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Weak solutions to rate-independent systems: Existence and Regularity

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Introduction

A rate-independent system is a specific case of quasistatic systems. It is time-dependent but its behavior is slow enough that the inertial effects can be ignored and the systems are affected only by external loadings. Moreover, in rate-independent systems, the rate of change of solutions to the systems depends only on the change of the velocity of the loading (it is independent of the velocity itself, and hence "rate-independent").

In this introduction, for simplicity, let us consider a point $x(t)$ with the initial position $x(0) = x_0$ in some finite dimensional normed vector space X, subject to a force defined by a smooth energy functional $\mathscr{E} : [0,T] \times X \to [0,+\infty)$ and a convex, positively 1-homogeneous dissipation potential $\Psi : X \to [0,+\infty)$. We say that $x(\cdot)$ is a solution to the rate-independent system $(\mathscr{E}, \Psi, x_0)$ if the following inclusion holds true,

$$0 \in \partial\Psi(\dot{x}(t)) + \partial_x \mathscr{E}(t, x(t)) \text{ for a.e. } t \in (0,T),$$

where $\partial\Psi$ is the sub-differential of the convex function Ψ.

Some specific examples of rate-independent systems were studied by many authors including Francfort, Marigo, Larsen, Dal Maso and Lazzaroni on brittle fractures [12, 14, 15, 9], Dal Maso, DeSimone and Solombrino on the Cam-Clay model [8], Dal Maso, DeSimone, Mora and Morini on plasticity with softening [6, 7], Alberti and DeSimone on capillary drops [1]. The reader is referred to the surveys [17, 19, 20] by Mielke for further references.

The case of a convex energy is quite classical and was considered long time ago by many authors. For instance, if the energy functional $\mathscr{E}(t, \cdot)$ is uniformly convex and satisfies some suitable smoothness conditions, then the system admits a unique solution $x(\cdot)$ which is Lipschitz continuous [28]. However, in the case that the energy functional $\mathscr{E}$ is not convex, uniqueness may be lost (see e.g. Example 1.6 in Chapter 1) and strong solutions may not exist [31]. Hence, the question of defining a suitable weak solution for (1.1) arises naturally. There are several ways to define a weak solution, such as the concept of energetic solution [27, 12], BV solution [25, 26], epsilon-neighborhood solution [17, 10], local solution [34], parametrized solution [26] and epsilon-stable solution [15]. In this thesis, we shall consider energetic solutions, BV solutions and epsilon-neighborhood solutions in a very abstract model.

The notion of *energetic solution* is the first attempt to answer the question of finding weak solutions. This notion was introduced by Mielke and Theil [27] for shape-memory alloys (Francfort and Marigo [12] developed a very similar of energetic solutions in the context of fracture mechanics at about the same time). Then the existence of energetic solutions was established in many other rate-independent systems as well as for the abstract model, see e.g. [28, 16, 18, 13, 19, 21]. A function $x : [0,T] \to X$ is called an energetic solution of the rate-independent system $(\mathscr{E}, \Psi, x_0)$, if it satisfies the initial condition $x(0) = x_0$ and $x(t)$, the *global stability*

$$\mathscr{E}(t, x(t)) \leq \mathscr{E}(t, x) + \Psi(x - x(t))$$

for all $(t,x) \in [0,T] \times X$, and the *energy-dissipation balance*

$$\mathscr{E}(t_2, x(t_2)) - \mathscr{E}(t_1, x(t_1)) = \int_{t_1}^{t_2} \partial_t \mathscr{E}(s, x(s)) ds - \mathscr{D}iss(x; [t_1, t_2])$$

for all $0 \leq t_1 \leq t_2 \leq T$.

If the energy functional $\mathscr{E}(t, \cdot)$ is convex, then the global stability ensures that at every time t, $x(t)$ is the optimal position, in the sense that any loss of energy when moving to another position will be compensated by dissipation. However, when the energy functional is non-convex, then the global stability is too strong. In fact, there are situations where the system admits a strong solution which is not an energetic solution (see e.g. [31]). Moreover, global minimizers make the energetic solutions jump easier than physically feasible and into far-apart energetic configurations, and hence fail to describe the related physical phenomena.

For general (non-convex) energy functionals, some notions based on local minimality are preferred to overcome this shortcoming of energetic solutions. One of these notions is BV solution constructed by vanishing viscosity. This notion was introduced by Mielke, Rossi and Savaré [25, 26]. The idea is to add a small viscosity term to the dissipation functional Ψ. This results in a new dissipation functional Ψ_ε, which has super-linear growth at infinity and which converges to Ψ as ε tends to zero in an appropriate meaning. The super-linear growth of Ψ_ε makes the limit $x(\cdot)$ of the sequence $\{x^\varepsilon(\cdot)\}$ when ε goes to 0 difficult to jump, and therefore $x(\cdot)$ should prefer a close-by state which is locally stable to a far-away state which is globally stable. Moreover, Mielke, Rossi and Savaré have proved in [26] that the limit $x(\cdot)$ is a BV solution to the system $(\mathscr{E}, \Psi, x_0)$, which means $x(0) = x_0$ and $x(t)$ satisfies the *weak local stability*,

$$|\partial_x \mathscr{E}(t, x(t))| \leq 1 \text{ provided that } t \mapsto x(t) \text{ is continuous at } t,$$

and the *new energy-dissipation balance*,

$$\mathscr{E}(t_2, x(t_2)) - \mathscr{E}(t_1, x(t_1)) = \int_{t_1}^{t_2} \partial_t \mathscr{E}(s, x(s)) ds - \mathscr{D}iss_{new}(x(t); [t_1, t_2])$$

for all $0 \leq t_1 \leq t_2 \leq T$.

Note that the weak local stability in BV solutions only holds at continuity points. The information at jump points is contained in the new energy-dissipation balance. Moreover, the new energy-dissipation balance also reveals the information of the solutions along the jump path. Indeed, if the BV solution $x(\cdot)$ jumps at time t, then there exists an absolutely continuous path $v : [0,1] \to X$ connecting $x(t^-)$ and $x(t^+)$ such that along this path, we have that $|\partial_x \mathscr{E}(t_0, v(s))| \geq 1$ for all $s \in [0,1]$ (see [25, 26]).

However, BV solutions constructed by vanishing viscosity depend heavily on the choice of the viscosity. There are examples (see Chapter 1) showing that different viscosities make BV solutions jump in different time.

Another way to avoid global minimality is to find the minimizer in a small neighborhood of order ε and obtain an *epsilon-neighborhood solution* $x^\varepsilon(\cdot)$. When taking the limit of $x^\varepsilon(\cdot)$ as $\varepsilon \to 0$, we get a function $x(\cdot)$ which satisfies both the weak local stability and the new energy-dissipation balance. We call the limit function $x(\cdot)$ *BV solution constructed by epsilon neighborhood*.

The epsilon-neighborhood approach was first suggested in [17] (Section 6) for a one-dimensional case when ε is chosen proportional to the square root of the time-step. The weak local stability was then obtained in [10]. However, it seems that the fact that the solution also satisfies the new energy-dissipation balance does not appear explicitly in the literature. In this thesis, we shall prove it in detail (see Section 2.3).

Another topic of this thesis is about regularity of weak solutions to rate-independent systems (see Chapter 3). The regularity for *energetic solutions* when the energy functional is *convex* was already considered by Mielke, Rossi and Thomas [19, 24, 33]. However, if the energy functional is *non-convex*, there are very few results on the regularity of energetic solutions.

The first regularity question is about the jump set of solutions. Since each solution has bounded variation, the number of jumps is at most countable. However, without the convexity assumption, there are examples showing that energetic solutions may have *dense jumps* (see Section 3.2). One of our results is to provide sufficient conditions to ensure that weak solutions have only finitely many jumps. More precisely, we consider one dimensional models, in which the energy functional $\mathscr{E}$ is C^2 and satisfies

$$\{(t,x) \in [0,T] \times \mathbb{R} \mid \partial_x \mathscr{E}(t,x) \in \{-1,1\}, \partial_{xx}\mathscr{E}(t,x) = 0\} = \emptyset$$

and the dissipation Ψ is the usual distance in $\mathbb{R}$. We show that every weak solution $x(\cdot)$ satisfying the weak local stability and the energy-dissipation upper bound

$$\mathscr{E}(t_2, x(t_2)) - \mathscr{E}(t_1, x(t_1)) \le \int_{t_1}^{t_2} \partial_t \mathscr{E}(s, x(s)) ds - \mathscr{D}iss(x(\cdot); [t_1, t_2])$$

for all $0 \le t_1 \le t_2 \le T$ (e.g., energetic solutions, epsilon-neighborhood solutions and BV solutions) has only finitely many jumps.

The second question is about the smoothness of solutions. In one-dimension, with the assumption that $\mathscr{E}(t,x)$ is of class C^3 and the set

$$\{(t,x) \in (0,T) \times \mathbb{R} \mid \partial_x \mathscr{E}(t,x) \in \{-1,1\}, \partial_{xx}\mathscr{E}(t,x) = \partial_{xxx}\mathscr{E}(t,x) = 0\}$$

is finite, we can show that any weak solution $x(\cdot)$ (in the same meaning as above) is of class SBV. Moreover, under some stronger conditions, we derive better smoothness properties for energetic solutions, such as the pointwise differentiability and the piecewise C^1-regularity.

In summary, this thesis contains four chapters. In Chapter 1 and 2, we review of the definitions of energetic and BV solutions in an abstract setting, with proofs in suitably simplified cases. In particular, we give a complete proof of the convergence of epsilon-neighborhood solutions to BV solutions, which seems to be not contained in the previous works [17, 10]. In Chapter 3, we consider the regularity of weak solutions and in particular energetic solutions. More precisely, we provide sufficient conditions to ensure that the weak solutions have finitely many jumps, or are of class SBV, and the pointwise differentiable, or piecewise C^1 for energetic solutions. These regularity results are taken from the preprint [22]. Finally, in Chapter 4, we explicitly compute the different types of weak solutions in many examples.

The research stated in this thesis is far from complete. First at all, we would like to extend the results of epsilon-neighborhood solutions in Section 2.3 to higher dimensions.

Moreover, we would like to improve the weak local stability for BV solutions constructed by epsilon-neighborhood. Note that this problem is not simple since there are examples (see Example 4.5) showing that BV solutions (both obtained by vanishing viscosity and epsilon neighborhood) do not satisfy the strong local stability. Second, the techniques we employed in Chapter 3 are rather specific for the one-dimensional case and some further work would be needed to obtain the results in higher dimensions. Finally, we would like to develop the theory of BV solutions in Chapter 2 and the regularity results in Chapter 3 in some concrete infinite-dimensional example, such as capillary drops.

Chapter 1

Weak solutions of rate-independent systems

1 General ideas about rate-independent systems

A rate-independent system is a specific case of a larger class of phenomena which are called quasistatic systems. Quasistatic systems are considered as the bridge between dynamic and static systems, and are used widely in mathematical frameworks as well as physical models to describe many phenomena involved in plasticity, phase transformation (e.g. electromagnetism, superconductivity or dry friction on surfaces), and some hysteresis models (e.g. shape-memory alloy, quasistatic delamination, fracture). Such systems are time-dependent, but unlike dynamic systems, their behavior is slow enough such that the inertial effects can be ignored.

Notice that quasistatic systems have no dynamics of their own. Hence, the changes of the systems are caused solely by the changes of the external conditions. Moreover, the time scales of the external conditions are much longer than the intrinsic time scales of the systems. This property makes the quasistatic systems close to equilibrium at almost every moment.

Although most of quasistatic systems are reversible, the presence of a dissipation effect makes rate-independent systems irreversible (The reader is referred to Section 1.3 below for the concept of irreversibility). On the other hand, in rate-independent systems, we do not allow for viscosity, while in a general quasistatic system, viscous effects may still be present. Another characterization of rate-independent systems is that, the rate of change of the solutions to the systems depends only on the change of the velocity of the loading. Due to this reason, these systems are called rate-independent. For example, if the loading acts twice faster, then the solutions also respond twice faster. Later on, we will see that this property is described by the positively 1-homogeneity of the dissipation potential.

In short, by rate-independent systems we denote those systems which have no inertial effects, no kinetic energy, no viscous effects, are irreversible, and are rate-independent. For a detailed discussion on the rate-independent systems, we refer to the book [18].

1.1 An abstract framework

We now give an abstract framework for rate-independent systems. Let us consider a point $x(t)$, dependent on the time t, in some finite-dimensional normed vector space X. Since the system is considered to be quasistatic, the energy functional $\mathscr{E}$ does not depend on the velocity $\dot{x}$, i.e., $\mathscr{E} : [0, T] \times X \to [0, +\infty)$ consists only the potential energy.

Apart from the energy, we have a dissipation in the model, which describes the loss of energy caused by the changing in position of the system. Usually, dissipation is characterized by the convex dissipation potential $\Psi : X \to [0, +\infty)$, which is supposed to be positively 1-homogeneous to make the system rate-independent.

More precisely, from now on, we shall use the following assumptions.

- X is finite-dimensional normed vector space.
- $\mathscr{E} : [0, T] \times X \to [0, +\infty)$ is of class C^2 and satisfies the following technical assumption: There exists $\lambda = \lambda(\mathscr{E})$ such that

$$|\partial_t \mathscr{E}(s, x)| \leq \lambda\, \mathscr{E}(s, x) \text{ for all } (s, x) \in [0, T] \times X.$$

- $\Psi : X \to [0, +\infty)$ is convex and positively 1-homogeneous, i.e., $\Psi(\gamma v) = \gamma \Psi(v)$ for all $\gamma > 0$.

Then the rate-independent system including the energy $\mathscr{E}$ and the dissipation Ψ can be written as the following differential inclusion

$$0 \in \partial\Psi(\dot{x}(t)) + \partial_x \mathscr{E}(t, x(t)) \text{ for a.e. } t \in (0, T). \tag{1.1}$$

Here $\partial\Psi$ is the sub-differential of the convex function Ψ,

$$\partial\Psi(v) := \{\eta \in X^* |\ \forall w \in X : \Psi(w) \geq \Psi(v) + \langle \eta, w - v \rangle\}.$$

(see [30] Section 23 for more detail about sub-differential of convex functions).

The following example is taken from [2].

Example 1.1. Let us consider a basic example of a small box pulled by a spring on a rough surface (see Figure 1). If we assume that the other end of the spring moves at a prescribed slow speed, then at every moment, the box is subject to two forces: the external force (f_e) due to the spring, and the frictional force (f_a) due to the rough surface.

More precisely, let us denote by $x(t)$ the center of the box, and by $y(t)$ the end of the spring. Here $x(t)$ and $y(t)$ are points of $\mathbb{R}^2$, but in the following we can think that they are point masses in $\mathbb{R}^d$. Now if we call l_0 the length at rest of the spring and c the constant of the spring, then the external force is

$$f_e = -\partial_x \mathscr{E}(t, x),$$

where the potential energy is

$$\mathscr{E}(t, x) := \frac{c}{2}(x - y(t) + l_0)^2.$$

On the other hand, since the dissipation force f_a in this case is caused by dry friction, it obeys the following law

$$f_a := \begin{cases} -k\frac{v}{|v|} & \text{if } v \neq 0, \\ -f_e \text{ and } |f_a| \leq k & \text{if } v = 0. \end{cases}$$

Here v is the velocity of the box, and k is the frictional coefficient.

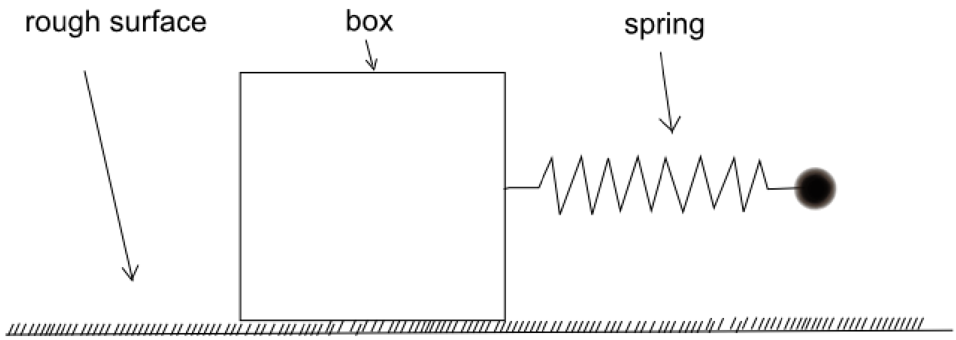

Figure 1. Small box on a rough surface (This picture is taken from [2]).

Therefore, if at the beginning the external force is small (that is, $|f_e| < k$), then it is cancelled by the frictional force (f_a). Therefore, the system is in static equilibrium and the box does not move. If we keep pulling the end of the spring, the external force becomes larger and larger, and when it reaches the critical value ($|f_e| = k$), the frictional force can no longer balance the external force. Equilibrium is broken and the box starts moving together with the spring.

The equation of dynamics is then

$$m\ddot{x} = f_a + f_e,$$

here we call m the mass of the box.

However, if everything is moving so slowly, we can neglect the term $m\ddot{x}$. Thus, we get the following force balance

$$0 = f_a + f_e. \tag{1.2}$$

Notice that this system is quasistatic only as a limit of approximation. And indeed, the body moves even if the total force is 0.

Now if we define the dissipation function $\Psi(v) := k|v|$, then by direct computation, we see that $-f_a \in \partial\Psi(v)$ (we denote by $\partial\Psi(v)$ the sub-differential of Ψ at v). Hence, equation (1.2) becomes

$$0 \in \partial\Psi(\dot{x}(t)) + \partial_x \mathscr{E}(t, x(t)),$$

which is exactly the equation (1.1).

1.2 Solutions to rate-independent systems

In the case that the energy functional $\mathscr{E}(t, \cdot)$ is uniformly convex and satisfies some suitable smoothness conditions, then (1.1) admits a unique solution which is Lipschitz continuous (see [28] Section 7). However, in the case that the energy functional $\mathscr{E}$ is not convex, uniqueness may be lost (see Example 1.6) and strong solutions may not exist [31]. Hence, the question on how to define a suitable weak solution for (1.1) arises naturally. In the following, we will introduce the notions of *weak solution*, *energetic solution*, *BV solution*, and *epsilon-neighborhood solution* to the system (1.1). Some other notions, e.g. *parameterized solution*, *epsilon stable solution*, can be found in [21, 15].

1.3 Some basic properties of weak solutions

Rate independence

If $x(t)$ is a solution to the system $(\mathscr{E}, \Psi)$, then it remains solution to this system after rescaling time.

More precisely, if we denote

$$\tilde{\mathscr{E}}(s, x) := \mathscr{E}(t(s), x),$$

with $s \mapsto t(s)$ is continuous and increasing, then $\tilde{x}(s) := x(t(s))$ is a solution to the system $(\tilde{\mathscr{E}}, \Psi)$.

Time irreversibility

If $x(t)$ is a solution to the system $(\mathscr{E}, \Psi)$, then it is no longer solution to the system after reversing the time.

More precisely, if we denote

$$\tilde{\mathscr{E}}(t, x) := \mathscr{E}(-t, x),$$

then in general $\tilde{x}(t) := x(-t)$ is not a solution to the system $(\tilde{\mathscr{E}}, \Psi)$.

Symmetry

Assume that $x(t)$ is a solution to the system $(\mathscr{E}, \Psi)$. Denote

$$\tilde{\mathscr{E}}(t, x) := \mathscr{E}(t, \Phi(x)),$$

where $\Phi : X \to X$ is an isometry w.r.t. Ψ, i.e., $\Psi(\Phi(x) - \Phi(y)) = \Psi(x - y)$, for any $x, y \in X$. Then $\tilde{x}(t) := -x(t)$ is a solution to the system $(\tilde{\mathscr{E}}, \Psi)$.

Concatenation

Let $x_1 : [a_1, a_2] \to X$ be a solution to the system $(\mathscr{E}, \Psi, x_1(a_1))$ on the time interval $[a_1, a_2]$ and $x_2 : [a_2, a_3] \to X$ be a solution to the system $(\mathscr{E}, \Psi, x_2(a_2))$ on the time interval $[a_2, a_3]$. If $x_1(a_2) = x_2(a_2)$, then

$$\tilde{x}(t) := \begin{cases} x_1(t) & \text{if } t \in [a_1, a_2], \\ x_2(t) & \text{if } t \in [a_2, a_3], \end{cases}$$

is a solution to the system $(\mathscr{E}, \Psi, x_1(a))$ on the time interval $[a_1, a_3]$.

Restriction

Let $x : [a, b] \to X$ be a solution to the system $(\mathscr{E}, \Psi, x(a))$ on the time interval $[a, b]$. For any subinterval $[c, d] \subset [a, b]$, the restriction of $x(\cdot)$ on $[c, d]$ is a solution to the system $(\mathscr{E}, \Psi, x(c))$ on the time interval $[c, d]$.

Remark. The above properties also hold for energetic solutions and BV solutions.

2 Energetic solutions

The first attempt to define a weak notion of solution for (1.1) was made in [27] by Mielke and Theil for a model for shape-memory alloys via the concept of *energetic solution*. Then the existence of energetic solutions was established in many other rate-independent systems as well as in the abstract case, see [28, 16, 13, 19].

2.1 Motivation and definition

The following argument is taken from [18, 20, 21].

Assume for the moment that the function $t \mapsto x(t)$ satisfies equation (1.1) for almost every $t \in (0, T)$. We denote $\partial_x \mathscr{E}(t, x(t))$ by $\xi(t)$, and denote by Ψ^* the Legendre-Fenchel transform of Ψ (see [30] Section 12),

$$\Psi^*(\eta) := \sup\{\langle \eta, v\rangle - \Psi(v) \mid v \in X\}.$$

By the Fenchel equivalence (see [30] Section 23)

$$x^* \in \partial F(x) \Longleftrightarrow F(x) + F^*(x^*) = \langle x^*, x\rangle\,,$$

equation (1.1) is equivalent to

$$\Psi(\dot{x}(t)) + \Psi^*(-\xi(t)) = \langle -\xi(t), \dot{x}(t)\rangle\,. \tag{1.3}$$

If we assume moreover that $\mathscr{E}(t, x)$ and $x(t)$ are smooth enough (e.g. $\mathscr{E} \in C^1$ and $x \in C^1$), we can apply the classical chain rule

$$\frac{d}{dt}\mathscr{E}(t, x(t)) = \langle \xi(t), \dot{x}(t)\rangle + \partial_t \mathscr{E}(t, x(t)),$$

and rewrite (1.3) as

$$\Psi(\dot{x}(t)) + \Psi^*(-\xi(t)) = -\frac{d}{dt}\mathscr{E}(t, x(t)) + \partial_t \mathscr{E}(t, x(t)).$$

Integrating this equation w.r.t. t, we get

$$\int_{t_1}^{t_2} [\Psi(\dot{x}(t)) + \Psi^*(-\xi(t))]\, dt = \mathscr{E}(t_1, x(t_1)) - \mathscr{E}(t_2, x(t_2)) + \int_{t_1}^{t_2} \partial_t \mathscr{E}(t, x(t))dt \tag{1.4}$$

for every $0 \le t_1 \le t_2 \le T$.

Since Ψ is positively 1-homogeneous, the value of its Legendre-Fenchel transform Ψ^* is

$$\Psi^*(\eta) = \begin{cases} 0 & \text{if } \eta \in \partial\Psi(0), \\ +\infty & \text{otherwise.} \end{cases}$$

From the equation (1.4), we have

$$\int_0^T \Psi^*(-\xi(t))dt < \infty.$$

Hence, $\Psi^*(-\xi(t)) < +\infty$ for a.e. $t \in (0, T)$. Thus, equation (1.4) can be rewritten in terms of the two following conditions

Weak local stability (w-LS)

$$\Psi^*(-\xi(t)) < +\infty \text{ a.e. in } (0,T), \text{ that is, } -\partial_x \mathscr{E}(t,x(t)) \in \partial\Psi(0) \text{ a.e. in } (0,T),$$

Energy-dissipation balance (ED)

$$\mathscr{E}(t_2, x(t_2)) - \mathscr{E}(t_1, x(t_1)) = \int_{t_1}^{t_2} \partial_t \mathscr{E}(t, x(t))dt - \int_{t_1}^{t_2} \Psi(\dot{x}(t))dt \qquad \text{for all } 0 \le t_1 \le t_2 \le T.$$

Note that the above formula (ED) involves the time derivative of the solution and therefore makes sense only for "smooth" $x(\cdot)$, while we know that in general solutions to the system $(\mathscr{E}, \Psi)$ may have jumps as a function of time and therefore are not smooth. To write this formula in a form that makes sense also for non-smooth functions, we need to introduce the notion of dissipation distance $\mathcal{D} : X \times X \to [0, +\infty]$, and the dissipation functional $\mathscr{D}iss(x(t); [t_1, t_2])$,

$$\mathcal{D}(x_0, x_1) := \inf \left\{ \int_0^1 \Psi(\dot{y}(t))dt \mid y \in W^{1,1}([0,1]; X), y(0) = x_0, y(1) = x_1 \right\}.$$

$$\mathscr{D}iss(x(t); [t_1, t_2]) := \sup \left\{ \sum_{i=1}^N \mathcal{D}(x(s_{i-1}), x(s_i)) \mid N \in \mathbb{N}, t_1 \le s_0 < s_1 < \cdots < s_N \le t_2 \right\}.$$

We have the following properties.

Proposition 1.2. *(i) $\mathcal{D}(x_0, x_1) = \Psi(x_1 - x_0)$ for all $x_0, x_1 \in X$.*

(ii) If $u : [0,T] \to X$ is absolutely continuous and satisfies

$$\int_0^T \Psi(\dot{u}(t))dt < +\infty,$$

then

$$\mathscr{D}iss(u(t); [t_1, t_2]) = \int_{t_1}^{t_2} \Psi(\dot{u}(t))dt.$$

The definition of absolutely continuous functions can be found in [11] (Section 1.7). The proof of Proposition 1.2 is given at the end of this chapter.

By Proposition 1.2, we have another definition of $\mathscr{D}iss(x(t); [t_1, t_2])$

$$\mathscr{D}iss(x(t); [t_1, t_2]) := \sup \left\{ \sum_{i=1}^N \Psi(x(s_i) - x(s_{i-1})) \mid N \in \mathbb{N}, t_1 \le s_0 < s_1 < \cdots < s_N \le t_2 \right\}.$$

Thanks to Proposition 1.2, we can deduce from the energy-dissipation balance (ED) the following equality

$$\mathscr{E}(t_2, x(t_2)) - \mathscr{E}(t_1, x(t_1)) = \int_{t_1}^{t_2} \partial_t \mathscr{E}(t, x(t))dt - \mathscr{D}iss(x(t); [t_1, t_2])$$

for all $0 \le t_1 \le t_2 \le T$, provided that $x(\cdot)$ is smooth enough. However, in general, the function $x(\cdot)$ is only of class BV. This leads us to the definition of weak solution.

Definition 1.1 (**Weak solution**). A function $x : [0,T] \to X$ is called a *weak solution* of the rate-independent system $(\mathscr{E}, \Psi, x_0)$, if it satisfies the initial condition $x(0) = x_0$, the *weak local stability* (w-LS)

$$-\partial_x \mathscr{E}(t, x(t)) \in \partial\Psi(0), \text{ for a.e. } t \in (0,T), \tag{w-LS}$$

and the *energy-dissipation upper bound* (ED-upper)

$$\mathscr{E}(t_2, x(t_2)) - \mathscr{E}(t_1, x(t_1)) \le \int_{t_1}^{t_2} \partial_t \mathscr{E}(t, x(t))dt - \mathscr{D}iss(x(t); [t_1, t_2]) \tag{ED-upper}$$

for all $0 \le t_1 \le t_2 \le T$.

The class of weak solutions is very large, since it contains all of the other notions of solutions, including energetic solutions [27], BV solutions [26], local solutions [34], parametrized solutions [26], epsilon-stable solutions [15], and epsilon-neighborhood solutions (introduced below). This also means that, the weak local stability (w-LS) and the energy-dissipation inequality (ED-ineq) seem to be too weak to characterize solutions for (1.1). However, if we require solution satisfies global minimality at every time instead of the local one (w-LS), then global minimality gives us the opposite side of energy-dissipation inequality (ED-ineq). This leads us to the notion of energetic solution. This notion was first introduced by Mielke and Theil in 1998 [27] and by Francfort and Marigo at the same time [12].

Definition 1.2 (**Energetic solution**). A function $x : [0,T] \to X$ is called an *energetic solution* of the rate-independent system $(\mathscr{E}, \Psi, x_0)$, if it satisfies the initial condition $x(0) = x_0$, the *global stability*

$$\mathscr{E}(t, x(t)) \le \mathscr{E}(t, x) + \Psi(x - x(t)) \tag{S}$$

for all $(t, x) \in [0,T] \times X$, and the *energy-dissipation balance*

$$\mathscr{E}(t_2, x(t_2)) - \mathscr{E}(t_1, x(t_1)) = \int_{t_1}^{t_2} \partial_t \mathscr{E}(s, x(s))ds - \mathscr{D}iss(x(t); [t_1, t_2]) \tag{ED}$$

for all $0 \le t_1 \le t_2 \le T$.

It can be seen that there is no time-dependence in the global stability (S), which means that rate-independent systems are quite "close" to static systems. And the energy balance (ED) can be understood as the conservation of energy, in the sense that, the released energy is always balanced by the difference between the work of external forces and the dissipated energy.

