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Renormalization and phenomenology of quantum electrodynamics with high-energy Lorentz violation

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Chapter 1

Introduction

Lorentz invariance is a key ingredient of the Standard Model of particle physics and it is experimentally verified with a high degree of precision (see for example [1, 2]), so that to date it appears to be one of the most precise symmetries in nature. Nevertheless, Lorentz breaking at high energies is explored as a possible feature of new physics beyond the Standard Model; in fact, as is well known, the Standard Model, although phenomenologically successful, leaves several unresolved issues (such as masses and charges of fermions, fine-tuning problems and so on).

One may think, for example, that there exists a fundamental underlying theory where the spontaneous breaking of Lorentz symmetry generates nonzero vacuum expectation values of Lorentz tensors; these ones would appear in a low-energy effective quantum field theory as coupling constants depending on the reference frame. So, Lorentz breaking, although suppressed, may be observable at energies much lower than the typical energy scales of the fundamental theory (e.g. the Planck scale if it has to include gravity), preferably by detecting an effect forbidden in the standard Lorentz-symmetric theory.

Such a perspective is adopted by Kostelecký and collaborators, who have constructed an effective low-energy theory extending the Standard Model with renormalizable and non-renormalizable terms that violate Lorentz symmetry and possibly CPT [3]. In this approach the underlying fundamental theory remains Lorentz invariant, thus keeping all the related features; its low-energy effective counterpart is supposed to inherit them. In particular it remains invariant under "observer" Lorentz transformations (which change both fields and background expectation values), while the presence of tensorial parameters affects only invariance under "particle" Lorentz transformations. We have stressed this kind of Standard Model extension, because it has been extensively studied and used as a framework for many investigations and experimental tests of Lorentz and CPT symmetries and several measurements of its coefficients have been set up. The result of these works is that, up to now, there is no observational evidence for Lorentz breaking, and very strict experimental bounds have been put on the parameters of the model (see [2] for a summary of experimental data).

From a totally different point of view, if one explores quantum field theories as possibly fundamental, one realizes that, relaxing some of the assumptions usually made, the set of renormalizable theories can be considerably enlarged. If one admits that at high energies Lorentz symmetry might be explicitly violated, the ultraviolet behaviour of propagators can be improved by introducing higher space derivatives, since space and time need no more to be related in the way fixed by Lorentz transformations. In such a way, theories that are not renormalizable by ordinary power counting become renormalizable by a modified "weighted" power-counting criterion, in which different weights are assigned to space and time coordinates. Each field acquires a weight different from its dimension in units of mass, allowing extra terms to be renormalizable [4]. In the common perturbative framework the theory remains local, polynomial, causal and unitary, since renormalization does not switch higher time derivatives on. The terms with higher mass dimension are multiplied by inverse powers of an energy scale Λ_L, which has to be interpreted as the scale of Lorentz violation *without* CPT violation.

In fact, the CPT theorem [5] states that in local Hermitian quantum field theories Lorentz violation is a necessary, but not sufficient condition for CPT violation. Therefore, if we intend to associate an energy scale with each violation, Λ_L and $\Lambda_{\rm CPT}$ are a priori different and $\Lambda_L \leqslant \Lambda_{\rm CPT}$. Different experimental data, such as bounds on antiproton lifetime [6] or measurements of light speed in gamma-ray bursts [7], clearly suggest that $\Lambda_{\rm CPT}$ is around or above the Planck scale, placing the limit $\Lambda_{\rm CPT} \geqslant 10^{18}$GeV. The parameters of dimension -1, which come from CPT-odd operators in the photon sector of the extended Standard Model of Kostelecký, have experimental bounds derived from astrophysical birefringence and reported in [2]; if we interpret these parameters as the inverse of the scale of CPT violation, the bound on $\Lambda_{\rm CPT}$ becomes much more stringent: $\Lambda_{\rm CPT} \geqslant 10^{32}$GeV. Thus, we assume from now on that CPT is conserved, at least in the region we are interested in.

Λ_L has to be large compared to the energies explored until now, as no effect of Lorentz violation has been detected yet. An estimate of Λ_L based on neutrino masses [8] put the bound $\Lambda_L \gtrsim 10^{14-15}$GeV, which is compatible with the values found in [2] for coefficients of dimension 6 operators. Only the analysis of ultrahigh-energy cosmic rays would lead to raise this bound: some authors [9] claim that Λ_L must be even larger than the Planck mass. Thus, we have investigated particularly this subject and we can argue that even ultrahigh-energy cosmic rays observations can be compatible with a value of Λ_L much smaller than the Planck scale [10].

Consistent models exist in arbitrary spacetime dimensions and, if they involve only scalars and fermions, for arbitrary splitting of "space" and "time" (i.e. the "time" submanifold can include some space coordinates). Instead, if one includes gauge fields this possibility is restricted, since it turns out that Feynman diagrams are free of certain "spurious" divergences only if spacetime is split in a one-dimensional time submanifold and a $\bar{d}$-dimensional space one [11, 12]. In this framework Lorentz-violating but rotational and CPT-invariant extended Standard Models have been proposed [8, 13]. Certain operators of dimension greater than four become renormalizable: these models contain, for example, two scalars-two fermions vertices and four fermions interactions at the fundamental level; the first one gives mass to left-handed neutrinos without introducing extra fields and the second can describe proton decay. A scalarless version of this model has also been considered, arguing that it can reproduce the known low-energy physics: the elementary Higgs boson is suppressed, while the four fermions vertices can trigger a Nambu–Jona-Lasinio mechanism, from which both gauge bosons and fermions acquire mass and a composite Higgs appears. It is important to stress that in the spirit of this construction these are viewed as fundamental theories, valid in principle at all energies, and not as mere effective low-energy limits of something more fundamental, such as a finite theory. Considering a fundamental theory as renormalizable and not finite leads to reconsider some typical assumptions.

Here we explicitly study the one-loop renormalization of the electromagnetic sector of this extended Standard Model: from the high-energy point of view the most characteristic property is that the electric charge is super-renormalizable, with a finite number of divergent diagrams at one and two loops only; thus the theory is asymptotically free. In the low-energy limit, obtained when the scale of Lorentz violation Λ_L goes to infinity, the model switches from weighted power counting to ordinary power counting, thus giving a power-counting renormalizable, but still Lorentz-violating, electrodynamics. Studying the interpolation between the renormalizations of the high-energy and low-energy theory, we realize that the power-like divergences of the low-energy theory (expressed as powers of Λ_L) become arbitrary, as they are multiplied by coefficients that incorporate an arbitrary renormalization scheme choice inherited from the high-energy theory. This is a very general property of high-energy Lorentz-violating theories, depending on the fact that they are not completely finite. Consequently, if the elementary Higgs field is present, this arbitrariness makes the hierarchy problem disappear.

Another interesting fact is that if we assume that Lorentz symmetry is not exact, several phenomena that are otherwise forbidden can occur. In fact, since one introduces higher space derivatives in the quadratic part of the Lagrangian, the dispersion relations are also modified by higher powers of momentum; therefore, phenomena which are kinematically, or even dynamically, forbidden in usual QED may be allowed, such as the Cherenkov radiation in vacuo and the photon decay into an electron-positron pair. Many of these phenomena, that are forbidden in Lorentz-invariant theories, but allowed in Lorentz-violating ones, have been studied in the literature, mainly using the modified dispersion relations of low-energy effective models. Instead, here we study some of them in the framework of a complete theory, where the dispersion relations are crucial for renormalizability, therefore more constrained and valid, in principle, at arbitrarily high energies. Comparing predictions with experimental data and observations, we can look for signs of Lorentz violation and put bounds on the values of Lorentz-violating parameters. In particular, we are interested in deriving bounds on the magnitude of Λ_L, in order to understand if effects of Lorentz violation may manifest as signals of new physics *before* any other. Indeed, we argue that the scale of Lorentz violation may be smaller than the Planck scale, and if this were true, the understanding of physics around the Planck scale, in particular the formulation of gravity, should be completely reconsidered.

This thesis is organized as follows. In chapter 2 we explain the theoretical motivations, from the point of view of renormalization, for relaxing the hypothesis of Lorentz symmetry. Then we argue how, once we assume that Lorentz invariance may be explicitly violated, higher space derivatives can be introduced in the Lagrangian, improving the ultraviolet behaviour of propagators. Therefore, a weighted power-counting criterion for renormalizability emerges, which weights differently space and time and, due to the faster decreasing of propagators at infinity, makes it possible to renormalize interactions that usually are not renormalizable. We briefly review how Lorentz-violating weighted power-counting renormalizable theories containing scalars, fermions and gauge fields can be constructed, as well as some Lorentz-violating Standard Model extensions with different features. The Standard Model extension may contain two scalars-two fermions and four fermions vertices at the fundamental level. In its simplest version it can also be scalarless.

In chapter 3 we study the electromagnetic sector of this Standard Model extension in detail, namely Lorentz-violating QED (LVQED). After illustrating its quantization by means of the functional integral, we show that the theory is super-renormalizable and has only one- and two-loops counterterms, then we explicitly work out its one-loop renormalization. We study the low-energy limit of the theory (lvQED) and compare and link the renormalizations of LVQED and lvQED; we prove the equivalence of two different regularizations of loop integrals up to a scheme-change, hence showing that the low-energy power-like divergences are multiplied by arbitrary coefficient inherited by the renormalizable, yet not finite, high-energy theory. Then we reconsider the hierarchy problem from this point of view, arguing that it is bypassed.

In chapter 4 we investigate some unusual phenomenological consequences of our Lorentz-violating electrodynamics and put bounds on the parameters, most importantly the scale of Lorentz violation Λ_L. We analyse in detail the emission of photons from moving fermions, namely Cherenkov radiation in vacuo. We study such phenomenon in the low-energy expansion, where it looks like the standard Cherenkov radiation in a medium and we investigate situations where the low-energy limit does not apply and the complete dispersion relations, containing higher powers of momentum, must be used. We find kinematic constraints for a very general set of dispersion relations and then derive the energy thresholds and radiation times for two cases, showing that, as in the low-energy limit, the energy loss is almost instantaneous. We compare our results with experimental data concerning ultrahigh-energy cosmic rays and give bounds on crucial parameters, arguing that, if they are sufficiently small, Λ_L can be lower than the Planck scale. A simple schematization of composite particles shows that compositeness favors larger energy thresholds and smaller values of parameters. Finally we discuss briefly the Cherenkov radiation from neutral particles, which is also allowed in our model.

Chapter 5 contains our conclusions. The appendices collect some details about calculations.

Chapter 2

Renormalizable Lorentz-violating theories

In this chapter we illustrate the connection between Lorentz violation and renormalizability and the motivations that have suggested to study the class of weighted power-counting renormalizable theories. In section 2.1 we briefly review the ordinary power-counting criterion and discuss how the class of renormalizable theories can be enlarged; in section 2.2 we describe how weighted power-counting renormalization works with both matter and gauge fields; finally in section 2.3 interesting Lorentz-violating extensions of the Standard Model that can be constructed within this method are presented.

2.1 Ordinary power-counting renormalization

As is well known, quantum field theories, if we take them as they come, contain, among others, ultraviolet divergence; naively one may think that this will make such theories unpredictive. However, if a specific theory is such that one can find a way to cancel those infinities redefining the fields and a finite number of parameters, the predictions of the theory are still meaningful and comparisons with experiments can be made. If this is the case, the theory is called renormalizable.

Thus, in order to construct a trustable theory it is important to demand that it be renormalizable, if not finite, at least if we want the results to be valid in principle at all energies. Ordinary particle quantum field theories also satisfy other key-properties: they are local (i.e. polynomial), unitary, causal and respect Lorentz symmetry, i.e. they are relativistic. When these requirements combine with renormalizability, a small set of interactions is selected.

According to the so-called power-counting criterion, a necessary condition for a theory to be renormalizable is that its coupling constants have non-negative mass dimension. Take for example a self-interacting scalar field with interactions of the general form $\lambda_i \partial^{\delta_i} \varphi^{n_i}$. Being d the number of space-time dimensions, consider a Feynman graph G and call L the number of independent d-dimensional momenta flowing internally (i.e. the number of loops of G) and I and E the number of internal and external legs, respectively, each carrying its own momentum. Moreover, call V_i the number of vertices of type i contained in G, each of which has n_i legs and δ_i derivatives and is multiplied by the coupling constant λ_i. We can write down the integral we have to evaluate in association with G, which is something like

$$\int (d^{\mathrm{d}}p)^L \frac{\prod_i \lambda_i^{V_i} p^{\delta_i V_i}}{(p^2+m^2)^I}.$$

Therefore, if we consider the region of integration where all the internal momenta go to infinity together, the integral diverges like $p^{\omega(\mathrm{G})}$, with

$$\omega(\mathrm{G}) = \mathrm{d}L + \sum_i \delta_i V_i - 2I; \tag{2.1}$$

$\omega(\mathrm{G})$ is referred to as the overall (or superficial) degree of divergence of the graph G. In order for equation (2.1) to be useful, we have to write it in terms of E instead of I and L. The number of independent internal momenta L is equal to the total number of independent momenta minus the independent external momenta, thus we have

$$L = (I + E - \sum_i V_i) - (E-1) = I + 1 - \sum_i V_i, \tag{2.2}$$

taking into account a momentum-conservation law in each of the vertices and the overall momentum conservation. Moreover, considering that each internal legs is connected with two vertices, if we sum the legs in all vertices we obtain

$$\sum_i n_i V_i = E + 2I. \tag{2.3}$$

Finally, imposing that the interaction term of type i has the same dimension of the Lagrangian density, we can write

$$n_i = \frac{2}{\mathrm{d}-2}(\mathrm{d} - [\lambda_i] - \delta_i),$$

where $[\lambda_i]$ is the mass dimension of the coupling constant of the vertex of type i. Expliciting I and L from (2.2) and (2.2), and substituting in (2.1) we get

$$\omega(\mathrm{G}) = \mathrm{d} - \frac{\mathrm{d}-2}{2}E - \sum_i [\lambda_i] V_i. \tag{2.4}$$

From equation (2.4) we see that, if a coupling constant has negative mass dimension, correlation functions with an arbitrary number of external legs and an arbitrary number of derivatives become divergent at some step of the perturbative expansion (i.e. adding up a sufficient number of vertices), so that the theory cannot be made convergent with a finite number of subtractions, thus it is unrenormalizable. On the other hand, if no coupling constant has negative dimension in units of mass, diagrams with more than a certain fixed number of external legs are superficially convergent, so that one needs to subtract terms with only a finite set of operators structures; then the theory is renormalizable. Moreover, if, for any i, $[\lambda_i] > 0$, diagrams with an arbitrary number of external legs become convergent at some order in perturbation theory, so that actually there is only a finite number of divergent diagrams: in such a case the theory is called super-renormalizable.

Now, $\omega < 0$ is a necessary but not sufficient condition for a diagram to be convergent: in fact there may be, and usually there are, divergent sub-diagrams nested into the integral. Nevertheless it has been demonstrated that divergences and subdivergences can be subtracted order by order in perturbation theory, and that the counterterms are local [14]; therefore, as long as the power-counting rule is satisfied, quantum field theories involving scalars, fermions and gauge vectors are renormalizable.

If we require a theory to satisfy together
-unitarity
-causality
-locality
-Lorentz symmetry
-renormalizability
it turns out that very few interaction terms are allowed. However, if one relaxes some of these assumptions, the set of acceptable theories gets enlarged. For example, every non-local theory, in principle, can be renormalized since the ultraviolet divergences arising from small-distance singularities are smoothed away by regular functions [15]. But such enlargement is so wide that we would remain with infinitely many potentially interesting theories and no hint on how to choose them. Analogous results can be obtained by improving the high energy behaviour of propagators by means of higher powers of covariant derivatives [16]. However, when more than two time derivatives are present, unitarity is lost. This is unacceptable, because it means that probability is not conserved, which is impossible to justify in a physical theory. Or, sometimes, unitarity violations due to higher time derivatives can be turned into causality violations, which are equally unacceptable [17]. On the contrary, relaxing the assumption of Lorentz symmetry seems not so hard, because it is, at the end, just a global symmetry (at least until we do not include gravity). Moreover, if we remove the hypotesis of Lorentz symmetry, it is possible to improve the UV behaviour of propagators by means of higher space derivatives, without being forced to introduce higher time derivatives and thus without spoiling unitarity. In fact, it has been demonstrated [4, 11] that if one relaxes the hypothesis of Lorentz symmetry, but preserves unitarity and locality, it is possible to construct a well-defined set of consistent theories including

fermions, scalars and gauge fields that are renormalizable by a modified power-counting criterion, that weights differently space and time. The set of considerable theories is enlarged with respect to the Lorentz invariant world, but still sufficiently selective. In the following section we are going to illustrate how this modified power-counting criterion works.

2.2 Weighted power-counting renormalizable theories

Assuming that Lorentz symmetry is explicitly violated implies that space and time need no more to be treated on the same footing; therefore the d-dimensional spacetime manifold M can be split in the product of the "time" submanifold $\hat{M}$, which is $\hat{\mathrm{d}}$-dimensional and contains time coordinates and possibly some of the space ones, and the $\bar{\mathrm{d}}$-dimensional space submanifold $\bar{M}$. In each submanifold rotational and Lorentz invariance are assumed, so that the d-dimensional Lorentz group $O(1,\mathrm{d}-1)$ is broken into a residual $O(1,\hat{\mathrm{d}}-1)\times O(\bar{\mathrm{d}})$ group. For the purpose of renormalization by means of the dimensional regularization technique, $\hat{\mathrm{d}}$ and $\bar{\mathrm{d}}$ will be independently continued to complex values. Coordinates, as well as momenta and derivatives, are similarly decomposed in time and space components which will be denoted by $x=(\hat{x},\bar{x})$, $k=(\hat{k},\bar{k})$ and $\partial=(\hat{\partial},\bar{\partial})$ respectively. The hatted components live in $\hat{M}$, while the barred ones live in $\bar{M}$; spacetime indices are also split into different components, such as $\mu=(\hat{\mu},\bar{\mu})$. When appearing, the Dirac gamma matrices will be decomposed similarly $\gamma_\mu=(\gamma_{\hat{\mu}},\gamma_{\bar{\mu}})$, or $\gamma=(\hat{\gamma},\bar{\gamma})$. The metric adopted, if not differently said, is the Euclidean one.

Scalars and fermions. For the purpose of illustrating how weighted power counting works, we can consider the simplest free scalar theory

$$\mathcal{L}_0=\frac{1}{2}(\hat{\partial}\varphi)^2+\frac{1}{2\Lambda_L^{2n-2}}\left(\bar{\partial}^n\varphi\right)^2,$$

where $n>1$ is an integer (up to total derivatives it is not needed to specify how derivatives contract among themselves). In order to restore the correct counting of dimensions, the term that contains higher space derivatives is divided by an energy scale Λ_L, which can be interpreted as the scale at which the effects of Lorentz violation start to manifest themselves. n has to be intended as the highest power of $\bar{\partial}$ appearing in the quadratic Lagrangian; mass and lower space derivatives could also be present or generated by renormalization, but they are not essential for the high-energy behaviour. In fact the main feature of this Lagrangian is that in the scalar propagator

$$\frac{1}{\hat{p}^2+\frac{\bar{p}^{2n}}{\Lambda_L^{2n-2}}} \tag{2.5}$$

the highest power of space momentum is $\bar{p}^{2n}$. Defining

$$\text{đ}=\hat{\mathrm{d}}+\frac{\bar{\mathrm{d}}}{n},$$

we see that the action is invariant under the rescaling

$$\hat{x} \to \lambda \hat{x}, \qquad \bar{x} \to \lambda^{\frac{1}{n}} \bar{x}, \qquad \varphi \to \frac{1}{\lambda^{\text{đ}/2-1}} \varphi, \tag{2.6}$$

where đ plays the same role that the spacetime dimension d plays in Lorentz-invariant theories.

If one introduces interactions it turns out that renormalization has the same features as usual provided that the ordinary dimension (in units of mass) of fields and parameters is replaced by a modified dimension, which we call weight. It is derived from the fundamental assignment for coordinates, which has the characteristic feature that the weight of space components is n times smaller than their usual mass dimension, according to (2.6). So, denoting the weight of $\mathcal{O}$ by $[\mathcal{O}]$, we have

$$\begin{array}{llll} [\hat{x}] = -1 & [\bar{x}] = -1/n & [\hat{\partial}] = 1 & [\bar{\partial}] = \dfrac{1}{n} \\ [\mathcal{L}] = \text{đ} & [\Lambda_L] = 0 & [\varphi] = \dfrac{\text{đ}}{2} - 1. & \end{array} \tag{2.7}$$

Following the same reasoning and notation of section 2.1, the integral associated with a Feynman diagram G can be simbolically written as

$$\int (d^{\hat{\text{d}}}\hat{p})^L (d^{\bar{\text{d}}}\bar{p})^L \frac{\prod_i \lambda_i^{V_i} \hat{p}^{\hat{\delta}_i V_i} \bar{p}^{\bar{\delta}_i V_i}}{(\hat{p}^2 + \frac{\bar{p}^{2n}}{\Lambda_L^{2n-2}})^I},$$

where the general interaction term with n_i legs, $\hat{\delta}_i$ time derivatives and $\bar{\delta}_i$ space derivatives is $\lambda_i \hat{\partial}^{\hat{\delta}_i} \bar{\partial}^{\bar{\delta}_i} \varphi^{n_i}$, λ_i is a dimensionless but possibly weightful coupling constant[1] and V_i is the number of vertices of this type i in G. Taking into account the rescaling (2.6), one can calculate the degree of divergence of the overall divergent part of G, namely

$$\omega(\text{G}) = \text{đ}L + \sum_i (\hat{\delta}_i + \frac{\bar{\delta}_i}{n}) V_i - 2I.$$

Using the topological identities (2.2) and (2.3), we find

$$\omega(\text{G}) = \text{đ} - \frac{\text{đ} - 2}{2} E - \sum_i (\text{đ} - (n_i \frac{\text{đ} - 2}{2} + \hat{\delta}_i + \frac{\bar{\delta}_i}{n})) V_i.$$

With the same arguments used for ordinary power counting, we see that interacting terms of weight đ are renormalizable, those of weight smaller than đ are super-renormalizable, while those of weight greater than đ are non-renormalizable. Thus renormalizability is ensured by weighted power counting if the theory does not contain any parameter of negative weight. It has also been

[1] The weight assigned to λ_i is the one needed to match the weight of the interaction to that of the Lagrangian, namely $[\lambda_i] = \text{đ} - (n_i \frac{\text{đ}-2}{2} + \hat{\delta}_i + \frac{\bar{\delta}_i}{n})$, while the dimension in units of mass is supposed to be adjusted by Λ_L.

demonstrated [4] that time derivatives are not turned on by renormalization, so that unitarity is preserved. Summarizing, thanks to the fact that the propagator (2.5), which has weight -2, decreases in the ultraviolet more rapidly than the usual relativistic one, certain operators that usually are non-renormalizable, can become renormalizable (since đ is always smaller than d). The exact value of đ depends on the spacetime dimensions and the kind of the theory one wants to construct, but $2 < đ \leqslant 6$, where the left-hand side bound is needed to ensure polinomiality ($[\varphi] > 0$), while the right-hand side to have nontrivial interactions.

One can introduce also fermions, with a free Lagrangian looking like

$$\mathcal{L}_{0f} = \bar{\psi}\hat{\partial\!\!\!/}\psi + \frac{1}{\Lambda_L^{n-1}}\bar{\psi}\bar{\partial\!\!\!/}^n\psi,$$

where n is again the maximum number of space derivatives; then

$$[\psi] = \frac{đ - 1}{2},$$

and polinomiality demands $đ > 1$. The propagator is

$$\frac{-i\hat{p\!\!\!/} + (-i)^n \frac{\bar{p\!\!\!/}^n}{\Lambda_L^{n-1}}}{\hat{p}^2 + \frac{(\bar{p}^2)^n}{\Lambda_L^{2n-2}}}$$

and has weight -1. It is possible also to couple scalars and fermions provided that both have the same n, to ensure the right ultraviolet behaviour.

