tau} \left\{ e^{-\frac{[\gamma-\gamma'-(A-A')]^2}{4(1-\rho^2)\tau}} - e^{-\frac{[\gamma+\gamma'-(A+A')]^2}{4(1-\rho^2)\tau}} \right\} \left\{ e^{-\frac{(\delta-\delta')^2}{4(1-\rho^2)\tau}} - e^{-\frac{(\delta+\delta')^2}{4(1-\rho^2)\tau}} \right\},
\end{aligned}$$

and solving the above integral we have the fair price of **down-knock-out** Call option:

$$\begin{aligned}
C_L^{out}(t,S,\nu) &= e^{-(b_\rho r(T-t))} \left[e^{c_\rho \nu(T-t)} N(h_1) - e^{-\frac{\rho\nu}{\alpha(1-\rho^2)}} N(h_2) \right] \times \\
&\left\{ S * \left[N(d_1) - \left(\frac{L}{S}\right)^{\frac{1-2\rho^2}{1-\rho^2}} N(d_2) \right] - e^{\frac{\nu(T-t)}{2(1-\rho^2)}} E * \left[N(\tilde{d}_1) - \left(\frac{S}{L}\right)^{\frac{1}{1-\rho^2}} N(\tilde{d}_2) \right] \right\}, \quad (5.18)
\end{aligned}$$

where

$$\begin{cases}
h_1 = \frac{\nu/\alpha+\rho\nu(T-t)/(1-\rho^2)}{\sqrt{\nu(T-t)}}, & h_2 = \frac{-\nu/\alpha+\rho\nu(T-t)/(1-\rho^2)}{\sqrt{\nu(T-t)}}, \\
d_1 = \frac{\ln(S/E)+(1-2\rho^2)(\nu/2)(T-t)}{\sqrt{(1-\rho^2)\nu(T-t)}}, & \tilde{d}_1 = \frac{\ln(S/E)-(\nu/2)(T-t)}{\sqrt{(1-\rho^2)\nu(T-t)}}, \\
d_2 = \frac{\ln(L^2/E*S)+(1-2\rho^2)(\nu/2)(T-t)}{\sqrt{(1-\rho^2)\nu(T-t)}}, & \tilde{d}_2 = \frac{\ln(L^2/E*S)-(\nu/2)(T-t)}{\sqrt{(1-\rho^2)\nu(T-t)}},
\end{cases}$$

and

$$b_\rho = \left(1 - \frac{1}{2}(1-\rho^2)\right) \qquad c_\rho = \rho^2\left(1 - \frac{1}{4(1-\rho^2)^2}\right).$$

By formula (5.18), we know that the price of a **down-and-out** Call option is cheaper than the price of a vanilla call option, as we are going to prove in the next section "Numerical experiments". Under the financial point of view, it is clear that a holder of a vanilla call option should pay less premium. However, if the price is always greater than L (which is what a holder of a Call Option expects), then it is the same as a Call. This is why a **down-and-out** Call option is so attractive for many people.

Let us now consider a **down-and-in** European Call option and let $C_L^{in}(t,S,\nu)$ be its value. The option value $C_L^{in}(t,S,\nu)$ satisfies the PDE associated to our model for $S > L$. We need to determine the correct final and boundary conditions. A **down-and-in** option expires worthless unless the asset price reaches the lower barrier L before expiry, i.e., if S has been greater than L up to time T, then the option is not activated. Thus for $S > L, \forall t \in [0,T]$, the final condition is:

$$C_L^{in}(T,S,\nu) = 0.$$

If the asset price S reaches L before expiry, then the option immediately turns into a Vanilla Call, and the boundary condition is:

$$C_L^{in}(t,S=L,\nu) = C(t,S,\nu).$$

Therefore, the fair value of a **down-and-in** option is the solution of the following final-boundary value problem

$$\begin{aligned}
&\frac{\partial f}{\partial t} + \frac{1}{2}\nu S^2\frac{\partial^2 f}{\partial S^2} + \rho\nu\alpha S\frac{\partial^2 f}{\partial S\partial\nu} + \frac{1}{2}\nu\alpha^2\frac{\partial^2 f}{\partial \nu^2} + rS\frac{\partial f}{\partial S} - rf = 0,\\
&f(T,S,\nu) = (S-E)^+ \times \mathbf{1}_{\{\forall t\in[0,T]:S>L\}},\\
&f(t,L,\nu) = C(t,L,\nu),
\end{aligned} \tag{5.19}$$

in which case, we have supposed $f(t,S,\nu) = C_L^{in}(t,S,\nu)$ consistently with previous notation, where $S \in [L,+\infty)$, $t \in [0,T]$. By the following identity:

$$C(t,S,\nu) = C_L^{out}(t,S,\nu) + C_L^{in}(t,S,\nu) \qquad S \geq L, \tag{5.20}$$

we have the fair value of a **down-and-in** Call option as follows:

$$C_L^{in}(t,S,\nu) = C(t,S,\nu) - C_L^{out}(t,S,\nu).$$

Obviously, $C_L^{out}(t,S,\nu) = 0$ and $C_L^{in}(t,S,\nu) = C(t,S,\nu)$ for $S < L$. Therefore the identity

$$C(S,\nu,t) = C_L^{out}(t,S,\nu) + C_L^{in}(t,S,\nu) \qquad S \geq L,$$

still holds for $S < L$.

For a European **up-and-out** option with $H > E$, the solution is similar to the formula (5.18). It can also be shown that the sum of a European **up-and-out** Put option and a European **up-and-in** Put option equals a European vanilla Put option.

5.2.1 Numerical experiments

In this section we compute the price of a **down-and-out** Call Barrier Option [26] using the approximation method described above. The correlation value is the same used by Baksi-Kao-Chen, but in our case the drift of the variance process is zero, and we do not have a direct comparison with an alternative method. Thus we report the corresponding European Call values which, by construction, should be larger; and also the prices obtained by Heston model, choosing a particular set of values: $k = 0$ and $\Theta = indeterminate$, in order to have the same market model (5.2) at stochastic volatility with drift zero for the variance process.

In table 5.2 we can read that the price of the **down-knock-out** Call options is lower than the European Call option obtained using the same market model and parameter values. This result is compatible with what we have expected for Financial Theory (see Hull, "Options , Futures and other Derivatives" pp. 483). In table 5.3, we can read the results from comparison with Fourier method in the Heston case (by Gauss-lobotto algorithm). As one can see, the **down-and-out** Call option prices given by both methods are close to each other with a difference lesser 1%.

We want to remark that the proposed methodology is an approximation method; this one can be used in a stochastic volatility market model with arbitrary drift term. In fact, it is sufficient to choose a price of volatility risk, such that the drift of volatility process is zero to obtain the PDE (5.13).