Since there is no derivative of $x(\cdot)$ appearing in the definition of energetic solution, this notion is well-suited even for solutions that are not smooth.

2.2 Construction of energetic solutions

A classical way to obtain energetic solutions is via time-discretization (see for example [21] for more details).

For every $\tau > 0$, we divide the time interval $[0,T]$ into smaller subintervals by a partition $0 = t_0 < t_1 < \cdots < t_N \le T$ such that $t_n - t_{n-1} = \tau$ for all $n \in \{1, 2, \ldots, N\}$ and $T - t_N < \tau$.

Denote $x_0 := x(0)$ the initial position. At every time t_n with $n \in \{1, \dots, N\}$, we require that the position x_n is a global minimizer of the energy plus the dissipation, i.e.

$$x_n \in \text{argmin}_{x \in X}\{\mathscr{E}(t_n, x) + \Psi(x - x_{n-1})\}. \tag{1.5}$$

The discretized solution x^τ is then defined by linear interpolation of $\{x_n\}_{n=0}^N$. It can be showed that $\{x^\tau\}$ has uniformly bounded variation. Therefore, Helly's selection theorem is applicable.

Proposition 1.3. *[Helly's selection theorem [1, 16, 29]] Let I be an interval, X a complete metric space, and $f_n : I \to X$ a sequence of maps with uniformly bounded variation, i.e.*

$$\sup_{n\in\mathbb{N}} \text{Var}(f_n; I) := \sup_{n\in\mathbb{N}} \sup \left\{ \sum_{k=1}^{N} d_X(f_n(t_{k-1}), f_n(t_k)) \mid \{t_k\}_{k=0}^N \textit{ is an partition of } I \right\} < \infty.$$

Assume moreover that for every $t \in I$, the set of values $\{f_n(t)\}$ is relatively compact in X. Then up to a subsequence, f_n converges pointwise to some limit $f : I \to X$ and

$$\text{Var}(f; I) \le \liminf_{n\to\infty} \text{Var}(f_n; I).$$

Thanks to Helly's selection theorem, we find a subsequence $\tau_n \to 0$ such that $x^{\tau_n}(\cdot)$ converges pointwise to some $x(\cdot)$, which also has bounded variation. Moreover, we can check that $x(\cdot)$ satisfies the global stability (S) and the energy-dissipation balance (ED) (see [21]). For the reader's convenience, a proof is presented in Chapter 2 below.

2.3 Some comments

Note that, at the moment, we still do not know whether or not the time-discretization method can characterize all energetic solutions given by Definition 1.2. However, given a BV function $t \mapsto x(t)$, it is easy to verify if $x(\cdot)$ is an energetic solution of the system $(\mathscr{E}, \Psi, x_0)$ by the following criterion (which generalizes Proposition 5.13 in [1]). This proposition is useful to verify an energetic solution in many examples. For the definition of BV functions see [11].

Proposition 1.4. *Let $x \in BV([0,T]; X)$ be a left-continuous function at every $t \in [0,T]$ and satisfy*

(i) $x(t) \in \text{argmin}\ \{\mathscr{E}(t,x) + \Psi(x - x(0))\}$ *for every* $t \in [0,T]$*;*

(ii) $\mathscr{D}iss(x(t); [0,T]) = \Psi(x(T) - x(0))$.

Here the dissipation functional $\Psi : X \to [0, +\infty)$ is convex and 1-positively continuous, the energy functional $\mathscr{E} : [0,T] \times X \to [0,+\infty)$ is C^2 and satisfies the following technical assumption:

There exists $\lambda = \lambda(\mathscr{E})$ such that

$$|\partial_t \mathscr{E}(s,x)| \le \lambda\, \mathscr{E}(s,x) \textit{ for all } (s,x) \in [0,T] \times X.$$

Then, $x(\cdot)$ is an energetic solution to the system $(\mathscr{E}, \Psi, x(0))$.

The condition on the left-continuity is only a technical assumption, since any energetic solution can be modified to be left- or right-continuous. More precisely, we have

Proposition 1.5. *Let $x : [0,T] \to X$ be an energetic solution of the system $(\mathscr{E}, \Psi, x_0)$. Here the energy functional $\mathscr{E} : [0,T] \times X \to [0,+\infty)$ is C^2 and the dissipation functional $\Psi : X \to [0,+\infty)$ is continuous.*

Since $x \in BV([0,T];X)$, the limits $x(t^+)$ and $x(t^-)$ always exist. Moreover, any function $\widetilde{x}(\cdot)$ satisfying $\widetilde{x}(0) = x(0)$, $\widetilde{x}(T) = x(T)$ and

$$\widetilde{x}(t) \in \{x(t^+), x(t^-)\} \quad \text{for all } t \in (0,T),$$

is also an energetic solution to the system $(\mathscr{E}, \Psi, x_0)$.

Moreover, if $x(\cdot)$ is a weak solution of the system $(\mathscr{E}, \Psi, x_0)$, then the function $\widetilde{x}(\cdot)$ defined as in Proposition 1.5 is also a weak solution of the system $(\mathscr{E}, \Psi, x_0)$.

For the reader's convenience, the proofs of Proposition 1.4 and 1.5 are given at the end of this chapter.

Example 1.6. Consider the system defined by the energy functional

$$\mathscr{E}(t,x) := x^2 - x^4 + 0.3\,x^6 + t\,(1 - x^2) - |x|$$

where $t \in [0,2]$, $x \in \mathbb{R}$, the dissipation functional $\Psi(x) := |x|$, and the initial value $x_0 := 0$.

Using Proposition 1.4, one can quickly check that this system admits two energetic solutions

$$x(t) = \begin{cases} 0 & \text{if } t \in [0,1/6), \\ \frac{\sqrt{10+\sqrt{10+90t}}}{3} & \text{if } t \in [1/6,2], \end{cases}$$

and

$$x(t) = \begin{cases} 0 & \text{if } t \in [0,1/6), \\ -\frac{\sqrt{10+\sqrt{10+90t}}}{3} & \text{if } t \in [1/6,2]. \end{cases}$$

In the case that the energy functional is convex, global stability is equivalent to local stability. Moreover, at every time, the position $x(t)$ defined by global minimality is the best one, in the sense that, any loss in energy when moving to another position will be compensated by dissipation. However, when the energy functional is non-convex, global stability turns out to be a too strong requirement. In fact, there are some situations where (1.1) admits a strong solution which is not globally stable (see, for example, in [31]). Moreover, global minimizers make the energetic solutions jump easier than physically feasible and into too far-apart energetic configurations. And hence sometimes, they fail in describing some natural phenomena.

For example, in nature, water always flows downhill, regardless of the fact that flowing uphill for a while may help it to reach a better final position. But if we force the water to satisfy the global minimality, then it would act so that the final position has the lowest altitude, no matter what in some moment it must go uphill.

More mathematically, let us return to Example 1.6. In this example, the energetic solutions jump at time $t = 1/6$ from $x = 0$ to $x = \pm\sqrt{15}/3$. However, this is not a reasonable jump, since the energy plus dissipation functional must increase a little bit when moving from $x = 0$ to $x = \pm\sqrt{15}/3$ (see Figure 2). Hence, the right position at time $t = 1/6$ should be $x = 0$, which is a local minimizer of energy plus dissipation functionals.

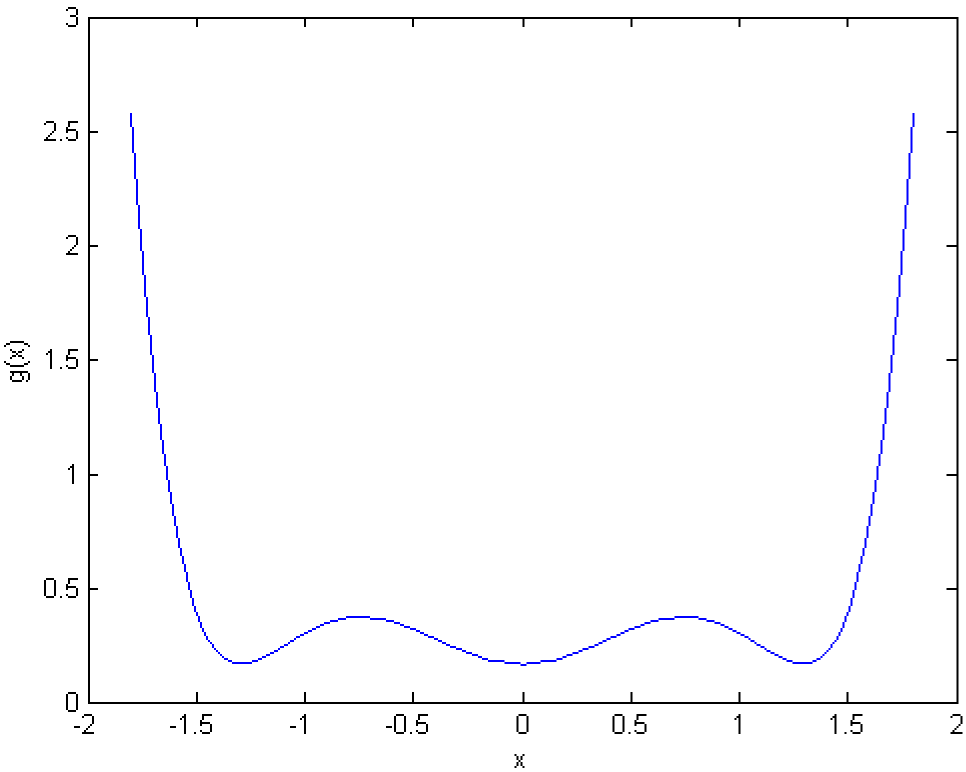

Figure 2. The function $\mathscr{E}(t,x) + \Psi(x) = x^2 - x^4 + 0.3\,x^6 + t\,(1 - x^2)$ at $t = 1/6$.

3 BV solutions

Since global minimality cannot characterize the "natural" solution trajectory in the case that the energy functional is nonconvex, alternative notions of solutions involving local minimality are preferred to overcome this shortcoming of energetic solutions. One of such approaches is vanishing viscosity, which produces the so-called BV solutions [25, 26].

The idea is to add a small viscosity term to the dissipation functional Ψ. This results in a new dissipation functional Ψ_ε, which has super-linear growth at infinity and which converges to Ψ as ε tends to zero in an appropriate meaning. The super-linear growth of Ψ_ε makes the limit $x(\cdot)$ of the sequence $\{x^\varepsilon(\cdot)\}$ when ε goes to 0 difficult to jump, and therefore $x(\cdot)$ should prefer a close-by state which is locally stable to a far-away state which is globally stable. Moreover, Mielke, Rossi and Savaré have proved in [26] that the limit $x(\cdot)$ also satisfies the new energy-dissipation balance, which is

$$\mathscr{E}(t_2, x(t_2)) - \mathscr{E}(t_1, x(t_1)) = \int_{t_1}^{t_2} \partial_t \mathscr{E}(s, x(s))ds - \mathscr{D}iss_{new}(x(t); [t_1, t_2])$$

for all $0 \le t_1 \le t_2 \le T$. Here $\mathscr{D}iss_{new}(xt); [t_1, t_2]) \ge \mathscr{D}iss(x(t); [t_1, t_2])$. The precise formula for the new dissipation function $\mathscr{D}iss_{new}$ will be given later.

3.1 Motivation and definition

The following argument follows that of Mielke, Rossi, and Savaré in [26].

In general, the viscous term can be chosen in the form $\varepsilon^{-1}\Psi_0(\varepsilon v)$, where $\varepsilon > 0$ and $\Psi_0 : X \to [0, \infty]$ is a convex function satisfying

$$(a) \quad \frac{\Psi_0(v)}{|v|} \to 0 \text{ as } |v| \to 0,$$
$$(b) \quad \frac{\Psi_0(v)}{|v|} \to \infty \text{ as } |v| \to \infty.$$

Then for each $\varepsilon > 0$, the dissipation with viscosity reads

$$\Psi_\varepsilon(v) := \Psi(v) + \varepsilon^{-1}\Psi_0(\varepsilon v), \text{ for all } v \in X.$$

Condition (a) guarantees that Ψ_ε converges to Ψ when ε goes to 0, while Ψ_ε has super-linear growth by condition (b). For simplicity, here we choose $\Psi(v) = |v|$ the usual distance in X, and $\Psi_0(v) = |v|^2/2$. Then

$$\Psi_\varepsilon(v) = |v| + \frac{\varepsilon}{2}|v|^2 \text{ for all } v \in X.$$

After adding viscosity, the evolution equation (1.1) becomes

$$0 \in \partial\Psi_\varepsilon(\dot{x}^\varepsilon(t)) + \partial_x\mathscr{E}(t, x^\varepsilon(t)) \text{ for a.e. } t \in (0, T). \tag{1.6}$$

Thanks to the results by Colli and Visintin [5, 4], this equation admits at least one absolutely continuous solution $x^\varepsilon \in AC([0, T]; X)$. We then repeat the argument for energetic solutions, and get an identity similar to (1.4)

$$\begin{aligned} \int_{t_1}^{t_2} [\Psi_\varepsilon(\dot{x}^\varepsilon(t)) + \Psi_\varepsilon^*(-\partial_x\mathscr{E}(t, x^\varepsilon(t)))]\, dt \quad &= \quad \mathscr{E}(t_1, x^\varepsilon(t_1)) - \mathscr{E}(t_2, x^\varepsilon(t_2)) \\ &\quad + \int_{t_1}^{t_2} \partial_t\mathscr{E}(t, x^\varepsilon(t))dt. \end{aligned} \tag{1.7}$$

In particular, we have

$$\int_0^T [\Psi_\varepsilon(\dot{x}^\varepsilon(t)) + \Psi_\varepsilon^*(-\partial_x\mathscr{E}(t, x^\varepsilon(t)))]\, dt \le C,$$

here we denote by C some constant independent of t.

Denote $x^+ := \max\{x, 0\}$. A direct computation gives $\Psi_\varepsilon^*(w) = \frac{1}{2\varepsilon}((|w| - 1)^+)^2$. Since Ψ_ε is non-negative, we get

$$\liminf_{n\to\infty} \int_0^T \frac{1}{2\varepsilon_n}((|\partial_x\mathscr{E}(t, x^{\varepsilon_n}(t))| - 1)^+)^2 dt = \liminf_{n\to\infty} \int_0^T \Psi_{\varepsilon_n}^*(-\partial_x\mathscr{E}(t, x^{\varepsilon_n}(t)))dt \le C.$$

Using Fatou's lemma and the fact that $\varepsilon_n \to 0$, we obtain

$$\int_0^T \liminf_{n\to\infty}((|\partial_x \mathscr{E}(t, x^{\varepsilon_n}(t))| - 1)^+)^2 dt \le \liminf_{n\to\infty} \int_0^T ((|\partial_x \mathscr{E}(t, x^{\varepsilon_n}(t))| - 1)^+)^2 dt = 0.$$

Therefore,

$$\liminf_{n\to\infty}(|\partial_x \mathscr{E}(t, x^{\varepsilon_n}(t))| - 1)^+ = 0 \text{ for a.e. } t \in (0, T). \tag{1.8}$$

Moreover, it follows from Proposition 1.2 that

$$\mathscr{D}iss(x^\varepsilon(t); [0, T]) \le \int_0^T \Psi(\dot{x}^\varepsilon(t))dt \le \int_0^T \Psi_\varepsilon(\dot{x}^\varepsilon(t))dt \le C. \tag{1.9}$$

Thus, the sequence $\{x^\varepsilon(\cdot)\}_{\varepsilon>0}$ has uniformly bounded variation. Then we can apply Helly's selection theorem (see Proposition 1.3) to get a subsequence $\varepsilon_n \to 0$ such that $x^{\varepsilon_n}(\cdot)$ converges pointwise to some BV function $x(\cdot)$. By the continuity of $\mathscr{E}$ we also have that $\partial_x \mathscr{E}(t, x^{\varepsilon_n}(t))$ converges pointwise to $\partial_x \mathscr{E}(t, x(t))$ as $n \to \infty$. Hence, we obtain from (1.8)

$$|\partial_x \mathscr{E}(t, x(t)))| \le 1 \text{ for a.e. } t \in (0, T).$$

Moreover, by the continuity of $x(\cdot)$, we get

$$|\partial_x \mathscr{E}(t, x(t)))| \le 1 \text{ if } x(\cdot) \text{ is continuous at } t.$$

This condition is in fact the weak local stability (w-LS), since we have chosen $\Psi(x) = |x|$ for all $x \in X$.

On the other hand, we have the lower semicontinuity of dissipation

$$\mathscr{D}iss(x(t); [t_1, t_2]) \le \liminf_{n\to\infty} \mathscr{D}iss(x^{\varepsilon_n}(t); [t_1, t_2]). \tag{1.10}$$

By Proposition 1.2, we also have

$$\mathscr{D}iss(x^{\varepsilon_n}(t); [t_1, t_2]) \le \int_{t_1}^{t_2} \left[\Psi_{\varepsilon_n}(\dot{x}^{\varepsilon_n}(t)) + \Psi^*_{\varepsilon_n}(-\partial_x \mathscr{E}(t, x^{\varepsilon_n}(t)))\right] dt. \tag{1.11}$$

Combining (1.10), (1.11) and (1.7), and taking into account the continuity of $\mathscr{E}$ and the convergence of $x^{\varepsilon_n}(t)$ to $x(t)$, we get the following energy-dissipation inequality (ED-ineq)

$$\mathscr{E}(t_2, x(t_2)) - \mathscr{E}(t_1, x(t_1)) \le \int_{t_1}^{t_2} \partial_t \mathscr{E}(t, x(t))dt - \mathscr{D}iss(x(t); [t_1, t_2]), \text{ for all } 0 \le t_1 \le t_2 \le T.$$

Thus, the BV function $x(\cdot)$ also satisfies the definition of weak solution (see Definition 1.1).

Moreover, Mielke, Rossi and Savaré have proved in [26] that we have an even better upper bound

$$\mathscr{E}(t_2, x(t_2)) - \mathscr{E}(t_1, x(t_1)) \le \int_{t_1}^{t_2} \partial_t \mathscr{E}(t, x(t))dt - \mathscr{D}iss_{new}(x(t); [t_1, t_2]) \tag{1.12}$$

for all $0 \le t_1 \le t_2 \le T$, where the new dissipation is defined by

$$\begin{aligned} \mathscr{D}iss_{new}(x(t);[t_1,t_2]) \;:=\; & \mathscr{D}iss(x(t);[t_1,t_2]) - \sum_{t\in J}\left[|x(t^-)-x(t)| + |x(t)-x(t^+)|\right] \\ & + \sum_{t\in J}\left[\Delta_{new}(t,x(t^-),x(t)) + \Delta_{new}(t,x(t),x(t^+))\right], \end{aligned}$$

J is the jump set of $x(\cdot)$ and

$$\begin{aligned} &\Delta_{new}(t;a,b) \\ &:= \inf\left\{\int_0^1 |\dot v(r)|\cdot \max\{1,|\partial_x\mathscr{E}(t,v(r))|\} \mid v\in AC([0,T];X), v(0)=a, v(1)=b\right\}. \end{aligned}$$

There is a general fact (see [25] Proposition 4.2) that if a BV function satisfies the weak local stability (w-LS), then it also satisfies the opposite of inequality (1.12). Hence, in (1.12) we have an equality. It leads us to the notion of BV solution.

Definition 1.3 (BV solution). A function $x:[0,T]\to X$ is called a *BV solution* of the rate-independent system $(\mathscr{E},\Psi,x_0)$ if it satisfies the initial condition $x(0)=x_0$, the *weak local stability* (w-LS)

$$|\partial_x\mathscr{E}(t,x(t))| \le 1 \text{ provided that } t\mapsto x(t) \text{ is continuous at } t, \tag{w-LS}$$

and the *new energy-dissipation balance* (ED-new)

$$\mathscr{E}(t_2,x(t_2)) - \mathscr{E}(t_1,x(t_1)) = \int_{t_1}^{t_2} \partial_t\mathscr{E}(s,x(s))ds - \mathscr{D}iss_{new}(x(t);[t_1,t_2]) \tag{ED-new}$$

for all $0\le t_1\le t_2\le T$.

Unlike energetic solutions, the stability condition for BV solutions (w-LS) does not hold for every t. Roughly speaking, (w-LS) can only tell us the information on solutions at continuity points. The information at jump points is contained in the new energy-dissipation balance (ED-new). More than that, (ED-new) also reveals the information of BV solutions along the jump path. Indeed, if the BV solution $x(\cdot)$ jumps at time t, then there exists an absolutely continuous path $v:[0,1]\to X$ connecting $x(t^-)$ and $x(t^+)$ such that along this path, we have that $|\partial_x\mathscr{E}(t_0,v(s))|\ge 1$ for all $s\in[0,1]$ (see [25, 26] for further details). However, we can see that the weak local stability (w-LS) is really weak, in the sense that it allows for both local minimizers and local maximizers. Moreover, we cannot expect a better local stability, since in some cases, the BV solutions are really local maximizer (see Example 1.7 below).

Example 1.7. Consider the system defined by the energy functional $\mathscr{E}(t,x) := t\,(x^6-x^4) - |x|$ where $t\in[0,1]$ and $x\in\mathbb{R}$, the dissipation function $\Psi(x):=|x|$, and the initial value $x_0:=0$. Then the BV solution corresponding to the dissipation with viscosity

$$\Psi_\varepsilon(x) = |x| + \frac{\varepsilon}{2}x^2,$$

is $x(t)=0$ for all $t\in[0,1]$.

In the figure below, we see that $x = 0$ is a local maximizer for the functional $x \mapsto \mathscr{E}(t, x) + |x|$ when $t > 0$. A detailed proof of this claim is given in Chapter 4.

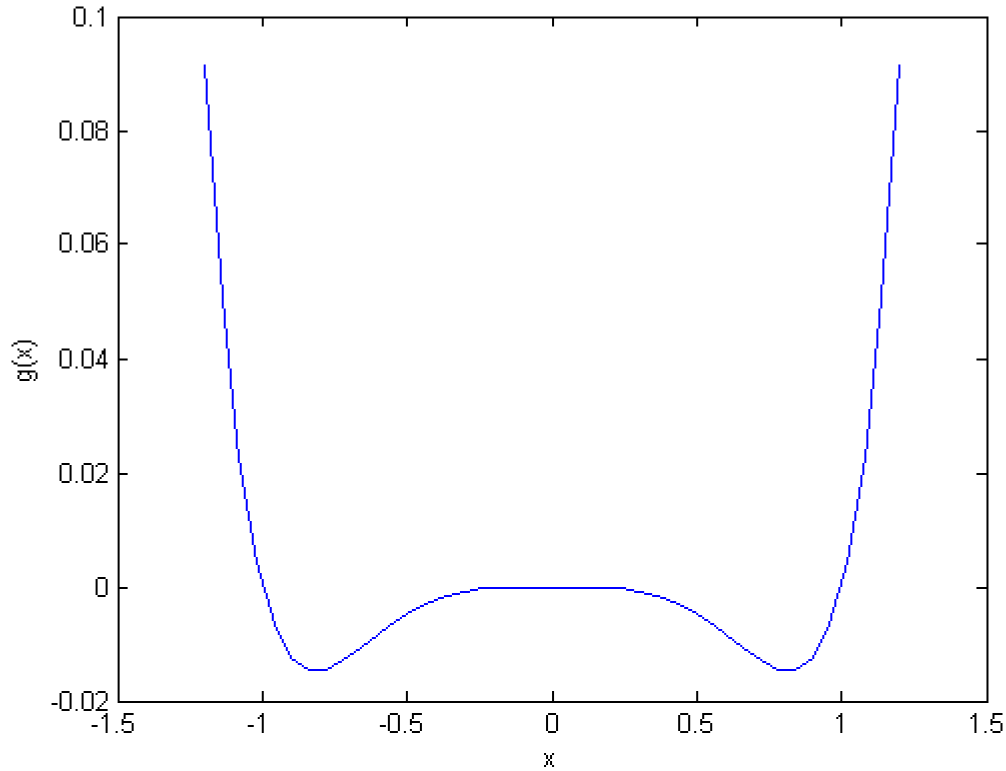

Figure 3. $\mathscr{E}(t,x) + \Psi(x) = t\,(x^6 - x^4)$ with $t = 0.1$.

3.2 Construction of BV solutions

Here we construct a BV solution by vanishing viscosity. This construction was given by Mielke, Rossi, and Savaré in [26]. For simplicity, here we choose the dissipation and dissipation with viscosity as follows

$$\Psi(v) := |v|,$$
$$\Psi_\varepsilon(v) := |v| + \frac{\varepsilon}{2}|v|^2,$$

for all $v \in X$.

For every $\varepsilon > 0, \tau > 0$, we choose the following partition of $[0, T] : 0 = t_0 < t_1 < \cdots < t_N \leq T$ such that $t_n - t_{n-1} = \tau$ for all $n \in \{1, 2, \ldots, N\}$ and $T - t_N < \tau$.

Denote $x_0 := x(0)$ the initial position. Then the approximation position $x_n^{\tau,\varepsilon}$ at every time t_n is defined by iteration as follows

$$x_n^{\tau,\varepsilon} \in \text{argmin}_{x \in X}\{\mathscr{E}(t_n, x) + |x - x_{n-1}^{\tau,\varepsilon}| + \frac{\varepsilon}{2\tau}|x - x_{n-1}^{\tau,\varepsilon}|^2\} \text{ for all } n \in \{1, 2, \ldots, N\}.$$

We shall assume that $\varepsilon \to 0$, $\tau \to 0$ and $\varepsilon/\tau \to \infty$. Hence, the appearance of the term $\frac{\varepsilon}{2\tau}|x - x_{n-1}^{\tau,\varepsilon}|^2$ ensures that $x_n^{\tau,\varepsilon}$ is very "close" to $x_{n-1}^{\tau,\varepsilon}$. In other words, even if we are looking for a minimizer over all X, only points close to $x_{n-1}^{\tau,\varepsilon}$ can be chosen. Then we also denote the discretized solution $x^{\tau,\varepsilon}$ by taking the interpolation of $\{x_n^{\tau,\varepsilon}\}_{n=0}^N$. The BV solution is obtained by taking the limit of $\{x^{\tau,\varepsilon}\}$ when $\varepsilon \to 0, \tau \to 0$ such that $\varepsilon/\tau \to \infty$. For the reader's convenience, a detailed proof is given in Chapter 2.

3.3 Some comments

Note that the BV solution defined above depends heavily on the choice of the viscosity. There are examples showing that for different choices of viscosity, we get different BV solutions that jump at different times.

Now we are back to Example 1.7, but here we consider the BV solution corresponding to the different viscosity term.

Example 1.8. Consider the system defined by the energy functional $\mathscr{E}(t,x) := t\,(x^6 - x^4) - |x|$, where $t \in [0,1]$ and $x \in \mathbb{R}$, the dissipation function $\Psi(x) := |x|$, and the initial value $x_0 := 0$.

As we know from Example 1.7 that the BV solution corresponding to viscosity $\frac{\varepsilon}{2}x^2$ is $x(t) = 0$ for all $t \in [0,1]$.

Now we choose the viscosity $\varepsilon^5 x^6$ with $\varepsilon^{-25/18}\tau \to \infty$ (where τ is the time step in the discretization). Then the BV solutions corresponding to this viscosity are

$$x(0) = 0,\ x(t) = \sqrt{2/3} \text{ for all } t \in (0,1]$$

and

$$x(0) = 0,\ x(t) = -\sqrt{2/3} \text{ for all } t \in (0,1].$$

A detailed proof of this claim is given in Chapter 4.

It can be seen in Figure 3 above that when $t > 0$, the BV solution corresponding to viscosity $\frac{\varepsilon}{2}x^2$ is a local maximizer of energy plus dissipation instead of a local minimizer, while the BV solutions corresponding to viscosity $\varepsilon^5 x^6$ is global minimizer of energy plus dissipation (see Figure 4 below). This fact implies that in some specific cases, a "too strong" viscosity could prevent the solution from jumping even when a jump might be expected. Besides, similarly to energetic solutions, we do not know if BV solutions obtained by vanishing viscosity can characterize all of BV solutions given by definition.

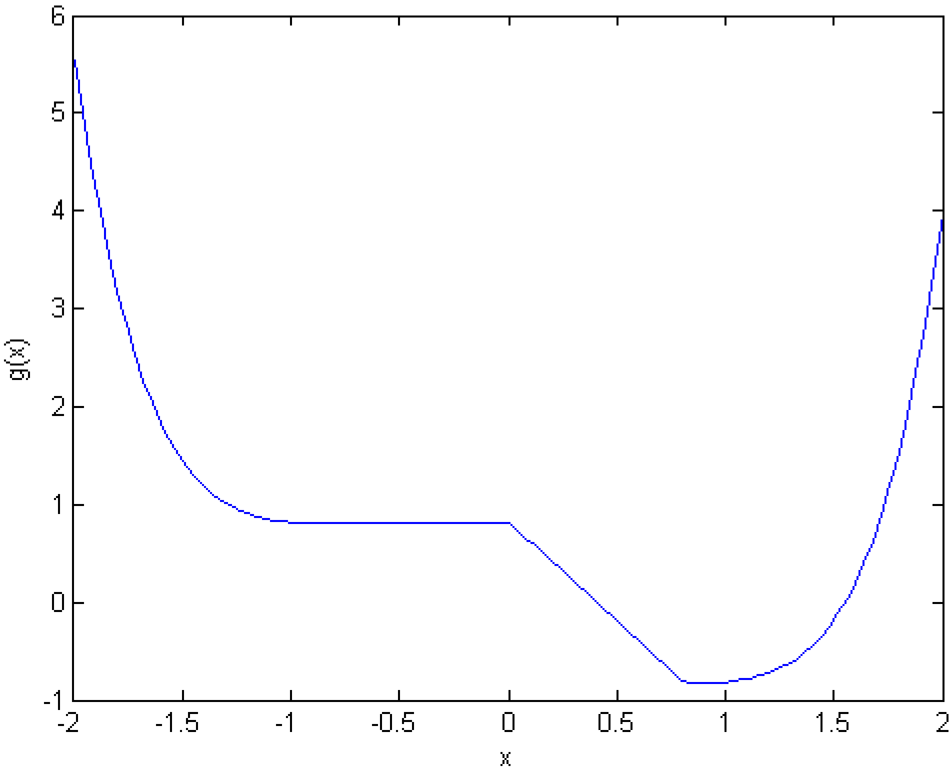

Figure 4. $\mathscr{E}(t,x) + \Psi(x - \sqrt{2/3}) = t(x^6 - x^4) - |x| + |x - \sqrt{2/3}|$ with $t = 0.1$.