Gauge fields. In this framework one can introduce also gauge fields and gauge interactions [11, 12]. The gauge field A has to be decomposed analogously to the partial derivative, $A = (\hat{A}, \bar{A})$. The covariant derivative then reads $D = (\hat{D}, \bar{D}) = (\hat{\partial} + g\hat{A}, \bar{\partial} + g\bar{A})$, being g the gauge coupling. Each component A_μ is intended to be a linear combination of the infinitesimal generators of the gauge symmetry Lie group $SU(N)$, namely $A_\mu = A_\mu^a T^a$, where T^a are $N^2 - 1$ anti-Hermitian traceless matrices; the structure constants of the algebra are f^{abc}. The field strength is then decomposed into three kinds of components, namely

$$\hat{F}_{\mu\nu} := F_{\hat{\mu}\hat{\nu}}, \qquad \tilde{F}_{\mu\nu} := F_{\hat{\mu}\bar{\nu}}, \qquad \bar{F}_{\mu\nu} := F_{\bar{\mu}\bar{\nu}}.$$

The kinetic Lagrangian must contain $\hat{F}^2$, that has the highest weight among the terms constructed with two F, i.e. it must contain $(\hat{\partial}\hat{A})^2$. Therefore the weight assigned to $\hat{A}$ is $đ/2 - 1$; since $[\hat{D}] = [\hat{\partial}] = 1$ and $[\bar{D}] = [\bar{\partial}] = 1/n$ one obtains

$$\begin{aligned} &[g] = 2 - \frac{đ}{2} \qquad &&[\hat{A}] = \frac{đ}{2} - 1 \qquad &&[\bar{A}] = \frac{đ}{2} - 2 + \frac{1}{n} \\ &[\hat{F}] = \frac{đ}{2} &&[\tilde{F}] = \frac{đ}{2} - 1 + \frac{1}{n} &&[\bar{F}] = \frac{đ}{2} - 2 + \frac{2}{n}. \end{aligned} \tag{2.8}$$

Since the weight of the gauge coupling cannot be negative, it must be

$$\text{đ} \leqslant 4.$$

The quadratic part of the gauge Lagrangian reads

$$\mathcal{L}_Q = \frac{1}{4}\left\{F_{\hat{\mu}\hat{\nu}}^2 + 2F_{\hat{\mu}\bar{\nu}}\eta(\bar{\Upsilon})F_{\hat{\mu}\bar{\nu}} + F_{\bar{\mu}\bar{\nu}}\tau(\bar{\Upsilon})F_{\bar{\mu}\bar{\nu}} + \frac{1}{\Lambda_L^2}(D_{\hat{\rho}}F_{\bar{\mu}\bar{\nu}})\xi(\bar{\Upsilon})(D_{\hat{\rho}}F_{\bar{\mu}\bar{\nu}})\right\}, \qquad (2.9)$$

where $\bar{\Upsilon} := -\bar{D}^2/\Lambda_L^2$ and η, τ and ξ are polynomials of degrees $n-1$, $2n-2$ and $n-2$ respectively, whose coefficients are dimensionless and eventually weightful. Precisely, each coefficient has the weight needed to match the weight of each monomial to đ, while the dimension in units of mass is matched by Λ_L, which is weightless. A sum over the gauge groups is understood. We have included the terms constructed with two field strengths and possibly some covariant derivatives, up to total derivatives.

Introducing ghost and antighost fields as well as Lagrange multipliers (C, $\bar{C}$ and B respectively), one can implement the usual BRST symmetry s [18]:

$$sA_\mu^a = D_\mu^{ab}C^b = \partial_\mu C^a + gf^{abc}A_\mu^b C^c, \qquad sC^a = -\frac{g}{2}f^{abc}C^bC^c,$$
$$s\bar{C}^a = B^a, \qquad sB^a = 0, \qquad s\psi^i = -gT_{ij}^a C^a\psi^j, \qquad s\bar{\psi}^i = -gT_{ij}^{a*}C^a\bar{\psi}^j$$

where the indices i, j are those of the fundamental representation of the gauge group. Then one can construct a convenient gauge-fixing Lagrangian. Since this one contains $\bar{C}\hat{\partial}^2 C$ and B^2 the weights assignment is

$$[C] = [\bar{C}] = \frac{\text{đ}}{2} - 1, \qquad [B] = \frac{\text{đ}}{2}, \qquad [s] = 1$$

and a suitable gauge-fixing Lagrangian reads

$$\mathcal{L}_{\text{gf}} = s\Psi, \qquad \Psi = \bar{C}^a\left(-\frac{\lambda}{2}B^a + \hat{\partial}\cdot\hat{A}^a + \zeta(\bar{\upsilon})\bar{\partial}\cdot\bar{A}^a\right), \qquad (2.10)$$

where $\bar{\upsilon} := -\bar{\partial}^2/\Lambda_L^2$, λ is a weightless and dimensionless constant and ζ is a polynomial of degree $n-1$.

In addition to the quadratic and gauge-fixing parts, the gauge Lagrangian may include interaction terms $\mathcal{L}_I$ containing three or more field strengths, so that the total action of the gauge fields is

$$S = \int d^{\text{d}}x(\mathcal{L}_Q + \mathcal{L}_I + \mathcal{L}_{\text{gf}}) = \int d^{\text{d}}x\mathcal{L}_G.$$

When the renormalization of such a theory is studied, it turns out that, due to the structure of the gauge field propagators, some spurious subdivergences of Feynman diagrams are generated unless

$$\hat{\text{d}} = 1;$$

this means that the splitting of the spacetime manifold M cannot be arbitrary, but it has to be split in space and time. This is also the most interesting physical case; moreover, in order not to have infrared divergences (at least for generic values of momenta) by ordinary power counting it must be d$\geqslant 4$. Then, we can consider the case of four dimensions

$$\text{đ} = 1 + \frac{3}{n},$$

which is assumed now on. As a consequence of $\hat{\mathrm{d}} = 1$ the $\hat{F}$ term drops out by antisymmetry. The following step is introducing interactions among gauge fields and fermions, and possibly a scalar Higgs field, which leads to the construction of a Lorentz-violating, weighted power-counting renormalizable extension of the Standard Model. This will be sketched in the following section.

2.3 Lorentz-violating extensions of the Standard Model

In the framework of weighted power-counting renormalization, one can construct different Lorentz-violating extensions of the Standard Model, accomodating various interaction terms at the fundamental level which can, for example, give mass to left-handed neutrinos without introducing right-handed ones nor other extra fields, or allow proton decay; it is also possible not to include elementary scalars [8, 13]. We are going to briefly illustrate these models. The first thing to observe is that, because of the first equation in (2.8), the gauge coupling g is always super-renormalizable in four dimensions, in any non-trivial case $n > 1$. Moreover, if we assume that CPT is preserved (or that it is violated at an energy scale much greater than the scale of Lorentz violation Λ_L) n must be odd. Indeed, if n is even the fermionic quadratic term $\bar{L}\bar{\partial}^n L$, which is essential for renormalizability, is ruled out (being L the left-handed lepton doublet); on the other hand terms like $\bar{L}a^\mu\gamma_\mu\bar{\partial}^n L$, which by the way violate maximally Lorentz symmetry, are forbidden by CPT conservation. Every odd n is allowed, so that infinitely many CPT invariant solutions exist, but clearly the simplest and most economic choice is

$$n = 3 \Longrightarrow \text{đ} = 2.$$

This choice is assumed henceforth. In addition to the gauge field Lagrangian, one has to include the quadratic gauge-invariant fermionic one

$$\mathcal{L}_{\mathrm{kin}f} = \sum_{a,b=1}^{3}\sum_{I=1}^{5}\bar{\chi}_I^a\left(\delta^{ab}\hat{\not{D}} - \frac{b_0^{Iab}}{\Lambda_L^2}\bar{\not{D}}^3 + b_1^{Iab}\bar{\not{D}}\right)\chi_I^b$$

where $\chi_1^a = L^a = (\nu_L^a, \ell_L^a)$, $\chi_2^a = Q_L^a = (u_L^a, d_L^a)$, $\chi_3^a = \ell_R^a$, $\chi_4^a = u_R^a$, $\chi_5^a = d_R^a$, while a and b are generations indices: $\nu^a = (\nu_e, \nu_\mu, \nu_\tau)$, $\ell^a = (e, \mu, \tau)$, $u^a = (u, c, t)$ and $d^a = (d, s, b)$; b_0^{Iab}, as well as b_1^{Iab}, are five 3×3 matrices.

The model constructed in [8] includes also the Higgs field Lagrangian $\mathcal{L}_H$, which contains kinetic and self-interaction terms, and the Yukawa coupling among fermions and Higgs field $\mathcal{L}_{\text{Yukawa}}$, namely

$$\mathcal{L}_{\text{Yukawa}} = \bar{g}\left(\sum_{a,b=1}^{3}(Y_1^{ab}\bar{L}_i^a\ell_R^b + Y_2^{ab}\bar{u}_R^a Q_{Lj}^b\varepsilon^{ji} + Y_3^{ab}\bar{Q}_{Li}^a d_R^b)H_i + \text{h.c.}\right),$$

where $Y_{1,2,3}^{ab}$ are 3×3 matrices of Yukawa couplings and the indices i, j denote the weak isospin components of the complex Higgs doublet and of the left-handed fermions doublets. $\bar{g}$ is a coupling constant of weight $\frac{1}{2}$ introduced because at đ= 2 the weight of the scalar field vanishes (see (2.7)); if one $\bar{g}$ is attached to each scalar leg, renormalizability is ensured anyway by weighted power counting. $\mathcal{L}_H$ and $\mathcal{L}_{\text{Yukawa}}$, with the usual Higgs mechanism, after the spontaneous symmetry breaking generate gauge bosons and fermions masses.

The model also contains all the allowed renormalizable interactions among fermions and gauge fields $\mathcal{L}_{IGf}$ and between the Higgs and gauge fields $\mathcal{L}_{IGH}$. In addition, one can include a vertex with two Higgs and two leptons, $\sim (LH)^2$, the unique one that, after symmetry breaking, can give masses to left-handed neutrinos without introducing extra fields [19]. Explicitly

$$\mathcal{L}_{LH} = \frac{\bar{g}^2}{4\Lambda_L}(LH)^2 := \frac{\bar{g}^2}{4\Lambda_L}\sum_{a,b=1}^{3} Y_{ab}\varepsilon_{ij}L_i^{\alpha a}H_j\varepsilon_{\alpha\beta}\varepsilon_{kl}L_k^{\beta b}H_l + \text{h.c.}, \tag{2.11}$$

where again i, j, k, l are $SU(2)_L$ indices , Y_{ab} is a constant matrix, while α, β are the Dirac indices of left-handed leptons spinors. The vertex (2.11) gives Majorana masses to left-handed neutrinos after symmetry breaking, but it is usually introduced as an effective vertex coming from fundamental interactions with right-handed neutrinos or extra fields, because it has a mass dimension of 5, hence it is not renormalizable by ordinary power counting. On the contrary, in the framework of weighted power counting it can be included at the fundamental level, since it has weight one and thus it is super-renormalizable.

Finally, one can include also usually non-renormalizable four fermions vertices, simbolically indicated as

$$\mathcal{L}_{4f} \sim \frac{Y_f}{\Lambda_L^2}\bar{\psi}\psi\bar{\psi}\psi, \tag{2.12}$$

which can provide proton decay. It has mass dimension 6 and weight 2, thus it is the only strictly renormalizable interaction; therefore its coupling constant Y_f must have a beta function proportional to Y_f itself, so that the four fermions interaction can be put consistently to zero without being generated back by renormalization. Then the total Lagrangian of the Lorentz violating extension of the Standard Model reads

$$\mathcal{L}_{LVSM} = \mathcal{L}_G + \mathcal{L}_H + \mathcal{L}_{\text{kin}f} + \mathcal{L}_{\text{Yukawa}} + \mathcal{L}_{IGH} + \mathcal{L}_{IGf} + \mathcal{L}_{LH} + \mathcal{L}_{4f}; \tag{2.13}$$

the complete list of terms in (2.13) is quite long and we do not report it here (see [8]). Such a model is not unique: indeed, there are actually infinitely many possible Standard Model extensions, one for every odd n (provided that CPT is conserved). Nevertheless, the one outlined above, with $n = 3$, appears to be the simplest (even if not quite simple) and the most interesting one.

A simplified version of the theory can be obtained dropping every vertex and quadratic term which, if not put in, is not switched on by renormalization; obviously, the quadratic terms that are crucial for the behaviour of propagators must be kept anyway, as well as the vertices that originate from them via covariantization, plus the interactions that we want in any case to be there (such as (2.11) and (2.12)). Note, for example, that in (2.9) ξ can be put to zero, because it is not essential for renormalization. The simplified Lagrangian reads

$$\begin{aligned}\mathcal{L}_{\text{simpl}} =&\frac{1}{4}\sum_{G}\left(2F^{G}_{\hat{\mu}\bar{\nu}}\eta^{G}(\bar{\Upsilon})F^{G}_{\hat{\mu}\bar{\nu}}+F^{G}_{\bar{\mu}\bar{\nu}}\tau^{G}(\bar{\Upsilon})F^{G}_{\bar{\mu}\bar{\nu}}\right)+\mathcal{L}_{\text{kin}f}\\&+|\hat{D}_{\hat{\mu}}H|^{2}+\frac{a_{0}}{\Lambda_{L}^{4}}|\bar{D}^{2}\bar{D}_{\bar{\mu}}H|^{2}+\frac{a_{1}}{\Lambda_{L}^{2}}|\bar{D}^{2}H|^{2}+a_{2}|\bar{D}_{\bar{\mu}}H|^{2}-\mu_{H}^{2}|H|^{2}+\frac{\lambda_{4}\bar{g}^{2}}{4}|H|^{4}\\&+\mathcal{L}_{\text{Yukawa}}+\sum_{I=1}^{5}\frac{1}{\Lambda_{L}^{2}}g\bar{D}\bar{F}\left(\bar{\chi}_{I}\bar{\Gamma}\chi_{I}\right)+\frac{g}{\Lambda_{L}^{2}}\bar{F}^{3}+\frac{\bar{g}^{2}}{4\Lambda_{L}}(LH)^{2}+\frac{Y_{f}}{\Lambda_{L}^{2}}\bar{\psi}\psi\bar{\psi}\psi.\end{aligned} \tag{2.14}$$

Here $\sum_G$ denotes the sum over the gauge groups $SU(3)_c, SU(2)_L, U(1)_Y$, a_i, μ_H, λ_4 are constants, $\bar{\Gamma}$ denotes a $\bar{\gamma}$ matrix or the product of three $\bar{\gamma}$ matrices. In the first line there are the quadratic Lagrangians of gauge fields (at $\xi = 0$) and fermions, in the second one the kinetic and simplified self-interaction parts of the Higgs Lagrangian, while in the last one are written simbolically the surviving interactions among gauge fields and fermions, the surviving vertices with gauge fields only, and the interaction accounting for neutrino mass, as well as the four fermions vertices. It has been proved that this model is power counting renormalizable, while renormalization does not turn on higher time derivatives, thus preserving unitarity; the gauge anomalies of this model, as well as of (2.13), are proved to coincide with those of the Standard Model, therefore they cancel out at all orders. Moreover, it has been shown [20] that the Källen-Lehman spectral decomposition and the cutting equations can be generalized to this kind of theories, as well as the unitarity relation and the Bogoliubov's concept of causality.

A further simplified model has been studied [13, 21], obtained suppressing the elementary scalar Higgs field in (2.14), and it has been argued that it can reproduce the known low-energy physics. The four fermion vertices, indeed, can trigger a Nambu–Jona-Lasinio mechanism: in a large N_c expansion, a dynamical symmetry breaking takes place that gives masses to fermions and gauge fields, and generates Higgs bosons as composite fields. This scalarless Lorentz-violating extension of the Standard Model reads

$$\mathcal{L}_{\text{no}H}=\mathcal{L}_{Q'}+\mathcal{L}_{\text{kin}f}+\sum_{I=1}^{5}\frac{1}{\Lambda_{L}^{2}}g\bar{D}\bar{F}\left(\bar{\chi}_{I}\bar{\gamma}\chi_{I}\right)+\frac{g}{\Lambda_{L}^{2}}\bar{F}^{3}+\frac{Y_{f}}{\Lambda_{L}^{2}}\bar{\psi}\psi\bar{\psi}\psi, \tag{2.15}$$

where

$$\mathcal{L}_{Q'} = \frac{1}{4}\sum_{G}\left(2F^{G}_{\hat{\mu}\bar{\nu}}F^{G}_{\hat{\mu}\bar{\nu}} + F^{G}_{\bar{\mu}\bar{\nu}}\tau'^{G}(\bar{\Upsilon})F^{G}_{\bar{\mu}\bar{\nu}}\right)$$

is an alternative simplified gauge sector obtained rearranging the weights assignment; in fact, η^G is replaced by 1 and τ^G, which had weight $2n-2 \to 4$, with τ'^G, $[\tau'^G] = n-1 \to 2$. We will show how this rearrangement works and see that this trick makes the gauge field propagators much simpler in the following chapter, where we are going to study in detail the QED $U(1)$ subsector of these Lorentz-violating versions of the Standard Model.

Chapter 3

Renormalization of Lorentz-violating QED

In this chapter we study in detail the electromagnetic sector of the Lorentz-violating CPT-invariant Standard Model extension described in chapter 2. The structure of the theory is discussed thoroughly in section 3.1, while in section 3.2 we study explicitly its renormalization. Lorentz-violating QED is super-renormalizable, thus it has a finite number of divergent diagrams at one and two loops only; we work out the one-loop counterterms and beta functions. In section 3.3 we study the low-energy limit of the model and compare its renormalization with the high-energy one; in section 3.4 we present the results for the renormalization of the low-energy theory. From this analysis it turns out that the power-like divergences contain arbitrariness inherited from the high-energy theory, thus leading to reconsider the hierarchy problem, as we explain in section 3.5. The results found here are published in [22].

3.1 Lorentz-violating QED

Starting from the most "economic" version of the Lorentz-violating extensions of the Standard Model outlined previously, namely (2.15), we concentrate on its simplest form, i.e. the Abelian $U(1)$ case with only one fermion. Recall that we have assumed

$$\hat{d} = 1 \qquad \bar{d} = 3 \qquad n = 3 \qquad đ = 2$$

As for the photon Lagrangian we take the quadratic one in (2.9), which in the Abelian case is really quadratic, since the covariant derivative acts on Abelian gauge fields as an ordinary derivative; namely

$$\mathcal{L}_q = \frac{1}{4}\left[2F_{\hat{\mu}\bar{\nu}}\eta(\upsilon)F_{\hat{\mu}\bar{\nu}} + F_{\bar{\mu}\bar{\nu}}\tau(\upsilon)F_{\bar{\mu}\bar{\nu}} + \frac{1}{\Lambda_L^2}(\partial_{\hat{\rho}}F_{\bar{\mu}\bar{\nu}})\xi(\upsilon)(\partial_{\hat{\rho}}F_{\bar{\mu}\bar{\nu}})\right], \tag{3.1}$$

where $\upsilon = -\bar{\partial}^2/\Lambda_L^2$ and $\eta(\upsilon)$, $\tau(\upsilon)$ and $\xi(\upsilon)$ are polynomials of degree 2, 4 and 1, respectively. The BRST symmetry coincides with that of ordinary Lorentz invariant QED,

$$sA_\mu = \partial_\mu C, \qquad sC = 0, \qquad s\bar{C} = B, \qquad sB = 0, \tag{3.2}$$

and the gauge-fixing Lagrangian is

$$\mathcal{L}_{\text{gf}} = s\Psi, \qquad \Psi = \bar{C}\left(-\frac{\lambda}{2}B + \mathcal{G}\right), \qquad \mathcal{G} := \hat{\partial}\cdot\hat{A} + \zeta(\bar{\upsilon})\bar{\partial}\cdot\bar{A} \tag{3.3}$$

being $\zeta(\upsilon)$ a polynomial of degree 2 and λ a weightless constant. Choosing a gauge-fixing $\mathcal{G}$ linear in the gauge potential, as in (2.10) and (3.3), gives the simplest and most convenient form of propagators. These can be worked out integrating the Lagrange multiplier B out, which amounts to add

$$\frac{1}{2}\mathcal{G}\frac{1}{\lambda}\mathcal{G} \tag{3.4}$$

to the quadratic free Lagrangian of gauge fields.

Simplified gauge sector. Now we want to show that the polynomial η appearing in (3.1) can be consistently set to 1: this choice considerably simplifies the gauge sector. Indeed, we can easily make a weights arrangement different from (2.8) in such a way that the gauge field components acquire higher weights and the gauge coupling a lower one, while the product gA maintains the same weight. In fact, if we require that the Lagrangian term $\tilde{F}^2$ be multiplied by one, it should be $[\tilde{F}] \equiv [\hat{\partial}] + [\bar{A}] \equiv [\bar{\partial}] + [\hat{A}] = \text{đ}/2$, then

$$\begin{aligned} [\bar{A}] &= \frac{\text{đ}}{2} - 1 \to 0 & \qquad [\tilde{F}] &= \frac{\text{đ}}{2} \to 1 \\ [\hat{A}] &= \frac{\text{đ}}{2} - \frac{1}{n} \to \frac{2}{3} & \qquad [\bar{F}] &= \frac{\text{đ}}{2} - 1 + \frac{1}{n} \to \frac{1}{3}; \end{aligned} \tag{3.5}$$

imposing the appropriate weight of the covariant derivative components we obtain for the electron charge e

$$[e] = 1 + \frac{1}{n} - \frac{\text{đ}}{2} \to \frac{1}{3}. \tag{3.6}$$

Checking the term containing $\tau\bar{F}^2$, we see that $[\tau] = 2(n-1)/n$, which means that for $\eta \equiv 1$, τ must be a polynomial of degree $n-1$ in υ; as for the term containing ξ, it is not possible to match its weight to đ, so we are forced to put $\xi \equiv 0$. Therefore, we can adopt instead of (3.1) the following simplified quadratic gauge Lagrangian

$$\mathcal{L}_{q'} = \frac{1}{2}F_{\hat{\mu}\bar{\nu}}F_{\hat{\mu}\bar{\nu}} + \frac{1}{4}F_{\bar{\mu}\bar{\nu}}\left(\tau_2 - \tau_1\frac{\bar{\partial}^2}{\Lambda_L^2} + \tau_0\frac{\left(-\bar{\partial}^2\right)^2}{\Lambda_L^4}\right)F_{\bar{\mu}\bar{\nu}}. \tag{3.7}$$

Gauge fixings and gauge propagators. At this point, we have to reconsider also the weight of the gauge-fixing term (3.3). In fact, we see that ζ has the same weight of τ, $[\zeta] = 2(n-1)/n$, while the ghost Lagrangian after the BRST transformation s contains $\bar{C}\zeta(\upsilon)\bar{\partial}^2 C$, from which it follows $[C] = [\bar{C}] = \text{đ}/2 - 1$. From the first equation of (3.2) we derive $[s] = 1/n$, and from the third one $[B] = \text{đ}/2 - 1 + 1/n$. Consequently, also the gauge parameter λ is not weightless anymore, indeed its weight is equal to $[\tau]$. Summarizing we have

$$\begin{aligned} &[\tau] = [\zeta] = [\lambda] = \frac{2}{n}(n-1) \to \frac{4}{3} \qquad &&[C] = [\bar{C}] = \frac{\text{đ}}{2} - 1 \to 0 \\ &[s] = \frac{1}{n} \to \frac{1}{3} &&[B] = \frac{\text{đ}}{2} - 1 + \frac{1}{n} \to \frac{1}{3}. \end{aligned} \tag{3.8}$$

We choose the gauge-fixing in order to find the most convenient propagator's form. Because of the weight assignements in (3.8) we can choose the "Feynman" gauge

$$\lambda = \tau = \zeta, \tag{3.9}$$

and the gauge-fixing term takes the form

$$\mathcal{L}_{\text{gf}} = s\left[\bar{C}\left(-\frac{\tau(-\bar{\partial}^2/\Lambda_L^2)}{2}B + \hat{\partial}\cdot\hat{A} + \tau(-\bar{\partial}^2/\Lambda_L^2)\bar{\partial}\cdot\bar{A}\right)\right], \tag{3.10}$$

where $\tau(x) = \tau_0 x^2 + \tau_1 x + \tau_2$. B can be integrated out giving

$$\mathcal{L}_{\text{gf}} \to (\hat{\partial}\cdot\hat{A} + \tau\bar{\partial}\cdot\bar{A})\frac{1}{2\tau}(\hat{\partial}\cdot\hat{A} + \tau\bar{\partial}\cdot\bar{A}) - \bar{C}(\hat{\partial}^2 + \tau\bar{\partial}^2)C. \tag{3.11}$$

As usual, in the Abelian case the ghosts decouple and do not interact, so in the following they may be neglected. Note that (3.11) is strictly speaking non-local, since the gauge condition $\lambda = \tau$ implies that the constant in the denominator of (3.4) is replaced by some derivatives. However, the replacement is legitimate as long as the term (3.10) originally included in the action is local before integrating B out, B is not propagating and the propagators are well behaved.