Table 5.2: **Down-and-out** Call Barrier Option prices and European Call option prices computed by Approximation method for $E = 100$, $L = 70$, $\alpha = 0.1$, $\rho = -0.64$. We have chosen $r = 3\%$, $\nu_0 = 0.04$ and three different maturities T. At-the-money options (ATM) have $S_0 = E$; in-the-money options (INM) have $S_0 = E(1 + 10\%\sqrt{\nu_0 T})$ and out-of-the money options (OTM) have $S_0 = E(1 - 10\%\sqrt{\nu_0 T})$.

$(T = 1/12)$		
	down-and-out Call	Vanilla Call
ATM	1.77384	2.4305
INM	2.0727	2.7337
OTM	1.5048	2.1503
$(T = 3/12)$		
	down-and-out Call	Vanilla Call
ATM	3.0715	4.3755
INM	3.5822	4.9037
OTM	2.6123	3.8871
$(T = 6/12)$		
	down-and-out Call	Vanilla Call
ATM	4.3145	6.3790
INM	5.0229	7.1214
OTM	3.6785	5.6925

Table 5.3: **Down-and-out** Call Barrier Option prices computed by Perturbative method and Fourier method, for $E = 100$, $L = 80$, $\alpha = 0.1$, $\rho = -0.64$. We have chosen: $r = 3\%$. At-the-money options (ATM) have $S_0 = E$; in-the-money options (INM) have $S_0 = E(1 + 10\%\sqrt{\nu_0 T})$ and out-of-the money options (OTM) have $S_0 = E(1 - 10\%\sqrt{\nu_0 T})$.

$(T = 6/12)$			
	Volatility	Perturbative method	Fourier method
ATM	20%	4.3361	4.3196
	30%	6.4678	6.4593
	40%	8.2098	8.4480
INM	20%	5.1092	4.9654
	30%	7.6807	7.6785
	40%	9.9626	9.9847
OTM	20%	3.6172	3.4234
	30%	5.7154	5.7209
	40%	6.5834	6.5061

Conclusions

The Thesis investigates the benefits of Spectral Methods, which are found to be an appealing numerical technique when the solution in closed form doesn't exist, but unfortunately it cannot be used in every case. A remarkable case in which it is possible to use the Spectral Methods is for pricing the Double Barrier Options as we have seen in Chapter 3.

The main achievement of this Thesis is the introduction of two methods, that we have called **Geometrical Approximation** and **Perturbative Method** respectively, by which is possible to evaluate the fair option prices in the Heston and SABR market model. Both proposed methods can be generalised to other market models and for pricing other derivatives contracts, although, in order to show the above methodologies, we have chosen to pricing Options of only two kinds: Vanilla Options and knock-out Barrier Options.

On the first, we have that the **G. A.** method intends to be an alternative method, which can be particularly convenient for sensible values of the model parameters, which allows computation of closed-form expressions of approximated option prices.

The option price is approximated since we can get closed-form solutions for the PDE at the cost of modifying the Cauchy's condition, rather than looking for a numerical solution to the PDE with the exact Cauchy's condition. The proposed method has the advantage to compute a solution in closed form, therefore, we do not have the problems which plague the numerical methods.

For example, one can consider the inverse Fourier transform method, in which we have to compute an integral between zero and infinity. In this case in fact, there is always some problem in order to define (in practice) the correct domain of integration; or equivalently, considering also the finite difference method, in which we have to define a suitable grid, in other words we have some problems about the choice of the grid's meshes.

In the present work we have used the Geometrical Approximation method in the Heston model and in the SABR model, comparing the Vanilla Option price obtained with these computed by inverse Fourier transform, Monte-Carlo simulation and Finite Difference method or again the Implied Volatility method.

The Geometrical Approximation method is more reliable for low values of the correlation between price and variance shocks. In this case, our numerical experiments in a specific but sensible case show that the difference with the Fourier method is of the order of 1%. Markets in which the price/volatility correlation is low, and thus the G. A. method seems more promising, are the Electricity Markets.

Besides it is possible, through the G. A. method, get the Vanilla Option price by a strategy, whose price at time zero is equal to the sum of the option price with modified payoff and a bond price, so that, this one is equal to the difference between $|S_0 e^{\varepsilon_0} - E|$ capitalised at rate r. This strategy gives us, for every correlation value, a price higher than the Heston price around some percent, but in this way the writer of the Options is fully hedged. This strategy can be very useful for Banks and Institutions that write derivatives contracts.

On the second, the **Perturbative Method**, we have elaborated another approximating approach, illustrated in Chapter 5, in which we have discussed a particular choice of the volatility price of risk in the Heston model, namely such that the drift term of the risk-neutral stochastic volatility process is zero. This allowed us to illustrate an alternative methodology for solving the pricing PDE in an approximate way, in which we have neglected some terms of the PDE, recovering a pricing formula which in this particular case, turn out to be simple, for Vanilla Options and Barrier Options. The approximating formulas give an accurate price close to that obtained by Fourier transform for the Vanilla options, and Down-knock-out Call options. The Perturbative method can be used for pricing several derivatives contracts and we are sure that manifold applications will follow.

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Appendix A

Numerical methods for the Heston and SABR model

A.1 Heston method

Let be given the Heston PDE:

$$\frac{\partial f}{\partial t} + \frac{1}{2}\nu S^2 \frac{\partial^2 f}{\partial S^2} + \alpha\rho\nu S \frac{\partial^2 f}{\partial S \partial \nu} + \frac{1}{2}\frac{\partial^2 f}{\partial \nu^2} + rS\frac{\partial f}{\partial S} + [\kappa(\Theta - \nu) - \lambda(t, S, \nu)]\frac{\partial f}{\partial \nu} - rf = 0; \tag{A.1}$$

with following Cauchy and Dirichlet conditions:

$$\begin{aligned}
&f(T, S\nu) = (S - E)^+,\\
&f(t, 0, \nu) = 0,\\
&\frac{\partial f}{\partial S}(t, \infty, \nu) = 1,\\
&rS\frac{\partial f}{\partial S} + \kappa\Theta\frac{\partial f}{\partial \nu}(t, S, 0) - rf(t, S, 0) + \frac{\partial f}{\partial t}(t, S, 0) = 0,\\
&f(t, S, \infty) = S.
\end{aligned}$$