4 Another construction of BV solutions

4.1 Motivation and construction

Another way to avoid global minimality is to find the minimizer x_n in (1.5) in a small neighborhood of x_{n-1}. More precisely, for any $\varepsilon > 0, \tau > 0$, consider the following partition of $[0,T] : 0 = t_0 < t_1 < \cdots < t_N \leq T$, where $t_n - t_{n-1} = \tau$ for all $n \in \{1, 2, \ldots, N\}$ and $T - t_n < \tau$. Set $x_0^{\varepsilon,\tau} := x(0)$. For all $n \in \{1, 2, \ldots, N\}$, define the sequence $\{x_n^{\varepsilon,\tau}\}$ as follows

$$x_n^{\varepsilon,\tau} \in \operatorname{argmin}_{x \in X}\{\mathscr{E}(t_n, x) + |x - x_{n-1}^{\varepsilon,\tau}| \mid |x - x_{n-1}^{\varepsilon,\tau}| \leq \varepsilon\}.$$

Here for simplicity, we focus on the case $\Psi(x) = |x|$ for all $x \in X$.

The discretized solution $t \mapsto x^{\varepsilon,\tau}(t)$ is then defined by interpolation as follows

$$x^{\varepsilon,\tau}(t) := x_{n-1}^{\varepsilon,\tau} \text{ for every } t \in [t_{n-1}, t_n), n \in \{1, 2, \ldots, N\}.$$

The *epsilon-neighborhood solution* $x^\varepsilon(t)$ is defined by the pointwise limit of $\{x^{\varepsilon,\tau}(t)\}$ when $\tau \to 0$ (such limit exists thanks to Helly's selection theorem). Then we can prove (see Chapter 2) that outside the jump set, $x^\varepsilon(t)$ satisfies the minimality in an epsilon neighborhood of $x^\varepsilon(t)$ (eps-LS) and the energy-dissipation inequality (ED-ineq). In particular, any epsilon-neighborhood solution satisfies the definition of weak solution (see Definition 1.1).

Once again, Helly's selection theorem gives us the pointwise limit $x(t)$ of $\{x^\varepsilon(t)\}$ when taking $\varepsilon \to 0$. In Chapter 2 (see Theorem 2.18), we will prove that $x(\cdot)$ fulfills the definition of BV solution, i.e., it satisfies the weak local stability (w-LS) outside the jump set, and the new energy-dissipation balance (ED-new) at every time. We call it *BV solution constructed by epsilon-neighborhood*.

This approach was first suggested in [17, Section 6] for one dimensional case when ε is chosen proportional to the square root of the time-step. The existence was then obtained in [10] via reparametrization. It was also proved in [10] that the solution satisfies the weak local stability (w-LS). However, to our knowledge, the fact that the solution satisfies the new energy-dissipation balance (ED-new) does not explicitly appear in the literature. In Chapter 2 (Section 2.3), we shall prove the existence and properties of epsilon-neighborhood solutions $x^\varepsilon(\cdot)$ and its limit $x(\cdot)$.

Remark. Roughly speaking, this approach is a special case of the vanishing viscosity approach when viscosity term is chosen as follows

$$\Psi_0(v) := \begin{cases} 0 & \text{if } |v| \leq 1, \\ +\infty & \text{if } |v| > 1. \end{cases}$$

4.2 Definition of epsilon-neighborhood solution

Definition 1.4 (Epsilon-neighborhood solution). For any fixed $\varepsilon > 0$, a function $x^\varepsilon : [0,T] \to X$ is called an *epsilon-neighborhood solution* of the rate-independent system $(\mathscr{E}, \Psi, x_0)$ if it satisfies the initial condition $x^\varepsilon(0) = x_0$, the *epsilon local stability* (eps-LS)

$$\mathscr{E}(t, x^\varepsilon(t)) \leq \mathscr{E}(t,x) + |x - x^\varepsilon(t)| \text{ for all } |x - x^\varepsilon(t)| \leq \varepsilon, \qquad \text{(eps-LS)}$$

provided that $x^\varepsilon(\cdot)$ is continuous at t, and the *energy-dissipation inequality* (ED-ineq)

$$\mathscr{E}(t_2, x^\varepsilon(t_2)) - \mathscr{E}(t_1, x^\varepsilon(t_1)) \leq \int_{t_1}^{t_2} \partial_t \mathscr{E}(s, x^\varepsilon(s))ds - \mathscr{D}iss(x^\varepsilon(t); [t_1, t_2]) \qquad \text{(ED-ineq)}$$

for all $0 \leq t_1 \leq t_2 \leq T$.

The epsilon local stability (eps-LS) is stronger than the weak local stability (w-LS). Hence, epsilon-neighborhood solutions belong to the class of weak solutions.

When ε becomes smaller and smaller, the epsilon-neighborhood solutions behave more and more like BV solutions. Besides, in many examples, we see that when ε is small enough, the epsilon-neighborhood solution $x^\varepsilon(\cdot)$ is independent of ε, and hence, $x^\varepsilon(\cdot)$ coincides with its limit $x(\cdot)$. Thus, $x(\cdot)$ satisfies the epsilon local stability (eps-LS) for some $\varepsilon > 0$. Therefore, in those cases, $x(\cdot)$ satisfies both the epsilon local stability (eps-LS) for some $\varepsilon > 0$ and the new energy-dissipation balance (ED-new).

Now we are back to the system given in Example 1.7, but here we are interested in the BV solutions constructed by epsilon-neighborhood.

Example 1.9. Consider the system defined by the energy function $\mathscr{E}(t,x) := t\,(x^6 - x^4) - |x|$ where $t \in [0,1]$ and $x \in \mathbb{R}$, the dissipation function $\Psi(x) := |x|$, and the initial value $x_0 := 0$.

It was already shown in Example 1.8 that the BV solution corresponding to the viscous dissipation $\Psi_\varepsilon(x) = |x| + \frac{\varepsilon}{2}x^2$ is $x(t) = 0$ for all $t \in [0,1]$.

When $\varepsilon \in (0, \sqrt{2/3}]$, the epsilon-neighborhood solutions are

$$x^\varepsilon(0) = 0,\; x^\varepsilon(t) = \sqrt{2/3} \text{ for all } t \in (0,1]$$

and

$$x^\varepsilon(0) = 0,\ x^\varepsilon(t) = -\sqrt{2/3} \ \text{ for all } t \in (0, 1].$$

The proof of this claim is given in Chapter 4.

In Example 1.8, the BV solution $x(t) = 0$ for all $t \in [0, 1]$ does not satisfy the strong local stability, namely

$$x(t) \text{ is a local minimizer of the functional } z \mapsto \mathscr{E}(t, z) + |z - x(t)| \text{ for a.e. } t. \qquad \text{(s-LS)}$$

However, the BV solutions constructed by epsilon-neighborhood in Example 1.9 satisfy the strong local stability (s-LS). Hence, the question on whether a BV solution constructed by epsilon neighborhood always satisfy (s-LS) arises naturally. Unfortunately, the answer is negative, as shown in the following example.

Example 1.10. Consider the system defined by the energy functional $\mathscr{E}(t, x) := t\, g(x) - x$ with $g(x) := x^5 \sin(1/x)$, $t \in [0, 1]$, the dissipation function $\Psi(x) := |x|$, and the initial value $x_0 := 0$. Note that $g(\cdot)$ has a unique global minimizer $z_1 = 0.2638367621...$ (see Figure 5 below). Moreover,

(i) The energetic solution constructed by time-discretization is

$$x(0) = 0 \ \text{ and } x(t) = z_1 \ \text{ for all } t \in (0, 1].$$

(ii) The BV solution constructed by epsilon-neighborhood is $x(t) = 0$ for all $t \in [0, 1]$. Here we can choose any neighborhood of the form $I_\varepsilon(a) = a + I_\varepsilon(0)$ where $I_\varepsilon(0)$ is a closed connected neighborhood of 0 with diameter of order $O(\varepsilon)$.

(iii) The BV solution constructed by vanishing viscosity is $x(t) = 0$ for all $t \in [0, 1]$. Here we can choose an arbitrary viscosity of the form $\varepsilon^{-1}\Psi_0(\varepsilon x)$ where $\Psi_0 : \mathbb{R} \to [0, \infty)$ and $\lim_{|x|\to\infty} \Psi_0(x)/|x| = \infty$.

A detailed proof of this claim is given in Chapter 4.

In this example, both the BV solution constructed by vanishing viscosity and the BV solution constructed by epsilon-neighborhood take the value 0 for every $t \in [0, 1]$. However, as we can see in Figure 5, for every $t > 0$, $x = 0$ is neither a local minimizer nor a local maximizer of the function $z \mapsto \mathscr{E}(t, z) + |z|$.

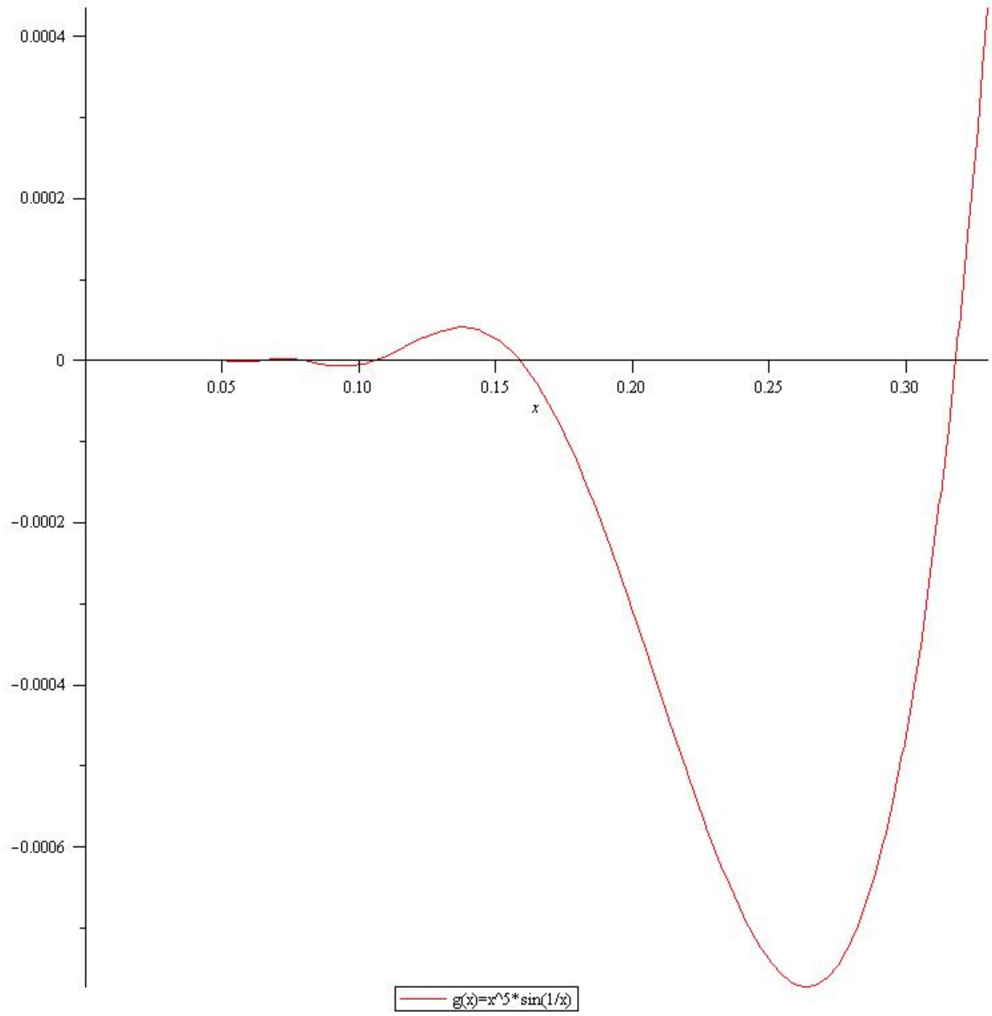

Figure 5. The function $\mathscr{E}(t,x) + |x| = tx^5 \sin(1/x)$ when $t = 1$ and $x > 0$.

Similarly to BV solutions constructed by vanishing viscosity, along the jump path of BV solutions constructed by epsilon-neighborhood, we have $|\partial_x \mathscr{E}(t,\cdot)| \geq 1$. Moreover, Lemma 2.24 in Chapter 2 tells us that the energy plus dissipation along each jump path is non-increasing. This fact makes the behavior of BV solutions constructed by epsilon-neighborhood at jumps more reasonable than the behavior of energetic solutions. However, BV solutions constructed by epsilon-neighborhood still depend on the way we choose the neighborhood, see the following example.

Example 1.11. Consider the system defined by the energy functional

$$\mathscr{E}(t,x) := t\left(\frac{x}{4} - \frac{3}{4}|x| + |x+1| + |x-2|\right) - |x|$$

where $t \in [0,1]$ and $x \in \mathbb{R}$, the dissipation function $\Psi(x) := |x|$, and the initial value $x_0 := 0$. We have

(i) The BV solution constructed by epsilon-neighborhood with the usual neighborhood $I_\varepsilon(a) = [a-\varepsilon, a+\varepsilon]$ is

$$x(0) = 0 \quad \text{and} \quad x(t) = -1 \quad \text{for all } t \in (0,1].$$

(ii) The BV solution constructed by epsilon-neighborhood with the neighborhood $I_\varepsilon(a) = [a-\varepsilon, a+3\varepsilon]$ is

$$x(0) = 0 \text{ and } x(t) = 2 \text{ for all } t \in (0,1].$$

(iii) The BV solutions constructed by epsilon-neighborhood with the neighborhood $I_\varepsilon(a) = [a-\varepsilon, a+2\varepsilon]$ are both solutions given before.

The proof of this claim can be found in Chapter 4.

5 Comparison of energetic and BV solutions

In this section, we will compare energetic solutions, BV solutions constructed by vanishing viscosity and BV solutions constructed by epsilon-neighborhood.

5.1 Energetic and BV solutions may be the same

We consider again the system given in Example 1.7.

Example 1.12. Consider the system defined by the energy functional $\mathscr{E}(t,x) := t\,(x^6 - x^4) - |x|$ where $t \in [0,1]$, the dissipation function $\Psi(x) := |x|$, and the initial value $x_0 := 0$. Then

(i) The energetic solutions constructed by time-discretization are

$$x(0) = 0,\ x(t) = \sqrt{2/3} \text{ for all } t \in (0,1]$$

and

$$x(0) = 0,\ x(t) = -\sqrt{2/3} \text{ for all } t \in (0,1].$$

These energetic solutions satisfy the definition of BV solution.

(ii) The BV solutions corresponding to the viscosity term $\varepsilon^5 x^6$ with $\varepsilon^{-25/18}\tau \to \infty$ (where τ is the time step in the discretization) are precisely the energetic solutions.

(iii) The BV solutions constructed by epsilon-neighborhood are precisely the energetic solutions.

5.2 BV solutions constructed by epsilon neighborhood may jump later than energetic solutions

Since the construction of a solution by epsilon neighborhood uses local minimizers, such solution is expected to jump later than energetic solutions. This fact is illustrated by the following example.

Example 1.13. Consider the system defined by the energy functional

$$\mathscr{E}(t,x) := x^2 - x^4 + 0.3\,x^6 + t\,(1-x^2) - |x|,\ t \in [0,2], x \in \mathbb{R},$$

the dissipation function $\Psi(x) := |x|$, and the initial value $x_0 := 0$. We have

(i) The energetic solutions constructed by time-discretization satisfy either

$$x(t) = 0 \text{ if } t < 1/6,\ x(1/6) \in \{0, \sqrt{5/3}\},\ x(t) = \frac{\sqrt{10 + \sqrt{10 + 90t}}}{3} \text{ if } t > 1/6$$

or

$$x(t) = 0 \text{ if } t < 1/6,\ x(1/6) \in \{0, -\sqrt{5/3}\},\ x(t) = -\frac{\sqrt{10 + \sqrt{10 + 90t}}}{3} \text{ if } t > 1/6.$$

(ii) The BV solutions constructed by epsilon-neighborhood satisfy either

$$x(t) = 0 \text{ if } t < 1,\ x(t) = \frac{\sqrt{10 + \sqrt{10 + 90t}}}{3} \text{ if } t > 1$$

or

$$x(t) = 0 \text{ if } t < 1,\ x(t) = -\frac{\sqrt{10 + \sqrt{10 + 90t}}}{3} \text{ if } t > 1.$$

As we can see from the example above, the energetic solutions jump at $t = 1/6$. This jump point is not reasonable since along the jump path, there are some moment the energy plus dissipation function is increased (see Figure 2 above). On the other hand, the BV solution constructed by epsilon-neighborhood jumps at $t = 1$. This is a reasonable jump time, since the functional $x \mapsto \mathscr{E}(t, x) + \Psi(x)$ admits $x = 0$ as a local minimizer when $t < 1$ (see Figure 2), and as a local maximizer when $t > 1$ (see Figure 6 below).

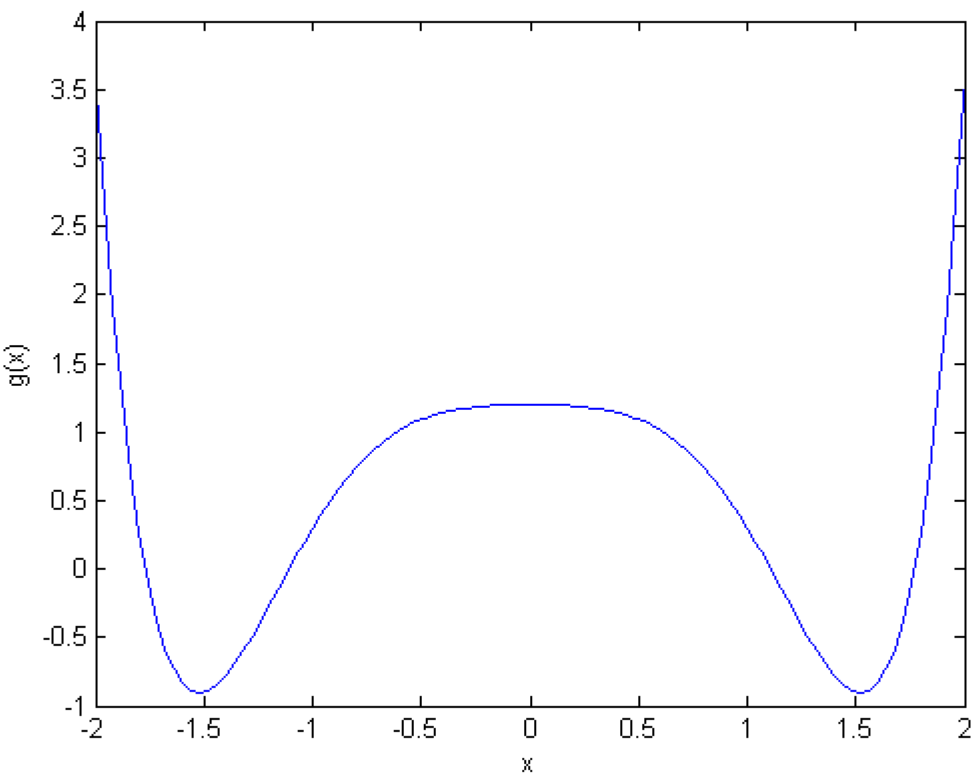

Figure 6. The function $\mathscr{E}(t, x) + \Psi(x)$ when $t = 1.2$.

Now we compare BV solutions constructed by epsilon-neighborhood with BV solutions constructed by vanishing viscosity.

5.3 BV solutions constructed by epsilon neighborhood may jump sooner than those constructed by vanishing viscosity

As we mentioned above, BV solutions constructed by epsilon-neighborhood belong to the class of BV solutions. Moreover, there are BV solutions that cannot be obtained by using epsilon-neighborhood. In fact, the following example points out that there exist BV solutions constructed by vanishing viscosity which are different from BV solutions constructed by epsilon-neighborhood. Now we are back to the system given in Example 1.6.

Example 1.14. Consider the system defined by the energy functional

$$\mathscr{E}(t,x) := x^2 - x^4 + 0.3\,x^6 + t\,(1-x^2) - |x|,\ t\in[0,2],\ x\in\mathbb{R},$$

the dissipation function $\Psi(x) := |x|$, and the initial value $x_0 := 0$. We have

(i) The BV solutions constructed by epsilon-neighborhood satisfy either

$$x(t) = 0 \text{ if } t<1,\ x(t) = \frac{\sqrt{10+\sqrt{10+90t}}}{3} \text{ if } t>1$$

or

$$x(t) = 0 \text{ if } t<1,\ x(t) = -\frac{\sqrt{10+\sqrt{10+90t}}}{3} \text{ if } t>1.$$

(ii) The BV solution with viscous dissipation $\Psi_\varepsilon(x) = |x| + \varepsilon x^2$ is $x(t) = 0$ for all $t\in[0,2]$.

The proof of this claim is given in Chapter 4.

Appendix A: Proofs of some technical lemmas

Proof of Proposition 1.2 (i). By the convexity of Ψ, the following inequality holds true for every function $x\in W^{1,1}([0,1];X)$ such that $x(0)=x_0$ and $x(1)=x_1$.

$$\int_0^1 \Psi(\dot{x}(t))dt \ge \Psi\left(\int_0^1 \dot{x}(t)dt\right) = \Psi(x_1-x_0).$$

Taking the infimum of the left-hand side of the above inequality over all functions $x\in W^{1,1}([0,1];X)$ such that $x(0)=x_0$ and $x(1)=x_1$, we get

$$\mathcal{D}(x_0,x_1) \ge \Psi(x_1-x_0). \tag{1.13}$$

On the other hand, choosing $y(t) = x_0 + t(x_1-x_0)$ then $y\in W^{1,1}([0,1];X)$ and $\dot{y}(t) = x_1-x_0$. Hence

$$\int_0^1 \Psi(\dot{y}(t))dt = \Psi(x_1-x_0).$$

Therefore, by definition of $\mathcal{D}(x_0,x_1)$ we obtain

$$\mathcal{D}(x_0,x_1) \le \Psi(x_1-x_0). \tag{1.14}$$

Thus, combining (1.13) and (1.14) we get the result. □

Proof of Proposition 1.2 (ii). **Step 1.** First, we prove that

$$\mathscr{D}iss(u(t);[t_1,t_2]) \le \int_{t_1}^{t_2} \Psi(\dot{u}(t))dt \quad \text{for every } u \in AC([0,T];X).$$

In fact, since $u \in AC([0,T];X)$, we have that $\dot{u} \in L^1(0,T)$ and

$$u(t_2) - u(t_1) = \int_{t_1}^{t_2} \dot{u}(t)dt \qquad \text{for all } 0 \le t_1 < t_2 \le T. \tag{1.15}$$

On the other hand, by Proposition 1.2 (i) and Jensen's inequality for the convex function Ψ, we have

$$\mathcal{D}(u(t_1),u(t_2)) = \Psi\left(u(t_2)-u(t_1)\right) = \Psi\left(\int_{t_1}^{t_2} \dot{u}(s)ds\right) \le \int_{t_1}^{t_2} \Psi(\dot{u}(s))ds. \tag{1.16}$$

Choosing an arbitrary partition $t_1 = s_0 < s_1 < \cdots < s_N = t_2$ of $[t_1,t_2]$ and applying (1.16) for $(u(s_{i-1}),u(s_i))$, we obtain

$$\sum_{i=1}^{N} \mathcal{D}(u(s_{i-1}),u(s_i)) = \sum_{i=1}^{N} \int_{s_{i-1}}^{s_i} \Psi(\dot{u}(s))ds \le \int_{t_1}^{t_2} \Psi(\dot{u}(s))ds.$$

Taking the supremum over all of the partitions of $[t_1,t_2]$, we get the desired result.

Step 2. Now, we prove the converse inequality

$$\mathscr{D}iss(u(t);[t_1,t_2]) \ge \int_{t_1}^{t_2} \Psi(\dot{u}(t))dt \quad \text{for every } u \in AC([0,T];X).$$

We need the following claims.

Claim 1: For any $0 \le s_1 \le s_2 \le T$, we have that

$$\begin{aligned}\int_{s_1}^{s_2} \Psi(\dot{u}(s))ds \quad &\le \quad \Psi(u(s_2)-u(s_1)) + \int_{s_1}^{s_2} \Psi(\dot{u}(s)-\dot{u}(s_0))ds \\ &\quad +\Psi\left(\int_{s_1}^{s_2} [\dot{u}(s_0)-\dot{u}(s)]ds\right) \text{ for every } s_0 \in [s_1,s_2].\end{aligned} \tag{1.17}$$

Proof of Claim 1. Since Ψ is convex and positively 1-homogeneous, the triangle inequality holds

$$\Psi(a+b) \le \Psi(a) + \Psi(b) \text{ for all } a,b \in X. \tag{1.18}$$

From the above triangle inequality, it follows that

$$\int_{s_1}^{s_2} \Psi(\dot{u}(s))ds \le \int_{s_1}^{s_2} \Psi(\dot{u}(s)-\dot{u}(s_0))ds + \int_{s_1}^{s_2} \Psi(\dot{u}(s_0))ds. \tag{1.19}$$

By the 1-homogeneity of Ψ and the triangle inequality (1.18), we can write

$$\begin{aligned}
\int_{s_1}^{s_2} \Psi(\dot{u}(s_0))ds &= (s_2 - s_1) \cdot \Psi\left(\dot{u}(s_0)\right) \\
&= \Psi\left((s_2 - s_1) \cdot \dot{u}(s_0)\right) \\
&= \Psi\left(\int_{s_1}^{s_2} \dot{u}(s_0)ds\right) \\
&= \Psi\left(\int_{s_1}^{s_2} [\dot{u}(s_0) - \dot{u}(s) + \dot{u}(s)]ds\right) \\
&\leq \Psi\left(\int_{s_1}^{s_2} [\dot{u}(s_0) - \dot{u}(s)]ds\right) + \Psi\left(\int_{s_1}^{s_2} \dot{u}(s)ds\right) \\
&= \Psi\left(\int_{s_1}^{s_2} [\dot{u}(s_0) - \dot{u}(s)]ds\right) + \Psi(u(s_2) - u(s_1)). \qquad (1.20)
\end{aligned}$$

In the last equality, we have employed the equality (1.15). Then, the estimate (1.17) follows by (1.19) and (1.20). □

Claim 2: For $\mathcal{L}^1$-a.e. $s \in [0, T]$ and every $\varepsilon > 0$, there exists $r_0 > 0$ (depending on ε and s) such that

$$\int_{s-r}^{s+r} \Psi(\dot{u}(t) - \dot{u}(s))dt \leq \varepsilon r, \text{ for all } r \leq r_0.$$

Proof of Claim 2. This proof follows by an argument in [11] (see Section 1.7, Corollary 1).