The propagators in this "Feynman" gauge turn out to be very simple:

$$\begin{aligned} \langle \hat{A}(k)\hat{A}(-k)\rangle &= \frac{\tau(\bar{k}^2/\Lambda_L^2)}{\hat{k}^2 + \tau(\bar{k}^2/\Lambda_L^2)\bar{k}^2} \qquad \langle \hat{A}(k)\bar{A}_{\bar{\mu}}(-k)\rangle = 0 \\ \langle \bar{A}_{\bar{\mu}}(k)\bar{A}_{\bar{\nu}}(-k)\rangle &= \frac{\delta_{\bar{\mu}\bar{\nu}}}{\hat{k}^2 + \tau(\bar{k}^2/\Lambda_L^2)\bar{k}^2}. \end{aligned} \tag{3.12}$$

They are found substituting (3.9), as well as $\eta \equiv 1$ and $\xi \equiv 0$, in the most general propagators worked out in [11] for $\hat{d} = 1$. The propagators (3.12) have explicitly a well behaved structure which falls down as $\bar{k}^6$ when $\bar{k} \to \infty$, thus exhibiting renormalizability. However they disguise the number of degrees of freedom. The really propagating fields are shown, as in ordinary QED, in the "Coulomb" gauge $\bar{\partial} \cdot \bar{A} = 0$, namely if we use the gauge-fixing $\mathcal{G} = \bar{\partial} \cdot \bar{A}$. This one can be obtained from the more general form (3.3) taking the limit $\lambda, \zeta \to \infty$, with $\lambda' := \lambda/\zeta^2$ fixed[1]; the gauge-fixing Lagrangian becomes

$$s\left[\bar{C}\left(-\frac{\lambda'}{2}B + \bar{\partial}\cdot\bar{A}\right)\right] \longrightarrow \frac{1}{2\lambda'}(\bar{\partial}\cdot\bar{A})^2 - \bar{C}\bar{\partial}^2 C.$$

The ghosts do not propagate, since their two-point function does not contain poles. The photon propagators read

$$\langle \hat{A}(k)\hat{A}(-k)\rangle = \frac{1}{\bar{k}^2} + \frac{\lambda'\hat{k}^2}{(\bar{k}^2)^2} \qquad \langle \hat{A}(k)\bar{A}_{\bar{\mu}}(-k)\rangle = \frac{\lambda'\hat{k}\bar{k}_{\bar{\mu}}}{(\bar{k}^2)^2}$$
$$\langle \bar{A}_{\bar{\mu}}(k)\bar{A}_{\bar{\nu}}(-k)\rangle = \frac{1}{\hat{k}^2 + \tau(\bar{k}^2/\Lambda_L^2)\bar{k}^2}\left(\delta_{\bar{\mu}\bar{\nu}} - \frac{\bar{k}_{\bar{\mu}}\bar{k}_{\bar{\nu}}}{\bar{k}^2}\right) + \frac{\lambda'\bar{k}_{\bar{\mu}}\bar{k}_{\bar{\nu}}}{(\bar{k}^2)^2}.$$

The poles structure shows that there are just the two right propagating degrees of freedom, thus exhibiting manifestingly unitarity. Because of gauge independence, both renormalizability and unitarity are guaranteed in any gauge choice.

The complete theory. The fermion Lagrangian, which has to be weighted power-counting renormalizable and gauge invariant, must include a term with three space (covariant) derivatives in order to provide a propagator that falls off rapidly enough for renormalizability; however, as long as we consider massive fermions, it accomodates also all terms of lower weight, i.e. with zero, one and two covariant derivatives:

$$\mathcal{L}_{\text{kin}F} = \bar{\psi}\left[\hat{\not{D}} - \frac{b_0}{\Lambda_L^2}\bar{\not{D}}^3 - \frac{b'}{\Lambda_L}\bar{\not{D}}^2 + b_1\,\bar{\not{D}} + m\right]\psi, \tag{3.13}$$

where the covariant derivative is defined as usual $D_\mu = \partial_\mu + ieA_\mu$. Inverting (3.13) one finds the electron propagator, which reads

$$\langle\psi(p)\bar{\psi}(-p)\rangle = \frac{-i\hat{\not{p}} - i\left(\frac{b_0}{\Lambda_L^2}\bar{p}^2 + b_1\right)\bar{\not{p}} + m + \frac{b'}{\Lambda_L}\bar{p}^2}{\hat{p}^2 + \left(\frac{b_0}{\Lambda_L^2}\bar{p}^2 + b_1\right)^2\bar{p}^2 + \left(m + \frac{b'}{\Lambda_L}\bar{p}^2\right)^2}. \tag{3.14}$$

The sum of (3.7) and (3.13) is not the most general renormalizable Lagrangian of a 1+3-dimensional model with $n = 1/3$. Other interaction vertices could be included. In fact, collecting the weights from (3.5), (3.6) and (3.13), one has

$$[\tilde{F}] = \frac{đ}{2} \to 1 \qquad [\bar{F}] = \frac{đ}{2} - \frac{2}{3} \to \frac{1}{3} \qquad [\psi] = \frac{đ}{2} - \frac{1}{2} \to \frac{1}{2} \qquad [e] = \frac{4}{3} - \frac{đ}{2} \to \frac{1}{3}.$$

[1]It is needed also a rescaling of B by a factor ζ and of C and s by a factor $\zeta^{1/2}$; consequently, in this "Coulomb" gauge the weights must be rearranged as follows: $[s] = 1$, $[C] = đ/2 - 1/n$, $[B] = đ/2 + 1 - 1/n$ and $[\lambda'] = 2/n - 2$. The fact that the weight of λ' is negative is unessential because renormalizability properties are gauge independent.

Since by weighted power counting each monomial of fields is renormalizable only if its weight is not greater than đ, the most general Lagrangian may include many other terms, such as $\bar{F}^6$, $\bar{\partial}^2\bar{F}^4$ and $\bar{F}^4$ (excluding odd terms by antisymmetry), $\bar{\psi}\tilde{F}\psi$, as well as the structures obtained from $\bar{\psi}\bar{D}^i\psi$, $i \leq 3$, by replacing one or more derivatives with $\bar{F}$ (since both $\bar{D}$ and $\bar{F}$ have the same weight) and even the four fermions coupling. Nevertheless, if we restrict our considerations to the so-called "$1/\alpha$" theories, namely those with Lagrangians of the usual form

$$\mathcal{L} = \frac{1}{e^2}\mathcal{L}\left(eA, e\psi\right),$$

we see that each of their parts has as many fields as powers of e minus 2, allowing only $e\bar{\psi}\bar{F}\psi$, $e\bar{\psi}\bar{D}\bar{F}\psi$ and $e^2\bar{F}^4$ to survive, since e carries its weight $1/3$. We drop the $\bar{F}^4$ term because, if not included, it is not generated back by renormalization. Therefore the interaction Lagrangian is

$$\mathcal{L}_{I\,gF} = \bar{\psi}\left[e\frac{b''}{\Lambda_L}\sigma_{\bar{\mu}\bar{\nu}}F_{\bar{\mu}\bar{\nu}} + ie\frac{b_0'}{\Lambda_L^2}(\partial_{\bar{\nu}}F_{\bar{\mu}\bar{\nu}})\gamma_{\bar{\mu}} + e\frac{b_0''}{\Lambda_L^2}F_{\bar{\mu}\bar{\nu}}\frac{\overleftrightarrow{D}_{\bar{\nu}}}{2}\gamma_{\bar{\mu}}\right]\psi,$$

where the convention $\sigma_{\bar{\mu}\bar{\nu}} = -\frac{i}{2}[\gamma_{\bar{\mu}}, \gamma_{\bar{\nu}}]$ is adopted and $\overleftrightarrow{D}_{\bar{\nu}}$ is intended to act only on ψ and $\bar{\psi}$.

Collecting everything together we have

$$\begin{aligned}
\mathcal{L}_{\mathrm{LVQED}} &= \mathcal{L}_{q'} + \mathcal{L}_{\mathrm{kin}F} + \mathcal{L}_{I\,gF} = \\
&= \frac{1}{2}F_{\hat{\mu}\bar{\nu}}F_{\hat{\mu}\bar{\nu}} + \frac{1}{4}F_{\bar{\mu}\bar{\nu}}\left(\tau_2 - \tau_1\frac{\bar{\partial}^2}{\Lambda_L^2} + \tau_0\frac{(-\bar{\partial}^2)^2}{\Lambda_L^4}\right)F_{\bar{\mu}\bar{\nu}} \\
&+ \bar{\psi}\left(\hat{\not{D}} - \frac{b_0}{\Lambda_L^2}\bar{\not{D}}^3 + b_1\bar{\not{D}} + m - \frac{b'}{\Lambda_L}\bar{\not{D}}^2\right)\psi \\
&+ \frac{e}{\Lambda_L}\bar{\psi}\left(b''\sigma_{\bar{\mu}\bar{\nu}}F_{\bar{\mu}\bar{\nu}} + \frac{ib_0'}{\Lambda_L}\gamma_{\bar{\mu}}\partial_{\bar{\nu}}F_{\bar{\mu}\bar{\nu}}\right)\psi + e\frac{b_0''}{\Lambda_L^2}F_{\bar{\mu}\bar{\nu}}\left(\bar{\psi}\gamma_{\bar{\mu}}\frac{\overleftrightarrow{\bar{D}}_{\bar{\nu}}}{2}\psi\right),
\end{aligned} \tag{3.15}$$

where for brevity the gauge-fixing contribution is understood. This Lagrangian contains interactions with one-, two- and three-photons vertices. The Feynman rules of this theory are summarized in Fig 3.1.

The Euclidean formulation (3.15) is the most convenient one to study renormalization; the Wick rotation works as in ordinary quantum field theories, since the time-derivative structure is the same. In the Minkowski framework the Lagrangian reads

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{2}F_{\hat{\mu}\bar{\nu}}F^{\hat{\mu}\bar{\nu}} - \frac{1}{4}F_{\bar{\mu}\bar{\nu}}\left(\tau_2 - \tau_1\frac{\bar{\partial}^2}{\Lambda_L^2} + \tau_0\frac{(-\bar{\partial}^2)^2}{\Lambda_L^4}\right)F^{\bar{\mu}\bar{\nu}} \\
&+ \bar{\psi}\left(i\hat{\not{D}} + \frac{ib_0}{\Lambda_L^2}\bar{\not{D}}^3 + ib_1\bar{\not{D}} - m - \frac{b'}{\Lambda_L}\bar{\not{D}}^2\right)\psi \\
&+ \frac{e}{\Lambda_L}\bar{\psi}\left(b''\sigma_{\bar{\mu}\bar{\nu}}F^{\bar{\mu}\bar{\nu}} + \frac{b_0'}{\Lambda_L}\gamma_{\bar{\mu}}\partial_{\bar{\nu}}F^{\bar{\mu}\bar{\nu}}\right)\psi + ie\frac{b_0''}{\Lambda_L^2}F^{\bar{\mu}\bar{\nu}}\left(\bar{\psi}\gamma_{\bar{\mu}}\frac{\overleftrightarrow{\bar{D}}_{\bar{\nu}}}{2}\psi\right)
\end{aligned} \tag{3.16}$$

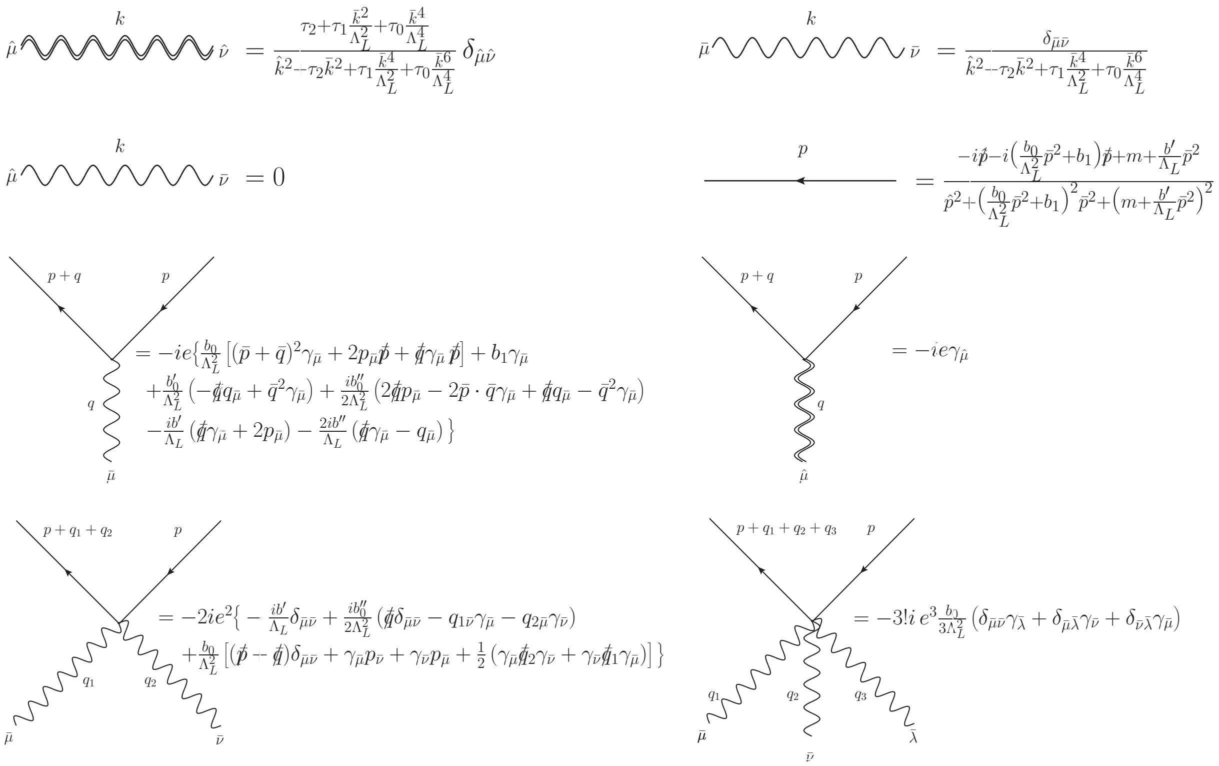

Figure 3.1: Feynman rules for $\mathcal{L}_{\text{LVQED}}$. The double curly lines denote $\hat{A}$, while the simple curly lines denote $\bar{A}$. Photons momenta are ingoing.

and the propagators (3.12) and (3.14) turn into

$$\begin{aligned}\langle A(k)A(-k)\rangle &= \frac{i}{\hat{k}^2-\tau(\bar{k}^2/\Lambda_L^2)\bar{k}^2+i0}\begin{pmatrix}-\tau(\frac{\bar{k}^2}{\Lambda_L^2}) & 0\\ 0 & \mathbb{1}\end{pmatrix}\\ \langle\psi(p)\bar{\psi}(-p)\rangle &= i\frac{\hat{\not{p}}+\left(\frac{b_0}{\Lambda_L^2}\bar{p}^2+b_1\right)\bar{\not{p}}+m+\frac{b'}{\Lambda_L}\bar{p}^2}{\hat{p}^2-\left(\frac{b_0}{\Lambda_L^2}\bar{p}^2+b_1\right)^2\bar{p}^2-\left(m+\frac{b'}{\Lambda_L}\bar{p}^2\right)^2+i0}.\end{aligned} \tag{3.17}$$

3.2 One-loop high-energy renormalization

In this section we want to perform the one-loop renormalization of Lorentz violating QED using the dimensional regularization. We start from the "bare" Lagrangian, that is (3.15) in which all fields and coupling constants have to be intended as bare. After dimensional continuation one has $\hat{d}=1-\hat{\varepsilon}$, $\bar{d}=3-\bar{\varepsilon}$ and hence $\text{đ}=2-\varepsilon\!\!\!-$, with $\varepsilon\!\!\!-=\hat{\varepsilon}+\bar{\varepsilon}/3$. The weights of bare fields and coupling constants thus read

$$[\hat{A}_B] = \frac{2}{3} - \frac{\varepsilon}{2} \qquad [\bar{A}_B] = -\frac{\varepsilon}{2} \qquad [\psi_B] = \frac{1}{2} - \frac{\varepsilon}{2} \qquad [m_B] = 1$$

$$[\tau_{2B}] = \frac{4}{3} \qquad [\tau_{1B}] = [b_{1B}] = \frac{2}{3} \qquad [e_B] = \frac{1}{3} + \frac{\varepsilon}{2}$$

$$[\tau_{0B}] = [b_{0B}] = [b'_{0B}] = [b''_{0B}] = [\Lambda_{LB}] = 0 \qquad [b'_B] = [b''_B] = \frac{1}{3}.$$

Since the electric charge is weightful the theory is super-renormalizable. For the same reason there are no wave function renormalization constants, so bare and renormalized fields coincide; by Ward identity the electric charge is also not renormalized. Being our Lagrangian of "$1/\alpha$" kind, each vertex is multiplied by a power of e equal to the number of its legs minus 2. Therefore each loop carries an additional factor e^2. In fact, a graph containing V_A vertices with N_A legs, V_B vertices with N_B legs etc. is multiplied by a number of factors e equal to

$$V_\mathrm{A}(N_\mathrm{A} - 2) + V_\mathrm{B}(N_\mathrm{B} - 2) + ... + V_\mathrm{K}(N_\mathrm{K} - 2).$$

The total number of legs in the vertices has to match the number of internal and external legs I and E:

$$V_\mathrm{A}N_\mathrm{A} + V_\mathrm{B}N_\mathrm{B} + ... + V_\mathrm{K}N_\mathrm{K} = E + 2I, \tag{3.18}$$

while the total number of independent momenta is given by the independent internal ones, which by definition is the number of loops L, plus the independent external ones that are $E-1$ (taking into account an overall momentum- conservation rule). But the same number has to be given by the total number of legs $E + I$ subtracted by one conservation rule in each vertex, so

$$E - 1 + L = E + I - V_\mathrm{A} - V_\mathrm{B} - ... - V_\mathrm{K}. \tag{3.19}$$

From (3.19) one derives I, which substituted in (3.18) gives

$$V_\mathrm{A}(N_\mathrm{A} - 2) + V_\mathrm{B}(N_\mathrm{B} - 2) + ... + V_\mathrm{K}(N_\mathrm{K} - 2) = E + 2L - 2.$$

Now, since each loop adds to a graph a factor e^2 with its weight 2/3 and there is no coupling of negative weight, it follows that only the parameters with weights $\geq 2/3$ are renormalized, that are τ_2, τ_1, b_1 and m. For the same reason, the renormalization of b_1, τ_1 and m has only one-loop contributions, while τ_2 has both one- and two-loops counterterms. Therefore, the only nontrivial relations among bare and renormalized parameters are

$$b_{1B} = b_1 + \delta^{(1)}b_1 \qquad \tau_{1B} = \tau_1 + \delta^{(1)}\tau_1 \qquad \tau_{2B} = \tau_2 + \delta^{(1)}\tau_2 + \delta^{(2)}\tau_2 \qquad m_B = m + \delta^{(1)}m,$$

where $\delta^{(1)}$ and $\delta^{(2)}$ denote one- and two-loops corrections respectively.

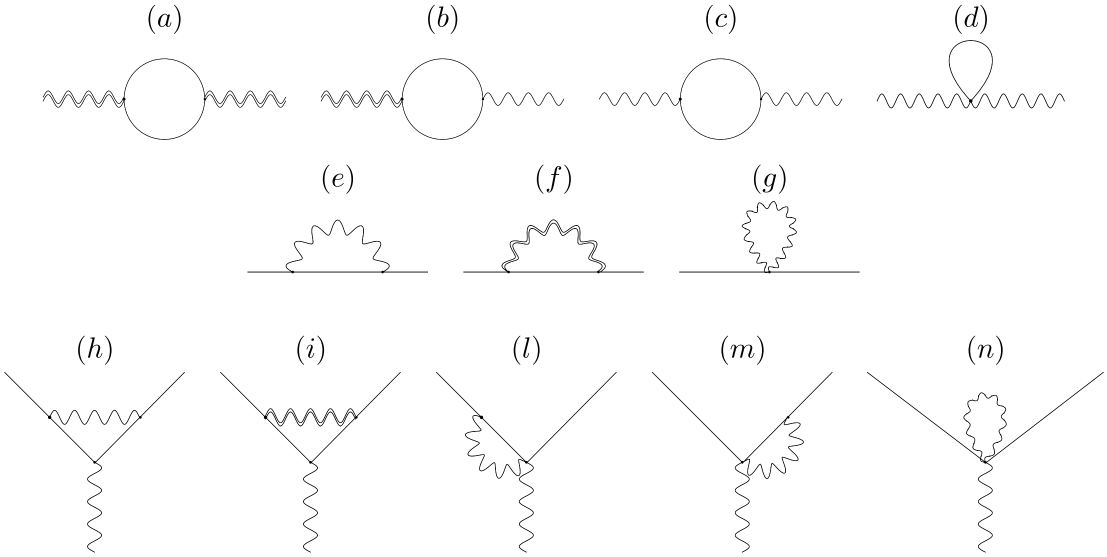

Figure 3.2: One-loop divergent diagrams.

The relation between the bare electric charge e_B and the physical one e is

$$e_B = e\mu^{\not\varepsilon/2}\Lambda_L^{\bar{\varepsilon}/3}, \tag{3.20}$$

where μ is the renormalization scale. Both Λ_L and μ have mass dimension one, but while the weight of Λ_L is zero, that of μ is chosen to be one. In fact, since e_B has dimension $\varepsilon/2$ and weight $1/3 + \not\varepsilon/2$, by making the choice (3.20) the renormalized electron charge e acquires $\not\varepsilon$-independent dimension and weight (zero and $1/3$ respectively).

We aim to calculate the counterterms needed to cancel the infinities arising from divergent one-loop diagrams, all represented in Fig. 3.2. The degree of divergence ω of one-loop graphs is easily computed by weighted power counting, taking into account that the measure $\mathrm{d}^{\hat{d}}\hat{p}\ \mathrm{d}^{\bar{d}}\bar{p}$ has weight two. One can see it also by making the change of variable $\bar{p}_i = \bar{p}'_i(\Lambda_L^2/\bar{p}'^2)^{\frac{1}{3}}$ in the integrals: this leads to the usual $\bar{p}'^2$ instead of $\bar{p}^6$ in the propagators, but an extra $\Lambda_L^2/(3\bar{p}'^2)$ factor comes from the Jacobian determinant of the transformation and lowers the four powers of the measure by two units.

For the purpose of renormalization the propagators can be expanded in powers of the external momenta as well as of the weightful parameters τ_1, τ_2, b_1, b' and m, since the divergent parts depend polynomially on them. As we are not interested in finite parts, to avoid infrared problems we can introduce a fictitious mass κ, which can be replaced by zero once the divergent parts have been calculated.

For example, the vertex renormalization diagrams have $\omega = 0$, so we can find their divergent part simply by putting external momenta and weightful parameters to zero, that means dropping all terms except the most divergent ones. At this point we remain with denominators like

$$\frac{1}{\left(\hat{p}^2+\frac{b_0^2}{\Lambda_L^4}\bar{p}^6+\kappa^2\right)^{n_1}\left(\hat{p}^2+\frac{\tau_0}{\Lambda_L^4}\bar{p}^6+\kappa^2\right)^{n_2}}.$$

Applying a Feynman parametrization we reduce to only one factor in the denominator and get an x-integral

$$\int_0^1 dx\,\frac{x^{n_1-1}(1-x)^{n_2-1}}{\left(\hat{p}^2+\frac{\tau_0+x\left(b_0^2-\tau_0\right)}{\Lambda_L^4}\bar{p}^6+\kappa^2\right)^{n_1+n_2}}.$$

In order to calculate integrals of powers of either $\hat{p}^2$ and $\bar{p}^2$ with such denominators it is sufficient to apply two times the standard formula for integrals in dimensional regularization and to perform the previously mentioned change of variable; the x-integral is then solvable in terms of an hypergeometric function. Finally, for the divergent part of a generic one-loop integral we obtain

$$\int\frac{\mathrm{d}^{1-\hat{\varepsilon}}\hat{p}\,\mathrm{d}^{3-\bar{\varepsilon}}\bar{p}}{(2\pi)^4}\frac{(\hat{p})^q\left(\bar{p}^2\right)^r\left(\bar{p}\cdot\bar{k}\right)^s}{\left(\hat{p}^2+\frac{b_0^2}{\Lambda_L^4}\bar{p}^6+\kappa^2\right)^{n_1}\left(\hat{p}^2+\frac{\tau_0}{\Lambda_L^4}\bar{p}^6+\kappa^2\right)^{n_2}}=\frac{\tau_0^{\frac{1+q}{2}-n_1-n_2}}{\Lambda_L^{2+2q-4n_1-4n_2}}\frac{\left(\bar{k}^2\right)^{\frac{s}{2}}}{3\,\not\varepsilon}\times \tag{3.21}$$
$$\times\frac{(1+(-1)^s)(1+(-1)^q)}{16\pi^3(s+1)\Gamma(n_1+n_2)}\Gamma(\frac{1+q}{2})\Gamma(n_1+n_2-\frac{1+q}{2}){}_2F_1(n_1,n_1+n_2-\frac{1+q}{2},n_1+n_2,1-\frac{b_0^2}{\tau_0})$$

for $n_1+n_2=1+q/2+(2r+s)/6$.