Suppose that the solution to the Heston PDE is like the form of Black-Scholes model:

$$f(t, S, \nu) = S\mathbf{P}_1 - Ee^{-r(T-t)}\mathbf{P}_2 \tag{A.2}$$

where $\mathbf{P}_1$ and $\mathbf{P}_2$ is what we are going to find. Make the transform $x = \ln S$ and substitute the formula $(A.2)$ into the PDE $(A.1)$. We will get

$$\frac{\partial P_j}{\partial t} + \frac{1}{2}\nu\frac{\partial^2 P_j}{\partial x^2} + \alpha\rho\nu\frac{\partial^2 P_j}{\partial x \partial \nu} + \frac{1}{2}\frac{\partial^2 P_j}{\partial \nu^2} + (r + u_j\nu)\frac{\partial P_j}{\partial x} + (a - b_j\nu)\frac{\partial P_j}{\partial \nu} = 0, \qquad j = 1, 2 \tag{A.3}$$

where $u_1 = \frac{1}{2}, \quad u_2 = -\frac{1}{2}, \quad a = \kappa\Theta, \quad b_1 = \kappa + \lambda - \alpha\rho, \quad b_2 = \kappa + \lambda.$

Considering the pay-off of the option, they are subject to the initial condition

$$\mathbf{P}_j(T, x, \nu; \ln E) = 1_{x \geq \ln E}$$

$\mathbf{P}_j$ are the conditional probability that the option will expire in the money. Heston showed that the characteristic function of $\mathbf{P}_j$ is:

$$f_j(t, x, \nu; \phi) = e^{C_j(T-t;\phi) + D_j(T-t;\phi)\nu + \imath\phi x}$$

where

$$C_j(T-t;\phi) = r\phi\imath(T-t) + \frac{a}{\alpha^2}\left[(b_j - \alpha\rho\phi\imath + d_j)(T-t) - 2\ln\left(\frac{1 - g_j e^{d(T-t)}}{1 - g_j}\right)\right],$$

$$D_j(T-t;\phi) = \frac{b_j - \alpha\rho\phi\imath + d_j}{\alpha^2}\left[\frac{1 - e^{d_j(T-t)}}{1 - g_j e^{d_j(T-t)}}\right],$$

and

$$g_j = \frac{b_j - \alpha\rho\phi\imath + d_j}{b_j - \alpha\rho\phi\imath - d_j},$$

$$d_j = \sqrt{(\alpha\rho\phi\imath - b_j)^2 - \alpha^2(2u_j\phi\imath - \phi^2)}.$$

Convert the characteristic function back, we can get

$$\mathbf{P}_j(T, x, \nu; \ln E) = \frac{1}{2} + \frac{1}{\pi} \int_0^{+\infty} Re \left[\frac{e^{-\imath\phi \ln E} f_j(t, x, \nu; \phi)}{\imath\phi} \right] d\phi$$

Then solution of Heston PDE is a direct result.

A.1.1 MatLab Code for the Heston model

```
function [call_value]=heston(T,S0,K,v0,theta,kappa,sigma,rho,lambda,r)\\

Heston.m calculates the heston option price by using 20 points\\
Gauss-Legendre Integration in every small interval.\\
\\
The algorithm is based on the analytic solution given by Heston (1993).\\
The same notation for each parameter as shown in Heston's paper is \\
being used (1993). \\
\\
T = 1;\\
r = 0.03;\\
S0 = 100;\\
K = 100;\\
kappa = 2;\\
theta= 0.04;\\
v0 = 0.4;\\
sigma = 0.038;\\
rho = -0.1;\\
lambda = 0;\\
\\
i = sqrt(-1);\\
x0 = log(S0);\\
a = kappa*theta;\\
\\
Gauss_Legendre20=[0.0176140070678 0.0406014298819 0.0626720482976\\
0.0832767415506  0.101930119822 0.118194531969 0.131688638458 \\
0.142096109327 0.149172986482 ...0.15275338714 0.15275338714 \\
0.149172986482 0.142096109327 0.131688638458  0.118194531969 \\
0.101930119826 0.0832767415506 0.0626720482976 0.0406014298819\\
0.0176140070678; -0.993128599185 -0.963971927278  -0.912234428251\\
-0.839116971822 -0.74633190646 -0.636053680727 -0.510867001951 ...\\
-0.373706088715 -0.227785851142 -0.0765265211335 0.0765265211335\\
0.227785851142  0.373706088715 0.510867001951 0.636053680727\\
0.74633190646 ...0.839116971822 0.912234428251 0.963971927278 \\
0.993128599185];   the first row is the weight, the second row is the abscissas\\
cutoff=200;    truncate phi at 200 \\
intvl=2;     length of each interval in each interval apply 20 points\\
  Gauss-Legendre Integration\\
\\
P1_int=zeros(1,cutoff/intvl);\\
P2_int=zeros(1,cutoff/intvl);\\
for j=1:cutoff/intvl\\
    for k=1:20\\
        phi=intvl*Gauss_Legendre20(2,k)/2+intvl*(2*j-1)/2;\\
        P1_int(j)=Gauss_Legendre20(1,k)*(real(exp(-i*phi*log(K))\\
        *feval('f1',x0,v0,T,phi,theta,kappa,sigma,rho,lambda,r,i,a)/(i*phi)))+P1_int(j);\\
        P2_int(j)=Gauss_Legendre20(1,k)*(real(exp(-i*phi*log(K))\\
        *feval('f2',x0,v0,T,phi,theta,kappa,sigma,rho,lambda,r,i,a)/(i*phi)))+P2_int(j);\\
    end\\
    P1_int(j)=intvl/2*P1_int(j);\\
    P2_int(j)=intvl/2*P2_int(j);\\
end\\
P1=0.5+sum(P1_int)/pi;\\
P2=0.5+sum(P2_int)/pi;\\
call_value=S0*P1-K*exp(-r*T)*P2;\\
\\
function y=f1(x,v,t,phi,theta,kappa,sigma,rho,lambda,r,i,a)\\
u1=0.5;\\
b1=kappa+lambda-rho*sigma;\\
d1=sqrt((rho*sigma*phi*i-b1).^2-sigma^2*(2*u1*phi*i-phi.^2));\\
g1=(b1-rho*sigma*phi*i+d1)/(b1-rho*sigma*phi*i-d1);\\
C1=r*phi*i*t+a*(b1-rho*sigma*phi*i+d1)*t/(sigma^2);\\
D1=(b1-rho*sigma*phi*i+d1)/(sigma^2)*((1-exp(d1*t))/(1-g1*exp(d1*t)));\\
y=exp(C1+D1.*v+i*phi*x).*((1-g1.*exp(d1.*t))./(1-g1)).^(-2*a/(sigma^2));\\
\\
function y=f2(x,v,t,phi,theta,kappa,sigma,rho,lambda,r,i,a)\\
u2=-0.5;\\
b2=kappa+lambda;\\
d2=sqrt((rho*sigma*phi*i-b2).^2-sigma^2*(2*u2*phi*i-phi^2));\\
g2=(b2-rho*sigma*phi*i+d2)/(b2-rho*sigma*phi*i-d2);\\
C2=r*phi*i*t+a*(b2-rho*sigma*phi*i+d2)*t/(sigma^2);\\
D2=(b2-rho*sigma*phi*i+d2)/(sigma^2)*((1-exp(d2*t))/(1-g2*exp(d2*t)));\\
y=exp(C2+D2.*v+i*phi*x).*((1-g2.*exp(d2.*t))./(1-g2)).^(-2*a/(sigma^2));\\
```