Since X is a finite dimensional normed vector space, we can choose a countable dense subset $\{x_i\}_{i=1}^{\infty}$ of X. Since $\Psi(\dot{u}) \in L^1$, we can apply the Lebesgue-Besicovitch Differentiation Theorem (see [11] page 43) to get

$$\lim_{r\to 0} \frac{1}{2r} \int_{s-r}^{s+r} \Psi(\dot{u}(t) - x_i)dt = \Psi(\dot{u}(s) - x_i)$$

for $\mathcal{L}^1$-a.e. $s \in (0, T)$ and $i = 1, 2, \ldots$. Thus there exists a set $A \subset [0, T]$ such that $\mathcal{L}^1(A) = 0$ and $s \in X \backslash A$ implies

$$\lim_{r\to 0} \frac{1}{2r} \int_{s-r}^{s+r} \Psi(\dot{u}(t) - x_i)dt = \Psi(\dot{u}(s) - x_i)$$

for all i. Fix $s \in [0, T]\backslash A$ and $\varepsilon > 0$. Since $\Psi(0) = 0$ and Ψ is convex, we have

$$\lim_{z\to 0} \Psi(z) = 0.$$

Moreover, since $\{x_i\}_{i=1}^{\infty}$ we can choose x_i such that

$$\max\{\Psi(\dot{u}(s) - x_i), \Psi(x_i - \dot{u}(s))\} < \frac{\varepsilon}{4}.$$

Then

$$\begin{aligned}\limsup_{r\to 0}\frac{1}{2r}\int_{s-r}^{s+r}\Psi(\dot u(t)-\dot u(s))dt &\le \limsup_{r\to 0}\frac{1}{2r}\int_{s-r}^{s+r}\Psi(\dot u(t)-x_i)dt\\ &\quad+\limsup_{r\to 0}\frac{1}{2r}\int_{s-r}^{s+r}\Psi(x_i-\dot u(s))dt\\ &= \Psi(\dot u(s)-x_i)+\Psi(x_i-\dot u(s))\le\frac{\varepsilon}{2}.\end{aligned}$$

□

Claim 3: For $\mathcal{L}^1$-a.e. $s\in(0,T)$, for any $\varepsilon>0$, there exists $r_0>0$ (depending on ε and s) such that

$$\Psi\left(\int_{s-r}^{s+r}[\dot u(s)-\dot u(t)]dt\right)\le\varepsilon r,\ \text{ for all } r\le r_0. \tag{1.21}$$

Proof of Claim 3. Apply the Lebesgue-Besicovitch Differentiation Theorem (see [11]) for $\dot u\in L^1$, we get

$$\lim_{r\to 0}\frac{1}{2r}\int_{s-r}^{s+r}[\dot u(s)-\dot u(t)]dt=0$$

for $\mathcal{L}^1$-a.e. $s\in(0,T)$. Since $\lim_{z\to 0}\Psi(z)=0$ and Ψ is positively 1-homogeneous, we also have

$$\lim_{r\to 0}\frac{1}{2r}\Psi\left(\int_{s-r}^{s+r}[\dot u(s)-\dot u(t)]dt\right)=0$$

for $\mathcal{L}^1$-a.e. $s\in(0,T)$. □

Given $\varepsilon>0$, for $\mathcal{L}^1$-a.e. $s\in(0,T)$, thanks to Claim 2 and 3, we can choose r (depending on ε and s) such that

$$\int_{s-r}^{s+r}\Psi(\dot u(t)-\dot u(s))dt\le\varepsilon r, \tag{1.22}$$

and

$$\Psi\left(\int_{s-r}^{s+r}[\dot u(s)-\dot u(t)]dt\right)\le\varepsilon r. \tag{1.23}$$

Now we can apply Besicovitch Covering Theorem (see [11]) to the family of these intervals $[s-r,s+r]$ and to the measure $\mu:=\Psi(\dot u)dt$ to find finitely many disjoint intervals $I_i:=[s_i-r_i,s_i+r_i]$ such that

$$\mu\left([t_1,t_2]\backslash\bigcup_i I_i\right)\le\varepsilon. \tag{1.24}$$

Hence, thanks to (1.17), (1.24), (1.22) and (1.23), we can write

$$\begin{aligned}
\int_{t_1}^{t_2} \Psi(\dot{u}(t))dt &= \int_{[t_1,t_2]\setminus\cup_i I_i} \Psi(\dot{u}(t))dt + \sum_i \int_{s_i-r_i}^{s_i+r_i} \Psi(\dot{u}(t))dt \\
&\leq \mu\left([t_1,t_2]\setminus\bigcup_i I_i\right) + \sum_i \Psi(u(s_i+r_i)-u(s_i-r_i)) \\
&\quad + \sum_i \int_{s_i-r_i}^{s_i+r_i} \Psi(\dot{u}(t)-\dot{u}(t_i))dt + \sum_i \Psi\left(\int_{s_i-r_i}^{s_i+r_i} [\dot{u}(t_i)-\dot{u}(t)]dt\right) \\
&\leq \varepsilon + \sum_i \Psi(u(s_i+r_i)-u(s_i-r_i)) + 2\sum_i \varepsilon r_i \\
&\leq (2T+1)\varepsilon + \mathscr{D}iss(u(t);[t_1,t_2]).
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we finish the proof of Step 2.

Step 3: Combine the inequalities in Step 1 and 2, we conclude that

$$\mathscr{D}iss(u(\cdot);[t_1,t_2]) = \int_{t_1}^{t_2} \Psi(\dot{u}(t))dt.$$

for every $u \in AC([0,T];X)$. This completes the proof of Proposition 1.2 (ii). □

Proof of Proposition 1.4. It is sufficient to prove that $x(\cdot)$ is a limit of a sequence of discretized solutions.

Step 1. First we check that, for every t and t' such that $0 \leq t' \leq t \leq T$ there holds

$$x(t) \in \text{argmin}_{x\in X}\{\mathscr{E}(t,x) + \Psi(x - x(t'))\}.$$

Indeed, since Ψ is convex, non-negative and positively 1-homogeneous, repeat the argument in the proof of Step 2 Proposition 1.2, we have the following triangle inequality for Ψ

$$\Psi(x - x(0)) \leq \Psi(x - x(t')) + \Psi(x(t') - x(0)) \text{ for all } x \in X. \tag{1.25}$$

Thanks to (i) we have the following inequality

$$\mathscr{E}(t,x(t)) + \Psi(x(t) - x(0)) \leq \mathscr{E}(t,x) + \Psi(x - x(0)) \text{ for all } x \in X. \tag{1.26}$$

On the other hand, thanks to assumption (ii) we get

$$\Psi(x(t) - x(0)) = \Psi(x(t) - x(t')) + \Psi(x(t') - x(0)) \text{ for any } t' \in [0,t]. \tag{1.27}$$

Combining (1.27), (1.26) and (1.25), we have that for all $t \in [0,T]$

$$\begin{aligned}
\mathscr{E}(t,x(t)) + \Psi(x(t) - x(t')) &= \mathscr{E}(t,x(t)) + \Psi(x(t) - x(0)) - \Psi(x(t') - x(0)) \\
&\leq \mathscr{E}(t,x) + \Psi(x - x(0)) - \Psi(x(t') - x(0)) \\
&\leq \mathscr{E}(t,x) + \Psi(x - x(t')) \text{ for all } x \in X.
\end{aligned}$$

Step 2. Construction of the discretized solution from $x(\cdot)$.
For every $\tau > 0$ fixed, we consider the following partition of $[0, T]$:

$$0 = t_0 < t_1 < \cdots < t_N \leq T, t_n - t_{n-1} = \tau \text{ for every } n \in \{1, 2, \ldots, N\} \text{ and } T - t_N < \tau.$$

Denote $y^\tau(t) := x(t_{n-1})$ if $t \in [t_{n-1}, t_n)$, then by Step 1

$$x(t_n) \in \text{argmin}_{x \in X}\{\mathscr{E}(t_n, x) + \Psi(x - x(t_{n-1}))\}.$$

Hence, $y^\tau(t)$ is one of the discretized solutions of $(\mathscr{E}, \Psi, x(0))$ corresponding to the partition $\{t_n\}_{n=0}^N$.

Step 3. Now we prove $x(\cdot)$ is the limit of the sequence of discretized solutions $\{y^{\tau_n}\}$ for some sequence $\{\tau_n\}$.

In fact, consider the sequence $\{\tau_n\}$ such that $\tau_n = 2^{-n}$, and denote $y^{\tau_n}(t)$ the discretized solution corresponding to τ_n.

If we call $\{t_k^{\tau_n}\}$ the partition of $[0, T]$ corresponding to τ_n, then we have

$$y^{\tau_n}(t) = x(t_k^{\tau_n}) \text{ for every } t \in [t_k^{\tau_n}, t_{k+1}^{\tau_n}).$$

On the other hand, for every $t \in [0, T]$ and for every $n \in \mathbb{N}$, there exists i such that $t \in [t_{i-1}^{\tau_n}, t_i^{\tau_n})$ and $t_{i-1}^{\tau_n} \to t$ as $n \to \infty$. Hence, by the left-continuity of $x(\cdot)$ we get

$$\lim_{n \to \infty} x(t_{i-1}^{\tau_n}) = x(t).$$

Thus,

$$\lim_{n \to \infty} y^{\tau_n}(t) = \lim_{n \to \infty} x(t_{i-1}^{\tau_n}) = x(t).$$

Step 4. Since $x(\cdot)$ is the limit of the sequence of discretized solutions $\{y^{\tau_n}\}$, we know that $x(\cdot)$ satisfies the definition of energetic solution for the system $(\mathscr{E}, \Psi, x(0))$ (see for example Theorem 2.1 in the forthcoming Chapter 2). □

Proof of Propositions 1.5. **Step 1.** First we prove the stability, namely

$$\mathscr{E}(t, \widetilde{x}(t)) \leq \mathscr{E}(t, z) + \Psi(z - \widetilde{x}(t))$$

for all $(t, z) \in [0, T] \times X$.

If $t \in \{0, T\}$ or $x(\cdot)$ is continuous at t, then $\widetilde{x}(t) = x(t)$ and we have the result. Now assume that $x(\cdot)$ is not continuous at $t \in (0, T)$ and $\widetilde{x}(t) = x(t^-)$ (the case $\widetilde{x}(t) = x(t^+)$ can be treated in the same way). Since $x(\cdot)$ has only at most countably many jumps, we can find a sequence $t_n \uparrow t$ such that $x(\cdot)$ is continuous at every t_n. Therefore, using the stability at t_n we have

$$\mathscr{E}(t, x(t_n)) \leq \mathscr{E}(t, z) + \Psi(z - x(t_n))$$

for all $z \in X$. Taking the limit as $n \to \infty$ we obtain

$$\mathscr{E}(t, x(t^-)) \leq \mathscr{E}(t, z) + \Psi(z - x(t^-))$$

for all $z \in X$. Since $\widetilde{x}(t) = x(t^-)$, we have the desired stability.

Step 2. Next, we verify the energy-dissipation balance

$$\mathscr{E}(t_2, \widetilde{x}(t_2)) - \mathscr{E}(t_1, \widetilde{x}(t_1)) = \int_{t_1}^{t_2} \partial_t \mathscr{E}(s, \widetilde{x}(s)) ds - \mathscr{D}iss(\widetilde{x}(\cdot); [t_1, t_2]).$$

From the proof of Step 2, Lemma 2.5, we know that the global stability implies

$$\mathscr{E}(t_2, \widetilde{x}(t_2)) - \mathscr{E}(t_1, \widetilde{x}(t_1)) \geq \int_{t_1}^{t_2} \partial_t \mathscr{E}(s, \widetilde{x}(s)) ds - \mathscr{D}iss(\widetilde{x}(\cdot); [t_1, t_2]). \tag{1.28}$$

Therefore, it remains to show the following inequality

$$\mathscr{E}(t_2, \widetilde{x}(t_2)) - \mathscr{E}(t_1, \widetilde{x}(t_1)) \leq \int_{t_1}^{t_2} \partial_t \mathscr{E}(s, \widetilde{x}(s)) ds - \mathscr{D}iss(\widetilde{x}(\cdot); [t_1, t_2]). \tag{1.29}$$

Since $\widetilde{x}(0) = x(0)$, $\widetilde{x}(T) = x(T)$ and $\mathscr{D}iss(\widetilde{x}(\cdot); [0, T]) \leq \mathscr{D}iss(x(\cdot); [0, T])$, we get immediately from the energy-dissipation balance of $x(\cdot)$ that

$$\begin{aligned} \mathscr{E}(T, \widetilde{x}(T)) - \mathscr{E}(0, \widetilde{x}(0)) &= \mathscr{E}(T, x(T)) - \mathscr{E}(0, x(0)) \\ &= \int_0^T \partial_t \mathscr{E}(s, x(s)) ds - \mathscr{D}iss(x(\cdot); [0, T]) \\ &\leq \int_0^T \partial_t \mathscr{E}(s, \widetilde{x}(s)) ds - \mathscr{D}iss(\widetilde{x}(\cdot); [0, T]). \end{aligned} \tag{1.30}$$

Now denote $I(t_1, t_2)$ by the difference between the left and right-hand side of (1.29). We have already know from (1.28) that $I(t_1, t_2) \geq 0$ for all $0 \leq t_1 \leq t_2 \leq T$. Moreover, thanks to (1.30), we also have $I(0, T) \leq 0$. Hence,

$$0 \geq I(0, T) = I(0, t_1) + I(t_1, t_2) + I(t_2, T).$$

Since each addendum in the last term is non-negative, all of them must be null. In particular, $I(t_1, t_2) = 0$.

□

Chapter 2

Existence of weak solutions to rate-independent systems

In this chapter, we prove the existence of *energetic solutions* and *BV solutions* to the rate-independent systems in the abstract one-dimensional framework.

For the sake of simplicity, we shall assume that the ambient space $X = \mathbb{R}$, the energy functional is non-negative, and the dissipation coincides with the usual distance in $\mathbb{R}$. Some of the following proofs do not work for higher dimensions as well as for the general dissipation functional.

Moreover, we always assume that the energy functional $\mathscr{E}$ is C^2, and satisfies the following technical assumption:

There exists $\lambda = \lambda(\mathscr{E})$ such that

$$|\partial_t \mathscr{E}(s, x)| \leq \lambda \mathscr{E}(s, x) \text{ for all } (s, x) \in [0, T] \times \mathbb{R}. \tag{E1}$$

Remark. The condition (E1) together with Gronwall's inequality imply that

$$\mathscr{E}(r, x) \leq \mathscr{E}(s, x)\, e^{\lambda|r-s|}, \quad |\partial_t \mathscr{E}(r, x)| \leq \lambda \mathscr{E}(s, x)\, e^{\lambda|r-s|} \tag{2.1}$$

for any r, s in $[0, T]$.

Under these assumptions, our equation (1.1) becomes

$$\partial_x \mathscr{E}(t, x(t)) + \partial|x'(t)| \ni 0 \text{ for a.e. } t \in (0, T). \tag{2.2}$$

1 Existence of energetic solutions

In this section we prove the existence of *energetic solutions* to (2.2) via time-discretization.

First, we divide $[0, T]$ into small intervals by the partition $0 = t_0 < t_1 < \cdots < t_N \leq T$ such that $t_n - t_{n-1} \leq \tau$ for every $n \in \{1, \ldots, N\}$ and $T - t_N < \tau$. Denote $x_0^\tau := x_0$ the initial position, then for $n \in \{1, \ldots, N\}$, we define the approximate position x_n^τ by iteration as follows

$$x_n^\tau \in \operatorname{argmin}_{x \in \mathbb{R}} \{\mathscr{E}(t_n, x) + |x - x_{n-1}^\tau|\}. \tag{IP}$$

Now we denote the discretized solution $x^\tau(t) := x_{n-1}^\tau$ for every $t \in [t_{n-1}, t_n)$. We will prove that the sequence $\{x^\tau(\cdot)\}_{\tau>0}$ is an approximation of some energetic solution $x(\cdot)$ in an appropriate meaning.

Here comes the main theorem of this section.

Theorem 2.1 (Energetic solution)**.** *Let $\mathscr{E} : [0,T] \times \mathbb{R} \to [0,+\infty]$ be of class C^2 and satisfy (E1). Given any initial data $x_0 \in \mathbb{R}$ such that x_0 is a minimizer for the function $x \mapsto \mathscr{E}(0,x) + |x - x_0|$ over $x \in \mathbb{R}$. Then the following statements hold true:*

(i) For any $\tau > 0$ and for any partition $0 = t_0 < t_1 < \cdots < t_N \leq T$ of $[0,T]$ such that $t_n - t_{n-1} = \tau$ for all $n \in \{1,\ldots,N\}$ and $T - t_N < \tau$, there exists a discretized solution $t \mapsto x^\tau(t)$ satisfying

$x_0^\tau = x_0$;
x_n^τ *minimizes the function* $x \mapsto \mathscr{E}(t_n,x) + |x_{n-1}^\tau - x|$, *for all* $n = 1,2,\ldots,N$;
$x^\tau(t) = x_{n-1}^\tau$ *if* $t \in [t_{n-1},t_n)$.

(ii) There exists a subsequence $\{\tau_k\}$ such that $x^{\tau_k}(\cdot)$ converges pointwise to some limit $x(\cdot)$ and $t \mapsto x(t)$ has bounded variation.

(iii) The limit $x(\cdot)$ is an energetic solution of (2.2), namely

(Global stability) For all $t \in [0,T]$ and $z \in \mathbb{R}$,

$$\mathscr{E}(t,x(t)) \leq \mathscr{E}(t,z) + |x(t) - z|.$$

(Energy-dissipation balance) For all $0 \leq s \leq t \leq T$, one has

$$\mathscr{E}(t,x(t)) - \mathscr{E}(s,x(s)) = \int_s^t \partial_t \mathscr{E}(r,x(r))dr - \mathscr{D}iss(x(\cdot);[s,t]).$$

The reader is referred to [11] for the definition of bounded variation functions. We recall here the definition of dissipation

$$\mathscr{D}iss(x;[s,t]) := \sup\left\{ \sum_{i=1}^N |x(t_i) - x(t_{i-1})| \mid N \in \mathbb{N}, s = t_0 < t_1 < \cdots < t_N = t \right\}.$$

for any mapping $x : [0,T] \to \mathbb{R}$ and any $0 \leq s \leq t \leq T$.

The proof of this theorem is given in Lemmas 2.2, 2.4 and 2.5 below.

1.1 Discretized solutions

Lemma 2.2 (Discretized solution)**.** *For any given initial state x_0, any $\tau > 0$ and any partition $0 = t_0 < t_1 < \cdots < t_N \leq T$ of $[0,T]$ such that $t_n - t_{n-1} = \tau$, there exists a discretized solution $t \mapsto x^\tau(t)$ satisfying the following two properties:*

(Minimizer) x_n^τ minimizes $x \mapsto \mathscr{E}(t_n,x) + |x_{n-1}^\tau - x|$, for all $n = 1,2,\ldots,N$; and

(Interpolation) $x^\tau(t) = x_{n-1}^\tau$ if $t \in [t_{n-1},t_n)$.

Proof. Since $x \mapsto \mathscr{E}(t_n, x) + |x - x^\tau_{n-1}|$ is continuous and converges to $+\infty$ as $x \to \pm\infty$, it is known that this function has a minimizer. $\square$

The following energy estimates will be useful.

Lemma 2.3 (Energy estimates)**.** *Let x^τ_n be as in Lemma 2.2. For any $n \in \{1, \dots, N\}$, we have*

$$\mathscr{E}(t_n, x^\tau_n) \le \mathscr{E}(0, x_0)\, e^{\lambda t_n} \text{ and } \mathscr{E}(0, x^\tau_n) \le \mathscr{E}(0, x_0)\, e^{2\lambda t_n}$$

Moreover, it holds that $\mathscr{D}iss(x^\tau; [0, T]) < \infty$, $\partial_t \mathscr{E}(\cdot, x^\tau(\cdot)) \in L^1(0, T)$ and, for all $0 \le s \le t \le T$,

$$\mathscr{E}(t, x^\tau(t)) - \mathscr{E}(s, x^\tau(s)) \le \int_s^t \partial_t \mathscr{E}(r, x^\tau(r)) dr - \mathscr{D}iss(x^\tau; [s, t]).$$

Proof. **Step 1.** By the minimality of x^τ_n at time t_n, we have

$$\begin{aligned} \mathscr{E}(t_n, x^\tau_n) + |x^\tau_{n-1} - x^\tau_n| &\le \mathscr{E}(t_n, x^\tau_{n-1}) \\ &= \mathscr{E}(t_{n-1}, x^\tau_{n-1}) + \int_{t_{n-1}}^{t_n} \partial_t \mathscr{E}(t, x^\tau_{n-1}) dt. \end{aligned}$$

here the last equality comes from the fact that $\mathscr{E}(t_{n-1}, x^\tau_{n-1}) < \infty$ and $\mathscr{E}(\cdot, x^\tau_{n-1})$ is of class C^1.

By (2.1),

$$\partial_t \mathscr{E}(t, x^\tau_{n-1}) \le \lambda \mathscr{E}(t_{n-1}, x^\tau_{n-1})\, e^{\lambda(t - t_{n-1})} \text{ for all } t \in [t_{n-1}, t_n].$$

Hence

$$\begin{aligned} \mathscr{E}(t_n, x^\tau_n) &\le \mathscr{E}(t_n, x^\tau_n) + |x^\tau_{n-1} - x^\tau_n| \\ &\le \int_{t_{n-1}}^{t_n} \lambda \mathscr{E}(t_{n-1}, x^\tau_{n-1})\, e^{\lambda(t - t_{n-1})} dt + \mathscr{E}(t_{n-1}, x^\tau_{n-1}) \\ &= \mathscr{E}(t_{n-1}, x^\tau_{n-1})(e^{\lambda(t_n - t_{n-1})} - 1) + \mathscr{E}(t_{n-1}, x^\tau_{n-1}) \\ &= \mathscr{E}(t_{n-1}, x^\tau_{n-1})\, e^{\lambda(t_n - t_{n-1})}. \end{aligned}$$

By induction,

$$\begin{aligned} \mathscr{E}(t_n, x^\tau_n) &\le \mathscr{E}(t_{n-1}, x^\tau_{n-1})\, e^{\lambda(t_n - t_{n-1})} \le \mathscr{E}(t_{n-2}, x^\tau_{n-2})\, e^{\lambda(t_{n-1} - t_{n-2})}\, e^{\lambda(t_n - t_{n-1})} \\ &\le \cdots \le \mathscr{E}(0, x_0)\, e^{\lambda(t_1 - t_0)}\, e^{\lambda(t_2 - t_1)} \dots e^{\lambda(t_n - t_{n-1})} = \mathscr{E}(0, x_0)\, e^{\lambda t_n}. \end{aligned}$$

Finally, by (2.1) again,

$$\mathscr{E}(0, x^\tau_n) \le \mathscr{E}(t_n, x^\tau_n)\, e^{\lambda t_n} \le \mathscr{E}(0, x_0)\, e^{2\lambda t_n}.$$

Step 2. Now we prove the integral bound. Assume that $t_{i-1} < s \le t_i < t_{i+1} < \cdots < t_j \le t < t_{j+1}$, where $\{t_n\}$ is the partition corresponding to x^τ. We start by writing

$$\begin{aligned} \mathscr{E}(t, x^\tau(t)) - \mathscr{E}(s, x^\tau(s)) &= \mathscr{E}(t, x^\tau(t)) - \mathscr{E}(t_j, x^\tau(t_j)) + \mathscr{E}(t_j, x^\tau(t_j)) - \mathscr{E}(t_{j-1}, x^\tau(t_{j-1})) \\ &\quad + \cdots + \mathscr{E}(t_i, x^\tau(t_i)) - \mathscr{E}(s, x^\tau(s)). \end{aligned} \tag{2.3}$$

Denote $x_k := x^\tau(t_k)$. By the minimality of x_k at time t_k, we have

$$
\begin{aligned}
\mathscr{E}(t_k, x_k) - \mathscr{E}(t_{k-1}, x_{k-1}) &\le \mathscr{E}(t_k, x_{k-1}) - |x_{k-1} - x_k| - \mathscr{E}(t_{k-1}, x_{k-1}) \\
&= \int_{t_{k-1}}^{t_k} \partial_t \mathscr{E}(r, x_{k-1}) dr - |x_{k-1} - x_k| \\
&= \int_{t_{k-1}}^{t_k} \partial_t \mathscr{E}(r, x^\tau(r)) dr - |x_{k-1} - x_k|.
\end{aligned}
$$

In the last equality we have used $x^\tau(r) = x_{k-1}$ for all $r \in [t_{k-1}, t_k)$.

Taking the sum for all k from $i+1$ to j, we get

$$
\begin{aligned}
\sum_{k=i+1}^{j} [\mathscr{E}(t_k, x_k) - \mathscr{E}(t_{k-1}, x_{k-1})] &\le \sum_{k=i+1}^{j} \int_{t_{k-1}}^{t_k} \partial_t \mathscr{E}(r, x^\tau(r)) dr \\
&\quad - \sum_{k=i+1}^{j} |x_k - x_{k-1}|. \qquad (2.4)
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\mathscr{E}(t, x^\tau(t)) - \mathscr{E}(t_j, x^\tau(t_j)) &= \mathscr{E}(t, x_j) - \mathscr{E}(t_j, x_j) \\
&= \int_{t_j}^{t} \partial_t \mathscr{E}(r, x_j) dr - |x_j - x_j| \\
&= \int_{t_j}^{t} \partial_t \mathscr{E}(r, x^\tau(r)) dr - |x^\tau(t) - x_j| \qquad (2.5)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathscr{E}(t_i, x^\tau(t_i)) - \mathscr{E}(s, x^\tau(s)) &= \mathscr{E}(t_i, x_i) - \mathscr{E}(s, x_{i-1}) \\
&\le \mathscr{E}(t_i, x_{i-1}) - |x_{i-1} - x_i| - \mathscr{E}(s, x_{i-1}) \\
&= \int_{s}^{t_i} \partial_t \mathscr{E}(r, x_{i-1}) dr - |x_{i-1} - x_i| \\
&= \int_{s}^{t_i} \partial_t \mathscr{E}(r, x^\tau(r)) dr - |x^\tau(s) - x_i|. \qquad (2.6)
\end{aligned}
$$

From (2.3), (2.4), (2.5) and (2.6), we get

$$
\begin{aligned}
\mathscr{E}(t, x^\tau(t)) - \mathscr{E}(s, x^\tau(s)) &\le \int_{s}^{t} \partial_t \mathscr{E}(r, x^\tau(r)) dr \\
&\quad - \left(|x^\tau(t) - x_j| + \sum_{k=i+1}^{j} |x_k - x_{k-1}| + |x^\tau(s) - x_i| \right) \\
&= \int_{s}^{t} \partial_t \mathscr{E}(r, x^\tau(r)) dr - \mathscr{D}iss(x^\tau; [s,t]).
\end{aligned}
$$

□

1.2 Existence and properties of the limit

Lemma 2.4 (Existence of the limit)**.** *Let x_n^τ be as in Lemma 2.2. Then $\mathscr{D}iss(x^\tau;[0,T]) \le C$ and $\mathscr{E}(t, x^\tau(t)) \le C$ for all $t \in [0,T]$, where C is a constant independent of τ.*

Consequently, there exists a subsequence $\tau_k \to 0$ such that $\{x^{\tau_k}(\cdot)\}$ converges pointwise to some function $x(\cdot)$ and

$$\mathscr{D}iss(x;[0,T]) \le \liminf_{k\to\infty} \mathscr{D}iss(x^{\tau_k};[0,T]) \le C.$$

Proof. By definition of $x^\tau(\cdot)$, condition (2.1), and Lemma 2.3, we have for all $t \in [t_{n-1}, t_n)$

$$\begin{aligned}
\mathscr{E}(t, x^\tau(t)) &= \mathscr{E}(t, x_{n-1}^\tau) \\
&\le \mathscr{E}(t_{n-1}, x_{n-1}^\tau)\, e^{\lambda(t-t_{n-1})} \\
&\le \mathscr{E}(0, x_0)\, e^{\lambda t_{n-1}}\, e^{\lambda(t-t_{n-1})} \\
&= \mathscr{E}(0, x_0)\, e^{\lambda t}.
\end{aligned} \tag{2.7}$$

Moreover, by Lemma 2.3 again, we get

$$\begin{aligned}
\mathscr{D}iss(x^\tau;[0,T]) &\le \mathscr{E}(0,x_0) - \mathscr{E}(T, x^\tau(T)) + \int_0^T \partial_t \mathscr{E}(t, x^\tau(t))dt \\
&\le \mathscr{E}(0,x_0) + \int_0^T \lambda \mathscr{E}(t, x^\tau(t))dt.
\end{aligned}$$

Here in the last inequality, we have used the fact that $\mathscr{E}(T, x^\tau(T))$ is non-negative, and condition (2.1).

Now taking into account (2.7), the last inequality becomes

$$\begin{aligned}
\mathscr{D}iss(x^\tau;[0,T]) &\le \mathscr{E}(0,x_0) + \int_0^T \lambda \mathscr{E}(0,x_0)\, e^{\lambda t} dt \\
&\le \mathscr{E}(0,x_0)\, e^{\lambda T}.
\end{aligned}$$

Finally, thanks to Helly's principle (see Proposition 1.3), we have a subsequence $\tau_k \to 0$ such that x^{τ_k} converges pointwise to some limit $x(\cdot)$. Moreover, we have

$$\mathscr{D}iss(x;[0,T]) \le \liminf_{k\to\infty} \mathscr{D}iss(x^{\tau_k};[0,T]) \le C,$$

□

Now we prove that the limit $x(\cdot)$ is an energetic solution of (2.2). More precisely, we prove

Lemma 2.5 (Properties of the limit)**.** *Let $x(\cdot)$ be as in Lemma 2.4. Then one has*

(i) (Global stability) For any $t \in [0,T]$,

$$\mathscr{E}(t, x(t)) \le \mathscr{E}(t, z) + |x(t) - z| \text{ for all } z \in \mathbb{R}.$$

(ii) (Energy-dissipation balance) For all $0 \leq s \leq t \leq T$,

$$\mathscr{E}(t, x(t)) - \mathscr{E}(s, x(s)) = \int_s^t \partial_t \mathscr{E}(r, x(r))dr - \mathscr{D}iss(x; [s, t]).$$

Proof. **Step 1: Global stability.** Recall that we have a sequence of discretized solutions $x^{\tau_k}(t)$ converging pointwise to $x(t)$ for every t. Now for every τ_k, we choose some n such that $t \in [t_{n-1}^{\tau_k}, t_n^{\tau_k})$. Using the minimality for $x^{\tau_k}(t_{n-1}^{\tau_k})$ at time $t_{n-1}^{\tau_k}$, we get

$$\mathscr{E}(t_{n-1}^{\tau_k}, x^{\tau_k}(t_{n-1}^{\tau_k})) + |x^{\tau_k}(t_{n-1}^{\tau_k}) - x^{\tau_k}(t_{n-2}^{\tau_k})| \leq \mathscr{E}(t_{n-1}^{\tau_k}, z) + |z - x^{\tau_k}(t_{n-2}^{\tau_k})| \text{ for all } z \in \mathbb{R}.$$

Hence,

$$\begin{aligned} \mathscr{E}(t_{n-1}^{\tau_k}, x^{\tau_k}(t_{n-1}^{\tau_k})) &\leq \mathscr{E}(t_{n-1}^{\tau_k}, z) + |z - x^{\tau_k}(t_{n-2}^{\tau_k})| - |x^{\tau_k}(t_{n-1}^{\tau_k}) - x^{\tau_k}(t_{n-2}^{\tau_k})| \\ &\leq \mathscr{E}(t_{n-1}^{\tau_k}, z) + |x^{\tau_k}(t_{n-1}^{\tau_k}) - z| \text{ for all } z \in \mathbb{R}. \end{aligned}$$

Taking the limit of the above inequality as $k \to \infty$ and using that $x^{\tau_k}(t_{n-1}^{\tau_k}) = x^{\tau_k}(t)$ whenever $t \in [t_{n-1}^{\tau_k}, t_n^{\tau_k})$, we get the global stability at t.