Diagram (a) of Fig. 3.2, which appears logarithmically divergent by power counting, would give therefore a mass term $\hat{A}^2$. However, its divergent part is proportional to

$$\int\frac{\mathrm{d}^{1-\hat{\varepsilon}}\hat{p}\,\mathrm{d}^{3-\bar{\varepsilon}}\bar{p}}{(2\pi)^4}\frac{\hat{p}^2-\frac{b_0^2}{\Lambda_L^4}\bar{p}^6}{\left(\hat{p}^2+\frac{b_0^2}{\Lambda_L^4}\bar{p}^6+m^2\right)^2},$$

which gives zero because the $\hat{p}$- and the $\bar{p}$-terms cancel each other. Diagram (b) is zero because the integrand surviving from the trace is odd in $\hat{p}$. Note that the electron self-energy receives a contribution from the tadpole (g), which is nonzero. As far as the photon self-energy is concerned, the tadpole (d) cancels exactly the mass term coming from the usual QED graph (c), ensuring gauge invariance. Finally, as a check, the divergences of the vertex and the electron self-energy turn out to be equal and reconstruct the covariant derivative, as required from gauge invariance.

In the end the one-loop counterterms in the minimal subtraction scheme are

$$\Delta^{(1)}\mathcal{L}=\frac{1}{4}\delta^{(1)}\tau_2F_{\bar{\mu}\bar{\nu}}^2-\frac{1}{4}\delta^{(1)}\tau_1F_{\bar{\mu}\bar{\nu}}\frac{\bar{\partial}^2}{\Lambda_L^2}F_{\bar{\mu}\bar{\nu}}+\bar{\psi}\left(\delta^{(1)}b_1\,\bar{\not{D}}+\delta^{(1)}m\right)\psi, \tag{3.22}$$

where, called $\delta^{(1)}\mathcal{O} = \Delta^{(1)}\mathcal{O}/(3\bar{\varepsilon})$ for $\mathcal{O} = \tau_2$, τ_1, b_1 or m,

$$
\begin{aligned}
\Delta^{(1)}\tau_2 &= \frac{e^2}{6\pi^2}\frac{|b_0|}{b_0}\left(-b_1 - 4\frac{b_0''^2 b_1}{b_0^2} - \frac{1}{2}\frac{b'^2}{b_0} - 2\frac{b_0''^2 b'^2}{b_0^3} - 12\frac{b'b''}{b_0} + 8\frac{b''^2}{b_0}\right)\\
\Delta^{(1)}\tau_1 &= -\frac{e^2|b_0|}{6\pi^2}\left(\frac{3}{10} + 2\frac{b_0'}{b_0} - 4\frac{b_0'^2}{b_0^2} + \frac{11}{5}\frac{b_0''^2}{b_0^2}\right)\\
\Delta^{(1)}b_1 &= \frac{e^2}{3\pi^2\left(|b_0| + \sqrt{\tau_0}\right)^2}\left(-\frac{9}{2}|b_0|b_0 + |b_0|b_0' - \frac{b_0 b_0''^2}{|b_0|} - \frac{1}{2}\frac{b_0}{|b_0|}\tau_0 + 4\frac{b_0^2 b_0'}{\sqrt{\tau_0}}\right.\\
&\quad \left. + \frac{3}{2}\frac{b_0 b_0'^2}{\sqrt{\tau_0}} - \frac{5}{8}\frac{b_0 b_0''^2}{\sqrt{\tau_0}} - \frac{7}{2}b_0\sqrt{\tau_0}\right)\\
\Delta^{(1)}m &= \frac{e^2\Lambda_L}{\pi^2\left(|b_0| + \sqrt{\tau_0}\right)}\left(-\frac{1}{2}\frac{b_0'^2 b'}{|b_0|\sqrt{\tau_0}} - \frac{1}{8}\frac{b_0''^2 b'}{|b_0|\sqrt{\tau_0}} + 2\frac{|b_0|b''}{\sqrt{\tau_0}} + 2\frac{b_0}{|b_0|}\frac{b_0' b''}{\sqrt{\tau_0}} - \frac{1}{4}\frac{b'\sqrt{\tau_0}}{|b_0|} - \frac{3}{4}b'\right).
\end{aligned}
\tag{3.23}
$$

Note that in (3.21) and consequently in (3.22) we can safely put $\hat{\varepsilon} = 0$. This means that we could have done our calculations without continuing the time manifold at all. This choice is understood from now on.

At this point it is trivial to derive the RG evolution of the coupling constants, because from (3.20) it follows that

$$\mu\frac{d}{d\mu}e^2 = -\bar{\varepsilon}\, e^2.$$

Thus the one-loop beta functions are

$$\beta_{\tau_2} = \frac{1}{3}\Delta^{(1)}\tau_2, \qquad \beta_{\tau_1} = \frac{1}{3}\Delta^{(1)}\tau_1, \qquad \beta_{b_1} = \frac{1}{3}\Delta^{(1)}b_1, \qquad \beta_m = \frac{1}{3}\Delta^{(1)}m.$$

The high-energy renormalization of LVQED can be completed evaluating a finite number of two-loops diagrams (which are represented in Fig. 3.3).

3.3 Low-energy limit

In this section we study the relation between the renormalization of the high-energy theory, already performed in the previous section, and the low-energy renormalization. The low-energy limit can be studied assuming that the scale Λ_L tends to infinity. In this limit the Lagrangian $\mathcal{L}_{\text{LVQED}}$ becomes

$$\mathcal{L}_{\text{lvQED}} = \frac{1}{2}F_{\hat{\mu}\bar{\nu}}F_{\hat{\mu}\bar{\nu}} + \frac{\tau_2}{4}F_{\bar{\mu}\bar{\nu}}F_{\bar{\mu}\bar{\nu}} + \bar{\psi}\left(\hat{\not{D}} + b_1\bar{\not{D}} + m\right)\psi. \tag{3.24}$$

To go from the renormalization of LVQED to that of lvQED one has to keep in mind that when $\Lambda_L \to \infty$, in addition to the $1/\bar{\varepsilon}$ divergences obtained for Λ_L fixed, other Λ_L-contributions arise, which have been disregarded previously, precisely because from the high-energy point of view they

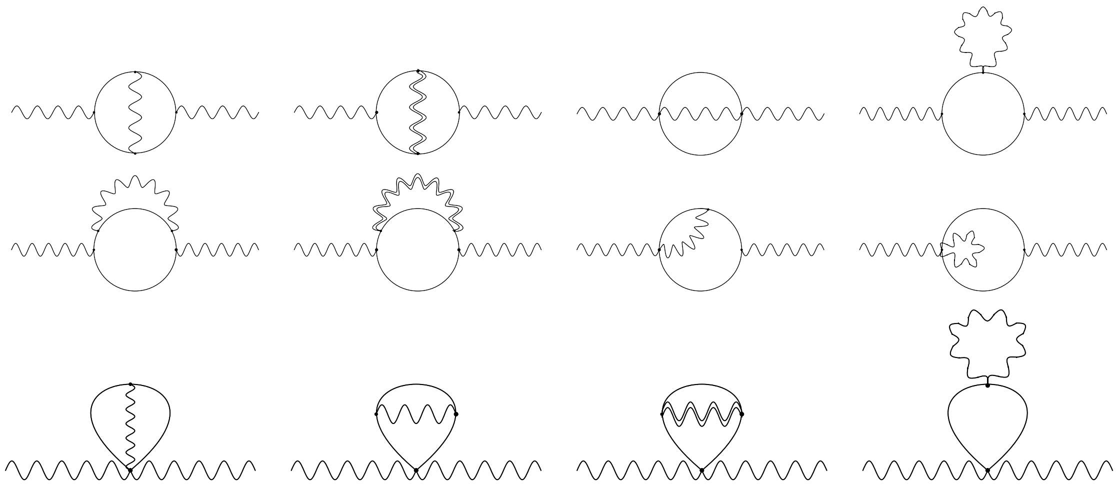

Figure 3.3: Two-loops divergent diagrams.

are finite. What we want to explain is how the $1/\bar{\varepsilon}$-divergences and Λ_L-divergences combine to reproduce the low-energy results.

In fact, from the low-energy point of view one can consider $\mathcal{L}_{\text{LVQED}}$ with its dimensional regularization as a particular regularization of $\mathcal{L}_{\text{lvQED}}$, by means of both $\bar{\epsilon}$ and Λ_L, which is regarded as a cut-off. In this approach the renormalized lvQED can be obtained subtracting the $1/\bar{\epsilon}$-divergences and then sending $\bar{\epsilon}$ to zero, next subtracting the Λ_L-divergences and finally sending Λ_L to infinity.

But we know that the difference between two different regularizations of the same quantity must be finite when the regularizations are removed; for example when a theory is regularized with two cut-offs the order of their removal can only affect intermediate results by a scheme change. Once physical normalization conditions are imposed, all physical quantities must coincide. Therefore, one can equivalently try to take the limit $\Lambda_L \to \infty$ before the $\bar{\varepsilon} \to 0$ limit, which would give simply the lvQED Lagrangian (3.24) equipped with its dimensional regularization, where only space is continued to complex dimensions.

Moreover, two different cut-offs can be identified up to an arbitrary constant, which has no universal meaning and can be chosen differently for every integral; indeed, changing it amounts simply to shift the pole subtraction, which is again a scheme change.

When the two methods explained above are considered as nothing more than different regularizations of (3.24), it is clear that the results obtained in the two ways differ at most by a scheme change. Since one-loop logarithmic divergences are scheme-independent, one can get the result directly from (3.24) instead of evaluating all the Λ_L-divergences coming from LVQED.

On the contrary, power-like divergences are scheme-dependent and so, if we consider LVQED as a fundamental theory and not simply as an unusual regularization of lvQED, we must calculate them directly, taking the $\bar{\varepsilon} \to 0$ limit before sending Λ_L to infinity.

Nevertheless, it turns out that the power-like Λ_L-divergences are multiplied by arbitrary and incalculable constants, coming from the arbitrariness involved in the regularization of the high-energy theory, which is renormalizable but not finite. Thus, such power-like divergences have no physical meaning, and therefore it is completely reliable to study the low-energy theory from (3.24), i.e. sending Λ_L to infinity at $\bar{\varepsilon} \neq 0$.

In the rest of this section we perform a detailed analysis of the problem and prove these statements.

3.3.1 Problem statement

A one-loop correlation function is the sum of contributions of the form I_r/Λ_L^r, where r is a non-negative integer and

$$I_r = \int \frac{d\hat{p} d^{3-\bar{\varepsilon}}\bar{p}}{(2\pi)^4} \frac{N_s(\hat{p}, \bar{p}, \hat{k}, \bar{k})}{\prod_{i=1}^n \left[(\hat{p} - \hat{k}_i)^2 + a_i(\bar{p} - \bar{k}_i)^2 + m_i^2 + (\bar{p} - \bar{k}_i)^2 \Delta_i((\bar{p} - \bar{k}_i)^2/\Lambda_L^2)\right]}, \tag{3.25}$$

where $\Delta_i(x)$ are polynomials with vanishing zero-th order coefficients, k_i denotes linear combinations of the external momenta, collectively called k, while the numerator N_s is a monomial of degree s in momenta. We can prove [22] that the integral I_r is equivalent to the cut-offed one

$$I'_{r<} = \int_{-\infty}^{\infty} \frac{d\hat{p}}{2\pi} \int_{|\bar{p}| \leq \Lambda_L} \frac{d^3\bar{p}}{(2\pi)^3} \frac{N_s(\hat{p}, \bar{p}, \hat{k}, \bar{k})}{\prod_{i=1}^n \left[(\hat{p} - \hat{k}_i)^2 + a_i(\bar{p} - \bar{k}_i)^2 + m_i^2\right]}. \tag{3.26}$$

up to a scheme change; in other words the divergent parts of (3.25) and (3.26) can differ only by local counterterms that have at most power-like divergences. Since $I'_{r<}$ is a one-loop integral, it can generate divergences containing logarithms and powers, but not powers times logarithms. By locality it must have the form

$$I'_{r<} = P(\Lambda_L, m, k) + P'(m, k) \ln \Lambda_L + \text{finite} + \mathcal{O}(1/\Lambda_L),$$

where P and P' are polynomials.

For $r > 0$ the contribution to the correlation function coming from $I'_{r<}$ (which has to be divided by Λ_L^r) and hence from I_r, is just a scheme change; thus, only the $r = 0$ term determines physical quantities. But the $I'_{0<}$ integrals are exactly those arising from the low-energy theory (3.24) regulated by means of a cut-off Λ_L; this proves that the low-energy limit of the LVQED renormalization can be studied, up to a scheme change, directly from (3.24), regulating it with a cut-off Λ_L on the space momenta. Precisely, the scheme-independent contributions are those arising from $I'_{0<}$, while the scheme-dependent ones have to be studied directly from the complete theory.

3.3.2 An example

As an illustrative example we can consider the tadpole integral

$$I = \int \frac{\mathrm{d}\hat{p}\ \mathrm{d}^{3-\bar{\varepsilon}}\bar{p}}{(2\pi)^4} \frac{1}{D(\hat{p},\bar{p},m) + \bar{p}^2\Delta(\bar{p}^2/\Lambda_L^2)},$$

where D is the quadratic part of the propagator and Δ is its extra higher-powers part, namely

$$D(\hat{p},\bar{p},m) = \hat{p}^2 + a_2\bar{p}^2 + m^2, \qquad \Delta(x) = a_0x^2 + a_1x,$$

and $a_0, a_2 > 0$. At Λ_L finite, this integral is logarithmically divergent. When $\Lambda_L \to \infty$, it becomes quadratically divergent.

We want to show that I is equivalent to

$$I'_< = \int_{-\infty}^{+\infty} \frac{\mathrm{d}\hat{p}}{2\pi} \int_{|\bar{p}|\leqslant\Lambda_L} \frac{\mathrm{d}^3\bar{p}}{(2\pi)^3} \frac{1}{D(\hat{p},\bar{p},m)}, \tag{3.27}$$

up to a scheme change.

It is convenient to split the $\bar{p}$-domain of integration in two regions: the sphere $|\bar{p}| \leqslant \Lambda_L$ and the crown $|\bar{p}| \geqslant \Lambda_L$. Rewriting all in terms of the adimensional variables $(\hat{p}/\Lambda_L, \bar{p}/\Lambda_L)$, and recalling them again $(\hat{p},\bar{p})$, we get

$$I = I_< + I_>, \qquad I_{\lessgtr} = \Lambda_L^{2-\bar{\varepsilon}} \int_{-\infty}^{+\infty} \frac{\mathrm{d}\hat{p}}{2\pi} \int_{|\bar{p}|\lessgtr 1} \frac{\mathrm{d}^{3-\bar{\varepsilon}}\bar{p}}{(2\pi)^3} \frac{1}{D(\hat{p},\bar{p},m/\Lambda_L) + \bar{p}^2\Delta(\bar{p}^2)}.$$

Consider first $I_>$. The integrand can be expanded in powers of m/Λ_L (there are no infrared problems, since $\bar{p}$ starts from one and cannot approach zero). Then we can write

$$I_> = \sum_{k=0}^{\infty} (-1)^k \Lambda_L^{2-\bar{\varepsilon}-2k} m^{2k} I_k, \tag{3.28}$$

where

$$I_k = \int_{-\infty}^{+\infty} \frac{\mathrm{d}\hat{p}}{2\pi} \int_{|\bar{p}|>1} \frac{\mathrm{d}^{3-\bar{\varepsilon}}\bar{p}}{(2\pi)^3} \frac{1}{[D(\hat{p},\bar{p},0) + \bar{p}^2\Delta(\bar{p}^2)]^{k+1}}. \tag{3.29}$$

When $\bar{\varepsilon} \to 0$ only I_0 diverges[2] We can write

$$I_0 = \frac{A_0}{\bar{\varepsilon}} + B_0 + \mathcal{O}(\bar{\varepsilon}),$$
$$I_k = B_k + \mathcal{O}(\bar{\varepsilon}) \qquad \text{for} \quad k > 0,$$

[2]For the sake of precision, in (3.28) and (3.29) we have neglected the m-dependence encoded in a_2; nevertheless, its contributions have the same structure of I_k, with some powers of $\bar{p}^2$ added, but they are convergent. Therefore nothing changes in our argument.

where A_0, B_i are constants, finite in the limit $\bar{\varepsilon} \to 0$. We have, for $\bar{\varepsilon} \sim 0$,

$$I_> = \Lambda_L^2 \left[A_0 \left(\frac{1}{\bar{\varepsilon}} - \ln \Lambda_L \right) + B_0 \right] - B_1 m^2 + \mathcal{O}(\bar{\varepsilon}, m^2/\Lambda_L^2).$$

Notice that if we had regulated the high-energy theory with a cut-off Λ instead of using the dimensional regularization, the coefficient of A_0 between the square brackets would be $\ln(\Lambda/\Lambda_L)$.

Using the fact that two cut-offs can be identified up to an arbitrary constant, i.e.

$$\frac{1}{\bar{\varepsilon}} = \ln \Lambda_L + c, \tag{3.30}$$

if we take the limit $\Lambda_L \to \infty$ after $\bar{\varepsilon} \to 0$ we find

$$I_> \to \Lambda_L^2 (cA_0 + B_0) - B_1 m^2.$$

We see that the contribution of the crown does not contain logarithmic divergences and it is polynomial in the mass. Moreover, the coefficient of the power-like divergence remains undetermined because of the arbitrary factor c.

Now we study $I_<$. Here we can immediately take the limit $\bar{\varepsilon} \to 0$, since the integral has no ultraviolet domain of integration. Thus it can be trivially checked that

$$\begin{aligned} I_< =& \Lambda_L^2 \int_{-\infty}^{+\infty} \frac{\mathrm{d}\hat{p}}{2\pi} \int_{|\bar{p}|<1} \frac{\mathrm{d}^3\bar{p}}{(2\pi)^3} \frac{1}{D(\hat{p}, \bar{p}, m/\Lambda_L) + \bar{p}^2 \Delta(\bar{p}^2)} = \\ =& \Lambda_L^2 \int_{-\infty}^{+\infty} \frac{\mathrm{d}\hat{p}}{2\pi} \int_{|\bar{p}|<1} \frac{\mathrm{d}^3\bar{p}}{(2\pi)^3} \frac{1}{D(\hat{p}, \bar{p}, m/\Lambda_L)} + \\ & - \Lambda_L^2 \int_{-\infty}^{+\infty} \frac{\mathrm{d}\hat{p}}{2\pi} \int_{|\bar{p}|<1} \frac{\mathrm{d}^3\bar{p}}{(2\pi)^3} \frac{\bar{p}^2 \Delta(\bar{p}^2)}{D(\hat{p}, \bar{p}, m/\Lambda_L)(D(\hat{p}, \bar{p}, m/\Lambda_L) + \bar{p}^2 \Delta(\bar{p}^2))}, \end{aligned} \tag{3.31}$$

The term in the second line is just the $I'_<$ contribution (3.27) rescaled in adimensional variables, while the one in the third line has no infrared divergences for $m = 0$. From (3.31) we can define a quantity X in such a way that

$$I_< = I'_< + \Lambda_L^2 J + m^2 X, \tag{3.32}$$

where

$$J = - \int_{-\infty}^{+\infty} \frac{\mathrm{d}\hat{p}}{2\pi} \int_{|\bar{p}|<1} \frac{\mathrm{d}^3\bar{p}}{(2\pi)^3} \frac{\bar{p}^2 \Delta(\bar{p}^2)}{D(\hat{p}, \bar{p}, 0)(D(\hat{p}, \bar{p}, 0) + \bar{p}^2 \Delta(\bar{p}^2))}.$$

J is finite, while X is regular in the limit $\Lambda_L \to \infty$; by making an expansion around $m^2/\Lambda_L^2 \sim 0$ we can find its limit $\bar{X}$, which reads

$$\bar{X} = \int_{-\infty}^{+\infty} \frac{\mathrm{d}\hat{p}}{2\pi} \int_{|\bar{p}|<1} \frac{\mathrm{d}^3\bar{p}}{(2\pi)^3} \frac{\bar{p}^2 \Delta(\bar{p}^2) \left(2D(\hat{p}, \bar{p}, 0) + \bar{p}^2 \Delta(\bar{p}^2) \right)}{D^2(\hat{p}, \bar{p}, 0) \left(D(\hat{p}, \bar{p}, 0) + \bar{p}^2 \Delta(\bar{p}^2) \right)^2}.$$

$\bar{X}$ has no infrared divergences and hence is finite[3]. Thus, the scheme-independent divergences are contained in $I'_<$.

Calculating $I'_<$ and collecting all together, we get

$$I = \Lambda_L^2 \left(\frac{1}{8\pi^2 a_2^{1/2}} + cA_0 + B_0 + J \right) - m^2 \left(\frac{\ln(4a_2\Lambda_L^2/m^2) - 1}{16\pi^2 a_2^{3/2}} + B_1 - \bar{X} \right) + \mathcal{O}(m^2/\Lambda_L^2). \quad (3.33)$$

The quadratic divergences remain arbitrary, due to the constant c inherited from the high-energy theory. Another argument to justify the identification (3.30) is that I cannot have divergences of the form $\Lambda_L^2/\bar{\varepsilon}$ or $\Lambda_L^2 \ln \Lambda_L$, because they can arise only at higher loops.

3.3.3 General proof

Now we give the general argument for the equivalence of (3.25) and (3.26) up to a scheme change. The degree of divergence ω of $I'_{r<}$ is $s + 4 - 2n$. If $\omega < 0$ the limits $\varepsilon_2 \to 0$ and $\Lambda_L \to \infty$ can be taken directly on the integrand of I_r and the result is equal to the limit $\Lambda_L \to \infty$ of $I'_{r<}$, which is finite. Thus, we can restrict our analysis to the case $\omega \geqslant 0$. As in the example, we can split the $\bar{p}$-domain of integration in two regions: the sphere $|\bar{p}| \leqslant \Lambda_L$ and the crown $|\bar{p}| \geqslant \Lambda_L$. We call $I_{r>}$ and $I_{r<}$ their two contributions to I_r. Rewriting all in terms of the adimensional variables $(\hat{p}/\Lambda_L, \bar{p}/\Lambda_L)$, and then recalling them again $(\hat{p}, \bar{p})$ we get

$$\frac{I_{r>}}{\Lambda_L^{\omega-\bar{\varepsilon}}} = \int_{-\infty}^{+\infty} \frac{\mathrm{d}\hat{p}}{2\pi} \int_{|\bar{p}|\geq 1} \frac{\mathrm{d}^{3-\bar{\varepsilon}}\bar{p}}{(2\pi)^3} \frac{N_s(\hat{p}, \bar{p}, \hat{k}/\Lambda_L, \bar{k}/\Lambda_L)}{\prod_{i=1}^n \left[(\hat{p} - \frac{\hat{k}_i}{\Lambda_L})^2 + a_i(\bar{p} - \frac{\bar{k}_i}{\Lambda_L})^2 + \frac{m_i^2}{\Lambda_L^2} + (\bar{p} - \frac{\bar{k}_i}{\Lambda_L})^2 \Delta_i((\bar{p} - \frac{\bar{k}_i}{\Lambda_L})^2) \right]}. \quad (3.34)$$

Now we expand $I_{r>}$ in expression (3.34) in powers of k/Λ_L and m/Λ_L, which is allowed because the integral has no infrared region. After a finite number of derivatives we get terms that are finite for $\bar{\varepsilon} \to 0$ and then disappear when later $\Lambda_L \to \infty$. Thus the result of these limits on $I_{r>}$ is a polynomial in k and m. The coefficients are powers Λ_L^i, possibly multiplied by simple poles $1/\bar{\varepsilon}$. Since

$$\frac{\Lambda_L^{i-\bar{\varepsilon}}}{\bar{\varepsilon}} = \Lambda_L^i \left(\frac{1}{\bar{\varepsilon}} - \ln \Lambda_L + \mathcal{O}(\bar{\varepsilon}) \right) \to \Lambda_L^i \left(c_i + \mathcal{O}(\bar{\varepsilon}) \right),$$

we see that all power-like divergences are multiplied by (different) arbitrary constants c_i and no $\ln \Lambda_L$ can appear.