A.2 Finite difference method

We present the finite difference schemes used in chapter4.

$$\frac{\partial f}{\partial t} = a(x,y)\frac{\partial^2 f}{\partial x^2} + 2b(x,y)\frac{\partial^2 f}{\partial x \partial y} + c(x,y)\frac{\partial^2 f}{\partial y^2} + d(x,y)\frac{\partial f}{\partial x} + e(x,y)\frac{\partial f}{\partial y} \tag{A.4}$$

on the domain $(x, y, t) \in \Omega = [L_x, H_x] \times [L_y, H_y] \times \mathbb{R}^+$ where the coefficients satisfy the following inequalities

$$ac - b^2 > 0, \quad a > 0, \quad c > 0, d \geq 0, \tag{A.5}$$

$$d \geq 0, \quad e(x, L_y) \geq 0 \quad \text{and} e(x, H_y) \leq 0 \tag{A.6}$$

and the following equalities

$$a(x; L_y) = b(x; L_y) = c(x; L_y) = 0. \tag{A.7}$$

The inequalities in $(A.5)$ must hold for the PDE to be parabolic. The inequalities in $(A.6)$ and the equalities in $(A.7)$ are general enough to allow for all PDEs that arise in stochastic volatility models. For the problem to be well posed an initial condition, $u(x; y; 0) = \phi(x; y)$, and four boundary conditions need to be specified. A Dirichlet condition will be used for the boundary at $x = L_x$ and a von Neumann condition will be used at $x = H_x$

$$f(t, L_x, y) = c_L \qquad \frac{\partial f}{\partial x}(t, H_x, y) = c_H. \tag{A.8}$$

Neither a Dirichlet nor a von Neumann condition is used as a boundary condition in the y-direction, the only requirement is that the PDE itself must be solved on the boundaries, see Zvan et al. [2003] and Duffy [2006]. Using equation $(A.7)$ and the assumption that

$$\frac{\partial f}{\partial t} = d(x,y)\frac{\partial f}{\partial x} + e(x,y)\frac{\partial f}{\partial y}, \qquad y = L_y$$

$$\frac{\partial f}{\partial t} = a(x,y)\frac{\partial^2 f}{\partial x^2} + d(x,y)\frac{\partial f}{\partial x} \qquad y = H_y.$$

The fully implicit and Crank-Nicolson schemes require the inversion of non tri-diagonal matrices. Such schemes turn out to be very slow, we will discuss how splitting methods can be employed to overcome this problem. Stability, consistency and convergence of these methods will be shown on a uniform grid. In the first section the general Taylor approximations used to obtain the discrete approximations to our continuous derivatives will be derived. Then we will investigate different types of finite difference schemes that can be used to approximate the solutions of two dimensional parabolic PDEs.

A.2.1 Discrete approximations

The finite difference method truncates the unbounded domain Ω to the bounded domain $\overline{\Omega} = [L_x, H_x] \times [L_y, H_y] \times [0, T]$. One can see that our aim is to obtain approximations to the true solution on the three dimensional mesh

$$\tilde{\Omega} = \{(t, x_i, y_i) | i = 0, 1, ..., m, j = 0, 1, ..., n, k = 0, 1, ..., l\}$$

.

with the approximation at each mesh point given by

$$f^k_{i,j} \approx f(t, x_i, y_j)$$

.

Assuming, for the moment, that the mesh points are uniformly spaced we can write

$$x_i = L_x + ih_x \qquad fori = 0, 1, ..., m$$

$$y_i = L_y + jh_y \qquad forj = 0, 1, ..., n$$

$$t_k = k\Delta t \qquad fork = 0, 1, ..., l$$

where

$$h_x = \frac{H_x - L_x}{m}, h_y = \frac{H_y - L_y}{m} \qquad \Delta t = \frac{T}{l}.$$

The finite difference approximations of the derivatives in $(A.4)$, excluding the approximation of the mixed derivative, one can be obtained in exactly way, as follows

$$\begin{aligned}
\Delta_x^+ u_{i,j}^k &= \frac{u_{i+1,j}^k - u_{i,j}^k}{h_x}\\
\Delta_y^+ u_{i,j}^k &= \frac{u_{i,j+1}^k - u_{i,j}^k}{h_y}\\
\\
\Delta_x^- u_{i,j}^k &= \frac{u_{i,j}^k - u_{i-1,j}^k}{h_x}\\
\Delta_y^- u_{i,j}^k &= \frac{u_{i,j}^k - u_{i,j-1}^k}{h_y}\\
\\
\Delta_x u_{i,j}^k &= \frac{u_{i+1,j}^k - u_{i-1,j}^k}{2h_x}\\
\Delta_y u_{i,j}^k &= \frac{u_{i,j+1}^k - u_{i,j-1}^k}{2h_y}\\
\\
\Delta_x^2 u_{i,j}^k &= \frac{u_{i+1,j}^k - 2u_{i,j}^k + u_{i-1,j}}{h_x^2}\\
\Delta_y^2 u_{i,j}^k &= \frac{u_{i,j+1}^k - 2u_{i,j}^k + u_{i,j-1}}{h_y^2}
\end{aligned} \tag{A.9}$$

We still need to derive the divided difference approximation of the mixed derivative. We give a derivation of the second order approximation to the mixed derivative proposed in Hout and Welfert [2006]. Second order approximations of the cross derivative can be derived with the aid of the following Taylor expansions about the reference point (x_i, y_j, τ_k)

$$\begin{aligned}
u(x_{i+1}, y_{j+1}, \tau_k) = {} & u + h_x \frac{\partial u}{\partial x} + h_y \frac{\partial u}{\partial y} + \frac{1}{2} h_x^2 \frac{\partial^2 u}{\partial x^2} + h_x h_y \frac{\partial^2 u}{\partial x \partial y} + \frac{1}{2} h_y^2 \frac{\partial^2 u}{\partial y^2}\\
& + \frac{1}{3!} h_x^3 \frac{\partial^3 u}{\partial x^3} + \frac{3}{3!} h_x^2 h_y \frac{\partial^3 u}{\partial x^2 \partial y} + \frac{3}{3!} h_x h_y^2 \frac{\partial^3 u}{\partial x \partial y^2} + \frac{1}{3!} h_y^3 \frac{\partial^3 u}{\partial y^3} + 0(h_x^4, h_x^3 h_y, h_x^2 h_y^2, h_x h_y^3, h_y^4)
\end{aligned} \tag{A.10}$$