Step 2: Energy-dissipation lower bound. By the continuity of $\partial_t \mathscr{E}$, for every $\varepsilon > 0$, we can find a partition $s = t_i < t_{i+1} < \cdots < t_j = t$ of $[s, t]$ such that

$$\sum_{k=i+1}^{j} \int_{t_{k-1}}^{t_k} \partial_t \mathscr{E}(r, x(t_k))dr \geq \int_s^t \partial_t \mathscr{E}(r, x(r))dr - \varepsilon.$$

Using the global stability of $x(t_{k-1})$ at time t_{k-1} we get

$$\mathscr{E}(t_{k-1}, x(t_{k-1})) \leq \mathscr{E}(t_{k-1}, x(t_k)) + |x(t_k) - x(t_{k-1})|.$$

Hence,

$$\begin{aligned} \mathscr{E}(t, x(t)) - \mathscr{E}(s, x(s)) &= \sum_{k=i+1}^{j} [\mathscr{E}(t_k, x(t_k)) - \mathscr{E}(t_{k-1}, x(t_{k-1}))] \\ &\geq \sum_{k=i+1}^{j} [\mathscr{E}(t_k, x(t_k)) - \mathscr{E}(t_{k-1}, x(t_k)) - |x(t_k) - x(t_{k-1})|] \\ &\geq \sum_{k=i+1}^{j} \int_{t_{k-1}}^{t_k} \partial_t \mathscr{E}(r, x(t_k)) - \mathscr{D}iss(x; [s, t]) \\ &\geq \int_s^t \partial_t \mathscr{E}(r, x(r))dr - \mathscr{D}iss(x; [s, t]) - \varepsilon. \end{aligned}$$

Since ε was arbitrary, we get the energy-dissipation lower bound

$$\mathscr{E}(t, x(t)) - \mathscr{E}(s, x(s)) \geq \int_s^t \partial_t \mathscr{E}(r, x(r))dr - \mathscr{D}iss(x; [s, t]) \text{ for all } 0 \leq s \leq t \leq T.$$

Step 3: Energy-dissipation upper bound. Using Lemma 2.3 we have

$$\mathscr{E}(t, x^{\tau_k}(t)) - \mathscr{E}(s, x^{\tau_k}(s)) \leq \int_s^t \partial_t \mathscr{E}(r, x^{\tau_k}(r))dr - \mathscr{D}iss(x^{\tau_k}; [s,t]),$$

here $\{x^{\tau_k}(t)\}$ is a sequence of discretized solutions of (2.2) that converges pointwise to $x(t)$.

We take the limit of the above inequality as $k \to \infty$. By the continuity of $\mathscr{E}$ and the fact that $x^{\tau_k}(r)$ converges pointwise to $x(r)$, we have

$$\lim_{k\to\infty} \mathscr{E}(t, x^{\tau_k}(t)) = \mathscr{E}(t, x(t)),$$
$$\lim_{k\to\infty} \mathscr{E}(s, x^{\tau_k}(s)) = \mathscr{E}(s, x(s)).$$

Moreover, by the continuity of $\partial_t \mathscr{E}$, we also get $\partial_t \mathscr{E}(r, x^{\tau_k}(r))$ converges pointwise to $\partial_t \mathscr{E}(r, x(t))$. Employing the dominated convergence theorem, we have

$$\lim_{k\to\infty} \int_s^t \partial_t \mathscr{E}(r, x^{\tau_k}(r))dr = \int_s^t \partial_t \mathscr{E}(r, x(r))dr.$$

Finally, thanks to Lemma 2.4, we get

$$\liminf_{k\to\infty} \mathscr{D}iss(x^{\tau_k}; [s,t]) \geq \mathscr{D}iss(x; [s,t]).$$

Now putting everything together, we get the following upper bound estimate

$$\mathscr{E}(t, x(t)) - \mathscr{E}(s, x(s)) \leq \int_s^t \partial_t \mathscr{E}(r, x(r))dr - \mathscr{D}iss(x; [s,t]) \text{ for all } 0 \leq s \leq t \leq T.$$

This ends the proof of the lemma. □

Thus, we have already constructed a solution to (2.2) satisfying global stability (S) and energy-dissipation balance (ED) in the one-dimensional case. For the proof in a more general framework, we refer to the paper of Mielke [21].

Remark. From the proof of Step 2, we also get the following result.

Let X be a finite dimensional normed space and $x : [0, T] \to X$ be any BV function satisfying the stability

$$\mathscr{E}(t, x(t)) \leq \mathscr{E}(t, z) + |z - x(t)| \text{ for all } (t, z) \in [0, T] \times X,$$

then it holds that

$$\mathscr{E}(t, x(t)) - \mathscr{E}(s, x(s)) \geq \int_s^t \partial_t \mathscr{E}(r, x(r))dr - \mathscr{D}iss(x; [s,t]) \text{ for all } 0 \leq s \leq t \leq T.$$

2 Existence of BV solutions

The existence of BV solutions is proved via the so-called vanishing-viscosity procedure. This method was developed by Mielke, Rossi and Savaré, see for instance [26]. The idea is to add a small viscosity into the dissipation term. BV solutions are then obtained in the limit of the sequence of the new discretized solutions when the viscosity term and the time-discretization step go to zero.

For simplicity, here we choose the dissipation with viscosity $\Psi_\varepsilon = |\cdot| + \frac{\varepsilon}{2}|\cdot|^2$. The incremental problem (IP) after adding the viscosity becomes

$$(\mathrm{IP}_\varepsilon)\ x_n^{\tau,\varepsilon} \in \operatorname{argmin}_{x\in\mathbb{R}} \{\mathscr{E}(t_n, x) + |x - x_{n-1}^{\tau,\varepsilon}| + \tfrac{\varepsilon}{2\tau}|x - x_{n-1}^{\tau,\varepsilon}|^2\} \text{ for every } n \in \{1, \ldots, N\}.$$

Here $\tau > 0$ and $\varepsilon > 0$ are fixed, $0 = t_0 < \cdots < t_N \leq T$ is a partition of $[0, T]$ satisfying $t_n - t_{n-1} = \tau$ for every $n \in \{1, \ldots, N\}$ and $T - t_N < \tau$.

Denote $\overline{u}_{\tau,\varepsilon}(t) := x_{n-1}^{\tau,\varepsilon}$ for every $t \in [t_{n-1}, t_n)$. We will prove that the sequence of $\{\overline{u}_{\tau,\varepsilon}\}$ is an approximation of some BV solution $u(\cdot)$ in an appropriate sense.

Theorem 2.6 (BV solution)**.** *Let $\mathscr{E} : [0, T] \times \mathbb{R} \to [0, +\infty]$ be of class C^2 and satisfy (E1). Given any initial data $x_0 \in \mathbb{R}$ such that x_0 is a local minimizer for the function $x \mapsto \mathscr{E}(0, x) + |x - x_0|$. Then the following statements hold true.*

(i) For any $\tau > 0$, for any $\varepsilon > 0$ and for any partition $0 = t_0 < t_1 < \cdots < t_N \leq T$ of $[0, T]$ such that $t_n - t_{n-1} = \tau$ and $T - t_N < \tau$, there exists a discretized solution $t \mapsto \overline{u}^{\tau,\varepsilon}(t)$ satisfying

$$x_n^{\tau,\varepsilon} \text{ minimizes } x \mapsto \mathscr{E}(t_n, x) + |x - x_{n-1}^{\tau,\varepsilon}| + \frac{\varepsilon}{2\tau}|x - x_{n-1}^{\tau,\varepsilon}|^2 \text{ for all } \quad n = 1, 2, \ldots, N;$$
$$\overline{u}^{\tau,\varepsilon}(t) = x_{n-1}^{\tau,\varepsilon} \text{ if } t \in [t_{n-1}, t_n).$$

(ii) There exists a subsequence $\{\overline{u}^{\tau_k,\varepsilon_k}\}$ such that $\overline{u}^{\tau_k,\varepsilon_k}(t)$ converges pointwise to some limit $u(t)$, and $u(\cdot)$ also has bounded variation.

(iii) The limit $u(t)$ is a BV solution of (2.2), namely

$$|\partial_x \mathscr{E}(t, u(t))| \leq 1 \text{ for a.e. } t \in (0, T),$$
$$\text{for all } 0 \leq s < t \leq T, \mathscr{E}(t, u(t)) - \mathscr{E}(s, u(s)) = \int_s^t \partial_t \mathscr{E}(r, u(r))dr - \mathscr{D}iss_{new}(u; [s, t]).$$

Here for simplicity, we always assume that $t \mapsto u(t)$ is right-continuous, then we can write

$$\mathscr{D}iss_{new}(u; [0, T]) := \mathscr{D}iss(u; [0, T]) - \sum_{t\in J}(\Delta(t, u(t^-), u(t^+))) + \sum_{t\in J}(\Delta_{new}(t, u(t^-), u(t^+))),$$

J is the jump set of u, the classical jump step is $\Delta(t, u(t^-), u(t^+)) := |u(t^-) - u(t^+)|$, and the new jump step is

$$\begin{aligned} &\Delta_{new}(t, u(t^-), u(t^+)) \\ := \ &\inf\{\int_0^1 |\dot{v}(r)| \cdot \max\{1, |\partial_x \mathscr{E}(t, v(r))|\}dr \mid v \in AC([0, 1]), v(0) = u(t^-), v(1) = u(t^+)\}. \end{aligned}$$

The notion of BV (bounded variation) functions could be found in [11]. The notion of AC (absolutely continuous) functions could be found in [32].

This theorem is proved thanks to Lemma 2.7, 2.9 and 2.10 below.

2.1 Discretized solutions

Lemma 2.7 (Discretized solution)**.** *For any given initial state x_0, any $\varepsilon > 0$, any $\tau > 0$ and any partition $0 = t_0 < t_1 < \cdots < t_N \le T$ of $[0, T]$ such that $t_n - t_{n-1} = \tau$ and $T - t_N < \tau$, there exists a discretized solution $t \mapsto \overline{u}_{\tau,\varepsilon}(t)$ defined by*

> *(Minimizer) $x_n^{\tau,\varepsilon}$ minimizes $x \mapsto \mathscr{E}(t_n, x) + |x - x_{n-1}^{\tau,\varepsilon}| + \frac{\varepsilon}{2\tau}|x - x_{n-1}^{\tau,\varepsilon}|^2$, for all $n = 1, 2, \ldots, N$; and*
>
> *(Interpolation) $\overline{u}_{\tau,\varepsilon}(t) = x_{n-1}^{\tau,\varepsilon}$ if $t \in [t_{n-1}, t_n)$.*

Proof. The proof is obvious from the general fact: Given $f : X \to [0, +\infty]$ such that f is continuous and $f(x) \to \infty$ as $|x| \to \infty$. Then f admits a minimizer. □

We shall need the following energy estimates for the discretized solutions.

Lemma 2.8 (Energy estimates)**.** *Let $x^{\tau,\varepsilon}$ be as in Lemma 2.7. Then for any n we have*

$$\mathscr{E}(t_n, x_n^{\tau,\varepsilon}) \le \mathscr{E}(0, x_0)\, e^{\lambda t_n} \quad \text{and} \quad \mathscr{E}(0, x_n^{\tau,\varepsilon}) \le \mathscr{E}(0, x_0)\, e^{2\lambda t_n}.$$

Moreover,

$$\sum_{n=1}^{N}(|x_n^{\tau,\varepsilon} - x_{n-1}^{\tau,\varepsilon}| + \frac{\varepsilon}{2\tau}|x_n^{\tau,\varepsilon} - x_{n-1}^{\tau,\varepsilon}|^2) \le \mathscr{E}(0, x_0)\, e^{\lambda T}$$

and for all $n = 1, 2, \ldots, N$,

$$|x_n^{\tau,\varepsilon} - x_{n-1}^{\tau,\varepsilon}| \le C\frac{\tau}{\varepsilon}$$

for some constant $C > 0$ independent of τ and ε.

Proof. **Step 1.** By the minimality of $x_n^{\tau,\varepsilon}$, we have

$$\mathscr{E}(t_n, x_n^{\tau,\varepsilon}) + |x_n^{\tau,\varepsilon} - x_{n-1}^{\tau,\varepsilon}| + \frac{\varepsilon}{2\tau}|x_n^{\tau,\varepsilon} - x_{n-1}^{\tau,\varepsilon}|^2 \le \mathscr{E}(t_n, x_{n-1}^{\tau,\varepsilon}).$$

By the C^1-continuity of $\mathscr{E}(\cdot, x_{n-1}^{\tau,\varepsilon})$, we can write

$$\begin{aligned}\mathscr{E}(t_n, x_{n-1}^{\tau,\varepsilon}) &= \mathscr{E}(t_n, x_{n-1}^{\tau,\varepsilon}) - \mathscr{E}(t_{n-1}, x_{n-1}^{\tau,\varepsilon}) + \mathscr{E}(t_{n-1}, x_{n-1}^{\tau,\varepsilon}) \\ &= \int_{t_{n-1}}^{t_n} \partial_t \mathscr{E}(t, x_{n-1}^{\tau,\varepsilon}) dt + \mathscr{E}(t_{n-1}, x_{n-1}^{\tau,\varepsilon}).\end{aligned}$$

Using condition (2.1), we get

$$\partial_t \mathscr{E}(t, x_{n-1}^{\tau,\varepsilon}) \le \lambda \mathscr{E}(t_{n-1}, x_{n-1}^{\tau,\varepsilon})\, e^{\lambda(t - t_{n-1})} \text{ for all } t \in [t_{n-1}, t_n].$$

Hence,

$$\begin{aligned}
\mathscr{E}(t_n, x_n^{\tau,\varepsilon}) &\leq \mathscr{E}(t_n, x_n^{\tau,\varepsilon}) + |x_n^{\tau,\varepsilon} - x_{n-1}^{\tau,\varepsilon}| + \frac{\varepsilon}{2\tau}|x_n^{\tau,\varepsilon} - x_{n-1}^{\tau,\varepsilon}|^2 \\
&\leq \mathscr{E}(t_n, x_{n-1}^{\tau,\varepsilon}) \\
&= \int_{t_{n-1}}^{t_n} \partial_t \mathscr{E}(t, x_{n-1}^{\tau,\varepsilon}) dt + \mathscr{E}(t_{n-1}, x_{n-1}^{\tau,\varepsilon}) \\
&\leq \int_{t_{n-1}}^{t_n} \lambda \mathscr{E}(t_{n-1}, x_{n-1}^{\tau,\varepsilon})\, e^{\lambda(t-t_{n-1})} dt + \mathscr{E}(t_{n-1}, x_{n-1}^{\tau,\varepsilon}) \\
&= \mathscr{E}(t_{n-1}, x_{n-1}^{\tau,\varepsilon})(e^{\lambda(t_n-t_{n-1})} - 1) + \mathscr{E}(t_{n-1}, x_{n-1}^{\tau,\varepsilon}) \\
&= \mathscr{E}(t_{n-1}, x_{n-1}^{\tau,\varepsilon})\, e^{\lambda(t_n-t_{n-1})}.
\end{aligned}$$

By induction,

$$\begin{aligned}
\mathscr{E}(t_n, x_n^{\tau,\varepsilon}) &\leq \mathscr{E}(t_{n-1}, x_{n-1}^{\tau,\varepsilon})\, e^{\lambda(t_n-t_{n-1})} \leq \mathscr{E}(t_{n-2}, x_{n-2}^{\tau,\varepsilon})\, e^{\lambda(t_{n-1}-t_{n-2})}\, e^{\lambda(t_n-t_{n-1})} \\
&\leq \cdots \leq \mathscr{E}(0, x_0)\, e^{\lambda(t_1-t_0)}\, e^{\lambda(t_2-t_1)} \ldots e^{\lambda(t_n-t_{n-1})} = \mathscr{E}(0, x_0)\, e^{\lambda t_n}.
\end{aligned}$$

Finally, by (2.1) again

$$\mathscr{E}(0, x_n^{\tau,\varepsilon}) \leq \mathscr{E}(t_n, x_n^{\tau,\varepsilon})\, e^{\lambda t_n} \leq \mathscr{E}(0, x_0)\, e^{2\lambda t_n}.$$

Thus,

$$\mathscr{E}(t_n, x_n^{\tau,\varepsilon}) \leq \mathscr{E}(0, x_0)\, e^{\lambda t_n} \text{ and } \mathscr{E}(0, x_n^{\tau,\varepsilon}) \leq \mathscr{E}(0, x_0)\, e^{2\lambda t_n}.$$

Step 2. Now we prove that

$$\sum_{n=1}^{N} (|x_n^{\tau,\varepsilon} - x_{n-1}^{\tau,\varepsilon}| + \frac{\varepsilon}{2\tau}|x_n^{\tau,\varepsilon} - x_{n-1}^{\tau,\varepsilon}|^2) \leq \mathscr{E}(0, x_0)\, e^{\lambda T}.$$

Since $x_n^{\tau,\varepsilon}$ is a minimizer of $y \mapsto \mathscr{E}(t_n, y) + |y - x_{n-1}^{\tau,\varepsilon}| + \frac{\varepsilon}{2\tau}|y - x_{n-1}^{\tau,\varepsilon}|^2$, we have

$$\mathscr{E}(t_n, x_n^{\tau,\varepsilon}) + |x_n^{\tau,\varepsilon} - x_{n-1}^{\tau,\varepsilon}| + \frac{\varepsilon}{2\tau}|x_n^{\tau,\varepsilon} - x_{n-1}^{\tau,\varepsilon}|^2 \leq \mathscr{E}(t_n, x_{n-1}^{\tau,\varepsilon}).$$

Thus

$$|x_n^{\tau,\varepsilon} - x_{n-1}^{\tau,\varepsilon}| + \frac{\varepsilon}{2\tau}|x_n^{\tau,\varepsilon} - x_{n-1}^{\tau,\varepsilon}|^2 \leq \mathscr{E}(t_n, x_{n-1}^{\tau,\varepsilon}) - \mathscr{E}(t_n, x_n^{\tau,\varepsilon}).$$

Taking the sum when n runs from 1 to N, we get

$$\begin{aligned}
\sum_{n=1}^{N}(|x_n^{\tau,\varepsilon}-x_{n-1}^{\tau,\varepsilon}|+\frac{\varepsilon}{2\tau}|x_n^{\tau,\varepsilon}-x_{n-1}^{\tau,\varepsilon}|^2) &\leq \sum_{n=1}^{N}\left[\mathscr{E}(t_n,x_{n-1}^{\tau,\varepsilon})-\mathscr{E}(t_n,x_n^{\tau,\varepsilon})\right] \\
&\leq \sum_{n=1}^{N}\left[\mathscr{E}(t_n,x_{n-1}^{\tau,\varepsilon})-\mathscr{E}(t_{n-1},x_{n-1}^{\tau,\varepsilon})\right]+ \\
&\quad +\sum_{n=1}^{N}\left[\mathscr{E}(t_{n-1},x_{n-1}^{\tau,\varepsilon})-\mathscr{E}(t_n,x_n^{\tau,\varepsilon})\right] \\
&= \sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}\partial_t\mathscr{E}(t,x_{n-1}^{\tau,\varepsilon})dt+\mathscr{E}(0,x_0)-\mathscr{E}(t_N,x_N^{\tau,\varepsilon}) \\
&\leq \sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}\partial_t\mathscr{E}(t,x_{n-1}^{\tau,\varepsilon})dt+\mathscr{E}(0,x_0).
\end{aligned}$$

By (2.1),

$$\partial_t\mathscr{E}(t,x_{n-1}^{\tau,\varepsilon})\leq\lambda\mathscr{E}(t_{n-1},x_{n-1}^{\tau,\varepsilon})\,e^{\lambda(t-t_{n-1})},\ \text{for all } t\in[t_{n-1},t_n).$$

By Step 1,

$$\mathscr{E}(t_{n-1},x_{n-1}^{\tau,\varepsilon})\leq\mathscr{E}(0,x_0)\,e^{\lambda t_{n-1}}.$$

Thus,

$$\partial_t\mathscr{E}(t,x_{n-1}^{\tau,\varepsilon})\leq\lambda\mathscr{E}(0,x_0)\,e^{\lambda t},\ \text{for all } t\in[t_{n-1},t_n).$$

Hence,

$$\begin{aligned}
\sum_{n=1}^{N}(|x_n^{\tau,\varepsilon}-x_{n-1}^{\tau,\varepsilon}|+\frac{\varepsilon}{2\tau}|x_n^{\tau,\varepsilon}-x_{n-1}^{\tau,\varepsilon}|^2) &\leq \sum_{n=1}^{N}\mathscr{E}(0,x_0)\int_{t_{n-1}}^{t_n}\lambda e^{\lambda t}dt+\mathscr{E}(0,x_0) \\
&= \sum_{n=1}^{N}\mathscr{E}(0,x_0)(e^{\lambda t_n}-e^{\lambda t_{n-1}})+\mathscr{E}(0,x_0) \\
&= \mathscr{E}(0,x_0)\,e^{\lambda t_N}\leq\mathscr{E}(0,x_0)\,e^{\lambda T}.
\end{aligned}$$

Step 3. Finally we show that for every n,

$$|x_n^{\tau,\varepsilon}-x_{n-1}^{\tau,\varepsilon}|\leq C\frac{\tau}{\varepsilon}$$

for some constant $C>0$ independent of τ and ε.

By the minimality of $x_n^{\tau,\varepsilon}$ we have

$$\mathscr{E}(t_n,x_n^{\tau,\varepsilon})+|x_n^{\tau,\varepsilon}-x_{n-1}^{\tau,\varepsilon}|+\frac{\varepsilon}{2\tau}|x_n^{\tau,\varepsilon}-x_{n-1}^{\tau,\varepsilon}|^2\leq\mathscr{E}(t_n,x_{n-1}^{\tau,\varepsilon}),$$

or equivalently,

$$\begin{aligned}
|x_n^{\tau,\varepsilon}-x_{n-1}^{\tau,\varepsilon}|+\frac{\varepsilon}{2\tau}|x_n^{\tau,\varepsilon}-x_{n-1}^{\tau,\varepsilon}|^2 &\leq \mathscr{E}(t_n,x_{n-1}^{\tau,\varepsilon})-\mathscr{E}(t_n,x_n^{\tau,\varepsilon}) \\
&= \int_{x_{n-1}^{\tau,\varepsilon}}^{x_n^{\tau,\varepsilon}}\partial_x\mathscr{E}(t_n,y)dy \\
&\leq C|x_n^{\tau,\varepsilon}-x_{n-1}^{\tau,\varepsilon}|.
\end{aligned}$$

Here, the constant C is independent of n, since the sequence $\{x_n^{\tau,\varepsilon}\}$ is uniformly bounded.

This ends the proof of Lemma 2.8. □

2.2 Existence and properties of the limit

Now we prove that there is a subsequence of $\{\overline{u}_{\tau,\varepsilon}(t)\}$ converging pointwise to some function $u(t)$ when $\varepsilon \to 0$ and $\tau \to 0$.

Lemma 2.9 (Existence of the limit)**.** *Let $x_n^{\tau,\varepsilon}$ be as in Lemma 2.7. Then $\mathscr{D}iss(\overline{u}_{\tau,\varepsilon};[0,T]) \leq C$ and $\mathscr{E}(t,\overline{u}_{\tau,\varepsilon}(t)) \leq C$ for all $t \in [0,T]$, where C is a constant independent of τ, ε.*

Consequently, there exists a subsequence $\{\overline{u}_{\tau_k,\varepsilon_k}\}$, $\tau_k \to 0, \varepsilon_k \to 0$, converging pointwise to some function $u(t)$ and

$$\mathscr{D}iss(u;[0,T]) \leq \liminf_{k\to\infty} \mathscr{D}iss(\overline{u}_{\tau_k,\varepsilon_k};[0,T]) \leq C$$

Proof. Thanks to Lemma 2.8, we have

$$\mathscr{D}iss(\overline{u}_{\tau,\varepsilon};[0,T]) = \sum_{n=1}^{N} |x_n^{\tau,\varepsilon} - x_{n-1}^{\tau,\varepsilon}| \leq \mathscr{E}(0,x_0)\, e^{\lambda T} = \text{const.}$$

By Helly's selection theorem, there exists a subsequence $\overline{u}_{\tau_k,\varepsilon_k}(t)$ converging to some $u(t)$ for all $t \in [0,T]$. Moreover,

$$\mathscr{D}iss(u;[0,T]) \leq \liminf_{k\to\infty} \mathscr{D}iss(\overline{u}_{\tau_k,\varepsilon_k};[0,T]) \leq C.$$

□

So far, we have proved there is a sequence $\{\overline{u}_{\tau,\varepsilon}\}$ converging pointwise to some function $u(t)$ for all $t \in [0,T]$ when τ and ε tend to 0. Now we will prove that $u(t)$ is a BV solution of (2.2) under the assumption that $\frac{\tau}{\varepsilon^2}$ tends to 0. More precisely, we prove that

Lemma 2.10 (Properties of the limit)**.** *Let $u(\cdot)$ be a limit of the sequence $\{\overline{u}_{\tau_k,\varepsilon_k}\}$ as in Lemma 2.9. If we choose the sequences $\tau_k \to 0, \varepsilon_k \to 0$ so that $\frac{\tau_k}{\varepsilon_k^2} \to 0$, then*

(Local stability) $|\partial_x \mathscr{E}(t,u(t))| \leq 1$ for a.e. $t \in (0,T)$.

(New energy-dissipation balance) For all $0 \leq s \leq t \leq T$,

$$\mathscr{E}(t,u(t)) - \mathscr{E}(s,u(s)) = \int_s^t \partial_t \mathscr{E}(r,u(r))dr - \mathscr{D}iss_{new}(u;[s,t]),$$

Remark. The condition $\tau_k/\varepsilon_k^2 \to 0$ is not optimal. In [26], Mielke, Rossi and Savaré show that the condition $\tau_k/\varepsilon_k \to 0$ is enough to obtain a BV solution. However, for simplicity, we use the stronger condition $\tau_k/\varepsilon_k^2 \to 0$, which allows for a simpler proof as below.

Now we introduce two more approximations of $u(t)$.