Now we consider $I_{r<}$: we can set $\bar{\varepsilon} = 0$, since here there is no ultraviolet region. To keep the notation simple, we collect both k's and m's in the same symbol K and leave index contractions implicit. Proceeding on the pattern of the exmple presented in the previous subsection we define

[3]Here again, for the sake of simplicity, we have considered a_2 as a constant. If one takes care of its m-dependence, (3.32) as well as the following (3.33) would contain also a term proportional to $m\Lambda_L$, but the crucial points of the reasoning would not be altered.

$K^\omega X$ as the difference between $I_{r<} - I'_{r<}$ and its expansion in k/Λ_L and m/Λ_L up to the order $\omega - 1$. We have

$$I_{r<} = I'_{r<} + \sum_{i=0}^{\omega-1} \Lambda_L^{\omega-i} K^i J_i + K^\omega X. \tag{3.35}$$

Now, by construction all J_i's are integrals of functions depending only on $\hat{p}$ and $\bar{p}$ and no other dimensionful quantities[4]. Power counting shows that they are also IR convergent, because they have dimensions $\omega - i$, which in (3.35) is contained in $\Lambda_L^{\omega-i}$; moreover, such integrals have by definition an ultraviolet cut-off ($|\bar{p}| \leqslant 1$). Next, we need to check that the $\Lambda_L \to \infty$ (or $K \to 0$) limit $\bar{X}$ of X is well defined. Again, there is no concern about the ultraviolet, but we must check IR convergence. Although X has dimension zero, we must recall that it is originated expanding the difference $I_{r<} - I'_{r<}$, whose integrand is proportional to a polynomial $\Delta(x) = \mathcal{O}(x)$. The factor Δ enhances the naive IR power counting by two units, just enough to make $\bar{X}$ well defined. This concludes the proof.

3.4 Low-energy renormalization

From the previous section we know that power-like divergences can be disregarded, while logarithmic ones can be calculated from (3.24), which gives the same Feynman rules and propagators of Fig. 3.1, where the limit $\Lambda_L \to \infty$ is taken.

The divergent diagrams to be calculated are the same as in ordinary QED, taking into account that there are two kinds of vertices and photon propagators. The integrands can be expanded as usual in powers of the external momenta. Then one remains with a sum of terms proportional to the integrals

$$\int_{-\infty}^{\infty} \frac{d\hat{k}}{2\pi} \int_{|\bar{k}|\leq\Lambda_L} \frac{d^3\bar{k}}{(2\pi)^3} \frac{(\hat{k}^2, \bar{k}^2)}{(\hat{k}^2 + b_1^2\bar{k}^2 + m^2)^2(\hat{k}^2 + \tau_2\bar{k}^2)}. \tag{3.36}$$

Performing the $\hat{k}$ integral before the $\bar{k}$ one and discarding all non-logarithmic terms, (3.36) gives

$$\frac{\ln(\Lambda_L/m)}{8\pi^2|b_1|(|b_1| + \sqrt{\tau_2})^2}(1, \frac{2|b_1| + \sqrt{\tau_2}}{b_1^2\sqrt{\tau_2}}).$$

Writing

$$\Delta\mathcal{L}_{\text{lvQED}} = \frac{1}{2}\delta Z_A\, F_{\hat{\mu}\bar{\nu}}F_{\hat{\mu}\bar{\nu}} + \frac{1}{4}\delta\tau_2\, F_{\bar{\mu}\bar{\nu}}F_{\bar{\mu}\bar{\nu}} + \delta Z_\psi\, \bar{\psi}\hat{\not{D}}\psi + \delta b_1\, \bar{\psi}\bar{\not{D}}\psi + \delta m\, \bar{\psi}\psi,$$

at one loop we obtain

$$\delta Z_A = -\frac{e^2}{6\pi^2|b_1|}\ln\frac{\Lambda_L}{\mu} \qquad \delta\tau_2 = b_1^2\delta^{(1)}Z_A \qquad \delta Z_\psi = \frac{e^2}{4\pi^2}\frac{(\tau_2 - 3b_1^2)}{\sqrt{\tau_2}(|b_1| + \sqrt{\tau_2})^2}\ln\frac{\Lambda_L}{\mu}$$

$$\delta b_1 = -b_1\frac{e^2}{12\pi^2}\frac{(\tau_2 + b_1^2)(|b_1| + 2\sqrt{\tau_2})}{|b_1|\sqrt{\tau_2}(|b_1| + \sqrt{\tau_2})^2}\ln\frac{\Lambda_L}{\mu} \qquad \delta m = -m\frac{e^2}{4\pi^2}\frac{(\tau_2 + 3b_1^2)}{|b_1|\sqrt{\tau_2}(|b_1| + \sqrt{\tau_2})}\ln\frac{\Lambda_L}{\mu}.$$

[4]Here $(\hat{p}, \bar{p})$ are meant as the original variables, not as the adimensional rescaled ones.

Since at one loop

$$\delta\tau_2 = \tau_2(\delta Z_A + \delta Z_{\tau_2}) \qquad \delta b_1 = b_1(\delta Z_\psi + \delta Z_{b_1}) \qquad \delta m = m(\delta Z_\psi + \delta Z_m),$$

the beta functions and anomalous dimensions are

$$\begin{aligned} \beta_e &= e\gamma_A = \frac{e^2}{12\pi^2|b_1|} \qquad \beta_{\tau_2} = \frac{e^2(\tau_2 - b_1^2)}{6\pi^2|b_1|} \quad \beta_m = -\frac{me^2}{4\pi^2}\frac{2|b_1|\sqrt{\tau_2}+\tau_2+3b_1^2}{|b_1|(|b_1|+\sqrt{\tau_2})^2} \\ \beta_{b_1} &= -\frac{e^2 b_1}{6\pi^2}\frac{2|b_1|(\tau_2 - 2b_1^2)+\sqrt{\tau_2}(\tau_2+b_1^2)}{|b_1|\sqrt{\tau_2}(|b_1|+\sqrt{\tau_2})^2} \qquad \gamma_\psi = -\frac{e^2(\tau_2 - 3b_1^2)}{8\pi^2\sqrt{\tau_2}(|b_1|+\sqrt{\tau_2})^2}. \end{aligned} \tag{3.37}$$

If we expand these results around $\tau_2 = 1$ and $b_1 = 1$ (which should be a very good approximation if lvQED has to agree with experiments) and keep only the first-order terms, our results are in agreement with those found in [23], once restricted to the subsector invariant for CPT, parity and rotations (see also [24]).

To recover QED one has to set

$$\tau_2 = b_1^2. \tag{3.38}$$

In fact, only if (3.38) holds it is possible to convert the Lagrangian (3.24) to the one of QED by rescaling the space coordinates, the fields and the electric charge e. This is equivalent to setting $\tau_2 = b_1 = 1$; doing so, one sees from (3.37) that β_{b_1} and β_{τ_2} vanish, while β_e, β_m and γ_ψ take their known values. This is another check of our results.

3.5 Power-like divergences and the hierarchy problem

After the detailed analysis worked out in section 3.3 we have seen that scheme-independent logarithmic divergences can be studied directly from the low energy theory (3.24); conversely, if one is interested in power-like divergences, they have to be studied from the limit $\Lambda_L \to +\infty$ of LVQED. Nevertheless, we have also found that some coefficients of the power-like divergences remain arbitrary (see eq. (3.33)), due to the divergences of the high energy theory. This fact acquires a special meaning when the scale Λ_L is regarded as physical (i.e. from the point of view of LVQED), and not as a cut-off (as done from the low-energy point of view). In fact, in the evaluation of power-like divergences the finite scale Λ_L mixes at low energy with the high-energy unphysical cut-off, say $1/\bar{\varepsilon}$, to make the coefficients of the Λ_L-powers arbitrary. That is, if the high-energy theory behind the low-energy one is not completely finite, but only renormalizable or even super-renormalizable, power-like divergences are arbitrary. These arguments are general and hence apply, in particular, to the Lorentz violating extended Standard Models of [8, 13] and lead to reconsider the hierarchy problem.

In fact, when one imagines new physics beyond the Standard Model, usually it is supposed to be described by a *finite* theory, depending on a physical scale, say Λ, much larger than the energies explored in the Standard Model, which therefore is recovered sending Λ to infinity. Thus, at energies much smaller than Λ, the Higgs mass is corrected by *physical* quadratic divergences, that pose a fine-tuning problem. On the contrary, if the theory beyond the Standard model is not assumed to be finite, but only renormalizable, as here, the quadratic divergences of the Higgs mass would have no physical meaning, but they would be only by-products of the scheme choice made during renormalization, hence they could be removed with the usual procedure of renormalization itself. Because of this, no fine-tuning problem arises, as we explain below.

More precisely, the renormalization of a mass, say the Higgs one, can be read for example from (3.33): at one loop it takes the general form

$$m_\Lambda^2 = m^2 + a\Lambda_L^2 \ln \frac{\Lambda^2}{\Lambda_L^2} + bm^2 \ln \frac{\Lambda_L^2}{m^2} + c\Lambda_L^2 + dm^2. \tag{3.39}$$

Here m_Λ denotes the bare mass, m is the low-energy mass, Λ is the ultraviolet cut-off (simply we have replaced $1/\bar{\varepsilon}$ with $\ln \Lambda$+constant), while a, b, c and d are coefficients, depending on the parameters of the theory. In LVQED the formula of the electron-mass renormalization has a form analogous to (3.39), but the squares m_Λ^2, m^2, Λ^2 and Λ_L^2 are replaced by m_Λ, m, Λ and Λ_L, respectively, and the coefficient a can be read from (3.23). b and d are also calculable, see (3.33).

Now, if Λ is physical, i.e. if it is thought as the energy scale of a finite underlying theory, all the coefficients would be calculable, including c. Since m is small, but Λ is large, the right-hand side of (3.39) is large, hence m_Λ has to be large, and a fine-tuning problem arises: two large quantities have nearly to cancel each other in order to reconstruct m, which is much smaller, namely

$$m^2 = \text{small} = \text{large} - \text{large}.$$

Instead, if our models are regarded as fundamental (here gravity is switched off), namely if we assume that they are true at any energy, then Λ is just an unphysical renormalization tool, which means that it must be sent to infinity as an ordinary cut-off, and c remains arbitrary. Thus the bare mass m_Λ is also infinite, so

$$m^2 = \infty - \infty.$$

But this is just what usually happens with renormalization: subtractions of infinities give finite quantities. So, if this is the case, the fine-tuning problem disappears, because m cannot be said to be small or large with respect to infinity.

Formula (3.39) incorporates also the (one-loop) running from energies Λ to energies Λ_L. In other words, if we substitute Λ with Λ_L formula (3.39) gives an expression for the Higgs mass m_L at the scale of Lorentz violation. We find

$$m_L^2 = m^2 + bm^2 \ln \frac{\Lambda_L^2}{m^2} + c\Lambda_L^2 + dm^2.$$

We see that the quadratic divergence $\sim \Lambda_L^2$ is multiplied by the meaningless arbitrary constant c. There is no reason why the quantity $c\Lambda_L^2$ should be large, even if Λ_L^2 is large. Actually, we can use the arbitrariness of c to make it disappear, and obtain

$$m_L^2 = m^2 + bm^2 \ln \frac{\Lambda_L^2}{m^2} + dm^2.$$

Again, we do not find any fine-tuning problem.

Our argument is very general. It does not depend on the particular high-energy completion of the theory, as long as it is not finite. Indeed, if the UV completion is not finite, at some point we do need an unphysical cut-off Λ, which brings some arbitrariness into the game and allows us to avoid fine-tuning.

In conclusion, the hierarchy problem is a true problem only if the ultimate theory of the Universe is completely finite. If the ultimate theory of the Universe is just renormalizable, or even super-renormalizable, the hierarchy problem disappears.

Chapter 4

Phenomenology

In this chapter we discuss how to extract phenomenological predictions from our theory and compare them with experimental data. In section 4.1 we give an outlook of the problem, then in section 4.2 we set up the general formulas needed for calculations. In section 4.3 we study a case which turns out to be entirely analogous to QED in a material medium with a refractive index. In section 4.4 we give some general kinematics results and introduce the more interesting case with higher derivatives, which allows to put bounds on the scale of Lorentz violation; two typical cases are fully studied in section 4.5. In section 4.6 we present an illustrative formulation involving composite particles and show how it seems to favor "small numbers". Finally, in section 4.7 we give some results on Cherenkov radiation from neutral particles, which is also allowed in our theory.

4.1 Lorentz violating phenomena

As well known from propagation of light in material media, when the dispersion relations are modified with respect to the purely relativistic ones, new phenomena may occur, such as Cherenkov effect, i.e. the emission of light from superluminal free electrons flying in a dielectric. In an analogous way, if we assume that Lorentz symmetry is not exact, several phenomena that are otherwise forbidden can occur. Examples are the Cherenkov radiation in vacuo and the photon decay into an electron-positron pair. Studying such phenomena, we can look for signs of Lorentz violation and comparing predictions with experimental data allows us to test our theories and put bounds on the values of Lorentz-violating parameters.

Various phenomena that are forbidden in Lorentz-invariant theories, but which are allowed in Lorentz-violating ones, have been studied in the literature, mainly using the modified dispersion relations of low-energy effective models. We have studied [10] some of those phenomena in the framework of the LVSM, where the dispersion relations have a deeper foundation, since they are crucial for renormalizability and valid, in principle, at arbitrarily high energies.

In the low-energy limit, when Λ_L goes to infinity, the Lorentz violation amounts to give to the vacuum a refractive index different from 1, which allows emission of photons from free electrons if $n > 1$, and pairs photoproduction if $n < 1$, phenomena that are both kinematically forbidden in the Loretz-invariant case. Cherenkov emission, for example, occurs very rapidly when the energy of the flying fermion is over a certain treshold $E_{\rm lim}$, so that no particle with energy greater than $E_{\rm lim}$ should be seen coming from space. Thus, since we *do* observe high-energy cosmic rays, stringent bounds can be posed on $n-1$; from the most energetic particle observed ($E = 3 \cdot 10^{11}$GeV), which is believed to be a proton, one obtains $n - 1 < 10^{-23}$ (see section 4.3).

Nevertheless, as the energy increases, the higher-dimension terms in the Lorentz-violating Lagrangian become greater, and it is not reliable to neglect them. It turns out that, because of those higher-dimension operators, Cherenkov emission still occurs (even if $n = 1$), as long as it is kinematically allowed; by the same reasoning as above one can pose bounds on the parameters of the theory. In particular, we want to derive bounds on the magnitude of the scale of Lorentz violation Λ_L, which may be smaller than the Planck scale, as we argue below. If this were true, our understanding of physics around the Planck scale, in particular quantum gravity, would have to be reconsidered anew.

We have assumed that CPT is preserved (or that it is violated at energies much larger than Λ_L). An estimate on the value of Λ_L can be made looking at the vertex $(LH)^2/\Lambda_L$ in (2.11), as originally suggested in ref. [8]. Indeed, since this vertex is the only dimension-5 vertex present in the Lorentz-violating extension of the Standard Model (2.13) or (2.14), it can be used to normalize the scale Λ_L. Assuming that the dimensionless couplings in front of it are of order one, after symmetry breaking each Higgs field in the vertex carries a factor $v/\sqrt{2}$ coming from its vacuum expectation value, hence

$$m_\nu \sim \frac{v^2}{8\Lambda_L}.$$

With a neutrino mass $m_\nu \lesssim 10^{-10}$GeV and $v^2 = (\sqrt{2}G_F)^{-1} \approx 6 \cdot 10^4 \text{GeV}^2$, we find $\Lambda_L \sim 10^{14}$-10^{15}GeV. Note that in the literature an identical estimate is made, though considering Λ_L the lepton number violation scale [19]. The four fermions vertices (2.12) can describe proton decay; the scale which appears there is normally a different one, related to baryon number violation. Anyway the existing bounds on proton decay constrain this scale, and, as figured in [8], if we interpret it as Λ_L, present data give $\Lambda_L \geqslant 10^{15}$GeV.

Here we show that such values are indeed compatible with experimental data on Lorentz-violating phenomena and even if to date there is no experimental evidence of Lorentz violation, the bounds on parameters obtained from direct measurements do not exclude values of Λ_L well below the Planck scale.

Experimental bounds on the parameters that multiply higher-dimensional operators can be read from the tables of Kostelecky and Russell [2]. At present, the best results belong to the photon sector, and concern the quadratic terms

$$F_{k\lambda}\partial_{\alpha_1}\cdots\partial_{\alpha_n}F_{\mu\nu}.$$

In particular, from astrophysical birefringence it is found that the upper bound on the coefficients of the dimension 6 and dimension 8 nonrenormalizable operators of the non-minimal photon sector in the Standard Model Extension constructed in [3] are

$$\lesssim 10^{-29}\text{GeV}^{-2} \text{ and } \lesssim 10^{-23}\text{GeV}^{-4},$$

respectively. If we interpret these coefficients as $\sim 1/\Lambda_L^2$ and $\sim 1/\Lambda_L^4$, respectively, the experimental data are consistent with any value of $\Lambda_L \gtrsim 10^{15}$GeV; actually, as we will see later, in our theory the coefficient of the operator of dimension 6 is not crucial and does not need to be of order 1, but could be smaller. In our model the crucial term is the one concerning the dimension 8 operator, on which the bound resulting for Λ_L is even weaker, allowing any value of $\Lambda_L > 10^6$GeV.

In ref. [9] it has been stated, under some assumptions, that ultrahigh-energy cosmic rays observations force to raise the bound on Λ_L well above the Planck scale. However, the nature of ultrahigh-energy cosmic rays has not been firmly established yet, so it is not obvious how to use them to put unambiguous bounds on the scale of Lorentz violation. Here we give several scenarios that are consistent with a value of Λ_L well below the Planck scale, assuming that ultrahigh-energy cosmic rays are protons or heavy nuclei. For our purposes, it will be sufficient to restrict to the minimal QED subsector of the LVSM, LVQED.

We focus on the Cherenkov radiation in vacuo. For a very general class of dispersion relations we prove that there exists an energy threshold above which radiation is emitted and below which it is not emitted. Quite interestingly, the threshold is enhanced in composite particles by a sort of kinematic screening mechanism. We study the energy loss as a function of time and prove that in all cases of our interest it is so rapid that the radiation is practically governed by pure kinematics. Our model predicts also the Cherenkov radiation of neutral particles.

4.2 General formulas

In our Lorentz-violating version of QED (3.16), which has been studied thoroughly in the previous chapter, the dispersion relations of particle energies are modified from the usual relativistically invariant ones, being

$$\begin{aligned} E(\bar{p}^2) &= \sqrt{\bar{p}^2\left(b_1+\frac{b_0}{\Lambda_L^2}\bar{p}^2\right)^2+\left(\frac{b'}{\Lambda_L}\bar{p}^2+m\right)^2}, \\ \omega(\bar{k}^2) &= \sqrt{\tau_2\bar{k}^2+\tau_1\frac{(\bar{k}^2)^2}{\Lambda_L^2}+\tau_0\frac{(\bar{k}^2)^3}{\Lambda_L^4}}, \end{aligned} \tag{4.1}$$

for the fermion and the photon respectively.

The low-energy limit of the Lagrangian is (3.24), which in the Minkowski framework can be written as

$$\mathcal{L}_{\rm low} = \frac{1}{2}F_{0i}^2 - \frac{\tau_2}{4}F_{ij}^2 + \bar{\psi}\left(i\gamma^0 D_0 + ib_1\bar{\not{D}} - m\right)\psi. \tag{4.2}$$

We can show easily that it formally coincides with the Lagrangian of QED in a medium. The parameters τ_2 and b_1 are related to the dielectric constant ε and the magnetic permeability μ by the formulas

$$\tau_2 = \frac{\varepsilon}{\mu}, \qquad b_1 = \varepsilon. \tag{4.3}$$

The refractive index n is

$$n = \sqrt{\varepsilon\mu} = \frac{b_1}{\sqrt{\tau_2}}$$

Performing the replacements (4.3) and the rescalings

$$x^i \to \varepsilon x^i, \qquad A_i \to \frac{A_i}{\varepsilon}, \qquad \psi \to \frac{\psi}{\varepsilon^{3/2}},$$

in the action of (4.2), we obtain the more common Lagrangian of electrodynamics in a medium

$$\mathcal{L}_{\rm medium} = \frac{\varepsilon}{2}F_{0i}^2 - \frac{1}{4\mu}F_{ij}^2 + \bar{\psi}\left(i\not{D} - m\right)\psi. \tag{4.4}$$

We use for (4.4) the gauge-fixing term of Lorenz type

$$\mathcal{L}_{\rm GF} = -\frac{1}{2\mu}(\varepsilon\mu\partial_0 A_0 - \partial_i A_i)^2. \tag{4.5}$$

At this point it is convenient to derive a general formula for the energy loss per unit time, without making specifical assumptions on the dispersion relations. It will be applied to both the complete theory (3.16) and its low energy limit (4.2). Consider a charged fermion of energy E and momentum p emitting a photon of frequency ω; calling E' and p' the energy and the momentum of the fermion after the emission, the expression of the differential width is

$$\mathrm{d}\Gamma = \frac{1}{2E}\overline{|\mathcal{M}|^2}(2\pi)\delta(E-\omega-E')(2\pi)^3\delta^3(\mathbf{p}-\mathbf{k}-\mathbf{p}')\frac{\mathrm{d}^3\mathbf{k}}{2\omega(2\pi)^3}\frac{\mathrm{d}^3\mathbf{p}'}{2E'(2\pi)^3},$$

where $\overline{|\mathcal{M}|^2}$ is the squared modulus of the transition amplitude, summed over the final states and averaged over the initial states.

As usual, the integral over $\mathbf{p}'$ is done eliminating the delta function associated with momentum conservation. The surviving integral is reduced to an integral over ω and $u = \cos\theta$, θ being the angle between the momentum of the incoming fermion and the momentum of the emitted photon. Next, the delta function of energy conservation can be used to perform the u-integral, giving u

as a function of p and k. Finally, the condition $|u(p,k)| \leqslant 1$ determines the range of the final k-integration. We then find

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}\omega} = \frac{\overline{|\mathcal{M}|^2}}{16\pi Ep}\frac{k}{\omega}\frac{\mathrm{d}k}{\mathrm{d}\omega}\sum_{u^*}\frac{1}{\left|\frac{E'}{p'}\frac{\mathrm{d}E'}{\mathrm{d}p'}\right|_{u=u^*}}, \tag{4.6}$$

where the sum is over the solutions $u^*(p,k)$ to the condition of energy conservation. In the case of (4.4), the solution is unique; conversely, the dispersion relations of complete Lorentz-violating models admit in general multiple solutions. Nevertheless, if one makes some quite reasonable assumptions the solution remains unique (see section 4.4). When the solution is unique the allowed k-range is of the standard form $0 \leqslant k \leqslant k_{\max}$, for some $k_{\max}$.

The differential width (4.6) can be used to calculate the energy loss per unit time, using the formula

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\int_0^{\omega_{\max}} \omega\frac{\mathrm{d}\Gamma}{\mathrm{d}\omega}\mathrm{d}\omega, \tag{4.7}$$

where $\omega_{\max} = \omega(k^2_{\max})$.

4.3 Cherenkov radiation in QED

In this section we study the energy loss of charged particles in a Lorentz violating vacuo in the low-energy limit (which is in turn the same as in standard QED in a medium), and apply it to ultrahigh-energy cosmic rays. Cherenkov radiation, i.e. emission of photons by non-accelerating charged particles, occurs if $n > 1$, while if $n < 1$ a sufficiently energetic photon can decay into an electron-positron pair, see for example [25]. However, we are not going to study this phenomenon and assume hereafter $n > 1$. We use the notation of (4.4) and work out exact formulas without assuming that n is close to 1, so our results can be also applied to the Cherenkov radiation of charged particles in true media. Some results of this section are already available in the literature [25, 26], others are new. The propagators derived from (4.4) and (4.5) are

$$\langle A_\mu(k)A_\nu(-k)\rangle = \frac{i}{\varepsilon}\frac{\mathrm{diag}\left(-1/n^2, \mathbb{1}\right)}{\omega^2 - (\mathbf{k}^2/n^2) + i0}, \qquad \langle\psi(p)\bar{\psi}(-p)\rangle = i\frac{\not{p} + m}{p^2 - m^2 + i0},$$

where $k = (\omega, \mathbf{k})$. From these expressions we can read the formulas for the sums over polarization states:

$$\sum_\lambda \varepsilon_\mu^{(\lambda)}\varepsilon_\nu^{(\lambda)*} = \frac{1}{\varepsilon}\mathrm{diag}\left(-1/n^2, \mathbb{1}\right), \qquad \sum_s u_s(p)\bar{u}_s(p) = \not{p} + m, \qquad \sum_s v_s(p)\bar{v}_s(p) = \not{p} - m. \tag{4.8}$$

After a small amount of work, using eq. (4.6) we find

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}\omega} = \frac{\mu\alpha}{2Ep}\left\{\frac{n^2-1}{n^2}\left[2E(E-\omega) + \frac{\omega^2}{2}(n^2+1)\right] - 2m^2\right\}, \tag{4.9}$$

with

$$\omega \leqslant \omega_{\max} = \frac{2(np - E)}{n^2 - 1}, \qquad \frac{1}{n} \leqslant v \equiv \frac{p}{E} < 1.$$

In the limit $v \ll 1$, $\omega \ll E$, formula (4.9) agrees with the classic one, see e.g. [27]. The energy loss (4.7) per unit time is

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\frac{\alpha m^2 \mu (nv - 1)^3 P(v)}{3n^2(n^2 - 1)^3 v(1 - v^2)}, \tag{4.10}$$

where

$$P(x) = 3n(3n^2 - 1)x - (5n^2 + 1).$$

The result (4.10) agrees with the one found by Klinkhamer and Schreck in ref.[26]. We can rewrite it as a differential equation for the velocity as a function of time:

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{\alpha m \mu (nv - 1)^3 \sqrt{1 - v^2} P(v)}{3n^2(n^2 - 1)^3 v^2}. \tag{4.11}$$

The energy of the radiating charged fermion decreases to the asymptotic limit

$$E_{\mathrm{lim}} = \frac{mn}{\sqrt{n^2 - 1}}, \tag{4.12}$$

which corresponds to the asymptotic velocity $v_{\mathrm{lim}} = 1/n$.