$$\begin{aligned}
u(x_{i-1}, y_{j+1}, \tau_k) = {} & u - h_x \frac{\partial u}{\partial x} + h_y \frac{\partial u}{\partial y} + \frac{1}{2} h_x^2 \frac{\partial^2 u}{\partial x^2} - h_x h_y \frac{\partial^2 u}{\partial x \partial y} + \frac{1}{2} h_y^2 \frac{\partial^2 u}{\partial y^2}\\
& - \frac{1}{3!} h_x^3 \frac{\partial^3 u}{\partial x^3} - \frac{3}{3!} h_x^2 h_y \frac{\partial^3 u}{\partial x^2 \partial y} + \frac{3}{3!} h_x h_y^2 \frac{\partial^3 u}{\partial x \partial y^2} + \frac{1}{3!} h_y^3 \frac{\partial^3 u}{\partial y^3} + 0(h_x^4, h_x^3 h_y, h_x^2 h_y^2, h_x h_y^3, h_y^4)
\end{aligned} \tag{A.11}$$

$$u(x_{i+1}, y_{j-1}, \tau_k) = u + h_x \frac{\partial u}{\partial x} - h_y \frac{\partial u}{\partial y} + \frac{1}{2} h_x^2 \frac{\partial^2 u}{\partial x^2} - h_x h_y \frac{\partial^2 u}{\partial x \partial y} + \frac{1}{2} h_y^2 \frac{\partial^2 u}{\partial y^2}$$
$$+ \frac{1}{3!} h_x^3 \frac{\partial^3 u}{\partial x^3} - \frac{3}{3!} h_x^2 h_y \frac{\partial^3 u}{\partial x^2 \partial y} + \frac{3}{3!} h_x h_y^2 \frac{\partial^3 u}{\partial x \partial y^2} - \frac{1}{3!} h_y^3 \frac{\partial^3 u}{\partial y^3} + 0(h_x^4, h_x^3 h_y, h_x^2 h_y^2, h_x h_y^3, h_y^4) \quad \text{(A.12)}$$

$$u(x_{i-1}, y_{j-1}, \tau_k) = u - h_x \frac{\partial u}{\partial x} - h_y \frac{\partial u}{\partial y} + \frac{1}{2} h_x^2 \frac{\partial^2 u}{\partial x^2} + h_x h_y \frac{\partial^2 u}{\partial x \partial y} + \frac{1}{2} h_y^2 \frac{\partial^2 u}{\partial y^2}$$
$$- \frac{1}{3!} h_x^3 \frac{\partial^3 u}{\partial x^3} - \frac{3}{3!} h_x^2 h_y \frac{\partial^3 u}{\partial x^2 \partial y} - \frac{3}{3!} h_x h_y^2 \frac{\partial^3 u}{\partial x \partial y^2} - \frac{1}{3!} h_y^3 \frac{\partial^3 u}{\partial y^3} + 0(h_x^4, h_x^3 h_y, h_x^2 h_y^2, h_x h_y^3, h_y^4) \quad \text{(A.13)}$$

$$u(x_{i-1}, y_j, \tau_k) = u - h_x \frac{\partial u}{\partial x} + \frac{1}{2} h_x^2 \frac{\partial^2 u}{\partial x^2} - \frac{1}{3!} h_x^3 \frac{\partial^3 u}{\partial x^3} + 0(h_x^4) \quad \text{(A.14)}$$

$$u(x_{i+1}, y_j, \tau_k) = u + h_x \frac{\partial u}{\partial x} + \frac{1}{2} h_x^2 \frac{\partial^2 u}{\partial x^2} + \frac{1}{3!} h_x^3 \frac{\partial^3 u}{\partial x^3} + 0(h_x^4) \quad \text{(A.15)}$$

$$u(x_i, y_{j-1}, \tau_k) = u - h_y \frac{\partial u}{\partial y} + \frac{1}{2} h_y^2 \frac{\partial^2 u}{\partial y^2} - \frac{1}{3!} h_y^3 \frac{\partial^3 u}{\partial y^3} + 0(h_y^4) \quad \text{(A.16)}$$

$$u(x_i, y_{j+1}, \tau_k) = u + h_y \frac{\partial u}{\partial y} + \frac{1}{2} h_y^2 \frac{\partial^2 u}{\partial y^2} + \frac{1}{3!} h_y^3 \frac{\partial^3 u}{\partial y^3} + 0(h_y^4). \quad \text{(A.17)}$$

To construct a general second order approximation the following linear combinations are of critical importance

$$u(x_{i+1}, y_{j+1}, \tau_k) + u(x_{i-1}, y_{j-1}, \tau_k) = 2u + h_x^2 \frac{\partial^2 u}{\partial x^2} + 2 h_x h_y \frac{\partial^2 u}{\partial x \partial y} + h_y^2 \frac{\partial^2 u}{\partial y^2}$$
$$+ 0(h_x^4, h_x^3 h_y, h_x^2 h_y^2, h_x h_y^3, h_y^4)$$
$$u(x_{i-1}, y_{j+1}, \tau_k) + u(x_{i+1}, y_{j-1}, \tau_k) = 2u + h_x^2 \frac{\partial^2 u}{\partial x^2} - 2 h_x h_y \frac{\partial^2 u}{\partial x \partial y} + h_y^2 \frac{\partial^2 u}{\partial y^2}$$
$$+ 0(h_x^4, h_x^3 h_y, h_x^2 h_y^2, h_x h_y^3, h_y^4)$$
$$u(x_{i-1}, y_j, \tau_k) + u(x_{i+1}, y_j, \tau_k) + u(x_i, y_{j-1}, \tau_k) + u(x_i, y_{j+1}, \tau_k)$$
$$= 4u + h_x^2 \frac{\partial^2 u}{\partial x^2} + h_y^2 \frac{\partial^2 u}{\partial y^2} + 0(h_x^4, h_y^4). \quad \text{(A.18)}$$

by making an appropriate linear combination of the first two equations we obtain

$$(1+\omega)[u(x_{i+1}, y_{j+1}, \tau_k) + u(x_{i-1}, y_{j-1}, \tau_k)] - (1-\omega)[u(x_{i-1}, y_{j+1}, \tau_k) + u(x_{i+1}, y_{j-1}, \tau_k)]$$
$$= 4\omega u + 2\omega h_x^2 \frac{\partial^2 u}{\partial x^2} + 4 h_x h_y \frac{\partial^2 u}{\partial x \partial y} + 2\omega h_x^2 \frac{\partial^2 u}{\partial x^2} + 0(h_x^4, h_x^3 h_y, h_x^2 h_y^2, h_x h_y^3, h_y^4) \quad \text{(A.19)}$$

where $\omega \in [-1, +1]$. The $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$ terms ca now be eliminate by adding -2ω of (5.18), after rearranging we obtain