Definition 2.1. Let $x_n^{\tau,\varepsilon}$ be as in Lemma 2.7. We define the *left-continuous piecewise constant interpolation*

$$\underline{u}_{\tau,\varepsilon}(t) = x_n^{\tau,\varepsilon} \text{ if } t \in (t_{n-1}, t_n],$$

and the *piecewise linear interpolation*

$$\begin{aligned} u_{\tau,\varepsilon}(t) &= \frac{t - t_{n-1}}{t_n - t_{n-1}} x_n^{\tau,\varepsilon} + \frac{t_n - t}{t_n - t_{n-1}} x_{n-1}^{\tau,\varepsilon} \\ &= \frac{x_n^{\tau,\varepsilon} - x_{n-1}^{\tau,\varepsilon}}{\tau} t + \frac{x_{n-1}^{\tau,\varepsilon} t_n - x_n^{\tau,\varepsilon} t_{n-1}}{\tau} \text{ if } t \in [t_{n-1}, t_n]. \end{aligned}$$

Lemma 2.11. *Let $x_n^{\tau,\varepsilon}$ be as in Lemma 2.7. Let $\overline{u}_{\tau,\varepsilon}, \underline{u}_{\tau,\varepsilon}$, and $u_{\tau,\varepsilon}$ be the right-continuous piecewise constant interpolation, left-continuous piecewise constant interpolation, and the piecewise linear interpolation of the sequence $\{x_n^{\tau,\varepsilon}\}$. Then , up to subsequences, all three sequences $\{\overline{u}_{\tau_k,\varepsilon_k}\}, \{\underline{u}_{\tau_k,\varepsilon_k}\}$, and $\{u_{\tau_k,\varepsilon_k}\}$ converge pointwise to the same limit $u(t)$ for all $t \in [0, T]$, when $\varepsilon_k \to 0$ and $\frac{\tau_k}{\varepsilon_k} \to 0$.*

Proof. Observe that

$$\begin{aligned} \|u_{\tau,\varepsilon} - \overline{u}_{\tau,\varepsilon}\| &\leq \sup_n |x_n^{\tau,\varepsilon} - x_{n-1}^{\tau,\varepsilon}| \\ \|u_{\tau,\varepsilon} - \underline{u}_{\tau,\varepsilon}\| &\leq \sup_n |x_n^{\tau,\varepsilon} - x_{n-1}^{\tau,\varepsilon}|. \end{aligned}$$

We also get from Lemma 2.8 that

$$|x_n^{\tau,\varepsilon} - x_{n-1}^{\tau,\varepsilon}| \leq C\frac{\tau}{\varepsilon}.$$

Thus, when τ and ε tend to 0 with constraint $\frac{\tau}{\varepsilon} \to 0$, we have

$$\lim_{\varepsilon\to 0, \frac{\tau}{\varepsilon}\to 0} |x_n^{\tau,\varepsilon} - x_{n-1}^{\tau,\varepsilon}| = 0.$$

On the other hand, by Lemma 2.9, we know that, up to a subsequence, $\{\overline{u}_{\tau_k,\varepsilon_k}\}$ converges pointwise to some $u(t)$. So we also have

$$\begin{aligned} &u_{\tau_k,\varepsilon_k}(t) \to u(t), \\ &\underline{u}_{\tau_k,\varepsilon_k}(t) \to u(t), \end{aligned}$$

as $\varepsilon_k \to 0$ and $\tau_k/\varepsilon_k \to 0$. □

Now we call Ψ the classical dissipation function and Ψ_ε the dissipation function with viscosity term,

$$\begin{aligned} &\Psi(x) := |x| \\ &\Psi_\varepsilon(x) := |x| + \frac{\varepsilon}{2}|x|^2 \end{aligned} \tag{2.8}$$

Lemma 2.12. *Let Ψ_ε be the dissipation function with viscosity as in (2.8), Ψ_ε^* be the Legendre transform of Ψ_ε (see [30] for the definition of Legrendre transform). Let $x_n^{\tau,\varepsilon}$ be as in Lemma 2.7, $u_{\tau,\varepsilon}$ be the piecewise linear interpolation of $\{x_n^{\tau,\varepsilon}\}$, $\underline{w}_{\tau,\varepsilon}$ be the left-continuous piecewise constant interpolation of $-\partial_x\mathscr{E}(t_n, x_n^{\tau,\varepsilon})$, i.e. $\underline{w}_{\tau,\varepsilon}(t) = -\partial_x\mathscr{E}(t_n, x_n^{\tau,\varepsilon})$ if $t \in (t_{n-1}, t_n]$, and $u(t)$ be the limit defined in Lemma 2.9. Then we have the following relation*

$$\lim_{\varepsilon\to 0, \frac{\tau}{\varepsilon^2}\to 0} \int_0^T \left[\Psi_\varepsilon(\dot{u}_{\tau,\varepsilon}(t)) + \Psi_\varepsilon^*(\underline{w}_{\tau,\varepsilon}(t))\right] dt = \mathscr{E}(0, x_0) - \mathscr{E}(T, u(T)) + \int_0^T \partial_t\mathscr{E}(t, u(t))dt,$$

Proof. From (IP_ε), we have that $x_n^{\tau,\varepsilon}$ is a minimizer of the following function

$$x \mapsto \mathscr{E}(t_n, x) + \tau\Psi_\varepsilon\left(\frac{x - x_{n-1}^{\tau,\varepsilon}}{\tau}\right).$$

Differentiating w.r.t. x, we get

$$0 \in \partial_x\mathscr{E}(t_n, x_n^{\tau,\varepsilon}) + \partial\Psi_\varepsilon\left(\frac{x_n^{\tau,\varepsilon} - x_{n-1}^{\tau,\varepsilon}}{\tau}\right). \tag{2.9}$$

Using the Fenchel equivalence,

$$x^* \in \partial F(x) \Longleftrightarrow F(x) + F^*(x^*) = \langle x^*, x\rangle,$$

(2.9) now becomes

$$\tau\Psi_\varepsilon\left(\frac{x_n^{\tau,\varepsilon} - x_{n-1}^{\tau,\varepsilon}}{\tau}\right) + \tau\Psi_\varepsilon^*(-\partial_x\mathscr{E}(t_n, x_n^{\tau,\varepsilon})) = \left\langle -\partial_x\mathscr{E}(t_n, x_n^{\tau,\varepsilon}), x_n^{\tau,\varepsilon} - x_{n-1}^{\tau,\varepsilon}\right\rangle. \tag{2.10}$$

Applying the chain rule for $\mathscr{E}(t, u_{\tau,\varepsilon}(t))$ where $\mathscr{E} \in C^1$ and $u_{\tau,\varepsilon}$ is piecewise linear, we have

$$\begin{aligned}\frac{d}{dt}\mathscr{E}(t, u_{\tau,\varepsilon}(t)) &= \partial_t\mathscr{E}(t, u_{\tau,\varepsilon}(t)) + \langle \partial_x\mathscr{E}(t, u_{\tau,\varepsilon}(t)), \dot{u}_{\tau,\varepsilon}(t)\rangle \\ &= \partial_t\mathscr{E}(t, u_{\tau,\varepsilon}(t)) + \left\langle \partial_x\mathscr{E}(t, u_{\tau,\varepsilon}(t)), \frac{x_n^{\tau,\varepsilon} - x_{n-1}^{\tau,\varepsilon}}{\tau}\right\rangle, \text{ for all } t \in (t_{n-1}, t_n).\end{aligned}$$

Then

$$\mathscr{E}(t_n, x_n^{\tau,\varepsilon}) - \mathscr{E}(t_{n-1}, x_{n-1}^{\tau,\varepsilon}) = \int_{t_{n-1}}^{t_n} \partial_t\mathscr{E}(t, u_{\tau,\varepsilon}(t))dt + \int_{t_{n-1}}^{t_n} \left\langle \partial_x\mathscr{E}(t, u_{\tau,\varepsilon}(t)), \frac{x_n^{\tau,\varepsilon} - x_{n-1}^{\tau,\varepsilon}}{\tau}\right\rangle dt,$$

or equivalently,

$$0 = \mathscr{E}(t_{n-1}, x_{n-1}^{\tau,\varepsilon}) - \mathscr{E}(t_n, x_n^{\tau,\varepsilon}) + \int_{t_{n-1}}^{t_n} \partial_t\mathscr{E}(t, u_{\tau,\varepsilon}(t))dt + \int_{t_{n-1}}^{t_n} \left\langle \partial_x\mathscr{E}(t, u_{\tau,\varepsilon}(t)), \frac{x_n^{\tau,\varepsilon} - x_{n-1}^{\tau,\varepsilon}}{\tau}\right\rangle dt.$$

Plugging the above equation into (2.10) and then taking the sum over n, we arrive at

$$\begin{aligned}&\sum_{n=1}^N \left[\tau\Psi_\varepsilon\left(\frac{x_n^{\tau,\varepsilon} - x_{n-1}^{\tau,\varepsilon}}{\tau}\right) + \tau\Psi_\varepsilon^*(-\partial_x\mathscr{E}(t_n, x_n^{\tau,\varepsilon}))\right] \\ = &-\sum_{n=1}^N \left\langle \partial_x\mathscr{E}(t_n, x_n^{\tau,\varepsilon}), x_n^{\tau,\varepsilon} - x_{n-1}^{\tau,\varepsilon}\right\rangle + \sum_{n=1}^N \left(\mathscr{E}(t_{n-1}, x_{n-1}^{\tau,\varepsilon}) - \mathscr{E}(t_n, x_n^{\tau,\varepsilon})\right) \\ &+ \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \partial_t\mathscr{E}(t, u_{\tau,\varepsilon}(t))dt + \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \left\langle \partial_x\mathscr{E}(t, u_{\tau,\varepsilon}(t)), \frac{x_n^{\tau,\varepsilon} - x_{n-1}^{\tau,\varepsilon}}{\tau}\right\rangle dt\end{aligned} \tag{2.11}$$

If we replace $\frac{x_n^{\tau,\varepsilon}-x_{n-1}^{\tau,\varepsilon}}{\tau}$ by $\dot{u}_{\tau,\varepsilon}(t)$ and $-\partial_x\mathscr{E}(t_n,x_n^{\tau,\varepsilon})$ by $\underline{w}_{\tau,\varepsilon}(t)$ for $t\in(t_{n-1},t_n]$, then the left-hand side of (2.11) becomes

$$\begin{aligned}\text{LHS(2.11)} &= \sum_{n=1}^N\int_{t_n-1}^{t_n}\Psi_\varepsilon\left(\frac{x_n^{\tau,\varepsilon}-x_{n-1}^{\tau,\varepsilon}}{\tau}\right)+\sum_{n=1}^N\int_{t_n-1}^{t_n}\Psi_\varepsilon^*(-\partial_x\mathscr{E}(t_n,x_n^{\tau,\varepsilon}))\\ &= \sum_{n=1}^N\int_{t_n-1}^{t_n}\Psi_\varepsilon(\dot{u}_{\tau,\varepsilon}(t))dt+\sum_{n=1}^N\int_{t_n-1}^{t_n}\Psi_\varepsilon^*(\underline{w}_{\tau,\varepsilon}(t))dt\\ &= \int_0^{t_N}\Psi_\varepsilon(\dot{u}_{\tau,\varepsilon}(t))dt+\int_0^{t_N}\Psi_\varepsilon^*(\underline{w}_{\tau,\varepsilon}(t))dt, \end{aligned}\tag{2.12}$$

The right-hand side of (2.11) can be written as

$$\begin{aligned}\text{RHS(2.11)} &= \mathscr{E}(0,x_0)-\mathscr{E}(t_N,x_{t_N}^{\tau,\varepsilon})+\int_0^{t_N}\partial_t\mathscr{E}(t,u_{\tau,\varepsilon}(t))dt\\ &\quad+\sum_{n=1}^N\int_{t_{n-1}}^{t_n}\left\langle\partial_x\mathscr{E}(t,u_{\tau,\varepsilon}(t)),\frac{x_n^{\tau,\varepsilon}-x_{n-1}^{\tau,\varepsilon}}{\tau}\right\rangle dt-\sum_{n=1}^N\left\langle\partial_x\mathscr{E}(t_n,x_n^{\tau,\varepsilon}),x_n^{\tau,\varepsilon}-x_{n-1}^{\tau,\varepsilon}\right\rangle\\ &= \mathscr{E}(0,x_0)-\mathscr{E}(t_N,x_{t_N}^{\tau,\varepsilon})+\int_0^{t_N}\partial_t\mathscr{E}(t,u_{\tau,\varepsilon}(t))dt+\sum_{n=1}^N M_n\end{aligned}\tag{2.13}$$

where

$$M_n:=\int_{t_{n-1}}^{t_n}\left\langle\partial_x\mathscr{E}(t,u_{\tau,\varepsilon}(t)),\frac{x_n^{\tau,\varepsilon}-x_{n-1}^{\tau,\varepsilon}}{\tau}\right\rangle dt-\left\langle\partial_x\mathscr{E}(t_n,x_n^{\tau,\varepsilon}),x_n^{\tau,\varepsilon}-x_{n-1}^{\tau,\varepsilon}\right\rangle.$$

Now we show that $\sum_{n=1}^N M_n\to 0$ as $\varepsilon\to 0$, and $\tau/\varepsilon^2\to 0$.

In fact, by C^2 continuity of $\mathscr{E}$ in both two variables t and x we can write

$$\begin{aligned}|M_n| &\le \left|\int_{t_{n-1}}^{t_n}\left\langle\partial_x\mathscr{E}(t,u_{\tau,\varepsilon}(t)),\frac{x_n^{\tau,\varepsilon}-x_{n-1}^{\tau,\varepsilon}}{\tau}\right\rangle dt-\int_{t_{n-1}}^{t_n}\left\langle\partial_x\mathscr{E}(t_n,x_n^{\tau,\varepsilon}),\frac{x_n^{\tau,\varepsilon}-x_{n-1}^{\tau,\varepsilon}}{\tau}\right\rangle dt\right|\\ &\le \int_{t_{n-1}}^{t_n}|\partial_x\mathscr{E}(t,u_{\tau,\varepsilon}(t))-\partial_x\mathscr{E}(t_n,x_n^{\tau,\varepsilon})|\cdot\left|\frac{x_n^{\tau,\varepsilon}-x_{n-1}^{\tau,\varepsilon}}{\tau}\right|dt\\ &\le \int_{t_{n-1}}^{t_n}|\partial_x\mathscr{E}(t,u_{\tau,\varepsilon}(t))-\partial_x\mathscr{E}(t,x_n^{\tau,\varepsilon})|\cdot\left|\frac{x_n^{\tau,\varepsilon}-x_{n-1}^{\tau,\varepsilon}}{\tau}\right|dt\\ &\quad+\int_{t_{n-1}}^{t_n}|\partial_x\mathscr{E}(t,x_n^{\tau,\varepsilon})-\partial_x\mathscr{E}(t_n,x_n^{\tau,\varepsilon})|\cdot\left|\frac{x_n^{\tau,\varepsilon}-x_{n-1}^{\tau,\varepsilon}}{\tau}\right|dt\\ &\le \int_{t_{n-1}}^{t_n}\frac{C}{\tau}|x_n^{\tau,\varepsilon}-x_{n-1}^{\tau,\varepsilon}|^2dt+\int_{t_{n-1}}^{t_n}C\tau\left|\frac{x_n^{\tau,\varepsilon}-x_{n-1}^{\tau,\varepsilon}}{\tau}\right|dt,\end{aligned}$$

with the constants C independent of τ,ε and n.

By Lemma 2.8 we have

$$|x_n^{\tau,\varepsilon}-x_{n-1}^{\tau,\varepsilon}|\le C\frac{\tau}{\varepsilon}.$$

Therefore, M_n can be estimated by

$$\begin{aligned} |M_n| &\leq \int_{t_{n-1}}^{t_n} C\frac{\tau}{\varepsilon^2}dt + \int_{t_{n-1}}^{t_n} C\frac{\tau}{\varepsilon}dt \\ &= C\frac{\tau^2}{\varepsilon^2} + C\frac{\tau^2}{\varepsilon}. \end{aligned}$$

Thus, taking the sum over n, and noticing that $n \sim \frac{T}{\tau}$, we get

$$\sum_{n=1}^{N} |M_n| \leq C\frac{T\tau}{\varepsilon^2} + C\frac{T\tau}{\varepsilon} \to 0 \text{ when } \varepsilon \to 0 \text{ and } \tau/\varepsilon^2 \to 0. \tag{2.14}$$

Finally, taking the limit in (2.11) and using (2.12), (2.13) and (2.14), we get the result. This ends the proof of Lemma 2.12. □

Now we are able to prove the Local Stability.

Proof of Lemma 2.10. **Local stability**

In particular, from Lemma 2.12, we have

$$\liminf_{\varepsilon\to 0, \frac{\tau}{\varepsilon^2}\to 0} \int_0^T \Psi_\varepsilon^*(\underline{w}_{\tau,\varepsilon}(t)) \leq \text{const.}$$

Denote $x^+ := \max\{x, 0\}$. A direct computation gives us that

$$\Psi_\varepsilon^*(w) = \frac{1}{2\varepsilon}((|w| - 1)^+)^2. \tag{2.15}$$

Hence, we have

$$\liminf_{\varepsilon\to 0, \frac{\tau}{\varepsilon^2}\to 0} \frac{1}{2\varepsilon} \int_0^T ((|\underline{w}_{\tau,\varepsilon}| - 1)^+)^2 \leq \text{const.}$$

Using Fatou's Lemma and the fact that $\varepsilon \to 0$, we obtain

$$0 = \liminf_{\varepsilon\to 0, \frac{\tau}{\varepsilon^2}\to 0} (|\underline{w}_{\tau,\varepsilon}| - 1)^+ = (|\partial_x \mathscr{E}(t, u(t))| - 1)^+ \text{ for a.e. } t \in (0, T).$$

Thus, we get the weak local stability (w-LS)

$$-\partial_x \mathscr{E}(t, u(t)) \in [-1, 1] \text{ for a.e. } t \in (0, T).$$

□

Remark. If we call J the jump set of $u(\cdot)$, then by the continuity of $\mathscr{E}(\cdot, \cdot)$ and $u(\cdot)$ outside J, we also have $-\partial_x \mathscr{E}(t, u(t)) \in [-1, 1]$ for all $t \in (0, T)\backslash J$.

Before proving the new Energy-Dissipation balance in Lemma 2.10, we need some preliminary lemmas.

Lemma 2.13. *Let $\Psi_\varepsilon, \Psi_\varepsilon^*, x_n^{\tau,\varepsilon}, u_{\tau,\varepsilon}, \underline{w}_{\tau,\varepsilon}$ as before. Then it holds that*

$$\liminf_{\varepsilon\to 0, \frac{\tau}{\varepsilon^2}\to 0} \int_0^T (\Psi_\varepsilon^*(\underline{w}_{\tau,\varepsilon}(t)) - \Psi_\varepsilon^*(-\partial_x \mathscr{E}(t, u_{\tau,\varepsilon}(t))))dt \geq 0.$$

Proof. **Step 1.** By (2.15), we can write

$$\begin{aligned}
&\int_0^{t_N} \left[\Psi_\varepsilon^*(\underline{w}_{\tau,\varepsilon}(t)) - \Psi_\varepsilon^*(-\partial_x\mathscr{E}(t,u_{\tau,\varepsilon}(t)))\right] dt \\
= &\sum_{n=1}^N \int_{t_{n-1}}^{t_n} \left[\frac{1}{2\varepsilon}((|\underline{w}_{\tau,\varepsilon}(t)|-1)^+)^2 - \frac{1}{2\varepsilon}((|\partial_x\mathscr{E}(t,u_{\tau,\varepsilon}(t))|-1)^+)^2\right] dt \\
= &\sum_{n=1}^N \int_{t_{n-1}}^{t_n} \left[\frac{1}{2\varepsilon}((|\partial_x\mathscr{E}(t_n,x_n^{\tau,\varepsilon})|-1)^+)^2 - \frac{1}{2\varepsilon}((|\partial_x\mathscr{E}(t,u_{\tau,\varepsilon}(t))|-1)^+)^2\right] dt \\
= &\sum_{n=1}^N \int_{t_{n-1}}^{t_n} (x_n^+)^2 - \sum_{n=1}^N \int_{t_{n-1}}^{t_n} (y(t)^+)^2 dt, \qquad (2.16)
\end{aligned}$$

here we denote

$$\begin{aligned}
x_n &:= \frac{|\partial_x\mathscr{E}(t_n,x_n^{\tau,\varepsilon})|-1}{\sqrt{2\varepsilon}}, \\
y(t) &:= \frac{|\partial_x\mathscr{E}(t,u_{\tau,\varepsilon}(t))|-1}{\sqrt{2\varepsilon}}.
\end{aligned}$$

From Lemma 2.12, we have

$$\int_0^T \Psi_\varepsilon^*(\underline{w}_{\tau,\varepsilon}(t))dt \le C.$$

Thus

$$\sum_{n=1}^N \int_{t_{n-1}}^{t_n} (x_n^+)^2 \le C. \qquad (2.17)$$

Step 2. Note that by the Cauchy-Schwarz inequality, for all $m > 0$ and $x, y \in \mathbb{R}$ we have

$$\begin{aligned}
(y^+)^2 - (x^+)^2 &= (y^+ - x^+)(y^+ + x^+) \\
&\le |y - x|.(y^+ + x^+) \\
&\le \left(\frac{1}{m}(x-y)^2 + m(x^+ + y^+)^2\right) \\
&\le \left(\frac{1}{m}(x-y)^2 + 2m(x^+)^2 + 2m(y^+)^2\right).
\end{aligned}$$

In particular, choosing $m = 1/4$ we obtain

$$(y^+)^2 \le 8(x-y)^2 + 3(x^+)^2.$$

Hence for all $m > 0$ and $x, y \in \mathbb{R}$ we get

$$(y^+)^2 - (x^+)^2 \le \left(\frac{1}{m} + 16m\right)(x-y)^2 + 8m(x^+)^2.$$

Step 3. Now applying the elementary inequality in Step 2, we can write

$$\begin{aligned}
& \int_{t_{n-1}}^{t_n} \left[\frac{1}{2\varepsilon}((|\partial_x \mathscr{E}(t_n, x_n^{\tau,\varepsilon})|-1)^+)^2 - \frac{1}{2\varepsilon}((|\partial_x \mathscr{E}(t, u_{\tau,\varepsilon}(t))|-1)^+)^2\right] dt \\
= & \int_{t_{n-1}}^{t_n} \left[(x_n^+)^2 - (y(t)^+)^2\right] dt \\
\geq & \int_{t_{n-1}}^{t_n} \left[-\left(\frac{1}{m}+16m\right)(x_n - y(t))^2 - 8m(x_n^+)^2\right] dt. \qquad (2.18)
\end{aligned}$$

Notice that for all $t \in (t_{n-1}, t_n)$, we have

$$\begin{aligned}
(x_n - y(t))^2 &= \left(\frac{|\partial_x \mathscr{E}(t_n, x_n^{\tau,\varepsilon})| - |\partial_x \mathscr{E}(t, u_{\tau,\varepsilon}(t))|}{\sqrt{2\varepsilon}}\right)^2 \\
&\leq \frac{1}{2\varepsilon}(\partial_x \mathscr{E}(t, u_{\tau,\varepsilon}(t)) - \partial_x \mathscr{E}(t_n, x_n^{\tau,\varepsilon}))^2 \\
&\leq \frac{1}{\varepsilon}\left[(\partial_x \mathscr{E}(t, u_{\tau,\varepsilon}(t)) - \partial_x \mathscr{E}(t, x_n^{\tau,\varepsilon}))^2 + (\partial_x \mathscr{E}(t, x_n^{\tau,\varepsilon}) - \partial_x \mathscr{E}(t_n, x_n^{\tau,\varepsilon}))^2\right] \\
&\leq \frac{1}{\varepsilon}C|x_n^{\tau,\varepsilon} - x_{n-1}^{\tau,\varepsilon}|^2 + \frac{1}{\varepsilon}C(t_n - t_{n-1})^2.
\end{aligned}$$

Thus by Lemma 2.8

$$\begin{aligned}
\int_{t_{n-1}}^{t_n} (x_n - y(t))^2 dt &\leq \int_{t_{n-1}}^{t_n} \left[\frac{1}{\varepsilon}C(x_n^{\tau,\varepsilon} - x_{n-1}^{\tau,\varepsilon})^2 + \frac{1}{\varepsilon}C(t_n - t_{n-1})^2\right] \\
&\leq \int_{t_{n-1}}^{t_n} \left[\frac{1}{\varepsilon}C\frac{\tau^2}{\varepsilon^2} + \frac{1}{\varepsilon}C\tau^2\right] \\
&\leq \int_{t_{n-1}}^{t_n} C\frac{\tau}{\varepsilon^2}\frac{\tau}{\varepsilon}. \qquad (2.19)
\end{aligned}$$

Combining (2.16), (2.17), (2.18) and (2.19), we get

$$\int_0^{t_N} (\Psi_\varepsilon^*(\underline{w}_{\tau,\varepsilon}(t))) - \Psi_\varepsilon^*(-\partial_x \mathscr{E}(t, u_{\tau,\varepsilon}(t))))dt \geq \left(\sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} -\left(\frac{1}{m}+16m\right)C\frac{\tau}{\varepsilon^2}\frac{\tau}{\varepsilon}\right) - 8mC.$$

Choosing $m = \tau/\varepsilon^2 \to 0$, we obtain the desired result in Lemma 2.13. □

Lemma 2.14. *Let $\{u_n\}$ be a sequence of Lipschitz functions. Suppose that u_n converges uniformly to some u in $C^0([0,S])$, and $\dot{u}_n$ is w^*-convergent to some v in $L^\infty(0,S)$. Then u is Lipschitz continuous and $v = \dot{u}$ for a.e. $s \in (0,S)$.*

Proof. By definition of the w^*-convergence in $L^\infty(0,S)$, we have for every g in $L^1(0,S)$,

$$\lim_{n\to\infty} \int_0^S \dot{u}_n(s)g(s)ds = \int_0^S v(s)g(s)ds. \qquad (2.20)$$

Now, we choose some $g \in W^{1,1}(0,S)$. Since the sequence u_n uniformly converges to u, the dominated convergence theorem gives us

$$\lim_{n\to\infty}\int_0^S u_n(s)\dot{g}(s)ds = \int_0^S u(s)\dot{g}(s)ds.$$

Itegrating by parts, we get from the above equality

$$\lim_{n\to\infty}\int_0^S \dot{u}_n(s)g(s)ds = \int_0^S \dot{u}(s)g(s)ds. \tag{2.21}$$

This equality is valid for every $g \in W^{1,1}(0,S)$.

Thus, we get from (2.20) and (2.21)

$$\int_0^S \dot{u}(s)g(s)ds = \int_0^S v(s)g(s)ds, \text{ for all } g \in W^{1,1}(0,S).$$

In particular, the above inequality also holds for all $g \in C_c^\infty$. Thus, we have that $\dot{u} = v$ a.e. in $(0,S)$. □

Lemma 2.15. *Let $\{u_n\}$ be a sequence of BV functions. If u_n converges to u in $L^\infty(0,S)$ and $|u_n|$ converges w^* to v in $L^\infty(0,S)$, then $v(s) \geq |u|(s)$ for a.e. $s \in (0,S)$.*

Proof. Let $\phi(x) = \mathrm{sign}(u(x))$. For all $\varphi \in L^1$, $\varphi \geq 0$ we have

$$\int_0^S |u|\varphi \;=\; \int_0^S u\phi\varphi = \lim_{n\to\infty}\int_0^S u_n\phi\varphi \leq \lim_{n\to\infty}\int_0^S |u_n|\cdot|\phi|\cdot|\varphi| = \lim_{n\to\infty}\int_0^S |u_n|\varphi = \int_0^S v\varphi.$$

Therefore,

$$\int_0^S v\varphi \geq \int_0^S |u|\varphi, \forall\varphi \in L^1, \varphi \geq 0.$$

This implies that $v \geq |u|$ a.e. □

Lemma 2.16. *Let $\Psi_\varepsilon, \Psi_\varepsilon^*, x_n^{\tau,\varepsilon}, u_{\tau,\varepsilon}, u(t)$ as before. Then we have the following estimate*

$$\liminf_{\varepsilon\to 0, \frac{\tau}{\varepsilon^2}\to 0}\int_0^T (\Psi_\varepsilon(\dot{u}_{\tau,\varepsilon}(t)) + \Psi_\varepsilon^*(-\partial_x\mathscr{E}(t,u_{\tau,\varepsilon}(t))))dt \geq \mathscr{D}iss_{new}(u;[0,T]).$$

Proof. The proof is divided into 6 steps.

Step 1. Change of variables.

We have

$$\begin{aligned}
&\int_0^T \Psi_\varepsilon(\dot{u}_{\tau,\varepsilon}(t)) + \Psi_\varepsilon^*(-\partial_x\mathscr{E}(t,u_{\tau,\varepsilon}(t)))\\
=&\int_0^T \left(|\dot{u}_{\tau,\varepsilon}(t)| + \frac{\varepsilon}{2}|\dot{u}_{\tau,\varepsilon}(t)|^2 + \frac{1}{2\varepsilon}((|-\partial_x\mathscr{E}(t,u_{\tau,\varepsilon}(t))|-1)^+)^2\right)\\
\geq&\int_0^T \left(|\dot{u}_{\tau,\varepsilon}(t)| + 2\sqrt{\frac{\varepsilon}{2}|\dot{u}_{\tau,\varepsilon}(t)|^2\cdot\frac{1}{2\varepsilon}(|-\partial_x\mathscr{E}(t,u_{\tau,\varepsilon}(t))|-1)^+)^2}\right)\\
=&\int_0^T \left(|\dot{u}_{\tau,\varepsilon}(t)| + |\dot{u}_{\tau,\varepsilon}(t)|\cdot(|-\partial_x\mathscr{E}(t,u_{\tau,\varepsilon}(t))|-1)^+\right)\\
=&\int_0^T |\dot{u}_{\tau,\varepsilon}(t)|\cdot\max\{1,|-\partial_x\mathscr{E}(t,u_{\tau,\varepsilon}(t))|\}dt.
\end{aligned}$$

Choose sequences $\{\tau_n\}$ and $\{\varepsilon_n\}$ such that $u_{\varepsilon_n,\tau_n}(t)$ converges to $u(t)$ pointwise for all $t \in [0,T]$ and denote

$$s_n(t) := t + \int_0^t \left[\Psi_{\varepsilon_n}(\dot{u}_{\tau_n,\varepsilon_n}(h)) + \Psi^*_{\varepsilon_n}(-\partial_x \mathscr{E}(h, u_{\tau_n,\varepsilon_n}(h)))\right] dh,$$

and $S_n := s_n(T)$. Since $s_n(t)$ is strictly increasing and continuous, we can define the inverse

$$\underline{t}_n(s) := s_n^{-1}(s), \text{ for all } s \in [0, S_n].$$

Choosing a subsequence of (τ_n, ε_n) if needed, by Lemma 2.13 we have

$$\lim_{\varepsilon_n \to 0, \tau_n/\varepsilon_n^2 \to 0} \int_0^T (\Psi^*_{\varepsilon_n}(\underline{w}_{\tau_n,\varepsilon_n}(t)) - \Psi^*_{\varepsilon_n}(-\partial_x \mathscr{E}(t, u_{\tau_n,\varepsilon_n}(t))))dt \geq 0.$$

Therefore, thanks to Lemma 2.12, the sequence $\{S_n\}_{n\in N}$ is uniformly bounded. Hence, we can find a number $\overline{S}$ such that $S_n \leq \overline{S}$, for all n. And we can also find a subsequence of $\{S_n\}$ (we also denote this subsequence by $\{S_n\}$) such that S_n converges to some $S > 0$.