Radiation time. The inverse of eq. (4.11) can be integrated to give the radiation time; it gives a finite result around $v = 1$, but not around $v = v_{\mathrm{lim}}$. This means that a particle with infinite energy and velocity 1 radiates to some final velocity v_f and energy $E_f = m/\sqrt{1 - v_f^2}$ in a finite amount of time $t(n, E_f)$, but it reaches v_{lim} and the energy limit (4.12) only after an infinite amount of time: $t(n, E_{\mathrm{lim}}) = \infty$. The explicit expression for the time taken by a particle to slow down from velocity 1 to v_f can be worked out; it is

$$\begin{aligned} t(n, E_f) \;=\;& \frac{3n(n^2 - 1)(3 - nv_f)}{16\alpha E_f \mu (nv_f - 1)^2} + \frac{3(25n^4 + 14n^2 - 3)}{64\alpha m \mu \sqrt{n^2 - 1}} \ln \frac{n - v_f + \sqrt{(n^2 - 1)(1 - v_f^2)}}{nv_f - 1} \\ & - \frac{9(3n^2 - 1)(5n^2 + 1)^2}{64\alpha m \mu \sqrt{n^2 - 1}\sqrt{P_+ P_-}} \ln \frac{v_f P(1/v_f) + \sqrt{P_+ P_- (n^2 - 1)(1 - v_f^2)}}{P(v_f)}, \end{aligned} \tag{4.13}$$

where

$$P_\pm = 9n^2 \pm 4n + 1.$$

Plotting (4.13) for various values of n close to 1, we can see that the energy decrease has a regular shape (see Fig.4.1). For all our practical purposes the particle looses “all” its energy during some finite effective radiation time. However, since the decay is not exponential, the radiation time must be defined in an unconventional way.

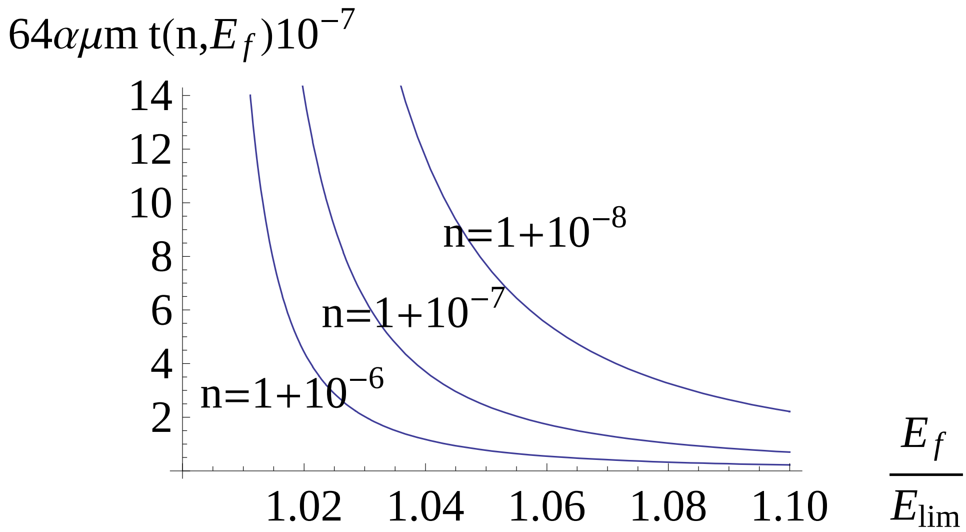

Figure 4.1: Plot of the radiation time obtained from eq. (4.13), for various values of n. Dimensionful quantities are intended in natural units.

Assume that the maximum observed energy of a certain class of particles is $E_{\rm obs} \gg m$. Then, if we knew that $E_{\rm obs} \leqslant E_{\rm lim}$ we would obtain the bound

$$n \leqslant \frac{1}{\sqrt{1 - \frac{m^2}{E_{\rm obs}^2}}}. \tag{4.14}$$

Since we cannot exclude that $E_{\rm lim}$ is smaller than our $E_{\rm obs}$ and that the particle observed is still radiating during the infinite time queue it spends reaching $E_{\rm lim}$, we must content ourselves with a worse bound. However, we can show that the decay is so fast that the “worse” bound is for all practical purposes as good as (4.14).

We consider ultrahigh-energy cosmic rays, for which we take the highest observed energy $E_{\rm obs} = 3 \cdot 10^{11}$GeV [28] and assume that they are protons or iron atoms moving in empty space. As far as the fine-structure constant α is concerned, we use the value $1/116$, namely the Standard-Model value of the running coupling at $E_{\rm obs}$, calculated using the beta functions of ref.s [29, 30], the value of $\alpha(M_Z)$ of [31] and the values of M_Z and $\sin\theta_W(M_Z)$ from Particle Data Group [32]. We neglect the running of α from $E_{\rm obs}$ to Λ_L, because it does not affect our estimates very much: indeed, $\alpha \sim 1/113$ at 10^{14}GeV, and $\alpha \sim 1/106$ at 10^{19}GeV. More details on this evaluation can be found in appendix A.

Writing $E_{\rm obs} = rE_{\rm lim}$, with $r > 1$, from (4.12) we get

$$n(r) = \frac{1}{\sqrt{1 - \frac{m^2 r^2}{E_{\rm obs}^2}}}.$$

The age of ultrahigh-energy cosmic rays cannot exceed

$$t_f(r) = t(n(r), E_{\rm obs}),$$

since when they were created they certainly had a finite energy. $t_f(r)$ is a decreasing function of r and tends to infinity for $r \to 1^+$. If ultrahigh-energy primaries are protons, it is easy to check that for $r^2 = 2$ and $\mu = 1$, for example, the time t_f is just $\sim 10^{-10}$ seconds, which means that the particle looses all its energy down to $E_{\rm obs}$ in a few centimeters. Since it certainly covers larger distances, we must have $r < \sqrt{2}$, therefore

$$n \sim 1 + \frac{r^2}{2} 10^{-23} < 1 + 10^{-23}.$$

Lowering r^2 does not improve this bound so much, so we do not need to struggle to make r as close as we can to 1 and $t_f(r)$ equal to the age of the Universe (or the time of some intergalactic travel). If ultrahigh-energy primaries are iron atoms we obtain the weaker bound

$$n < 1 + 3 \cdot 10^{-20},$$

and $t_f \sim 4 \cdot 10^{-14}$s.

In summary, for our purposes the energy loss is so rapid that we do not make any relevant mistake if we use (4.14).

$1/\Lambda_L$-corrections. Our model (3.16) predicts corrections to the results found above, which can be calculated expanding all the ingredients of calculation (matrix element, dispersion relations etc.) in powers of m/Λ_L. To illustrate integrability properties we consider $\mathrm{d}t/\mathrm{d}v$, instead of $\mathrm{d}v/\mathrm{d}t$. The first correction to $\mathrm{d}t/\mathrm{d}v$ is

$$\Delta \frac{\mathrm{d}t}{\mathrm{d}v} = \frac{3\mu v^2 (n^2-1)^2 (48 b'' n^4 (nv-1)^2 + b' P_2(v))}{\alpha \Lambda_L n^2 (nv-1)^4 P(v)^2 \sqrt{1-v^2}}, \tag{4.15}$$

where

$$P_2(x) = -3n^2(3n^4 + 8n^2 - 3)x^2 + 2n(23n^4+1)x - 25n^4 + 1$$

and v still stands for the uncorrected expression

$$v = \sqrt{1 - \frac{m^2}{E^2}}. \tag{4.16}$$

$\Delta(\mathrm{d}t/\mathrm{d}v)$ can be integrated analytically from $v = 1$ to any v_f greater than $v_{\rm lim}$; we do not report the lengthy result since it is not so useful. On the other hand, higher corrections to $\mathrm{d}t/\mathrm{d}v$ cannot be integrated around $v = 1$, because they contain factors $(1-v^2)^k$ with $k > 1$ in the denominator.

The $1/\Lambda_L$-corrections are meaningful if they are much smaller than the main effect, so an expansion in powers of $1/\Lambda_L$ is meaningful only if n is not too close to one, otherwise there is no radiation at the zeroth order. In this section we have assumed that the powers of $n-1$ are dominant, and we have seen that the energy loss is so rapid that the phenomenon is governed by pure kinematics, so corrections such as (4.15) are unnecessary. When n is equal to 1, or sufficiently close to 1, there is no radiation to the zeroth order, or almost none, and we cannot make a standard low-energy expansion. In the next sections we study the case when the $1/\Lambda_L$-effects are dominant.

4.4 Effects of higher space derivatives

The LVSM, of which LVQED (3.16) is a subsector, contains terms of higher dimensions. Under certain conditions those terms can cause Cherenkov radiation in vacuo even if n is exactly one or smaller than one; some of them can even cause the radiation of neutral particles. In this section we begin to study those effects. We first discuss the definition of Λ_L and present our work hypothesis. Then we study the kinematics of the Cherenkov process.

Definition of Λ_L. Each term of higher dimension contained in the LVSM can be used to define a scale of Lorentz violation. Normalizing dimensionless coefficients to one, we can write a term of this type as

$$\frac{1}{\Lambda_{iL}^{d_i-4}}\mathcal{O}^i$$

where $\mathcal{O}^i$ is a local operator of dimension $d_i > 4$ constructed with the fields and their derivatives and Λ_{iL} is an energy scale, which can be regarded as the scale of Lorentz violation associated with $\mathcal{O}^i$.

As far as we know, the values of such Λ_{iL}'s may significantly differ from one another. So we have to ask ourselves: which is *the* scale of Lorentz violation Λ_L? The answer that comes naturally is: the smallest Λ_{iL}, namely the smallest energy scale at which Lorentz violation may manifest itself. Since the LVSM contains a finite number of parameters, this definition is meaningful in our approach; nevertheless, at the moment it is a purely theoretical definition, because no sign of Lorentz violation has been observed so far.

Anyway, not all parameters of the LVSM are on the same footing from the theoretical point of view: most of them could be set to zero without affecting the consistency of the model; some parameters, on the other hand, must necessarily be nonzero, because they are crucial for renormalizability. These are the coefficients that multiply the quadratic terms of largest dimensions of each particle: the τ_0's for the gauge groups and the b_0's for fermions. In the model (3.16) the crucial terms are

$$-\frac{\tau_0}{4\Lambda_L^4}F_{ij}(-\bar{\partial}^2)^2F_{ij}, \qquad \frac{ib_0}{\Lambda_L^2}\bar{\psi}\bar{\not{D}}^3\psi, \tag{4.17}$$

while parameters such as τ_1, τ_2-1, b_1-1, etc. are not crucial.

We would like to set the noncrucial parameters to zero, to better isolate the effects of the crucial ones. In most cases this is compatible with renormalization, because the running of couplings is only logarithmic, and even if we need to cover a huge energy range, from, say, $E = M_Z$ to $E = \Lambda_L$, the factor $\ln(\Lambda_L/M_Z)$ is just a few tens, which can be easily beaten by the small numbers that are present everywhere in our game. It is also interesting to study cases where particular relations among parameters hold, but then the effects of renormalization on those relations need to be studied carefully. In the next section we provide explicit examples. To summarize, the parameters of the Lorentz violating extended Standard Model can be arranged according to a hierarchy of conceptual importance, which may or may not correspond to a hierarchy of magnitude. We take it as a work hypothesis to organize our analysis. We assume that the absolute values of the non-crucial parameters are as small as possible, and concentrate on the crucial ones.

The values of the crucial parameters themselves can significantly differ from one another. The largest of them defines Λ_L. For example, if the scale of Lorentz violation Λ_L is defined by the crucial term belonging to the photon sector, namely

$$-\frac{1}{4}F_{ij}\frac{(-\bar{\partial}^2)^2}{\Lambda_L^4}F_{ij}, \tag{4.18}$$

then we can set $\tau_0 = 1$ for the photon, and assume that all other τ_0's, and the b_0's, are not greater than 1. This choice sounds reasonable, indeed, because the photon sector contains the best measured parameters among those multiplying operators of higher dimensions [2]. Under these assumptions, our plan is to study how small the parameters b_0's have to be to explain data, in particular ultrahigh-energy cosmic rays.

In the rest of this section we study the kinematics of a large class of dispersion relations. In particular, we study the threshold for Cherenkov radiation and the range of frequencies of the emitted photon.

General kinematics. As before, p denotes the momentum of the incoming fermion, p' the momentum of the fermion after the emission of the photon, k is the momentum of the emitted photon, θ is the angle between the trajectory of the incoming fermion and the photon and $u = \cos\theta$.

We consider a general dispersion relation, only assuming that at $p, k \neq 0$ the dispersion relations $E(p)$ and $\omega(k)$ are non-negative, have positive first derivatives (namely velocities are always positive) and non-negative second derivatives, and that at least one dispersion relation is convex:

$$E \geqslant 0, \qquad \omega \geqslant 0, \qquad \frac{\mathrm{d}E}{\mathrm{d}p} > 0, \qquad \frac{\mathrm{d}\omega}{\mathrm{d}k} > 0, \qquad \frac{\mathrm{d}^2E}{\mathrm{d}p^2} > 0, \qquad \frac{\mathrm{d}^2\omega}{\mathrm{d}k^2} \geqslant 0. \tag{4.19}$$

These properties are obeyed by the usual relativistic and non-relativistic dispersion relations and appear realistic for any reasonable dispersion relation. In relativistic dispersion relations convexity holds any time the mass is non-vanishing.

Energy and momentum conservations imply

$$E(p) = \omega(k) + E(p'), \qquad p' = \sqrt{p^2 + k^2 - 2pku}. \tag{4.20}$$

The condition (4.20) is involved, but some inequalities that are useful for the calculation can be derived straightforwardly. For example, we have

$$k < 2p.$$

This information is quite redundant (the precise k-range is determined below), but enough for the moment. It can be proved observing that $E(p) - E(p') \geqslant 0$ implies $p \geqslant p'$, by the monotonicity of $E(p)$, while $k \geqslant 2p$ would give $p' \geqslant p$ (using $u \leqslant 1$).

Next, consider the condition of energy conservation (4.20) in the (k, p')-plane and call its solution $p'(k)$. For a given k the equation for p' reads $E(p')$=constant. Since the function $E(p')$ is monotonic, the solution $p'(k)$, when it exists, is unique. Second, $p' = |\mathbf{p} - \mathbf{k}|$ and $p' \leqslant p$ tell us that we must focus on the region

$$|p - k| \leqslant p' \leqslant p.$$

Third, at $k = 0$, $p' = p$ is a solution of (4.20), so $p'(0) = p$.

Finally, $p'(k)$ is monotonically decreasing and concave. These properties are proved differentiating (4.20) with respect to k once and twice and using the hypotheses (4.19): we find, at $k \neq 0$,

$$\frac{\mathrm{d}p'}{\mathrm{d}k} = -\frac{\mathrm{d}\omega}{\mathrm{d}k}\left(\left.\frac{\mathrm{d}E}{\mathrm{d}p}\right|_{p'}\right)^{-1} < 0, \qquad \frac{\mathrm{d}^2p'}{\mathrm{d}k^2} = -\left[\frac{\mathrm{d}^2\omega}{\mathrm{d}k^2} + \left.\frac{\mathrm{d}^2E}{\mathrm{d}p^2}\right|_{p'}\left(\frac{\mathrm{d}p'}{\mathrm{d}k}\right)^2\right]\left(\left.\frac{\mathrm{d}E}{\mathrm{d}p}\right|_{p'}\right)^{-1} < 0.$$

Using these pieces of information, we can draw the picture in Fig. 4.4. We see that a non-trivial range of solutions exists if and only if the first derivative of $p'(k)$ at $k = 0$ is smaller than one in modulus, namely

$$-\left.\frac{\mathrm{d}p'}{\mathrm{d}k}\right|_0 < 1 \qquad \text{or, equivalently,} \qquad \left.\frac{\mathrm{d}\omega}{\mathrm{d}k}\right|_0 < \frac{\mathrm{d}E}{\mathrm{d}p}, \tag{4.21}$$

which means that the velocity of the charged particle must be greater than a certain threshold determined by the photon dispersion relation, as in the usual case. Moreover, the k-range is the segment

$$0 \leqslant k \leqslant k_{\max}(p), \tag{4.22}$$

where $k_{\max}(p)$ is the solution of $p'(k_{\max}) = |p - k_{\max}|$, namely it is obtained from the forward emission $u = 1$.

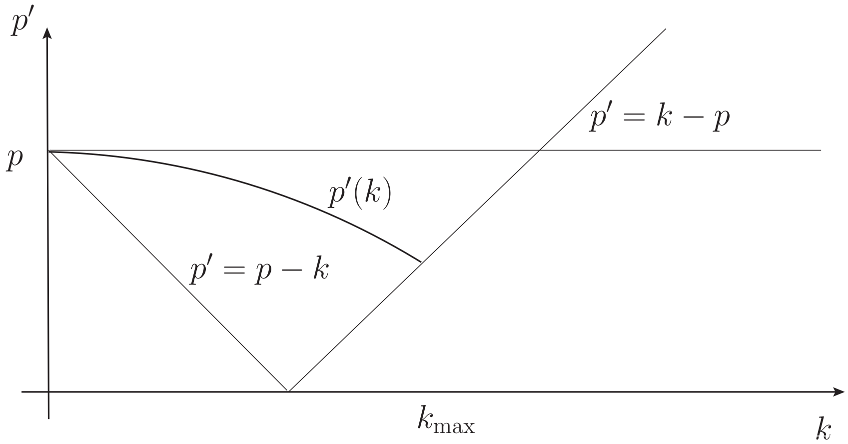

Figure 4.2: Allowed region for the solution $p'(k)$ of the general energy-momentum conservation equation.

Observe that the condition (4.21) does not depend on most parameters of $\omega(k)$. When the dispersion relations are those in eq. (4.1), the condition (4.21) does not depend on τ_0 and τ_1, but only on τ_2 and the parameters of the fermion dispersion relation.

4.5 Typical scenarios

In this section we study two scenarios and their compatibility with the observation of ultrahigh-energy cosmic rays and other experimental data. Our purpose is to show that there exist reasonable scenarios where the scale of Lorentz violation can be smaller than the Planck scale.

From the propagators given in (3.17), we can derive the following sums over polarization states, to be used in formula (4.6):

$$\begin{aligned}
&\sum_\lambda \varepsilon_\mu^{(\lambda)} \varepsilon_\nu^{(\lambda)*} = \mathrm{diag}\left(-\omega^2(\bar{k}^2)/\bar{k}^2, \mathbb{1}\right) \\
&\sum_s u_s(p)\bar{u}_s(p) = \not{p} + m + \bar{\not{p}}\left(b_1 - 1 + \frac{b_0}{\Lambda_L^2}\bar{p}^2\right) + \frac{b'}{\Lambda_L}\bar{p}^2 \\
&\sum_s v_s(p)\bar{v}_s(p) = \not{p} - m + \bar{\not{p}}\left(b_1 - 1 + \frac{b_0}{\Lambda_L^2}\bar{p}^2\right) - \frac{b'}{\Lambda_L}\bar{p}^2.
\end{aligned} \tag{4.23}$$

4.5.1 First scenario

As a first example we consider the case in which all non-crucial parameters are set to be zero, keeping only the highest dimension terms, essential for renormalizability:

$$\tau_2 = 1, \qquad \tau_1 = 0, \qquad b_1 = 1, \qquad b' = 0, \qquad b_0 > 0. \tag{4.24}$$

Substituting (4.24) in (4.1), the dispersion relations reduce to[1]

$$E(p^2) = \sqrt{m^2 + p^2 \left(1 + \frac{b_0 p^2}{\Lambda_L^2}\right)^2}, \qquad \omega(k^2) = \sqrt{k^2 + \tau_0 \frac{(k^2)^3}{\Lambda_L^4}}. \tag{4.25}$$

The inequality $b_0 > 0$ is assumed in order to ensure that $E(p^2)$ is monotonic. Note that in this case the refractive index n would be 1, but due to higher derivative corrections Cherenkov radiation is still allowed. The condition (4.21) is fullfilled if

$$\xi^2 \equiv \frac{m^2 \Lambda_L^2}{6 b_0 p^4} < \left(1 + \frac{b_0 p^2}{\Lambda_L^2}\right)^2 \left(1 + \frac{3 b_0 p^2}{2\Lambda_L^2}\right). \tag{4.26}$$

We are interested in the case

$$m \ll p \ll \Lambda_L, \tag{4.27}$$

which can help us solve the kinematic constraints in an approximate way. Specifically, we have

$$p \leqslant 3 \cdot 10^{11}\text{GeV}, \qquad \Lambda_L \geqslant 10^{14}\text{GeV}.$$

Within our approximation the right-hand side of (4.26) is practically 1, so the condition for the emission of Cherenkov radiation is

$$\xi < 1,$$

which can also be expressed as an energy threshold, namely

$$E > E_{\text{lim}} \sim \frac{m^{1/2} \Lambda_L^{1/2}}{6^{1/4} b_0^{1/4}}.$$

A particle above threshold radiates and looses energy till it reaches the limit value E_{lim}. When $\Lambda_L \to \infty$ at m and p fixed, the condition (4.20) admits no solution, because it reduces to the kinematic relation of the Lorentz-invariant theory. It is more convenient to study the limit $\Lambda_L \to \infty$ at ξ and p fixed, because in such a limit (4.20) becomes

$$p = k + \sqrt{p^2 + k^2 - 2pku},$$

so its solution is $u = 1$, $k \leqslant p$. For $\Lambda_L < \infty$ we can find an approximate solution of the form

$$u = 1 - \varepsilon, \qquad 0 < \varepsilon \ll 1,$$

with k belonging to a certain range of values that has to be worked out. Expanding in ε and p/Λ_L around the solution at $\Lambda_L = \infty$, we find

$$\varepsilon = \frac{b_0}{pk\Lambda_L^2} \left\{ (p-k) \left[p^3 - (p-k)^3 \right] - 3\xi^2 p^3 k \right\}. \tag{4.28}$$

[1] Here p and k stand for $\bar{p}$ and $\bar{k}$, respectively, as in section 4.4.

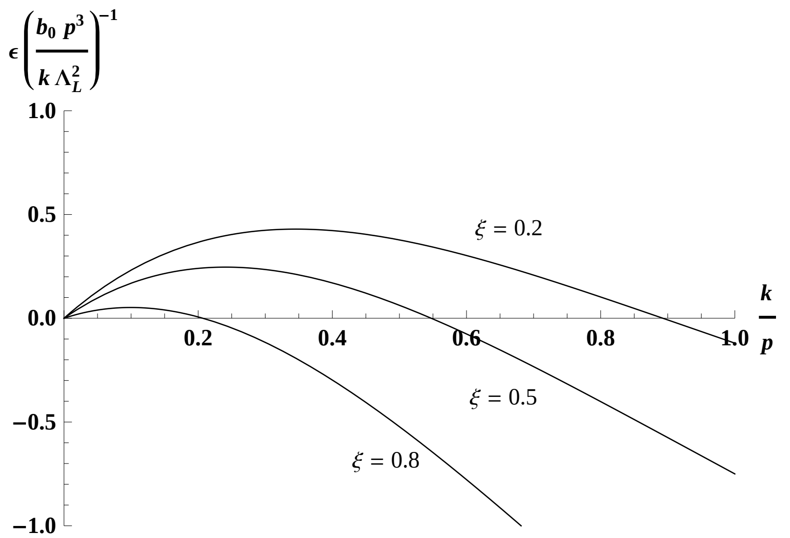

Figure 4.3: The plot represents the sign of ε, eq. (4.28), for various values of ξ. The region where ϵ is positive determines the k-range, $0 \leqslant k \leqslant k_{\max}$.