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{(1+\omega)[u(x_{i+1}, y_{j+1}, \tau_k) + u(x_{i-1}, y_{j-1}, \tau_k)]}{4 h_x h_y} - \frac{(1-\omega)[u(x_{i-1}, y_{j+1}, \tau_k) + u(x_{i+1}, y_{j-1}, \tau_k)]}{4 h_x h_y}$$
$$+ \frac{4\omega u - 2\omega[u(x_{i-1}, y_j \tau_k) + u(x_{i+1}, y_j, \tau_k) + u(x_i, y_{j-1}, \tau_k) + u(x_i, y_{j+1}, \tau_k)]}{4 h_x h_y} + 0(h_x^2, h_x h_y, h_y^2) \quad \text{(A.20)}$$

The difference operator for the mixed derivative can be written as

$$\Delta_{xy}^{\omega} u_{i,j}^{k} = \frac{(1+\omega)[u_{i+1,j+1}^{k} + u_{i-1,j-1}^{k}] - (1-\omega)[u_{i-1,j+1}^{k} + u_{i+1,j-1}^{k}]}{4h_x h_y} + \frac{4\omega u_{i,j}^{k} - 2\omega[u_{i-1,j}^{k} + u_{i+1,j}^{k} + u_{i,j-1}^{k} + u_{i,j+1}^{k}]}{4h_x h_y} \quad \text{(A.21)}$$

A.2.2 MatLab Code for CranKNicolson method

```
function [P] = CrankNicolsonEuro(OptionType,SO,K,r,q,T,sig,Smin,Smax,Ds,Dt)
This function prices a Vanilla European Call/Put using the Crank Nicolson
Scheme of the Finite Difference Method.

Parameters are as follows:
OptionType = 1 for a Call or 0 for a Put
SO = initial asset price ;
K = strike price ;
r = risk free rate ;
q = dividend rate;
T = time to maturity ;
sig = volatility ; Smin = minimum stock price ;
Smax = maximum stock price ;
Ds = stock price step size ;
Dt = time step size.

This function keeps track of the time required to price the option.
Note: When pricing using Finite Difference Methods, you increase accuracy
by making the mesh finer and finer. In our case adequate pricing results are
obtained with Ds = 1 and Dt = 1/600 with Smin = 20 and Smax = 300.

Please note that the use of this code is not restricted in any way.
However, referencing the author of the code would be appreciated.
To run this program, simply use the function defined in the 1st line.
http://www.global-derivatives.com
info@global-derivatives.com
Olivier Rochet (February 2006)

tic Keep track of time
Smax = 300;
Smin = 0;
 Calculate number of stock price steps and take care of rounding.
N = round((Smax - Smin)/Ds);
Ds = (Smax - Smin)/;
 Calculate number of time steps and take care of rounding.
M = round(T/Dt);
Dt = (T/M);

ME=zeros(N,N);  define ME matrix
MI=zeros(N,N);  define MI matrix
S=zeros(N,1);   stock price vector
V=zeros(N,1);   option value vector
matsol=zeros(N,M+1);  solution matrix

for i=1:1:N Generate S and V vector
    S(i)=Smin + i*Ds;
    if OptionType == 1
        V(i)=max(S(i)-K,0);  Call: Payoff that is initial condition
    else
        V(i)=max(K-S(i),0); Put: Payoff that is initial condition
    end
end

 Build ME matrix
for i=1:1:N
    Set up coefficients
    Alpha = 0.5*(sig^2)*(S(i)^2)*(Dt/(Ds^2));
    Betha = (r-q)*S(i)*(Dt/(2*Ds));
    Bde = Alpha - Betha;
    De = 1-r*Dt-2*Alpha;
    Ade = Alpha + Betha;
     Fill ME matrix
    if i==1
        ME(i,i) = 1 + De + 2*Bde;
        ME(i,i+1) = Ade - Bde;
    elseif i==N
        ME(i,i-1) = Bde - Ade;
        ME(i,i) = 1 + De + 2*Ade;
    else
        ME(i,i-1) = Bde;
        ME(i,i) = 1 + De;
```

```
        ME(i,i+1) = Ade;
    end
end

matsol(:,1)=V;  Initiate first column of matrix solution with payoff that
                is initial condition

 Build MI matrix
for i=1:1:N
    Set up coefficients
    Alpha = 0.5*(sig^2)*(S(i)^2)*(Dt/(Ds^2));
    Betha = (r-q)*S(i)*(Dt/(2*Ds));
    Bdi= -Alpha + Betha;
    Di= 1+r*Dt+2*Alpha;
    Adi= -Alpha - Betha;
    Fill MI matrix
    if i==1
        MI(i,i) = 1 + Di + 2*Bdi;
        MI(i,i+1) = Adi - Bdi;
    elseif i==N
        MI(i,i-1) = Bdi - Adi;
        MI(i,i) = 1 + Di + 2*Adi;
    else
        MI(i,i-1) = Bdi;
        MI(i,i) = 1 + Di;
        MI(i,i+1) = Adi;
    end
end

invMI = MI^-1;  Invert matrix MI before performing calculations

for k=1:M  Generate solution matrix
    matsol(:,k+1)= invMI*ME*matsol(:,k);
end

 find closest point on the grid and return price
 with a linear interpolation if necessary

DS = SO-Smin;

indexdown = floor(DS/Ds);
indexup = ceil(DS/Ds);

if indexdown == indexup
    P = matsol(indexdown,M+1);
else
    P = matsol(indexdown,M+1)+ (SO - S(indexdown))/(S(indexup)-S(indexdown))...
       *(matsol(indexup,M+1) - matsol(indexdown,M+1));
end

toc  Keep track of time
```

A.3 SDE approximation

Monte Carlo simulation was performed by discretising the stochastic processes using the Euler-Maruyama method. This resulted in,

$$S_t = S_{t-1} + rS_{t-1}dt + \sqrt{\nu_{t-1}}S_{t-1}d\tilde{W}_t^{(1)}$$
$$\nu_t = \nu_{t-1} + \kappa(\Theta - \nu_{t-1})dt + \alpha\sqrt{\nu_{t-1}}d\tilde{W}_t^2$$

where $\tilde{W}_t^{(1)}$ and $\tilde{W}_t^{(2)}$ are standard normal random variables with correlation ρ. The above can be made computationally easier by expressing $\tilde{W}_t^{(1)}$ and $\tilde{W}_t^{(2)}$, as a function of independent standard normal random variables, using the Cholesky decomposition,

$$\tilde{W}_t^{(1)} = \xi_t^{(1)}$$
$$\tilde{W^{(2)}}_t = \rho\xi_t^{(1)} + \xi_t^{(2)}\sqrt{1-\rho^2}$$

where $\xi_t^{(1)}$ and $\xi_t^{(2)}$ are independent standard normal random variables.