Now we extend the sequence of functions $\{\underline{t}_n\}$ over $[0, \overline{S}]$ by defining $\underline{t}_n(s) := \underline{t}_n(S_n)$ for all $s \geq S_n$ and denote for every $s \in [0, S]$

$$\begin{aligned} \underline{u}_n(s) &:= u_{\tau_n,\varepsilon_n}(\underline{t}_n(s)), \\ \underline{w}_n(s) &:= -\partial_x \mathscr{E}(\underline{t}_n(s), \underline{u}_n(s)). \end{aligned}$$

Notice that since $s_n(t)$ is strictly increasing, $\underline{t}_n(s)$ is also strictly increasing on $[0, S_n]$. So $\dot{\underline{t}}_n(s) > 0$ for all $s \in [0, S_n]$. By changing variables in the integral, we get

$$\int_0^T |\dot{u}_{\tau_n,\varepsilon_n}(t)| \cdot \max\{1, |-\partial_x \mathscr{E}(t, u_{\tau_n,\varepsilon_n}(t))|\}dt = \int_0^{\overline{S}} 1_{[0,S_n]} \cdot |\dot{\underline{u}}_n(s)| \cdot \max\{1, |\underline{w}_n(s)|\}ds.$$

In the next steps we find convergent subsequences of $\underline{t}_n(s)$, $\underline{u}_n(s)$ and $\underline{w}_n(s)$.

Step 2. Convergent subsequence of $\{\underline{t}_n\}$.

Notice that the sequence $\{\underline{t}_n\}$ is uniformly bounded by a constant: $|\underline{t}_n(s)| \leq T$, for all $s \in [0, \overline{S}]$. Moreover, the sequence $\{\underline{t}_n\}$ is equicontinuous: $\dot{\underline{t}}_n(s) < 1$ for all $s \in (0, S_n)$ since $|\dot{s}_n(t)| = \dot{s}_n(t) > 1$ for all $t \in (0,T)$, and $\dot{\underline{t}}_n(s) = 0$ for all $s \in (S_n, \overline{S})$ by definition of $\underline{t}_n$. Hence we can apply the Arzelà-Ascoli theorem to get a subsequence (still denoted by $\{\underline{t}_n\}$) converging uniformly to some $\underline{t}$.

Moreover, we also have $||\dot{\underline{t}}_n||_{L^\infty(0,\overline{S})} \leq 1$. Thus, (up to a subsequence) we can assume that $\dot{\underline{t}}_n$ w^*-converges to some $\underline{s}$ in $L^\infty(0, \overline{S})$. By Lemma 2.14, we have $\dot{\underline{t}} = \underline{s}$.

Step 3. Convergent subsequence of $\{\underline{u}_n\}$.

First, we prove that the sequence $\{\dot{\underline{u}}_n\}$ is uniformly bounded in L^∞-norm. In fact, by the definition of $\underline{u}_n$ we have

$$\dot{\underline{u}}_n(s) = \dot{u}_{\tau_n,\varepsilon_n}(\underline{t}_n(s))\, \dot{\underline{t}}_n(s) = \frac{\dot{u}_{\tau_n,\varepsilon_n}(t)}{\dot{s}_n(t)},$$

here we denote $t := \underline{t}_n(s)$.

Now we call $\{t_k\}$ the partition corresponding to τ_n. Since $s_n(t)$ is strictly increasing, we can find two numbers s_1, s_2 such that if $s_1 < s < s_2$ then $t_{k-1} < t < t_k$. Notice that both s_1, s_2 and t_{k-1}, t_k depend on τ_n.

By the definition of $u_{\tau_n,\varepsilon_n}(t)$, we know that

$$\dot{u}_{\tau_n,\varepsilon_n}(t) = \frac{x_k^{\tau_n,\varepsilon_n} - x_{k-1}^{\tau_n,\varepsilon_n}}{\tau_n} \text{ if } t \in (t_{k-1}, t_k).$$

By the definition of $s_n(t)$, we get

$$\begin{aligned}
\dot{s}_n(t) &= 1 + \Psi_{\varepsilon_n}(\dot{u}_{\tau_n,\varepsilon_n}(t)) + \Psi^*_{\varepsilon_n}(-\partial_x \mathscr{E}(t, u_{\tau_n,\varepsilon_n}(t))) \\
&= 1 + \Psi_{\varepsilon_n}\left(\frac{x_k^{\tau_n,\varepsilon_n} - x_{k-1}^{\tau_n,\varepsilon_n}}{\tau_n}\right) + \Psi^*_{\varepsilon_n}(-\partial_x \mathscr{E}(t, u_{\tau_n,\varepsilon_n}(t))) \\
&= 1 + \left|\frac{x_k^{\tau_n,\varepsilon_n} - x_{k-1}^{\tau_n,\varepsilon_n}}{\tau_n}\right| + \frac{\varepsilon_n}{2}\left|\frac{x_k^{\tau_n,\varepsilon_n} - x_{k-1}^{\tau_n,\varepsilon_n}}{\tau_n}\right|^2 + \frac{1}{2\varepsilon_n}\left((|\partial_x \mathscr{E}(t, u_{\tau_n,\varepsilon_n}(t))| - 1)^+\right)^2.
\end{aligned}$$

Hence, for all $s_1 < s < s_2$ we have

$$\begin{aligned}
|\underline{\dot{u}}_n(s)| &= \left|\frac{x_k^{\tau_n,\varepsilon_n} - x_{k-1}^{\tau_n,\varepsilon_n}}{\tau_n}\right| \times \\
&\times \frac{1}{1 + \left|\frac{x_k^{\tau_n,\varepsilon_n} - x_{k-1}^{\tau_n,\varepsilon_n}}{\tau_n}\right| + \frac{\varepsilon_n}{2}\left|\frac{x_k^{\tau_n,\varepsilon_n} - x_{k-1}^{\tau_n,\varepsilon_n}}{\tau_n}\right|^2 + \frac{1}{2\varepsilon_n}\left((|\partial_x \mathscr{E}(t, u_{\tau_n,\varepsilon_n}(t))| - 1)^+\right)^2} \\
&\leq 1.
\end{aligned}$$

Thus, $|\underline{\dot{u}}_n(s)| \leq 1$ for all $s \in (s_1, s_2)$. This implies $||\underline{\dot{u}}_n||_{L^\infty(0,\overline{S})} \leq 1$.

Therefore, we can apply the Arzelà-Ascoli theorem to get a subsequence (also denoted by $\{\underline{u}_n\}$) converging uniformly to some $\underline{u}$ in $C^0([0, \overline{S}])$. Moreover, since $||\underline{\dot{u}}_n||_{L^\infty(0,\overline{S})} \leq 1$, up to subsequence, we can assume that $\underline{\dot{u}}_n$ converges in w^*-sense to some $\underline{v}$ in $L^\infty(0, \overline{S})$. Applying Lemma 2.14, we have $\underline{\dot{u}} = \underline{v}$ a.e.

Moreover, we can check that $\underline{u}(s) = u(\underline{t}(s))$, for all $s \in [0, S]$ such that $\underline{t}(s) \notin J$.

In fact, let $\phi : [0, T] \to \mathbb{R}$ be an arbitrary continuous function. Since u_{τ_n,ε_n} converges to u pointwise and $|u_{\tau_n,\varepsilon_n}| \leq C$, we can apply the Dominated Convergence Theorem to get u_{τ_n,ε_n} converges to u strongly in $L^1(0, T)$. Consequently,

$$\lim_{n\to\infty} \int_0^T u_{\tau_n,\varepsilon_n}(t)\, \phi(t)\, dt = \int_0^T u(t)\, \phi(t)\, dt. \tag{2.22}$$

On the other hand, for every $n \in \mathbb{N}$, using change of variable $t = \underline{t}_n(s)$, we get

$$\begin{aligned}
\lim_{n\to\infty} \int_0^T u_{\tau_n,\varepsilon_n}(t)\, \phi(t)\, dt &= \int_0^{\overline{S}} 1_{[0,S_n]}\, u_{\tau_n,\varepsilon_n}(\underline{t}_n(s))\, \phi(\underline{t}_n(s))\, \underline{\dot{t}}_n(s)\, ds \\
&= \int_0^{\overline{S}} 1_{[0,S_n]}\, \underline{u}_n(s)\, \phi(\underline{t}_n(s))\, \underline{\dot{t}}_n(s)\, ds. \qquad (2.23)
\end{aligned}$$

Since $\underline{u}_n$ converges to $\underline{u}$ uniformly in $C^0([0,\overline{S}])$, ϕ is continuous, $\underline{t}_n$ converges to $\underline{t}$ uniformly in $C^0([0,\overline{S}])$, S_n converges to S in $\mathbb{R}$, and $\dot{\underline{t}}_n$ converges to $\dot{\underline{t}}$ weakly-* in $L^\infty(0,\overline{S})$, we have

$$\begin{aligned}\lim_{n\to\infty}\int_0^{\overline{S}} 1_{[0,S_n]}\,\underline{u}_n(s)\,\phi(\underline{t}_n(s))\,\dot{\underline{t}}_n(s)\,ds &= \int_0^{\overline{S}} 1_{[0,S]}\,\underline{u}(s)\,\phi(\underline{t}(s))\,\dot{\underline{t}}(s)\,ds\\ &= \int_0^{S} \underline{u}(s)\,\phi(\underline{t}(s))\,\dot{\underline{t}}(s)\,ds. \end{aligned} \tag{2.24}$$

Now, using change of variable $t=\underline{t}(s)$, we get

$$\int_0^T u(t)\,\phi(t)\,dt = \int_0^S u(\underline{t}(s))\,\phi(\underline{t}(s))\,\dot{\underline{t}}(s)\,ds. \tag{2.25}$$

Combining (2.22), (2.23), (2.24), and (2.25), we get

$$\int_0^S u(\underline{t}(s))\,\phi(\underline{t}(s))\,\dot{\underline{t}}(s)\,ds = \int_0^S \underline{u}(s)\,\phi(\underline{t}(s))\,\dot{\underline{t}}(s)\,ds. \tag{2.26}$$

Denote by J the jump set of $u(\cdot)$. For any $t\in J$, we denote the set $A(t) := \{s\in[0,S]\mid \underline{t}(s)=t\}$. Since $\underline{t}$ is increasing, $A(t)$ is an interval in $\mathbb{R}$. Since $u(t)$ has bounded variation, J is at most countable. Thus, we can write $J=\{t_1,t_2,\dots,t_n,\dots\}$. Now denote by $A := \cup_{n\in\mathbb{N},t_n\in J}A(t_n)$, we can see that $\underline{t}$ is strictly increasing outside the set A. Together with the fact that $\underline{t}$ is absolutely continuous, we get that $\dot{\underline{t}}>0$ for a.e. $s\in[0,S]\backslash A$.

Hence, by (2.26) and the fact that $\dot{\underline{t}}>0$ for a.e. $s\in[0,S]\backslash A$, we obtain

$$u(\underline{t}(s)) = \underline{u}(s) \text{ for a.e. } s\in[0,S]\backslash A.$$

By continuity of $\underline{u}$, we also get

$$u(\underline{t}(s)) = \underline{u}(s) \text{ for all } s\in[0,S]\backslash A.$$

Step 4. Convergent subsequence of $\{\underline{w}_n\}$.

Since $\underline{u}_n$ converges uniformly to $\underline{u}$ in $C^0([0,\overline{S}])$, $\underline{t}_n$ converges uniformly to $\underline{t}$ in $C^0([0,\overline{S}])$, and $\mathscr{E}$ is C^2, we have immediately that $\underline{w}_n$ to $\underline{w}$ in $C^0([0,\overline{S}])$, and

$$\underline{w}(s) = -\partial_x\mathscr{E}(\underline{t}(s),\underline{u}(s)).$$

Step 5. Now we show that

$$\lim_{n\to\infty}\int_0^{\overline{S}} 1_{[0,S_n]}\cdot|\dot{\underline{u}}_n(s)|\cdot\max\{1,|\underline{w}_n(s)|\}ds \geq \mathscr{D}iss_{new}(u;[0,T]).$$

In fact, denote $a_n(s) := 1_{[0,S_n]}(s)\cdot\max\{1,|\underline{w}_n(s)|\}$, then $a_n(s)$ converges to

$$a(s) = 1_{[0,S]}(s)\cdot\max\{1,|\underline{w}(s)|\}$$

strongly in $L^1(0,\overline{S})$. Since $\|\dot{\underline{u}}_n\|_{L^\infty} \le C$, up to a subsequence we can assume that $|\dot{\underline{u}}_n|$ w^*-converges to some v in $L^\infty(0,\overline{S})$. Apply Lemma 2.15 we have $v \ge |\dot{\underline{u}}|$ a.e. Thus

$$\begin{aligned}
\lim_{n\to\infty}\int_0^{\overline{S}} 1_{[0,S_n]}\cdot|\dot{\underline{u}}_n(s)|\cdot\max\{1,|\underline{w}_n(s)|\}ds &= \lim_{n\to\infty}\int_0^{\overline{S}} a_n(s)\cdot|\dot{\underline{u}}_n(s)|ds\\
&= \int_0^{\overline{S}} a(s)\cdot v(s)ds\\
&= \int_0^{\overline{S}} 1_{[0,S]}\cdot\max\{1,|\underline{w}(s)|\}\cdot v(s)ds\\
&\ge \int_0^{S}\max\{1,|\underline{w}(s)|\}\cdot|\dot{\underline{u}}(s)|ds.
\end{aligned}$$

By the local stability, for all $t \notin J$, we know that $|\partial_x\mathscr{E}(t,u(t))| \le 1$. Hence, for all $s \notin \cup_{t\in J}A(t)$, we have $|\underline{w}(s)| \le 1$. Thus, we can write

$$\int_0^S \max\{1,|\underline{w}(s)|\}\cdot|\dot{\underline{u}}(s)|ds = \int_{[0,S]\setminus\cup_{t\in J}A(t)}|\dot{\underline{u}}(s)| + \sum_{t\in J}\int_{A(t)}\max\{1,|\underline{w}(s)|\}\cdot|\dot{\underline{u}}(s)|ds.$$

First, we prove that

$$\begin{aligned}
\int_{[0,S]\setminus\cup_{t\in J}A(t)}|\dot{\underline{u}}(s)|\,ds &\ge \mathscr{D}iss(u;[0,T]) - \sum_{t\in J}|u(t^+)-u(t^-)|\\
&= \mathscr{D}iss(u;[0,T]) - \sum_{t\in J}\Delta(t,u(t^-),u(t^+)).
\end{aligned}$$

In fact, consider $t_1 \in J$, we can write that $A(t_1) = [a_1,b_1]$. For any $\delta > 0$ such that $a_1-\delta, b_1+\delta \notin \cup_{t\in J}A(t)$, we have

$$\begin{aligned}
\int_{[0,S]\setminus[a_1-\delta,b_1+\delta]}|\dot{\underline{u}}(s)|\,ds &= \mathscr{D}iss(\underline{u};[0,S]\setminus[a_1-\delta,b_1+\delta])\\
&= \mathscr{D}iss(\underline{u};[0,a_1-\delta]) + \mathscr{D}iss(\underline{u};[b_1+\delta,S])\\
&\ge \mathscr{D}iss(u;[0,\underline{t}(a_1-\delta)]) + \mathscr{D}iss(u;[\underline{t}(b_1+\delta),T])\\
&= \mathscr{D}iss(u;[0,T]) - \mathscr{D}iss(u;[\underline{t}(a_1-\delta),\underline{t}(b_1+\delta)]).
\end{aligned}$$

Here the inequalities

$$\mathscr{D}iss(\underline{u};[0,a_1-\delta]) \ge \mathscr{D}iss(u;[0,\underline{t}(a_1-\delta)])$$

and

$$\mathscr{D}iss(\underline{u};[b_1+\delta,S]) \ge \mathscr{D}iss(u;[\underline{t}(b_1+\delta),T])$$

come from the fact that $a_1-\delta, b_1+\delta \notin \cup_{t\in J}A(t)$, and that $\underline{u}(s) = u(\underline{t}(s))$ for all $s \in [0,S]\setminus A$. Now choosing a sequence $\delta_k \to 0$ such that $a_1-\delta_k, b_1+\delta_k \notin \cup_{t\in J}A(t)$ for all k. Taking $k\to\infty$, we get

$$\int_{[0,S]\setminus[a_1,b_1]}|\dot{\underline{u}}(s)|\,ds \ge \mathscr{D}iss(u;[0,T]) - |u(t_1^-)-u(t_1^+)|.$$

By induction, the above inequality also holds for finitely many jumps

$$\int_{[0,S]\setminus\cup_{i=1}^{n}A(t_i)} |\underline{\dot{u}}(s)|\,ds \geq \mathscr{D}iss(u;[0,T]) - \sum_{i=1}^{n} |u(t_i^-) - u(t_i^+)|.$$

Taking $n \to \infty$, we get the result.

Now we verify that

$$\sum_{t\in J} \int_{A(t)} \max\{1, |\underline{w}(s)|\} \cdot |\underline{\dot{u}}(s)| ds \geq \sum_{t\in J} \Delta_{new}(t, u(t^-), u(t^+)).$$

In fact, for any $t \in J$, since $A(t)$ is an interval in $\mathbb{R}$, we can denote $A(t) = [s_0, s_1]$. Now using the change of variable to replace $\underline{u}(s)$ by some $v(r)$ satisfying $v \in AC([0,1]), v(0) = \underline{u}(s_0) = u(t^-), v(1) = \underline{u}(s_1) = u(t^+)$, we have

$$\begin{aligned}
\int_{A(t)} \max\{1, |\underline{w}(s)|\} \cdot |\underline{\dot{u}}(s)| &= \int_{s_0}^{s_1} \max\{1, |\underline{w}(s)|\} \cdot |\underline{\dot{u}}(s)| \\
&= \int_{s_0}^{s_1} \max\{1, |\partial_x \mathscr{E}(\underline{t}(s), \underline{u}(s))|\} \cdot |\underline{\dot{u}}(s)| \\
&= \int_{0}^{1} \max\{1, |\partial_x \mathscr{E}(t, v(r))|\} \cdot |\dot{v}(r)| dr \\
&\geq \Delta_{new}(t, u(t^-), u(t^+)\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_0^S \max\{1, |\underline{w}(s)|\} \cdot |\underline{\dot{u}}(s)| ds &\geq \mathscr{D}iss(u;[0,T]) - \sum_{t\in J} \Delta(t, u(t^-), u(t^+)) + \sum_{t\in J} \Delta_{new}(t, u(t^-), u(t^+)) \\
&= \mathscr{D}iss_{new}(u;[0,T]).
\end{aligned}$$

This completes the proof of Lemma 2.16. □

Lemma 2.17 (Lower bound of the new Energy - Dissipation balance)**.** *For any BV function $u : [0,T] \to \mathbb{R}$, for any energy functional $\mathscr{E} \in C^2(\mathbb{R}^2)$ satisfying the constraint $|\partial_x \mathscr{E}(t, u(t))| \leq 1$ for a.e. $t \in (0,T)$, it holds that*

$$\mathscr{E}(t_1, u(t_1)) - \mathscr{E}(t_0, u(t_0)) \geq \int_{t_0}^{t_1} \partial_t \mathscr{E}(s, u(s)) ds - \mathscr{D}iss_{new}(u, [t_0, t_1]).$$

Proof. Since u is BV, the distributional derivative Du can be split into three parts: the absolutely continuous part w.r.t. Lebesgue measure $D^a u$, the jump part $D^j u$ and the Cantor part $D^c u$. Now we denote $u'_{co} = D^a u + D^c u$, then applying the chain rule formula for $\mathscr{E} \in C^2$ and $u \in BV$ (see [3]), we get

$$\begin{aligned}
&\mathscr{E}(t_1, u(t_1)) - \mathscr{E}(t_0, u(t_0)) \\
= &\int_{t_0}^{t_1} \partial_t \mathscr{E}(s, u(s)) ds + \int_{t_0}^{t_1} \langle \partial_x \mathscr{E}(s, u(s)), u'_{co}(s) \rangle \, ds + \sum_{t\in J\cap(t_0,t_1)} [\mathscr{E}(t, u(t^+)) - \mathscr{E}(t, u(t^-))] \\
&+\mathscr{E}(t_0, u(t_0^+)) - \mathscr{E}(t_0, u(t_0)) + \mathscr{E}(t_1, u(t_1)) - \mathscr{E}(t_1, u(t_1^-)) \\
\geq &\int_{t_0}^{t_1} \partial_t \mathscr{E}(s, u(s)) ds - \int_{t_0}^{t_1} |u'_{co}(s)| ds - \sum_{t\in J\cap(t_0,t_1)} |\mathscr{E}(t, u(t^+)) - \mathscr{E}(t, u(t^-))| \\
&-|\mathscr{E}(t_0, u(t_0^+)) - \mathscr{E}(t_0, u(t_0))| - |\mathscr{E}(t_1, u(t_1)) - \mathscr{E}(t_1, u(t_1^-))|.
\end{aligned}$$

Notice that

$$\begin{aligned}\int_{t_0}^{t_1} |u'_{co}(s)|ds &= \mathscr{D}iss(u;[t_0,t_1]) - \sum_{t\in J\cap(t_0,t_1)} |u(t^+)-u(t^-)| - |u(t_0^+)-u(t_0)| - |u(t_1)-u(t_1^-)| \\ &= \mathscr{D}iss(u;[t_0,t_1]) - \sum_{t\in J\cap(t_0,t_1)} \Delta(t,u(t^-),u(t^+)) \\ &\quad -\Delta(t_0,u(t_0),u(t_0^+)) - \Delta(t_1,u(t_1^-),u(t_1)).\end{aligned} \tag{2.27}$$

Moreover, for every absolutely continuous curve v in $AC([0,1])$ such that $v(0)=u(t^-)$, $v(1)=u(t^+)$ we have the following formula

$$\mathscr{E}(t,u(t^+)) - \mathscr{E}(t,u(t^-)) = \int_0^1 \partial_x \mathscr{E}(t,v(x))\cdot \dot v(x)dx.$$

Thus

$$\begin{aligned}|\mathscr{E}(t,u(t^+)) - \mathscr{E}(t,u(t^-))| &\le \int_0^1 |\partial_x \mathscr{E}(t,v(x))|\cdot |\dot v(x)|dx \\ &\le \int_0^1 \max\{1,|\partial_x \mathscr{E}(t,v(x))|\}\cdot |\dot v(x)|dx.\end{aligned}$$

The above inequality holds for every absolutely continuous curve v connecting $u(t^-)$ and $u(t^+)$. Thus we can write

$$\begin{aligned}&|\mathscr{E}(t,u(t^+)) - \mathscr{E}(t,u(t^-))| \\ \le\ & \inf_v\{\int_0^1 \max\{1,|\partial_x \mathscr{E}(t,v(x))|\}\cdot |\dot v(x)|dx : v\in AC([0,1]), v(0)=u(t^-), v(1)=u(t^+)\} \\ =\ & \Delta_{new}(t,u(t^-),u(t^+)).\end{aligned} \tag{2.28}$$

Therefore, it follows from (2.27) and (2.28)

$$\begin{aligned}\mathscr{E}(t_1,u(t_1)) - \mathscr{E}(t_0,u(t_0)) &\ge \int_{t_0}^{t_1} \partial_t \mathscr{E}(s,u(s))ds - \mathscr{D}iss(u;[t_0,t_1]) \\ &\quad + \sum_{t\in J\in(t_0,t_1)} \Delta(t,u(t^-),u(t^+)) + \Delta(t_0,u(t_0),u(t_0^+)) \\ &\quad +\Delta(t_1,u(t_1^-),u(t_1)) - \sum_{t\in J\cap(t_0,t_1)} \Delta_{new}(t,u(t^-),u(t^+)) \\ &\quad -\Delta_{new}(t_0,u(t_0),u(t_0^+)) - \Delta_{new}(t_1,u(t_1^-),u(t_1)) \\ &= \int_{t_0}^{t_1} \partial_t \mathscr{E}(s,u(s))ds - \mathscr{D}iss_{new}(u,[t_0,t_1]).\end{aligned}$$

This ends the proof of the lower bound. □

Now we are able to prove the upper bound of the new Energy-Dissipation balance.

Proof of Lemma 2.10. From Lemma 2.12, we know that

$$\lim_{\varepsilon\to 0, \frac{\tau}{\varepsilon^2}\to 0} \int_0^T (\Psi_\varepsilon(\dot{u}_{\tau,\varepsilon}(t)) + \Psi_\varepsilon^*(\underline{w}_{\tau,\varepsilon}(t)))dt = \mathscr{E}(0,x_0) - \mathscr{E}(T,u(T)) + \int_0^T \partial_t\mathscr{E}(t,u(t))dt.$$

On the other hand, we see that

$$\begin{aligned}\int_0^T (\Psi_\varepsilon(\dot{u}_{\tau,\varepsilon}(t)) + \Psi_\varepsilon^*(\underline{w}_{\tau,\varepsilon}(t)))dt &= \int_0^T (\Psi_\varepsilon^*(\underline{w}_{\tau,\varepsilon}(t)) - \Psi_\varepsilon^*(-\partial_x\mathscr{E}(t,u_{\tau,\varepsilon}(t))))dt \\ &\quad + \int_0^T (\Psi_\varepsilon(\dot{u}_{\tau,\varepsilon}(t)) + \Psi_\varepsilon^*(-\partial_x\mathscr{E}(t,u_{\tau,\varepsilon}(t))))dt.\end{aligned}$$

By Lemma 2.13,

$$\liminf_{\varepsilon\to 0, \frac{\tau}{\varepsilon^2}\to 0} \int_0^T (\Psi_\varepsilon^*(\underline{w}_{\tau,\varepsilon}(t)) - \Psi_\varepsilon^*(-\partial_x\mathscr{E}(t,u_{\tau,\varepsilon}(t))))dt \geq 0.$$

Thanks to Lemma 2.16,

$$\liminf_{\varepsilon\to 0, \frac{\tau}{\varepsilon^2}\to 0} \int_0^T (\Psi_\varepsilon(\dot{u}_{\tau,\varepsilon}(t)) + \Psi_\varepsilon^*(-\partial_x\mathscr{E}(t,u_{\tau,\varepsilon}(t))))dt \geq \mathscr{D}iss_{new}(u;[0,T]).$$

Hence,

$$\begin{aligned}\mathscr{D}iss_{new}(u;[0,T]) &\leq \liminf_{\varepsilon\to 0, \frac{\tau}{\varepsilon^2}\to 0} \int_0^T (\Psi_\varepsilon^*(\underline{w}_{\tau,\varepsilon}(t)) - \Psi_\varepsilon^*(-\partial_x\mathscr{E}(t,u_{\tau,\varepsilon}(t))))dt \\ &\quad + \liminf_{\varepsilon\to 0, \frac{\tau}{\varepsilon^2}\to 0} \int_0^T (\Psi_\varepsilon(\dot{u}_{\tau,\varepsilon}(t)) + \Psi_\varepsilon^*(-\partial_x\mathscr{E}(t,u_{\tau,\varepsilon}(t))))dt \\ &\leq \liminf_{\varepsilon\to 0, \frac{\tau}{\varepsilon^2}\to 0} \int_0^T (\Psi_\varepsilon(\dot{u}_{\tau,\varepsilon}(t)) + \Psi_\varepsilon^*(\underline{w}_{\tau,\varepsilon}(t))dt \\ &\leq \mathscr{E}(0,x_0) - \mathscr{E}(T,u(T)) + \int_0^T \partial_t\mathscr{E}(t,u(t))dt.\end{aligned}$$

Thus, we get

$$\mathscr{E}(T,u(T)) - \mathscr{E}(0,x_0) \leq \int_0^T \partial_t\mathscr{E}(t,u(t))dt - \mathscr{D}iss_{new}(u;[0,T]). \tag{2.29}$$

Now denote by $I(t_1,t_2)$ the difference between the left-hand side and the right-hand side of (2.29). Then it follows from Lemma 2.17 that $I(t_1,t_0) \geq 0$ for every $0 \leq t_0 \leq t_1 \leq T$. On the other hand, from Lemma 2.10, we have that $I(0,T) \leq 0$. Thus,

$$0 \geq I(0,T) = I(0,t_0) + I(t_0,t_1) + I(t_1,T).$$

Since each addendum of the above formula is nonnegative, we get that $I(t_0,t_1) = 0$, which is the new energy-dissipation balance.

$$\mathscr{E}(t_1,u(t_1)) - \mathscr{E}(t_0,u(u_0)) \leq \int_{t_0}^{t_1} \partial_t\mathscr{E}(s,u(s))ds - \mathscr{D}iss_{new}(u,[t_0,t_1]) \tag{2.30}$$

for every $0 \leq t_0 \leq t_1 \leq T$. This ends the proof of the new energy-dissipation balance. □

Remark. For a proof in a more general setting, we refer to the paper of Mielke, Rossi, and Savaré [26].

3 Another construction of BV solutions

The idea is to find the minimizer x_i of the discretized problem in some small neighborhood of x_{i-1}, instead of finding the minimizer over all $\mathbb{R}$. To be precise, let $\varepsilon > 0$, $\tau > 0$ and let $N \in \mathbb{N}$ satisfy $1 \in [\tau N, \tau(N+1))$. We define a sequence $\{x^{\varepsilon,\tau}\}_{i=0}^{N}$ by
(IP$'_\varepsilon$) $x_0^{\varepsilon,\tau} = x_0$ (initial position) and

$$x_i^{\varepsilon,\tau} \in \operatorname{argmin}\{\mathscr{E}(t_n, x) + |x - x_{i-1}^{\varepsilon,\tau}| \mid |x - x_{i-1}^{\varepsilon,\tau}| \leq \varepsilon\} \text{ for every } i \in \{1, \ldots, N\}.$$

We define the discretized solution $x^{\varepsilon,\tau}(\cdot)$ by interpolation

$$x^{\varepsilon,\tau}(t) := x_{i-1}^{\varepsilon,\tau} \text{ for every } t \in [t_{i-1}, t_i), i \in \{1, \ldots, N\}.$$

The limit $x(\cdot)$ of $x^{\varepsilon,\tau}(\cdot)$ as $\varepsilon \to 0$ and $\tau \to 0$ is a solution to (2.2) in an appropriate sense.