We see that ε is indeed much smaller than one, as needed for consistency; the k-range can be found from the condition $\varepsilon \geqslant 0$, which, by the way, requires again $b_0 > 0$. Plotting the function appearing in (4.28) it is easy to show that for $\xi < 1$ a range of the form (4.22) exists and has $k_{\max} < p$ (see Fig. 4.3). The energy losses (4.7) can be worked out starting from the differential width (4.6). For the analysis of ultrahigh-energy cosmic rays it is sufficient to consider the situations $\xi^2 \ll 1$ and $1 - \xi^2 \ll 1$. For $\xi^2 \ll 1$ we obtain the range

$$0 \leqslant k \leqslant p(1 - 3\xi^2)$$

and the energy loss

$$\left.\frac{\mathrm{d}E}{\mathrm{d}t}\right|_{\xi^2 \ll 1} = -\frac{\alpha p^4}{\Lambda_L^2}\left(\frac{11 b_0}{12} + \frac{2b''}{5}\right), \tag{4.29}$$

which neglects contribution of order ξ. For $1 - \xi^2 \ll 1$ we obtain the range

$$0 \leqslant k \leqslant \frac{p}{2}(1 - \xi^2),$$

and the energy loss

$$\left.\frac{\mathrm{d}E}{\mathrm{d}t}\right|_{1-\xi^2 \ll 1} = -\frac{\alpha p^4 \left(1 - \xi^2\right)^3}{4\Lambda_L^2} b_0. \tag{4.30}$$

Formula (4.29) depends also on b''; however, in estimating numerical results one can set eventually $b'' = 0$, since b'' is not in the list of crucial parameters.

Recall that, since we have used the approximation (4.27), we cannot use (4.29) and (4.30) above $E = \Lambda_L$. As in the case of QED in a medium, the radiating particle takes an infinite amount of time to reach the energy limit. For our purposes it is sufficient to calculate the time the particle takes to radiate from energy Λ_L to, say, 1.3-1.1$E_{\rm lim}$. It is not meaningful to approach the energy limit further, since the energies we are considering are not measured so precisely.

Now we apply our results to ultrahigh-energy cosmic rays. If $\Lambda_L = 10^{14}$GeV, protons of $3 \cdot 10^{11}$GeV emit Cherenkov radiation if $b_0 > 1.8 \cdot 10^{-19}$. If $\Lambda_L = 10^{14}$GeV and $b_0 = 1.8 \cdot 10^{-19}$ we can use (4.29), with b'' set to zero, as long as ξ is small, for example down to $2E_{\rm lim}$ ($\xi^2 = 1/16$). The time spent to radiate from Λ_L to $2E_{\rm lim}$ is

$$t' \sim 7 \cdot 10^{-12}\text{s}.$$

When the energy approaches further $E_{\rm lim}$ we have to use (4.30). The particle radiates from energy $2E_{\rm lim}$ to $1.1E_{\rm lim}$ in about

$$t'' \sim 8 \cdot 10^{-10}\text{s}.$$

The radiation time $t_f = t' + t''$ is too short to be compatible with the observation of ultrahigh-energy cosmic rays. Therefore, as in section 4.3, we may assume that the energy loss down to $E_{\rm lim}$ occurs instantaneously any time it is allowed by kinematics.

With the same magnitude of Λ_L, larger values of b_0 would give smaller energy tresholds and shorter radiation times. For example, if $b_0 \sim 1$ and $\Lambda_L = 10^{14}$GeV particles would radiate down to $E_{\rm lim} = 6 \cdot 10^6$GeV in a very short time. In fact, integrating (4.29) from Λ_L to $E_{\rm obs} = 3 \cdot 10^{11}$GeV, when we see the particle, we obtain

$$t_{\rm obs} \sim 10^{-29}\text{s},$$

while continuing down to $1.1E_{\rm lim}$ we have to use both (4.29) and (4.30), and get

$$t_f \sim 2 \cdot 10^{-14}\text{s}.$$

This means that, with these values of parameters, we could not observe high-energy cosmic rays coming from large distance. In order to forbid the radiation of $3 \cdot 10^{11}$GeV protons with $b_0 \simeq 1$, it would be necessary to have $\Lambda_L = 2.4 \cdot 10^{23}$GeV, well beyond the Planck scale. Thus, if we take $\Lambda_L = 10^{14}$GeV only values of $b_0 \leqslant 1.8 \cdot 10^{-19}$ are consistent with data, if we consider cosmic rays as protons.

The limiting value on b_0 can be raised increasing Λ_L; for various values of Λ_L the bounds on b_0 are collected in the first two lines of table 4.1; when Λ_L is varied between 10^{14}GeV and the Planck scale the radiation time t_f does not change very much.

If the ultrahigh-energy cosmic rays are iron atoms we obtain greater values of b_0 in corrispondence with equal values of Λ_L, therefore a better situation, from our point of view. These bounds are summarized in the second two lines of tabel 4.1.

We have also considered a variant of (4.24), with $\tau_1 = 2\sqrt{\tau_0}$ instead of $\tau_1 = 0$ (see appendix B). The radiation times are still too short and the threshold condition is exactly the same, therefore

Table 4.1: Bounds on b_0 for various values of Λ_L; in the first two lines for $m = 0.938$GeV, in the second two for $m = 55.8u$, in the first scenario.

Λ_L	10^{14}GeV	10^{15}GeV	10^{16}GeV	10^{17}GeV	10^{19}GeV	$2.4 \cdot 10^{23}$GeV
b_0	$1.8 \cdot 10^{-19}$	$1.8 \cdot 10^{-17}$	$1.8 \cdot 10^{-15}$	$1.8 \cdot 10^{-13}$	$1.8 \cdot 10^{-9}$	1
Λ_L	10^{14}GeV	10^{15}GeV	10^{16}GeV	10^{17}GeV	10^{19}GeV	$4.2 \cdot 10^{21}$GeV
b_0	$5.6 \cdot 10^{-16}$	$5.6 \cdot 10^{-14}$	$5.6 \cdot 10^{-12}$	$5.6 \cdot 10^{-10}$	$5.6 \cdot 10^{-6}$	1

table 4.1 remains unchanged. The bounds we have found on the crucial parameter b_0 are very small; however, they can be raised taking into account of compositeness. Before that, we study a second scenario with different dispersion relations.

4.5.2 Second scenario

The procedure just used is quite general, and can be applied to examine other cases. We illustrate a second scenario taking the dispersion relations

$$E(p^2) = \sqrt{m^2 + p^2 + b_0^2 \frac{(p^2)^3}{\Lambda_L^4}}, \qquad \omega(k^2) = \sqrt{k^2 + \tau_0 \frac{(k^2)^3}{\Lambda_L^4}}, \tag{4.31}$$

for the fermion energy and photon frequency. Here we are assuming that the parameters of the Lagrangian (3.16) satisfy

$$b_0 = -\frac{b'^2}{2b_1}, \qquad b_1 = \sqrt{1 - 2\frac{mb'}{\Lambda_L}}, \qquad \tau_2 = 1, \qquad \tau_1 = 0, \tag{4.32}$$

namely they are such that only the highest powers of momentum, which are the crucial ones for renormalization, correct the relativistic dispersion relations.

Here the condition for emission obtained from (4.21) is

$$\xi^2 \equiv \frac{\Lambda_L^4 m^2}{5b_0^2 p^6} < 1 + \frac{9b_0^2 p^4}{5\Lambda_L^4}, \tag{4.33}$$

using a more convenient definition of ξ. In the range of our interest, $m \ll p \ll \Lambda_L$, (4.33) reduces again to the approximate condition

$$\xi < 1,$$

while the limit energy is

$$E_{\text{lim}} = \frac{\Lambda_L^{2/3} m^{1/3}}{5^{1/6} |b_0|^{1/3}}.$$

On the same path of the previous subsection, we set $u = 1 - \varepsilon$ (being $u = 1$ the solution in the limit $\Lambda_L \to \infty$ at ξ fixed) and solving the energy-momentum conservation equation we find

$$\varepsilon = \frac{b_0^2}{2pk\Lambda_L^4} \left\{ (p-k) \left[p^5 - (p-k)^5 - \frac{k^5}{\zeta} \right] - 5\xi^2 p^5 k \right\},$$

having defined

$$\zeta \equiv b_0^2/\tau_0.$$

By imposing $\varepsilon \geqslant 0$, we obtain the k-range; the mainly interesting situation are $\zeta \ll 1$ or $\zeta \sim 1$, which we analyse in detail. For $\zeta \ll 1$ we have to consider two cases, in which we find the following results

$$0 \leqslant k \leqslant p(5\zeta(1-\xi^2))^{1/4}, \quad \text{for} \quad \zeta \ll (1-\xi^2)^3$$

and

$$0 \leqslant k \leqslant \frac{p}{3}(1-\xi^2), \quad \text{for} \quad \zeta \gtrsim (1-\xi^2)^3.$$

Then we calculate, to the lowest order in $1/\Lambda_L$ and ζ (at fixed τ_0), the energy losses

$$\left.\frac{\mathrm{d}E}{\mathrm{d}t}\right|_{\zeta \ll 1, \zeta \ll (1-\xi^2)^3} = -\frac{5\alpha(1-\xi^2)\zeta^{3/2}\sqrt{\tau_0}p^4}{4\Lambda_L^2}, \tag{4.34}$$

$$\left.\frac{\mathrm{d}E}{\mathrm{d}t}\right|_{\zeta \ll 1, \zeta \gtrsim (1-\xi^2)^3} = -\frac{\alpha(1-\xi^2)^4\sqrt{\zeta\tau_0}p^4}{324\Lambda_L^2}. \tag{4.35}$$

In these formulas we have already set $b'' = 0$. Instead b' must be kept, because it is related to b_0 from (4.32). Note that since $b_1 \sim 1$, b_0 must be negative; thus we have $b'^2 \sim -2b_0 = 2\sqrt{\zeta\tau_0}$.

If $\zeta = 1$ we need to distinguish the cases $\xi^2 \ll 1$ and $1 - \xi^2 \ll 1$, obtaining the k-ranges

$$0 \leqslant k \leqslant p(1-\xi) \quad \text{for} \quad \zeta = 1 \quad \text{and} \quad \xi^2 \ll 1$$

and

$$0 \leqslant k \leqslant \frac{p}{3}(1-\xi^2) \quad \text{for} \quad \zeta = 1 \quad \text{and} \quad \xi^2 \sim 1.$$

The corrispondent energy losses (for $b'' = 0$) are respectively

$$\left.\frac{\mathrm{d}E}{\mathrm{d}t}\right|_{\zeta = 1, \xi^2 \ll 1} = -\frac{\alpha p^4 |b_0|}{20\Lambda_L^2}, \tag{4.36}$$

$$\left.\frac{\mathrm{d}E}{\mathrm{d}t}\right|_{\zeta = 1, \xi^2 \sim 1} = -\frac{\alpha(1-\xi^2)^4|b_0|p^4}{324\Lambda_L^2}. \tag{4.37}$$

With $\Lambda_L = 10^{14}$GeV, protons of $3 \cdot 10^{11}$GeV emit Cherenkov radiation if $|b_0| > 1.6 \cdot 10^{-7}$. We take $\tau_0 = 1$, which means that we assume that the scale of Lorentz violation Λ_L is defined by the photon sector, precisely by the first term of (4.17). If we take $b_0 = -1.6 \cdot 10^{-7}$ the approximation

$\zeta \ll (1-\xi^2)^3$ holds in the entire energy range from Λ_L down to $1.1E_{\rm lim}$, so we do not need to use (4.35), and we can just integrate (4.34). It can be integrated exactly; we obtain

$$E^3(t)\big|_{\zeta\ll 1,\zeta\ll(1-\xi^2)^3} = E^3_{\rm lim}\frac{\Lambda_L^3\cosh(\kappa t)+E^3_{\rm lim}\sinh(\kappa t)}{\Lambda_L^3\sinh(\kappa t)+E^3_{\rm lim}\cosh(\kappa t)}, \tag{4.38}$$

where

$$\kappa = \frac{3\sqrt{5}}{4}\alpha m\zeta.$$

and the initial condition is fixed setting $E(0)=\Lambda_L$ (recall that our calculations are valid only for energies below Λ_L). Formula (4.38) allows us to define a radiation time in a familiar way, since it contains only exponentials; it can be inverted and then approximate assuming $\Lambda_L \gg E_{\rm lim}$ (which is true in the cases studied here), giving

$$\kappa t \simeq \tanh^{-1}(E^3_{\rm lim}/E^3). \tag{4.39}$$

Therefore the radiation time is

$$t_f|_{\zeta\ll 1,\zeta\ll(1-\xi^2)^3} \sim \frac{1}{\kappa}, \tag{4.40}$$

which is approximately the time taken to reach the energy $\sim 1.1E_{\rm lim}$. Then the typical radiation time of protons above threshold is

$$t_f \sim 2\cdot 10^{-9}{\rm s}.$$

During this time the particle looses most of its energy, then it continues radiating slowly; only values of $|b_0|$ equal or smaller than $1.6\cdot 10^{-7}$ are consistent with observations. In fact, at fixed Λ_L larger values of b_0 give smaller t_f's and lower tresholds. For example, if $|b_0|\sim 1$ and $\Lambda_L = 10^{14}$GeV a proton of energy $E_{\rm obs} = 3\cdot 10^{11}$GeV has $\xi \sim 10^{-7}$, and it keeps radiating down to $E_{\rm lim} \sim 1.6\cdot 10^9$GeV. The time it spends to radiate from energy Λ_L to the observed energy $E_{\rm obs} = 3\cdot 10^{11}{\rm GeV} \gg E_{\rm lim}$ can be calculated using formula (4.36). We find

$$t_f|_{\zeta=1,\xi^2\ll 1} \sim \frac{40\Lambda_L^2}{3\alpha b'^2 E_f^3}.$$

Numerically, taking $b_0=-1$ and $b'\sim\sqrt{2}$, we have

$$t_{\rm obs} \sim 2\cdot 10^{-28}{\rm s}.$$

After this time, the cosmic rays keep radiating till they reach the limit energy $E_{\rm lim} = 1.6\cdot 10^9$GeV. We can use formula (4.36) as long as ξ^2 is small, for example down to $2E_{\rm lim}$ ($\xi^2 = 1/64$). The time spent to radiate from Λ_L to $2E_{\rm lim}$ is

$$t' \sim 1.6\cdot 10^{-22}{\rm s}.$$

Table 4.2: Bounds on $|b_0|$ for various values of Λ_L, for $m = 0.938$GeV, in the second scenario.

Λ_L	10^{14}GeV	10^{15}GeV	10^{16}GeV	10^{17}GeV	$2.5 \cdot 10^{17}$GeV	10^{18}GeV
$\lvert b_0\rvert$	$1.6 \cdot 10^{-7}$	$1.6 \cdot 10^{-5}$	$1.6 \cdot 10^{-3}$	0.16	1	16

When the energy approaches $E_{\rm lim}$ we have to use formula (4.37). The particle radiates from energy $2E_{\rm lim}$ to $1.1E_{\rm lim}$ during

$$t'' \sim 6 \cdot 10^{-20}\text{s},$$

which is still very short.

Summarizing, we may assume that the energy loss is instantaneous, so only the values $|b_0| \leqslant 1.6 \cdot 10^{-7}$ are consistent with data at $\Lambda_L = 10^{14}$GeV. The limiting value of $|b_0|$ can be raised increasing Λ_L and becomes 1 for $\Lambda_L = 2.5 \cdot 10^{17}$GeV; this means that for $|b_0| \sim 1$ and $\Lambda_L \geq 2.5 \cdot 10^{17}$GeV, protons of $3 \cdot 10^{11}$GeV do not emit Cherenkov radiation and can reach the earth. In this case, with the same procedure as above, from (4.36) and (4.37), we find that protons above treshold have a radiation time $t_f = t' + t'' \simeq 6 \cdot 10^{-20}$s. Bounds on $|b_0|$ for different values of Λ_L are collected in table 4.2.

Again, if the ultrahigh-energy cosmic rays are instead iron atoms their observation can be explained with lower values of Λ_L. For example, with $|b_0| \sim 1$, $|b'| \sim \sqrt{2}$ Λ_L can be as small as $3.4 \cdot 10^{16}$GeV, or with $|b_0| \sim 9 \cdot 10^{-6}$, $|b'| \sim 4 \cdot 10^{-3}$, Λ_L can be lowered to 10^{14}GeV.

Since we have assumed that relations among parameters hold (see (4.32)), we must check the compatibility of b' and b_1 with present data. Using $b_1 \sim 1$, $p \ll \Lambda_L$ and eq.s (4.32), (4.33), for a cosmic ray of mass m and momentum p which is known not to radiate, we get

$$-\frac{\Lambda_L^2 m}{\sqrt{5}p^3} \leqslant b_0 < 0, \qquad |b'| \leqslant \left(\frac{4}{5}\right)^{\frac{1}{4}} \frac{\Lambda_L m^{1/2}}{p^{3/2}}, \qquad 1 - \left(\frac{4}{5}\right)^{\frac{1}{4}} \left(\frac{m}{p}\right)^{\frac{3}{2}} \leqslant n \leqslant 1 + \left(\frac{4}{5}\right)^{\frac{1}{4}} \left(\frac{m}{p}\right)^{\frac{3}{2}}. \tag{4.41}$$

$n = b_1/\sqrt{\tau_2}$ is the refractive index of the vacuum "as seen by the proton". Observe that the bound on the refractive index is independent of Λ_L, so it cannot be improved changing the scale of Lorentz violation.

Is it reliable to accept for $|b_0|$ the largest bound as possible $|b_0| = \Lambda_L^2 m/(\sqrt{5}p^3)$? In fact, since the three inequalities (4.41) are equivalent to one another, if $|b_0|$ is not small enough, n may be too far from one, which may contradict existing bounds. We search for the largest $|b_0|$ compatible with data, considering that the highest-energy cosmic ray observed up to now has a momentum of $3 \cdot 10^{11}$GeV, and it is supposed to be a proton.

If $b' > 0$ we find $1 - 5 \cdot 10^{-18} \leqslant n < 1$; at present no bounds contradict this range [2]. Instead, if $b' < 0$ we have $1 < n \leqslant 1 + 5 \cdot 10^{-18}$. In this case, a more stringent bound exists in literature,

$n < 1 + 6 \cdot 10^{-20}$ [26, 2]. If it has been derived independently, it would have forced us to choose a maximum value of n below our limiting value on the right-hand side of the third formula in (4.41). As a consequence we should have assumed a maximum acceptable value of $|b_0|$ lower than the largest possible one $|b_0| = \Lambda_L^2 m/(\sqrt{5}p^3)$. However, this is not the case, because the cited bound on n is not derived in an independent way, but it is based on ultrahigh-energy cosmic rays themselves, which we are explaining with a different approach; so it cannot be applied here. Thus, the largest $|b_0|$ we can take is given by

$$b_0 = -\frac{\Lambda_L^2 m}{\sqrt{5}p^3}.$$

Now we have to discuss the consistency of the dispersion relations (4.31) with renormalization. The first condition of (4.32) demands that the combination

$$\epsilon \equiv 2b_1 b_0 + b'^2$$

vanishes, in order to ensure that the dispersion relations do not contain terms proportional to the fourth power of momentum. We have to ask ourselves if or how much such condition is preserved under renormalization. A typical case with ϵ different from zero is the first scenario already studied, thus we can take as a reference the typical values of b_0 of the first scenario to understand how small ϵ has to be to make the second scenario meaningful and consistent. We can use the b_0-bounds of table 4.1; for example, for protons in the first scenario $\epsilon \sim 4 \cdot 10^{-19}$ at $\Lambda_L = 10^{14}$GeV. Instead, the results found in the second scenario tell us $2b_1 b_0 \sim b'^2 \sim 3.2 \cdot 10^{-7}$, which is 12 orders of magnitude larger, unless cancellations occur.

We may assume that the relations (4.32) are valid at the scale Λ_L, or anyway just at one energy scale, but the scale we need to work with is $E_{\text{obs}} = 3 \cdot 10^{11}$GeV. The b_0- and b'-runnings contain, among the others, terms proportional to

$$\alpha b_0 \ln \frac{\Lambda_L}{E_{\text{obs}}}, \qquad \alpha b' \ln \frac{\Lambda_L}{E_{\text{obs}}}.$$

So, assuming that the cancellation $\epsilon = 0$ occurs at Λ_L it will not necessarily occur at E_{obs}, where instead we find

$$\epsilon \sim \alpha b_0 \ln \frac{\Lambda_L}{E_{\text{obs}}},$$

which is about 1/10 of b_0. Then, the cancellation covers one order of magnitude, not the 12 needed; therefore, renormalization forces us to take values of b_0 much smaller than the ones given in table 4.2. Precisely, we can take bounds just a factor

$$\frac{1}{\alpha \ln \frac{\Lambda_L}{E_{\text{obs}}}} \tag{4.42}$$

larger than the bounds of table 4.1, improving the first scenario by about one order of magnitude.

4.6 Composite particles

In the previous section we have used the dispersion relations predicted by our models (2.15) and (3.16) for elementary particles, but we have applied them to composite particles, such as protons and iron atoms. In this section we investigate the dispersion relations of composite particles and discuss some phenomenological consequences. In particular, we show that in composite particles lower values of b_0 are favored.

We consider the simplest picture, where all constituents move with the same velocity $\mathbf{v}$. We first keep only the crucial parameters, namely we assume that the constituents are elementary particles with dispersion relations

$$E_i = |\mathbf{p}_i|\sqrt{1+\left(\frac{\eta_i^2\mathbf{p}_i^2}{\Lambda_L^2}\right)^{n-1}}. \tag{4.43}$$

Their velocities are

$$\mathbf{v}_i = \frac{\mathrm{d}E_i}{\mathrm{d}\mathbf{p}_i} = \frac{\mathbf{p}_i}{E_i}\left(1+n\left(\frac{\eta_i^2 p_i^2}{\Lambda_L^2}\right)^{n-1}\right), \tag{4.44}$$

where $p_i = |\mathbf{p}_i|$. Setting $\mathbf{v}_i = \mathbf{v}$ for every i it is easy to derive the dispersion relation of the composite particle. Calling

$$x_i = \left(\frac{\eta_i^2 p_i^2}{\Lambda_L^2}\right)^{n-1}$$

and squaring (4.44), we get the equations

$$v^2(1+x_i) = (1+nx_i)^2.$$

Their solutions are

$$x_i = \frac{v^2-2n+v\sqrt{v^2+4n(n-1)}}{2n^2} \equiv x(v).$$

(It is easy to check that the other solution of the quadratic equation is not acceptable). Then we have

$$p_i = \frac{x^{1/(2n-2)}}{\eta_i}\Lambda_L, \qquad \mathbf{p}_i = \mathbf{v}\frac{E_i}{1+nx}, \qquad E_i = \frac{x^{1/(2n-2)}}{\eta_i}\Lambda_L\sqrt{1+x},$$

and therefore the total momentum and total energy are

$$\mathbf{P} = \sum_i \mathbf{p}_i = \mathbf{v}\frac{E}{1+nx}, \qquad E = \sum_i E_i = \frac{x^{1/(2n-2)}}{\eta}\Lambda_L\sqrt{1+x},$$

where η is defined by

$$\frac{1}{\eta} = \sum_i \frac{1}{\eta_i}. \tag{4.45}$$

Moreover, since $\mathbf{p}_i = \mathbf{v}p_i/v$, we have also

$$P = \sum_i p_i = \frac{x^{1/(2n-2)}}{\eta}\Lambda_L, \qquad x = \left(\frac{\eta^2\mathbf{P}^2}{\Lambda_L^2}\right)^{n-1}.$$

Thus, we find that E and $\mathbf{P}$ are related by the collective dispersion relation

$$E = |\mathbf{P}|\sqrt{1 + \left(\frac{\eta^2\mathbf{P}^2}{\Lambda_L^2}\right)^{n-1}},$$

which has the same form as the dispersion relations (4.43) of the constituents.

The crucial result is the composition rule (4.45), which states that "the weakest wins", namely if one constituent has a $\eta_i \equiv \bar{\eta}_i$ much smaller than the η_i's of the other constituents, then the composite particle has a η practically equal to $\bar{\eta}_i$. For $n = 0$, $m_i = \Lambda_L/\eta_i$, we get the dispersion relation of relativistic theories, with the usual composition rule for the mass, namely $\sum_i m_i = \sum_i \Lambda_L/\eta_i = \Lambda_L/\eta = M$.