A.3.1 C++ Code for Monte-Carlo method

```
#include <iostream>
#include <valarray>

#define BOOST_NO_STDC_NAMESPACE
#include <Scrivania/cpp/boost/random.hpp>



std::pair<double,double> meanVariance(const std::valarray<double>& x)
{
  //std::cout << x.size() << std::endl;
  double mean = 0;
  double M2   = 0.0;
  size_t n=0;
  while(n<x.size())
  {
    double valore = x[n++];
    double delta = valore-mean;
    mean += delta/n;
    M2 += delta*(valore-mean);
  }
  double variance = M2/(n-1);
  //std::cout << mean << ' ' << variance << std::endl;
  return std::make_pair(mean,variance);
}


std::valarray<double> partePositiva(const std::valarray<double>& x)
{
  std::valarray<double> result = x;
  for(size_t i=0; i<x.size(); ++i)
  {
    if(result[i]<0.0) result[i]=0.0;
  }
  return result;
}


class StateVariable : public std::pair<double,double>
{
  public:
    StateVariable()
    : std::pair<double,double>(), S(this->first), V(this->second) {}
    StateVariable(const double& S0, const double& V0)
    : std::pair<double,double>(S0,V0), S(this->first), V(this->second) {}
    StateVariable& operator=(const StateVariable& s)
    {
      if(this!=&s)
      {
        S = s.S;
        V = s.V;
      }
      return *this;
    }
    double& S;
    double& V;
};


class RandomVector
{
  public:
    RandomVector(size_t n0) : n(n0), v(n) {}
    const StateVariable& operator[] (size_t i) const
    {
      return v[i];
    }
    StateVariable& operator[] (size_t i)
    {
      return v[i];
    }
    const double& S(size_t i) const
    {
      return v[i].S;
    }
    const double& V(size_t i) const
    {
      return v[i].V;
    }
    double& S(size_t i)
    {
      return v[i].S;
    }
    double& V(size_t i)
    {
      return v[i].V;
    }
    std::valarray<double> S() const
```

```
    {
      std::valarray<double> result(n);
      for(size_t i=0; i<n; ++i)
        result[i] = S(i);
      return result;
    }
    std::valarray<double> V() const
    {
      std::valarray<double> result(n);
      for(size_t i=0; i<n; ++i)
        result[i] = V(i);
      return result;
    }
  private:
    const size_t n;
    std::valarray<StateVariable> v;
};


const std::pair<double,double> operator*(const RandomVector& v1, const RandomVector& v2)
{
  return std::make_pair((v1.S()*v2.S()).sum(),(v1.V()*v2.V()).sum());
}


class HestonModel
{
  public:
    HestonModel(const double& r0, const double& kappa0, const double& theta0,
     const double& lambda0, const double&corr0, const double& S00, const double& V00)
     : S0(S00), V0(V00), r(r0), kappa(kappa0), theta(theta0), lambda(lambda0), corr(corr0),
     rad1MenoCorr2(std::sqrt(1.0-corr0*corr0)), kappaTheta(kappa*theta)
    {}
    double S0;
    double V0;
    std::pair<double,double> callMonteCarlo(const double& T, const double& K, const double& deltaT, size_t N, size_t seed) const
    {
      std::valarray<double> S = distributionAtMaturity(T,deltaT,N,seed).S();
      std::valarray<double> payoff = partePositiva(S-K);
      std::pair<double,double> stat = meanVariance(payoff*std::exp(-r*T));
      stat.second = std::sqrt(stat.second/N);
      return stat;
    }
    RandomVector distributionAtMaturity(const double& T, const double& deltaT, size_t N, size_t seed) const
    {
      RandomVector result(N);
      boost::mt19937 motore(static_cast<boost::uint32_t>(seed)); // tutti i processi hanno lo stesso seed ed esiste almeno il processo 0
      boost::normal_distribution<double> distribuzione;
      boost::variate_generator<boost::mt19937,boost::normal_distribution<double> > rng(motore,distribuzione);
      for(size_t n=0; n<N; ++n)
      {
        double S = S0;
        double V = V0;
        for(double t=0.0; std::abs(t-T)>deltaT/2; t += deltaT)
        {
          double epsS = rng();
          double epsV = rng();
          chol(epsS,epsV);
          double driftS = r*S*deltaT;
          double diffS  = sqrt(V*deltaT)*epsS;
          double driftV = (kappaTheta-kappa*V)*deltaT;
          double diffV  = lambda*sqrt(V*deltaT)*epsV;
          S += driftS + diffS;
          V += driftV + diffV;
          if(V<0) V = -V;
          //std::cout << n << ' ' << t << ' ' << T << ' ' << std::abs(t-T) << ' ' << S << ' ' << V << std::endl;
          //system("PAUSE");
        }
        //std::cout << n << ' ' << S << ' ' << V << std::endl;
        result[n] = StateVariable(S,V);
      }
      return result;
    }
  private:
    const double r;
    const double kappa;
    const double theta;
    const double lambda;
    const double corr;
    const double rad1MenoCorr2;
    const double kappaTheta;
    void chol(double& e1, double& e2) const
    {
      e2=corr*e1 + rad1MenoCorr2*e2;
    }
};

int main()
{
  double r      = 0.05;
  double kappa  = 10.;
  double theta  = 0.16;
```

```
  double lambda = 0.10;
  double corr   = -0.8;
  double S0     = 100;
  double V0     = 0.16;
  HestonModel hestonModel(r,kappa,theta,lambda,corr,S0,V0);
  double deltaT = 1./(250);
  size_t N      = 10000;
  size_t seed   = 32421;
  double T      = 0.5;
  double K      = S0*0.8;
  std::pair<double,double> priceEss = hestonModel.callMonteCarlo(T,K,deltaT,N,seed);
  std::cout.precision(12);
  int dec = int(std::log10(priceEss.first));
  std::cout << "Prezzo = " << priceEss.first  << std::endl;
  std::cout << "Ess    = ";
  for(int i = 1; i<=dec; ++i) std::cout << ' ';
  std::cout << priceEss.second << std::endl;
  return 0;
}
```