The result just found can be extended to more general dispersion relations of the form

$$E_i = |\mathbf{p}_i| f\left(x_i\right), \qquad x_i = \left(\frac{\eta_i^2\mathbf{p}_i^2}{\Lambda_L^2}\right)^{n-1}. \tag{4.46}$$

Squaring the velocities

$$\mathbf{v}_i = \frac{\mathrm{d}E_i}{\mathrm{d}\mathbf{p}_i} = \frac{\mathbf{p}_i}{|\mathbf{p}_i|}\left(f + 2(n-1)x_i f'\right)$$

and equating them to $\mathbf{v}$, we get the equations

$$v^2 = \left(f(x_i) + 2(n-1)x_i f'(x_i)\right)^2.$$

Let us assume that the solution is unique, $x_i = x(v)$. Then, proceeding as above, we easily find that E and $\mathbf{P}$ are related by the collective dispersion relation

$$E = |\mathbf{P}|f(x), \qquad x = \left(\frac{\eta^2\mathbf{P}^2}{\Lambda_L^2}\right)^{n-1},$$

where η is still given by (4.45). Again, the dispersion relation of the composite particle has the same form as the dispersion relations of its constituents.

Although the procedure just outlined is general, few dispersion relations can be treated so simply. More complicated relations generate polynomial equations of high degree, and the dispersion relation of the composite particle does not have the form of the dispersion relations of its constituents. To convince oneself of this, it is sufficient to repeat the derivation adding mass terms to (4.43) and (4.46). Yet, masses are important for the Cherenkov effect, because they determine the energy threshold. To apply our results to ultrahigh-energy cosmic rays we argue as follows.

The dispersion relations (4.43) and (4.46) are good approximations at high energies, namely when the Lorentz violating corrections start to become important and the mass becomes negligible with respect to them. These are precisely the energies above threshold. Indeed, the emission of radiation is the first effect of the Lorentz violation in the phenomenon we are considering. Instead, at energies much smaller than the threshold the Lorentz violating corrections become negligible with respect to the mass, and the usual relativistic dispersion relation $E = \sqrt{M^2 + p^2}$ holds, where M is the mass of the composite particle. The full dispersion relation of the composite particle can be well approximated pasting the low- and high-energy dispersion relations.

Now, consider ultrahigh-energy cosmic rays. In our model, setting all non-crucial parameters but the mass to zero as in (4.24), or relating the parameters as in (4.32), at high energies quarks have dispersion relations (4.46) with $n = 2$, $f(x) = 1 + x$, or (4.43) with $n = 3$, respectively. In both cases $\eta_i^2 = |b_{0i}|$. Thus, the dispersion relation of the composite particle can be approximated by the formulas

$$E = \sqrt{M^2 + p^2 \left(1 + \eta^2 \frac{p^2}{\Lambda_L^2}\right)^2}, \qquad E = \sqrt{M^2 + p^2 + \eta^4 \frac{(p^2)^3}{\Lambda_L^4}},$$

in the first and second scenarios, respectively, where η is determined by equation (4.45).

Thus, for example, if we take a proton, its dispersion relation has the same form as the dispersion relations of its constituents, with

$$|b_{0p}| = \left(\frac{2}{|b_{0u}|^{1/2}} + \frac{1}{|b_{0d}|^{1/2}}\right)^{-2},$$

where b_{0u} and b_{0d} are the b_0-parameters of the quarks u and d, respectively. If $|b_{0d}| \ll |b_{0u}|$ then $|b_{0p}| \sim |b_{0d}|$, while if $|b_{0u}| \ll |b_{0d}|$ then $|b_{0p}| \sim |b_{0u}|/4$. This means that in composite particles smaller values of $|b_0|$ are favored and the energy threshold for Cherenkov radiation is enhanced. In practice, compositeness creates a sort of screening for the emission of radiation and makes it easier to justify the small numbers found in the previous section.

We have no reason to assume that $|b_{0u}|$ and $|b_{0d}|$ are of the same order. Let us normalize τ_0 to one, as usual, and assume for example $|b_{0d}| \ll |b_{0u}|$. If ultrahigh-energy cosmic rays are protons, we have

$$|b_{0p}| \sim |b_{0d}|.$$

If they are iron atoms we gain an extra factor 7396:

$$b_{0\text{iron}} = \left(\frac{82}{|b_{0u}|^{1/2}} + \frac{86}{|b_{0d}|^{1/2}}\right)^{-2} \sim \frac{|b_{0d}|}{7396}.$$

Consider for example the first scenario described in the previous section. If ultrahigh-energy cosmic rays are made of iron atoms and we take into account their compositeness, the observations

Table 4.3: Improved bounds on $|b_0|$ for various values of Λ_L, for iron atoms taking into account of compositeness.

Λ_L	10^{14}GeV	10^{15}GeV	10^{16}GeV	10^{17}GeV	10^{19}GeV	$4.9 \cdot 10^{19}$GeV
b_{0d}	$4.1 \cdot 10^{-12}$	$4.1 \cdot 10^{-10}$	$4.1 \cdot 10^{-8}$	$4.1 \cdot 10^{-6}$	.04	1

can be explained with the bounds reported in table 4.3, which are quite larger than those in the second two lines of table 4.1.

We see that when the composite structure gets more complex it becomes easier to generate small numbers from larger ones. In summary, patterns like e.g.

$$\begin{aligned} \tau_0 &= 1, \qquad b_{0u} \sim 10^{-6}, \qquad b_{0d} \sim 4 \cdot 10^{-12}, \qquad \Lambda_L \sim 10^{14}\text{GeV}, \\ \tau_0 &= 1, \qquad b_{0u} \sim 10^{-3}, \qquad b_{0d} \sim 4 \cdot 10^{-6}, \qquad \Lambda_L \sim 10^{17}\text{GeV}, \end{aligned}$$

are compatible with a scale of Lorentz violation smaller than the Planck scale. The values of b_{0u} have been chosen to lie somewhere in the middle between those of τ_0 and those of b_{0d} for illustrative purposes.

In the second scenario we can gain an extra factor (4.42) and can explain the same $b_{0\text{iron}}$'s with slightly larger b_{0d}'s:

$$\begin{aligned} \tau_0 &= 1, \qquad b_{0u} \sim 10^{-6}, \qquad b_{0d} \sim 5 \cdot 10^{-11}, \qquad \Lambda_L \sim 10^{14}\text{GeV}, \\ \tau_0 &= 1, \qquad b_{0u} \sim 10^{-3}, \qquad b_{0d} \sim 3 \cdot 10^{-5}, \qquad \Lambda_L \sim 10^{17}\text{GeV}, \end{aligned}$$

If we assume $b_{0u} \sim b_{0d}$ we gain another factor 4:

$$|b_{0p}| \sim |b_{0n}| \sim \frac{|b_{0u}|}{9}, \qquad b_{0\text{iron}} \sim \frac{|b_{0u}|}{28224}.$$

4.7 Cherenkov radiation of neutral particles

We know that when Lorentz symmetry is violated, several otherwise forbidden phenomena are allowed. In this section we describe the Cherenkov radiation of neutral particles. In fact, our model (3.16) contains Pauli-like terms at the fundamental level, which couple neutral particles to the electromagnetic field; in Lorentz-invariant theories these terms could not be there because they are not renormalizable. Moreover, because of Lorentz violation, photon emission is allowed by kinematics. We consider neutrons and neutrinos and sketch an analysis of both the Cherenkov radiation in the low-energy limit (similar to the Cherenkov effect in a medium) and the effects of higher-derivative terms.

The case of neutrons. The neutron Lagrangian, as long as the neutron is considered elementary is obtained simply from (3.16) putting $D = \partial$, since the neutron has no gauge transformation,

$$\begin{aligned}\mathcal{L}_{\text{neutron}} &= \mathcal{L}_F + \bar{\psi}_n \left(i\gamma^0\partial_0 + \frac{ib_{0n}}{\Lambda_L^2}\bar{\partial\!\!\!/}^{\,3} + ib_{1n}\bar{\partial\!\!\!/} - m_n - \frac{b_n'}{\Lambda_L}\bar{\partial\!\!\!/}^{\,2} \right)\psi_n \\ &\quad + \frac{e}{\Lambda_L}\bar{\psi}_n \left(b_n''\sigma_{ij}F^{ij} + \frac{b_{0n}'}{\Lambda_L}\gamma_i\partial_j F_{ij} \right)\psi_n + ie\frac{b_{0n}''}{2\Lambda_L^2}F_{ij}\left(\bar{\psi}_n\gamma_i \overleftrightarrow{\partial}_j\psi_n \right);\end{aligned}$$

the kinematics of the Cherenkov process is the one of section 4.4. The Cherenkov radiation then can be studied adapting the results found for the proton. Indeed, after replacements of the form

$$b'' = \frac{\tilde{b}''}{e}, \qquad b_0' = \frac{\tilde{b}_0'}{e}, \qquad b_0'' = \frac{\tilde{b}_0''}{e}, \tag{4.47}$$

the neutron Lagrangian matches the proton Lagrangian at $e = 0$. As far as the Cherenkov radiation in a medium is concerned, we must evaluate formulas up to $\mathcal{O}(1/\Lambda_L^2)$ corrections, perform the replacements (4.47), followed by the limit $e \to 0$ and then by the converse replacements. We find

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{16\alpha m_n^3\mu^5 b_n''^2(nv-1)^4}{15\Lambda_L^2 n^8(n^2-1)^5 v^2\sqrt{1-v^2}}\left[(5n^2-1)(6n^2+1) - 4n(n^2+1)v - 5n^2(5n^2-1)v^2\right],$$

with the velocity v defined as in (4.16).

We can also make an analysis similar to the one of section 4.5, taking into account the effects of higher derivatives. In the limit $b_{0n}^2 \ll \tau_0$, the analogue of (4.40) gives the typical radiation time

$$t_{fn}|_{b_{0n}^2 \ll \tau_0} \sim \frac{\tau_0}{6\sqrt{5}b_n''^2|b_{0n}|\alpha m_n}.$$

In both cases we see that the Cherenkov radiation of neutrons crucially depends on the parameter b_n'', besides τ_0 and b_{0n}. Thus, measurements cannot say much about the scale of Lorentz violation, on which we have concentrated, but can be useful to put bounds on the values of the parameters b_n''. Nevertheless, some aspects of the neutron Cherenkov radiation may deserve further study, since it is known that in some cases the Lorentz violation makes protons decay into neutrons [25]. Then ultrahigh-energy cosmic rays could be regarded as a mixture of protons and neutrons and both particles would contribute to the emission of Cherenkov radiation.

The case of neutrinos. If neutrinos are taken to be massive, their case is entirely analogous to the case of the neutron. Instead, if we neglect their mass (or assume that they are massless and that neutrino oscillations have a different explanation) there are two main differences: kynematics changes because of the absence of mass and the vertex contains only the $1/\Lambda_L^2$ contributions. In fact the Lagrangian that describes interactions of neutrinos with the electromagnetic field is

$$\mathcal{L}_{\text{neutrino}} = \mathcal{L}_{q'} + \bar{\nu}\left(i\gamma^0\partial_0 + \frac{ib_{0\nu}}{\Lambda_L^2}\bar{\partial\!\!\!/}^{\,3} + ib_{1\nu}\bar{\partial\!\!\!/} \right)\nu + \frac{eb_{0\nu}'}{\Lambda_L^2}\partial_j F_{ij}(\bar{\nu}\gamma_i\nu) + ie\frac{b_{0\nu}''}{2\Lambda_L^2}F_{ij}\left(\bar{\nu}\gamma_i \overleftrightarrow{\partial}_j\nu \right). \tag{4.48}$$

Here the effect is of higher order, because b'', which carried the only interaction term of dimension 5, is absent; in fact no term mixing left- and right-handed solution can be present. The dispersion relation for the neutrino derived from the Lagrangian (4.48) is

$$E(p) = p(b_{1\nu} + b_{0\nu}\frac{p^2}{\Lambda^2})$$

assuming $b_{0\nu}$ positive to ensure monotonicity. Note that for neutrinos it is not possible to use a dispersion relation of the form of the second example studied for protons in section 4.4, since there b' and b_0 are related. Taking for the photons the complete dispersion relation in (4.1) and following the same reasoning as in section 4.4 one finds the condition for the existence of kinematics solutions to Cherenkov radiation from neutrinos

$$\sqrt{\tau_2} < b_{1\nu} + 3b_{0\nu}^2\frac{p^2}{\Lambda_L^2},$$

which reduces to

$$\sqrt{\tau_2} < b_{1\nu} \qquad \text{for} \quad p << \Lambda_L.$$

In this case, following the same path and notations of section 4.5, we find that a solution for the energy-momentum conservation equation exists for $\Lambda_L \to \infty$, being

$$u = \frac{k}{2p}(1 - \frac{\tau_2}{b_{1\nu}^2}) + \frac{\sqrt{\tau_2}}{b_{1\nu}},$$

and thus the limiting momentum for the outgoing photon is

$$0 \leq k \leq \frac{2p}{(1 + \frac{\sqrt{\tau_2}}{b_{1\nu}})}.$$

We can calculate the energy loss up to order $\mathcal{O}(1/\Lambda_L^4)$

$$\frac{dE}{dt} = -\frac{2\alpha}{21}\frac{p^6}{\Lambda_L^4}\frac{b_{1\nu}(b_{1\nu} - \sqrt{\tau_2})}{(b_{1\nu} + \sqrt{\tau_2})^7}(49b_{1\nu}^2 + 8b_{1\nu}\sqrt{\tau_2} + \tau_2)(4b_{0\nu}'^2 b_{1\nu}^2 + b_{0\nu}''^2\tau_2).$$

Nevertheless, as already explained, the crucial coefficients are $b_{0\nu}$ and τ_0, while $b_{1\nu}$ and τ_2 can be taken to be 1, in which case the formulas just found are not valid. For $b_{1\nu}$ and τ_2 equal to 1 there is always a solution of the form $u = 1 - \epsilon$ and, up to $\mathcal{O}(p^2/\Lambda_L^2)$, we find

$$\epsilon = \frac{p^2}{2\Lambda_L^2}(1 - \frac{k}{p})[2b_{0\nu}(\frac{k^2}{p^2} - 3\frac{k}{p} + 3) - \tau_1\frac{k^2}{p^2}].$$

Combining the conditions $\epsilon \geqslant 0$ and $E(p) - \omega(k) > 0$ we obtain the k-range

$$0 \leqslant k \leqslant p$$

assuming $\tau_1 \leqslant 2b_{0\nu}$, which is the interesting case, since τ_1 is not a crucial parameter. Now, by integration, it is straightforward to calculate the energy loss

$$\frac{dE}{dt} = -\frac{\alpha p^8\left(4b_{0\nu}'^2 + b_{0\nu}''^2\right)(517b_{0\nu} - 161\tau_1)}{10080\Lambda_L^6}.$$

Chapter 5

Conclusions

Lorentz symmetry violation is an interesting matter of investigation, both from the theoretical and the phenomenological perspective. If one assumes explicit Lorentz breaking in the fundamental high-energy theory, an enlargement of the set of renormalizable interactions is obtained as a consequence. This theoretical framework is complete and consistent and allows to construct a Standard Model extension, which can be studied thoroughly.

From the theoretical side, the main fact is that the gauge coupling is super-renormalizable, thus leading to a closed renormalization: here we have calculated in detail the one-loop counterterms and beta functions of the $U(1)$ gauge subsector, whose renormalization has further contributions from two-loops diagrams only. Understanding the running of couplings from low energy up to energies near the scale Λ_L, and then, from Λ_L up to much higher energies is an important issue in order to understand if, and up to which energy, the noncrucial parameters, multiplying lower powers of momentum, can be neglected. From the study we have made of the connection between the high- and low-energy theory renormalizations, another aspect comes out and deserves to be stressed: the high-energy theory transfers incalculable arbitrary factors to the low-energy theory and puts them in front of power-like divergences. This feature is completely general and does not depend on the particular high-energy fundamental theory considered, provided that it is renormalizable, or even super-renormalizable, anyway whenever it is not completely finite. From this point of view, the hierarchy problem concerning the Higgs mass disappears, since the quadratically divergent corrections can be removed, using the arbitrariness of their coefficients.

Even phenomenologically the model studied here suggests interesting investigations; in fact, various phenomena forbidden in standard electrodynamics can take place if Lorentz violation occurs. These could be the first and clearest signs of Lorentz violation that experimentalists may possibly observe. Moreover, from actual experimental data we can put bounds on the parameters, and mainly on the scale of violation. From the point of view adopted here, the order of magnitude of Λ_L is the most important quantity to estimate, since it scales the higher-dimension terms, which are the most relevant ones. This approach is different from the normally adopted one, where modified low-energy dispersion relations are used and low-energy parameters are those to be evaluated first. Our model predicts many unusual interactions, such as photon decay and radiation of light from free fermions, both charged and neutral. The last phenomenon has been briefly explored here and could be studied further. We have mostly focused on the Cherenkov radiation in vacuo by charged particle, and we have found that in general an energy treshold exists above which flying fermions do radiate photons; the time during which this energy loss occurs is very rapid, so that the process is governed by mere kinematics. In practice ultrahigh-energy cosmic rays (which are rather puzzling themselves, see for example [33]) can be considered to be below the energy treshold, each time they have arrived to us and have been observed. Values of Λ_L below the Planck scale are compatible with data if the parameters in our model are sufficiently small; from a simple schematization of composite particles, it seems that such small values of parameters are favored in composite objects. This analysis shows that there is indeed the possibility that the scale of Lorentz violation with preserved CPT is smaller than the Planck scale. If this is confirmed, it would mean that the construction of gravity should be reconsidered from the beginning.

Appendices

A. Value of α

The result for the one-loop beta function of the gauge coupling g_i associated with each of the three gauge groups of the Standard Model, given in [29], eq.4, is

$$\beta_{g_i} = \frac{a_i}{16\pi^2} g_i^3,$$

where $i = 1, 2, 3$ refers to $U(1)_Y$, $SU(2)_L$, and $SU(3)_c$, respectively, and

$$\begin{aligned} a_1 &= \frac{4}{3} N_{gen} + \frac{1}{10} N_H \\ a_2 &= \frac{4}{3} N_{gen} + \frac{1}{6} N_H - \frac{22}{3} \\ a_3 &= \frac{4}{3} N_{gen} - 11, \end{aligned}$$

where N_{gen} is the number of generations of fermions, and N_H is the number of complex Higgs doublets. For $N_{gen} = 3$ and $N_H = 1$ as in the Standard Model, we get $a_1 = 41/10$, $a_2 = -19/6$ and $a_3 = -7$. Thanks to the electroweak unification, the Weinberg angle θ_W relates the coupling constants of hypercharge and weak isospin, and defines also the fine-structure constant $\alpha = e^2/(4\pi)$, namely

$$\frac{g'}{g} = \tan\theta_W \qquad \alpha = \frac{g^2 \sin^2\theta_W}{4\pi}, \qquad \text{with} \quad g \equiv g_2 \quad \text{and} \quad g' \equiv \sqrt{\frac{3}{5}} g_1.$$

The factor $(3/5)^{1/2}$ is due to the normalization of the hypercharge (see ref.[30]). Then, differentiating

$$\alpha = \frac{1}{4\pi} \frac{g^2 g'^2}{g^2 + g'^2}$$

we get

$$\frac{\mathrm{d}\alpha}{dt} = \beta_\alpha = \frac{11}{6\pi}\alpha^2, \qquad t = \ln\frac{E}{\mu}.$$

The solution for the running coupling is the usual one

$$\alpha(E)=\frac{\alpha(E_0)}{1-\frac{11}{6\pi}\alpha(E_0)\ln\frac{E}{E_0}}.$$

Starting from the value found in [31], $\alpha(M_Z)=(128.957)^{-1}$ and taking $M_Z=91.1876\text{GeV}$ from [32], we get $\alpha(3\cdot 10^{11}\text{GeV})=(116)^{-1}$, which is the value adopted in the estimates of chapter 4.

B. Modified first scenario

Here we report results about a case slightly different from that worked out in section 4.5.1. Notations are those defined there. In this example we choose parameters to satisfy

$$\tau_2=1,\qquad \tau_1=2\sqrt{\tau_0},\qquad b_1=1,\qquad b'=0,\qquad b_0>0.$$

Thus, the dispersion relations become

$$E(p^2)=\sqrt{m^2+p^2\left(1+\frac{b_0p^2}{\Lambda_L^2}\right)^2},\qquad \omega(k)=k\left(1+\frac{\sqrt{\tau_0}k^2}{\Lambda_L^2}\right),$$

namely the proton and photon energies have both the same structure, differently from (4.25). Applying the kinematic constraint (4.21) we obtain identical conditions for Cherenkov emission as in section 4.5.1. Solving the energy-momentum conservation equation for $p\ll\Lambda_L$, $m\ll p$, with ξ^2 fixed, we find

$$\varepsilon=3\frac{b_0p^2}{\Lambda_L^2}\left[1-\xi^2-2\frac{k}{p}+\frac{k^2}{3p^2}\left(4-\frac{1}{\sqrt{\zeta}}\right)-\frac{k^3}{3p^3}\left(1-\frac{1}{\sqrt{\zeta}}\right)\right]\tag{B.1}$$

with $u=1-\varepsilon$ and $\zeta\equiv b_0^2/\tau_0$.

The k-range is deduced imposing $\varepsilon\geqslant 0$; supposing that the gauge field parameter τ_0 is the one defining Λ_L, we focus on the cases $\zeta\ll 1$ and $\zeta=1$.

For $\zeta\ll 1$ we need to distinguish when $\zeta\ll(1-\xi^2)^2$ or $\zeta\gtrsim(1-\xi^2)^2$. We obtain the ranges

$$k<p\left(3\sqrt{\zeta}(1-\xi^2)\right)^{\frac{1}{2}},\quad \text{for}\quad \zeta\ll 1\quad\text{and}\quad \zeta\ll(1-\xi^2)^2$$

while

$$k<p\frac{1-\xi^2}{2},\quad \text{for}\quad \zeta\ll 1\quad\text{and}\quad \zeta\gtrsim(1-\xi^2)^2.$$

The energy losses to the lowest order in $1/\Lambda_L$ and ζ come out to be

$$\begin{aligned}\left.\frac{\mathrm{d}E}{\mathrm{d}t}\right|_{\zeta\ll1,\zeta\ll(1-\xi^2)^2}&=-\frac{9\alpha p^4\left(1-\xi^2\right)^2b_0\sqrt{\zeta}}{2\Lambda_L^2},\\ \left.\frac{\mathrm{d}E}{\mathrm{d}t}\right|_{\zeta\ll1,\zeta\gtrsim(1-\xi^2)^2}&=-\frac{\alpha p^4\left(1-\xi^2\right)^3b_0}{4\Lambda_L^2}.\end{aligned}\tag{B.2}$$

For $\zeta = 1$ the condition $\varepsilon > 0$ imposed on (B.1) gives the range

$$k < p(1-\xi),$$

and the energy loss is

$$\left.\frac{dE}{dt}\right|_{\zeta=1} = -\frac{\alpha p^4 (1-\xi)^3}{20\Lambda_L^2}\left[b_0\left(13+39\xi-12\xi^2\right)+20b''\sqrt{6b_0}\xi(1-\xi)+8b''(1+3\xi-4\xi^2)\right]. \tag{B.4}$$

If we take $\Lambda_L = 10^{14}$GeV, protons of $3\cdot 10^{11}$GeV, would be above the threshold for Cherenkov radiation, unless $b_0 < 1.8\cdot 10^{-19}$, as in section 4.5.1. With the same Λ_L, if $b_0 \simeq 1$, particles would radiate photons loosing energy down to $E_{\rm lim} = 6\cdot 10^6$GeV in a very short time. In fact, integrating (B.4) from $\Lambda_L$to $E_{\rm obs} = 3\cdot 10^{11}$GeV, we obtain, putting $b'' = 0$,

$$t_{\rm obs} \approx 10^{-29}{\rm s},$$

while, continuing down to $1.1E_{\rm lim}$

$$t_f \approx 10^{-14}{\rm s}.$$

If we take $b_0 = 1.8\cdot 10^{-19}$ and $\Lambda_L = 10^{14}$GeV, we should integrate the first line of eq.(B.2), which is valid throughout the interval from Λ_L to $1.1E_{\rm lim}$, obtaining

$$t_f \approx 4\cdot 10^8{\rm s}.$$

All values $b_0 \leq 1.8\cdot 10^{-19}$ are compatible with data. The values of b_0 with the corresponding bounds on Λ_L are the same given in table 4.1.

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