A.4 Geometrical Approximation Code: Heston

```
function V_Hcall = C(S_t,E,nu_t,T,t,r)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%     Heston Model
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc

t = 0;
n = 6;
T = n/12;                           %%%%% Maturity date  %%%%%
r = 0.03;   q=0.0;                  %%%%% Interest rate  %%%%%

nu_t = 0.03;
B_t = exp(r*(T-t));


K = 1.15;
theta = 0.04;
alpha = 0.39;
rho   = -0.64;

lambda = 0;
kappa =  (K-lambda*alpha);
Theta =  K*theta/(K-lambda*alpha);

var = (alpha^2/kappa)*(1-exp(-kappa*T))*[(nu_t-Theta/2)*exp(-kappa*T)+Theta/2];   %%%%             Variance                 %%%%%%%%%%
var_infinity  = alpha^2*Theta/2*kappa;                                            %%%% Variance for T that goes to infinity %%%%%%%%%%


a   =  -(1/2-kappa*rho/alpha)^2/(2*(1-rho^2));
b   =   (1/2-kappa*rho/alpha)/(1-rho^2);

V  =  ((nu_t-Theta)*exp(-kappa*(T-t))+Theta);
epsilon_t = (rho*nu_t/alpha);
epsilon_medio = ((rho/alpha)*((nu_t-Theta)*exp(-kappa*(T-t))+ Theta));
Stochastic_Err = (epsilon_medio-epsilon_t);


E =   100;                                                             %%%%% Stike Price %%%%%%%%
S_t = E;%*(1-0.1*sqrt(Theta*T));                                       %%%%% Underlying value %%%

(E*exp(Stochastic_Err+(rho/alpha)^2*var));

d1 =  [log(S_t*exp(Stochastic_Err)/E) + ((r-q-(kappa*rho*Theta/alpha))+(1-rho^2)*(1-b)*nu_t)*(T-t)]/sqrt((1-rho^2)*nu_t*(T-t));
d2 =  d1 - sqrt((1-rho^2)*nu_t*(T-t));

delta = -[(kappa*rho*Theta/alpha)-(a+(1-b)^2*(1-rho^2)/2)*nu_t]*(T-t);

                                                                 %%%%%% Pricing Formula %%%%%%%

    V_Hcall = (S_t*exp(Stochastic_Err + (rho/alpha)^2*var))*exp(delta)*normcdf(d1,0,1)-E*(exp(-r*(T-t))*normcdf(d2,0,1)) + (E-E*exp(Stochastic_Err))*exp(r*(T-t));

if V_Hcall<0;
    V_Hcall=0;
end

if (2*kappa*Theta/alpha^2) > 1                     %%%%%%%%%%%%%%%%%% Feller's condition %%%%%%%%%%%%%%%%%%%%%
    V_Hcall=0;
end
```

A.5 Generating volatility surfaces and skews

Prices of the European call options are calculated, using the Heston closed-form solution (Heston,1993), at different maturities and strikes. These prices are then equated to the Black-Scholes-Merton (BSM) solution and the volatility that satisfies this equation is evaluated. The function blsimpv returns the BSM implied volatility for a given set of parameters. The following script file, VolSmile.m generates the volatility surfaces of (Figures 4.3, 4.4, 4.5).

```
strikes = linspace(.8,1.2,11);
mats = linspace(.3,3,11);
for i = 1:11
for j = 1:11
price = HestonCallQuad(2,.04,.1,0.5,.04,.01,mats(i),1...,strikes(j));
prices(i,j) = price;
Volatility(i,j) = blsimpv(1, strikes(j), 0.01 , ...mats(i), price);
end
end

[strike mat] = meshgrid(strikes,mats);
surf(mat,strike,Volatility),xlabel('Maturity(years)'), ... ylabel('Strike'),Title('\rho = 0.5'),zlabel('Implied ...Volatility');
figure;
for i = 1:11
for j = 1:11
price = HestonCallQuad(2,.04,.1,0,.04,.01,mats(i),1... ,strikes(j));
prices(i,j) = price;
Volatility(i,j) = blsimpv(1, strikes(j), 0.01 , ...mats(i), price);
end
end

surf(mat,strike,Volatility),xlabel('Maturity(years)'),..ylabel('Strike'),Title('\rho = 0'),zlabel('Implied ...Volatility');
figure;
for i = 1:11
for j = 1:11
price = HestonCallQuad(2,.04,.1,-0.5,.04,.01,mats(i),...1,strikes(j));

prices(i,j) = price;
Volatility(i,j) = blsimpv(1, strikes(j), 0.01 ,...mats(i), price);
end
end

surf(mat,strike,Volatility),xlabel('Maturity(years)'), ylabel('Strike'),Title('\rho = -0.5'),zlabel('Implied... Volatility');
```

A.6 Matlab Code for the Hagan method in the case of $\beta = 1$

```
function C_BS=SABR(sigma, F, E, t , alpha, beta, rho)
clc

t = 0;
n = 3;
T = n/12;
r = 0.03;

sigma_0 = 0.2;

E = 100*(1+0.1*sqrt(sigma_0*T));
F = 100;

zeta =  alpha/sigma_0*log(F/E);
xi = log((sqrt(1-2*rho*zeta+zeta^2)+zeta-rho)/(1-rho));

sigma_t = sigma_0*(zeta/xi)*(1+(rho*sigma_0*alpha/4+(2-3*rho^2)*(alpha^2)/24)*T)

d1 = (log(F/E)+(0.5*sigma_t^2))/sigma_t*sqrt(T-t);
d2 = d1-sigma_t*sqrt(T-t);

C_BS = exp(-r*(T-t))*F*normcdf(d1)-E*exp(-r*(T-t))*normcdf(d2);
```

A.7 Geometrical Approximation Code: SABR

```
function V_call = C(S_t,E,sigma_t,T,t,r, epsilon_t)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%     SABR Model   %%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%   dF_t = sigma_tF_tdw_t %%%%%%%%%%%%%%%%%%
%%%%%%%%%%%   dsigma_t = alphasigmadW_t %%%%%%%%%%%%%%%
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc

t = 0;
n = 1;
T = n/12;                                        %%%%% Maturity date %%%%%%
r = 0.03;                                        %%%%% Interest rate  %%%%%
sigma_t = 0.2;                                   %%%%% Variance %%%%%%%%%%%%

E = 99.9999;          %%%%% Stike Price %%%%%%%%
F_t = E;%*(1+0.1*sqrt(sigma_t*T)                                     %%%%% Underlying value %%%

alpha = 0.29;
rho = -0.71;

epsilon_t = (rho/alpha)*sigma_t;
epsilon_medio = (rho/alpha)*(sigma_t*exp(0.5*alpha^2*(T-t)));
Stochastic_Err = (epsilon_medio-epsilon_t);
(E*exp(Stochastic_Err)-E)

d1 =  [log(F_t*exp(Stochastic_Err)/E) + 0.5*(1-rho^2)*(sigma_t^2)*(T-t)]/sqrt((1-rho^2)*(sigma_t^2)*(T-t));
d2 =  [log(F_t*exp(Stochastic_Err)/E) - 0.5*(sigma_t^2)*(T-t)]/sqrt((1-rho^2)*(sigma_t^2)*(T-t));


delta_1 =  -(r-(1-2*rho^2)*(sigma_t^2)/8*(1-rho^2))*(T-t);
delta_2 =  -(r-(sigma_t^2)/8*(1-rho^2))*(T-t);

 %%%%%%%%%% Pricing Formula %%%%%%%%%%%%%
V_call = F_t*exp(Stochastic_Err)*exp(delta_1)*normcdf(d1,0,1)-E*exp(delta_2)*normcdf(d2,0,1);

if V_call<=0;
   V_call=0;
end
```