Theorem 2.18 (BV solutions constructed by epsilon-neighborhood)**.** *Let $\mathscr{E} : [0,T] \times \mathbb{R} \to [0, +\infty]$ be of class C^2 and satisfy (E1). Let us consider an initial datum $x_0 \in \mathbb{R}$ such that x_0 is a local minimizer for the functional $x \mapsto \mathscr{E}(0,x) + |x - x_0|$. Then the following statements hold true.*

- *(i) For any $\varepsilon > 0$ and $\tau > 0$, there exists a discretized solution $t \mapsto x^{\varepsilon,\tau}(\cdot)$ as described above. For any $\varepsilon > 0$, there exists a subsequence $\tau_n \to 0$ such that $x^{\varepsilon,\tau_n}(\cdot)$ converges pointwise to some limit $x^{\varepsilon}(\cdot)$. There exists a subsequence $\varepsilon_n \to 0$ such that $x^{\varepsilon_n}(\cdot)$ converges pointwise to some BV function $x(\cdot)$.*
- *(ii) (Local stability) If $t \mapsto x(t)$ is continuous at t, then*

$$|\partial_x \mathscr{E}(t, x(t))| \leq 1.$$

- *(iii) (New energy-dissipation balance) For all $0 \leq s \leq t \leq T$, one has*

$$\mathscr{E}(t, x(t)) - \mathscr{E}(s, x(s)) = \int_s^t \partial_t \mathscr{E}(r, x(r)) dr - \mathscr{D}iss_{new}(x; [s,t]).$$

This theorem is proved in the following subsections.

3.1 Discretized solutions

We start by considering the discretized solution $x^{\varepsilon,\tau}$.

Lemma 2.19 (Discretized solution)**.** *For any given initial state x_0, any $\tau > 0$ and any partition $0 = t_0 < t_1 < \cdots < t_N \leq T$ of $[0,T]$ such that $t_n - t_{n-1} = \tau$, there exists a discretized solution $t \mapsto x^{\varepsilon,\tau}(t)$ satisfying the following two properties:*

- *(i) (Minimizer) We have $x_0^{\varepsilon,\tau} = x_0$ and for every $i = 1, 2, ..., N$, $x_i^{\varepsilon,\tau}$ minimizes $x \mapsto \mathscr{E}(t_n, x) + |x_{i-1}^{\varepsilon,\tau} - x|$ over $x \in \mathbb{R}$, $|x - x_{i-1}^{\varepsilon,\tau}| \leq \varepsilon$; and*
- *(ii) (Interpolation) $x^{\varepsilon,\tau}(t) = x_{i-1}^{\varepsilon,\tau}$ if $t \in [t_{i-1}, t_i)$, $i \in \{1, ..., N\}$.*

Proof. Since $x \mapsto \mathscr{E}(t_n, x) + |x - x_{i-1}^{\varepsilon,\tau}|$ is continuous, this functional has a minimizer $x_i^{\varepsilon,\tau}$ in the compact set $|x - x_{i-1}^{\varepsilon,\tau}| \leq \varepsilon$. □

By the same argument as for energetic solutions (cf. Lemma 2.3), we have the following estimates.

Lemma 2.20 (Energy estimates). *Let $x^{\varepsilon,\tau}$ be as in Lemma 2.19. Then we have*

(i) (Discrete bound) For any $n \in \{1, \dots, N\}$ we have

$$\mathscr{E}(t_n, x_n^{\varepsilon,\tau}) \le \mathscr{E}(0, x_0)\, e^{\lambda t_n} \text{ and } \mathscr{E}(0, x_n^{\varepsilon,\tau}) \le \mathscr{E}(0, x_0)\, e^{2\lambda t_n}.$$

(ii) (Integral bound) For all $0 \le s \le t \le T$, it holds that $\mathscr{D}iss(x^{\varepsilon,\tau}; [s,t]) < \infty$, $\partial_t \mathscr{E}(\cdot, x^{\varepsilon,\tau}(\cdot)) \in L^1(0,T)$ and

$$\mathscr{E}(t, x^{\varepsilon,\tau}(t)) - \mathscr{E}(s, x^{\varepsilon,\tau}(s)) \le \int_s^t \partial_t \mathscr{E}(r, x^{\varepsilon,\tau}(r)) dr - \mathscr{D}iss(x^{\varepsilon,\tau}; [s,t]).$$

3.2 The epsilon-neighborhood solutions

Lemma 2.21 (epsilon-neighborhood solution). *Given an initial datum $x_0 \in \mathbb{R}$ such that $\mathscr{E}(0, x_0) < \infty$ and x_0 is a local minimizer for the functional $x \mapsto \mathscr{E}(0,x) + |x - x_0|$, let $x^{\varepsilon,\tau}$ be as in Lemma 2.19. Then there exists a subsequence $\tau_n \to 0$ such that $x^{\varepsilon,\tau_n}(t) \to x^\varepsilon(t)$ for all $t \in [0,T]$. Moreover, the epsilon-neighborhood solution $x^\varepsilon(\cdot)$ satisfies the following properties:*

(i) (Epsilon local stability) If $x^\varepsilon(\cdot)$ is right-continuous at t, namely $\lim_{t' \to t^+} x^\varepsilon(t') = x^\varepsilon(t)$, then $x^\varepsilon(t)$ satisfies the epsilon local stability

$$\mathscr{E}(t, x^\varepsilon(t)) \le \mathscr{E}(t, x) + |x - x^\varepsilon(t)| \quad \text{for all } |x - x^\varepsilon(t)| \le \varepsilon.$$

(ii) (Energy-dissipation inequalities) We have $\mathscr{D}iss(x^\varepsilon; [0,T]) \le C$ (independent of ε), $\partial_t \mathscr{E}(\cdot, x^\varepsilon(\cdot)) \in L^1(0,T)$ and for all $0 \le s \le t \le T$,

$$-\mathscr{D}iss_{new}(x^\varepsilon; [s,t]) \le \mathscr{E}(t, x^\varepsilon(t)) - \mathscr{E}(s, x^\varepsilon(s)) - \int_s^t \partial_t \mathscr{E}(r, x^\varepsilon(r)) dr \le -\mathscr{D}iss(x^\varepsilon; [s,t]).$$

Proof. **Step 1. Existence.** By Lemma 2.20 and the condition (E1), we see that $\{x^{\varepsilon,\tau}(\cdot)\}$ has uniformly bounded variation and it is uniformly bounded. Therefore, applying Helly's selection principle we can find a subsequence $\tau_n \to 0$ and a BV function $x^\varepsilon(\cdot)$ such that $x^{\varepsilon,\tau_n}(t) \to x^\varepsilon(t)$ as $n \to \infty$ for all $t \in [0,T]$.

Step 2. A consequence of the right-continuity. Let us denote by $\{t_i^n\}_{i=0}^{N_n}$ the partition corresponding to τ_n and assume that $t \in [t_{i-1}^n, t_i^n]$. It is obvious that

$$x_{i-1}^{\varepsilon,\tau_n} = x^{\varepsilon,\tau_n}(t) \to x^\varepsilon(t)$$

as $n \to \infty$. Now we show that if $x^\varepsilon(\cdot)$ is right-continuous at t, then

$$x_i^{\varepsilon,\tau_n} = x^{\varepsilon,\tau_n}(t_i^n) \to x^\varepsilon(t).$$

Let $t' > t$. Due to the integral bound in Lemma 2.20, we have

$$\mathscr{E}(t, x^{\varepsilon,\tau_n}(t)) - \mathscr{E}(t', x^{\varepsilon,\tau_n}(t')) + \mathscr{D}iss(x^{\varepsilon,\tau_n}; [t,t']) \le \int_t^{t'} \partial_t \mathscr{E}(r, x^{\varepsilon,\tau_n}(r))dr \le C|t'-t|. \tag{2.31}$$

For n large enough, we have $t < t_i^n < t'$. Therefore,

$$|x_i^{\varepsilon,\tau_n} - x_{i-1}^{\varepsilon,\tau_n}| \le \mathscr{D}iss(x^{\varepsilon,\tau_n}; [t,t']).$$

Moreover, when $n \to \infty$, we have

$$x^{\varepsilon,\tau_n}(t) \to x^{\varepsilon}(t) \text{ and } x^{\varepsilon,\tau_n}(t') \to x^{\varepsilon}(t').$$

Thus it follows from (2.31) that

$$\mathscr{E}(t, x^{\varepsilon}(t)) - \mathscr{E}(t', x^{\varepsilon}(t')) + \limsup_{n\to\infty} |x_i^{\varepsilon,\tau_n} - x_{i-1}^{\varepsilon,\tau_n}| \le C|t'-t|.$$

Since this inequality holds for all $t' > t$, we can take $t' \to t$ and use the assumption $x^{\varepsilon}(t^+) = x^{\varepsilon}(t)$ to obtain

$$\limsup_{n\to\infty} |x_i^{\varepsilon,\tau_n} - x_{i-1}^{\varepsilon,\tau_n}| \le 0.$$

Since we have already known that $x_{i-1}^{\varepsilon,\tau_n} \to x(t)$, we can conclude that $x_i^{\varepsilon,\tau_n} \to x(t)$.

Step 3. Stability. We show that for all $t \in [0,T]$,

$$\mathscr{E}(t, x^{\varepsilon}(t)) \le \mathscr{E}(t,z) + |z - x^{\varepsilon}(t)| \text{ for all } |z - x^{\varepsilon}(t)| \le \varepsilon.$$

First, we prove the result for $z \in \mathbb{R}$ such that $|z - x^{\varepsilon}(t)| < \varepsilon$. Since $\lim_{n\to\infty} x^{\varepsilon,\tau_n}(t) = x^{\varepsilon}(t)$, we get

$$|z - x^{\varepsilon,\tau_n}(t)| < \varepsilon$$

for n large enough. Using the notation in Step 2, from the definition of x_i^{ε,τ_n} and condition $|z - x_{i-1}^{\varepsilon,\tau_n}| < \varepsilon$, we obtain

$$\mathscr{E}(t_i^n, x_i^{\varepsilon,\tau_n}) + |x_i^{\varepsilon,\tau_n} - x_{i-1}^{\varepsilon,\tau_n}| \le \mathscr{E}(t_i^n, z) + |z - x_{i-1}^{\varepsilon,\tau_n}|.$$

Taking the limit as $n \to \infty$ and using the fact that both $x_{i-1}^{\varepsilon,\tau_n}$ and x_i^{ε,τ_n} converge to $x(t)$ (see Step 2), we obtain

$$\mathscr{E}(t, x^{\varepsilon}(t)) \le \mathscr{E}(t,z) + |z - x^{\varepsilon}(t)| \text{ for all } |z - x^{\varepsilon}(t)| < \varepsilon. \tag{2.32}$$

Now for any z such that $|z - x^{\varepsilon}(t)| = \varepsilon$, we can choose a sequence z_n converges to z such that $|z_n - x^{\varepsilon}(t)| < \varepsilon$. Applying (2.32) for z_n, we get

$$\mathscr{E}(t, x^{\varepsilon}(t)) \le \mathscr{E}(t, z_n) + |z_n - x^{\varepsilon}(t)|. \tag{2.33}$$

Notice that the mappings $z \mapsto \mathscr{E}(t,z)$ and $z \mapsto |z - x^{\varepsilon}(t)|$ are continuous. Hence, we can take the limit in (2.33) and get the result also for $|z - x^{\varepsilon}(t)| = \varepsilon$.

Step 4. Energy-dissipation inequalities.

From Lemma 2.20 we have, for all $0 \le s \le t \le T$,

$$\mathscr{E}(t, x^{\varepsilon,\tau_n}(t)) - \mathscr{E}(s, x^{\varepsilon,\tau_n}(s)) \le \int_s^t \partial_t \mathscr{E}(r, x^{\varepsilon,\tau_n}(r))dr - \mathscr{D}iss(x^{\varepsilon,\tau_n}; [s,t]).$$

Since $x^{\varepsilon,\tau_n}(r) \to x^\varepsilon(r)$ for all $r \in [0,T]$, we have

$$\mathscr{E}(t, x^{\varepsilon,\tau_n}(t)) - \mathscr{E}(s, x^{\varepsilon,\tau_n}(s)) \to \mathscr{E}(t, x^\varepsilon(t)) - \mathscr{E}(s, x^\varepsilon(s))$$

and

$$\int_s^t \partial_t \mathscr{E}(r, x^{\varepsilon,\tau_n}(r))dr \to \int_s^t \partial_t \mathscr{E}(r, x^\varepsilon(r))dr$$

as $n \to \infty$. Moreover, one has

$$\liminf_{n\to\infty} \mathscr{D}iss(x^{\varepsilon,\tau_n}; [s,t]) \ge \mathscr{D}iss(x^\varepsilon; [s,t]).$$

Thus we can derive one energy-dissipation inequality

$$\mathscr{E}(t, x^\varepsilon(t)) - \mathscr{E}(s, x^\varepsilon(s)) \le \int_s^t \partial_t \mathscr{E}(r, x^\varepsilon(r))dr - \mathscr{D}iss(x^\varepsilon; [s,t]).$$

We shall use Lemma 2.17 to obtain the other energy-dissipation inequality,

$$\mathscr{E}(t, x^\varepsilon(t)) - \mathscr{E}(s, x^\varepsilon(s)) \ge \int_s^t \partial_t \mathscr{E}(r, x^\varepsilon(r))dr - \mathscr{D}iss_{new}(x^\varepsilon; [s,t]).$$

It is suffices to verify that $|\partial_x \mathscr{E}(t, x^\varepsilon(t))| \le 1$ for a.e. $t \in (0,T)$. In fact, for every $t \in [0,T]$ such that $x^\varepsilon(\cdot)$ is right-continuous at t, we have proved in Step 3 the ε-stability

$$\mathscr{E}(t, x^\varepsilon(t)) \le \mathscr{E}(t,x) + |x - x^\varepsilon(t)| \;\text{ for all } |x - x^\varepsilon(t)| \le \varepsilon.$$

This inequality implies that $\partial_x \mathscr{E}(t, x^\varepsilon(t)) \in [-1,1]$. On the other hand, since $x^\varepsilon(\cdot)$ is a BV function, it is continuous except at most countably many points. Therefore, we can apply Lemma 2.17 to derive the desired inequality. □

3.3 Existence and properties of the limit

Lemma 2.22 (Limit of epsilon-neighborhood solutions)**.** *Let us consider an initial datum $x_0 \in \mathbb{R}$ such that $\mathscr{E}(0, x_0) < \infty$ and x_0 is a local minimizer for the functional $x \mapsto \mathscr{E}(0,x) + |x - x_0|$. Let x^ε be as in Lemma 2.21. Then there exists a subsequence $\varepsilon_n \to 0$ and a BV function $x(\cdot)$ such that $x^{\varepsilon_n}(t) \to x(t)$ for all $t \in [0,T]$. Moreover, the function $x(\cdot)$ satisfies the following properties*

(i) (Local stability) If $t \mapsto x(t)$ is continuous at t, then

$$|\partial_x \mathscr{E}(t, x(t))| \le 1.$$

(ii) (New energy-dissipation balance) For all $0 \le s \le t \le T$, one has

$$\mathscr{E}(t, x(t)) - \mathscr{E}(s, x(s)) = \int_s^t \partial_t \mathscr{E}(r, x(r))dr - \mathscr{D}iss_{new}(x; [s,t]).$$

Proof. **Step 1. Existence.** Since $\mathscr{D}iss(x^\varepsilon;[0,T]) \le C$ independent of ε, by Helly's selection principle we can find a subsequence $\varepsilon_n \to 0$ and a BV function $x(\cdot)$ such that $x^{\varepsilon_n}(t) \to x(t)$ as $n \to \infty$ for all $t \in [0,T]$.

Step 2. Stability. Let

$$A := \{t \in [0,T] \,|\, x^{\varepsilon_n}(\cdot) \text{ is right continuous at } t \text{ for all } n \ge 1\}.$$

Then $[0,T]\backslash A$ is at most countable. Moreover, for $t \in A$, by Lemma 2.21 we have

$$\mathscr{E}(t, x^{\varepsilon_n}(t)) \le \mathscr{E}(t,z) + |z - x^{\varepsilon_n}(t)| \;\text{ for all } |z - x^{\varepsilon_n}(t)| \le \varepsilon_n$$

for all $n \ge 1$. Therefore,

$$|\partial_x \mathscr{E}(t, x^{\varepsilon_n}(t))| \le 1 \;\text{ for all } n \ge 1.$$

Taking $n \to \infty$, we obtain

$$|\partial_x \mathscr{E}(t, x(t))| \le 1$$

for all $t \in A$.

Moreover, by continuity, we also get $|\partial_x \mathscr{E}(t,x(t))| \le 1$ provided $x(\cdot)$ is continuous at t.

Step 3. New energy-dissipation balance. First, similarly to the proof of energy inequalities in Lemma 2.21, we have

$$-\mathscr{D}iss_{new}(x(\cdot);[s,t]) \le \mathscr{E}(t,x(t)) - \mathscr{E}(s,x(s)) - \int_s^t \partial_t \mathscr{E}(r,x(r))dr \le -\mathscr{D}iss(x(\cdot);[s,t]).$$

(More precisely, the second inequality is a consequence of the corresponding inequality of x^ε in Lemma 2.21 and Fatou's lemma, while the first inequality follows from Lemma 2.17.)

If $x(\cdot)$ has no jumps in $[s,t]$, then we have immediately the energy-dissipation balance

$$\mathscr{E}(t,x(t)) - \mathscr{E}(s,x(s)) - \int_s^t \partial_t \mathscr{E}(r,x(r))dr = -\mathscr{D}iss(x(\cdot);[s,t]) = -\mathscr{D}iss_{new}(x(\cdot);[s,t]).$$

Therefore, it remains to consider jump points. More precisely, we need to show that if $x(\cdot)$ jumps at $t \in (0,T)$, namely $x(t^-) \ne x(t^+)$, then

$$\mathscr{E}(t,u(t^+)) - \mathscr{E}(t,u(t^-)) = -\Delta_{new}(t,u(t^-),u(t^+)).$$

This fact follows from Lemma 2.23 and 2.24 below. □

The following lemma is similar to Theorem 4.7 in [26].

Lemma 2.23 (New energy-dissipation balance at a jump)**.** *Let be given a BV function $u:[0,T]\to\mathbb{R}$ and an energy functional $\mathscr{E} \in C^2(\mathbb{R}^2)$. If $u(\cdot)$ jumps at $t \in (0,T)$ and*

$$\operatorname{sign}(u(t^+) - u(t^-)) \cdot \partial_x \mathscr{E}(t,z) \le -1$$

for all z between $u(t^-)$ and $u(t^+)$, then we have the equality

$$\mathscr{E}(t,u(t^+)) - \mathscr{E}(t,u(t^-)) = -\Delta_{new}(t,u(t^-),u(t^+)).$$

Here recall that $\Delta_{new}(t, u(t^-), u(t^+))$ is defined by

$$\inf\left\{\int_0^1 \max\{1, |\partial_x \mathscr{E}(t, v(x))|\} \cdot |\dot{v}(x)| dx \mid v \in AC([0,1]), v(0) = u(t^-), v(1) = u(t^+)\right\}.$$

Proof. For any BV function u one has

$$\begin{aligned}
\mathscr{E}(t, u(t^+)) - \mathscr{E}(t, u(t^-)) &= \int_0^1 \partial_x \mathscr{E}(t, v(x)) \cdot \dot{v}(x) dx \\
&\geq -\int_0^1 |\partial_x \mathscr{E}(t, v(x))| \cdot |\dot{v}(x)| dx \\
&\geq -\int_0^1 \max\{1, |\partial_x \mathscr{E}(t, v(x))|\} \cdot |\dot{v}(x)| dx.
\end{aligned}$$

for any $v \in AC([0,1])$ such that $v(0) = u(t^-)$ and $v(1) = u(t^+)$. Therefore,

$$\mathscr{E}(t, u(t^+)) - \mathscr{E}(t, u(t^-)) \geq -\Delta_{new}(t, u(t^-), u(t^+)).$$

Now we show the reverse inequality. We shall consider the case $u(t^+) > u(t^-)$ (the other case can be treated by the same way). Since $\partial_x \mathscr{E}(t, z) \leq -1$ for all z between $u(t^-)$ and $u(t^+)$, then by choosing

$$v(s) = u(t^-) + s(u(t^+) - u(t^-)), \quad s \in [0,1],$$

we obtain $\dot{v}(s) = u(t^+) - u(t^-) \geq 0$ and

$$\begin{aligned}
\mathscr{E}(t, u(t^+)) - \mathscr{E}(t, u(t^-)) &= \int_0^1 \partial_x \mathscr{E}(t, v(s)) \cdot \dot{v}(s) ds \\
&= -\int_0^1 \max\{1, |\partial_x \mathscr{E}(t, v(s))|\} \cdot |\dot{v}(s)| ds \\
&\leq -\Delta_{new}(t, u(t^-), u(t^+)).
\end{aligned}$$

This completes the proof. □

To obtain the energy-dissipation balance in Lemma 2.22, it remains to verify the second condition in Lemma 2.23.

Lemma 2.24 (Monotonicity in jump intervals)**.** *Let $x(\cdot)$ be the function as in Lemma 2.22. If $x(t^-) < x(t^+)$, then the function*

$$z \mapsto \mathscr{E}(t, z) + |z - x(t^-)|$$

is decreasing on $[x(t^-), x(t^+)]$.

On the other hand, if $x(t^-) > x(t^+)$, then the function

$$z \mapsto \mathscr{E}(t, z) + |z - x(t^-)|$$

is increasing on $[x(t^+), x(t^-)]$.

Consequently, if $x(t^-) \neq x(t^+)$, then we always have

$$\operatorname{sign}(x(t^+) - x(t^-)) \cdot \partial_x \mathscr{E}(t, z) \leq -1$$

for all z between $x(t^-)$ and $x(t^+)$.

Proof. We shall consider the case $x(t^-) < x(t^+)$ (the other case can be treated in the same way). Assume by contradiction that there exist $z_1, z_2 \in (x(t^-), x(t^+))$ such that $z_1 < z_2$ and

$$\mathscr{E}(t, z_1) + |z_1 - x(t^-)| < \mathscr{E}(t, z_2) + |z_2 - x(t^-)|.$$

Denote

$$\delta := \mathscr{E}(t, z_2) - \mathscr{E}(t, z_1) + z_2 - z_1 > 0.$$

By the continuity of $(t, z_1, z_2) \mapsto \mathscr{E}(t, z_2) - \mathscr{E}(t, z_1) + z_2 - z_1$, we can choose $t_1 < t < t_2$ and $\varepsilon' \in (0, z_2 - z_1)$ such that

$$x(t_1) < z_1 < z_2 < x(t_2)$$

and

$$\mathscr{E}(s_2, y_2) - \mathscr{E}(s_1, y_1) + y_2 - y_1 \geq \delta/2 \tag{2.34}$$

for all $s_1, s_2 \in [t_1, t_2]$, $y_1 \in [z_1 - \varepsilon'/2, z_1 + \varepsilon'/2]$, $y_2 \in [z_2 - \varepsilon'/2, z_2 + \varepsilon'/2]$.

By the definition of $x(\cdot)$, there exist $\varepsilon \in (0, \varepsilon')$ and $\tau > 0$ such that

$$x^{\varepsilon,\tau}(t_1) < z_1 < z_2 < x^{\varepsilon,\tau}(t_2).$$

Since the function $x^{\varepsilon,\tau}(\cdot)$ has no jump step bigger than ε, there exist $t_1 < s_1 < s_2 < t_2$ such that

$$x^{\varepsilon,\tau}(s_1) \in [z_1 - \varepsilon/2, z_1 + \varepsilon/2] \text{ and } x^{\varepsilon,\tau}(s_2) \in [z_2 - \varepsilon/2, z_2 + \varepsilon/2].$$

The inequality (2.34) implies that

$$\mathscr{E}(s_2, x^{\varepsilon,\tau}(s_2)) - \mathscr{E}(s_1, x^{\varepsilon,\tau}(s_1)) + x^{\varepsilon,\tau}(s_2) - x^{\varepsilon,\tau}(s_1) \geq \delta/2.$$

On the other hand, using the integral bound in Lemma 2.20 we get

$$\begin{aligned} \mathscr{E}(s_2, x^{\varepsilon,\tau}(s_2)) - \mathscr{E}(s_1, x^{\varepsilon,\tau}(s_1)) &\leq \int_{s_1}^{s_2} \partial_t \mathscr{E}(r, x^{\varepsilon,\tau}(r))dr - \mathscr{D}iss(x^{\varepsilon,\tau}; [s_1, s_2]) \\ &\leq C(s_2 - s_1) - (x^{\varepsilon,\tau}(s_2) - x^{\varepsilon,\tau}(s_1)) \end{aligned}$$

since $\partial_t \mathscr{E}$ is continuous and $x^{\varepsilon,\tau}(\cdot)$ is bounded uniformly w.r.t. ε and τ. Therefore,

$$\delta/2 \leq \mathscr{E}(s_2, x^{\varepsilon,\tau}(s_2)) - \mathscr{E}(s_1, x^{\varepsilon,\tau}(s_1)) + x^{\varepsilon,\tau}(s_2) - x^{\varepsilon,\tau}(s_1) \leq C(s_2 - s_1).$$

Since $0 \leq s_2 - s_1 \leq t_2 - t_1$, we can conclude that

$$\delta \leq 2C(t_2 - t_1).$$

This is a contradiction since $\delta > 0$ is fixed while the difference $t_2 - t_1$ can be chosen arbitrary small. □

Chapter 3

Regularity of weak solutions to rate-independent systems

In this chapter, we deal with the regularity results for *weak solutions* (in particular, *energetic solutions* and *BV solutions*) to the rate-independent systems in one-dimension.

The regularity for *energetic solutions* to rate-independent systems when the energy functional is *convex* was already considered by Mielke and Thomas [19, 33]. It was shown that if the energy functional $\mathscr{E}(t,x)$ is α-convex in $x \in X$, where X is a finite dimensional normed space, for any fixed $t \in [0,T]$ and satisfies some technical assumptions like

$$\begin{aligned} \exists\lambda : \quad & \forall (t,x) \text{ such that } \mathscr{E}(t,x) < \infty, \text{ then} \\ & \mathscr{E}(\cdot,x) \in C^1([0,T]) \text{ and } \partial_s \mathscr{E}(s,x) \leq \lambda \mathscr{E}(s,x) \quad \forall s \in [0,T], \end{aligned}$$

then the energetic solutions are Lipschitz continuous (or Hölder continuous) provided that $\partial_t \mathscr{E}(t,\cdot)$ is Lipschitz continuous (or Hölder continuous, respectively).

However, if the energy functional is *non-convex*, there are very few results on the regularity of the energetic solutions as well as the other classes of weak solutions. In general, we cannot expect a regularity to be better than BV as we can see in Example 3.1 below. But under some assumptions on $\mathscr{E}$, we can obtain the SBV regularity, or even piecewise C^1.

Example 3.1. Let $u : [0,1] \to [0,1]$ be any increasing and left-continuous function. Then $u(t)$ is an energetic solution of the rate-independent system defined by $(\mathscr{E}, \Psi, x_0)$, where the energy functional is $\mathscr{E}(t,x) := \frac{1}{2}(u(t)+1-x)^2$, the dissipation function is $\Psi(x) := |x|$ and the initial position is $x_0 := u(0)$.

Proof. Applying Cauchy's inequality of the form $a + b \geq 2\sqrt{ab}$ with $a = (u(t)+1-x)^2$ and $b = 1$, we get

$$\begin{aligned} \frac{1}{2}(u(t)+1-x)^2 + |x-x_0| &= \frac{1}{2}\big((u(t)+1-x)^2+1\big) + |x-x_0| - \frac{1}{2} \\ &\geq |u(t)+1-x| + |x-x_0| - \frac{1}{2} \\ &\geq u(t) - x_0 + \frac{1}{2} \\ &= |u(t)-x_0| + \frac{1}{2}. \end{aligned}$$

Therefore, $u(t)$ minimizes $x \mapsto \{\mathscr{E}(t,x) + |x - x_0|\}$ among all $x \in \mathbb{R}$. Moreover, since $u(t)$ is increasing in t, we have $\mathscr{D}iss(u;[0,1]) = |u(1) - u(0)| = |u(1) - x_0|$. Applying Proposition 1.4, we obtain that $u(t)$ is an energetic solution of the system $(\mathscr{E}, \Psi, x_0)$. □

In general, we have the following result, which will be proved later.

Theorem 3.2 (Any increasing function is an energetic solution)**.** *Let $u : [0,T] \to \mathbb{R}$ be an arbitrary increasing and left-continuous function. Then u is an energetic solution of some rate-independent system with smooth energy functional.*

Thus, an energetic solution may not belong to the class SBV, even if the energy functional is smooth. In this chapter, we prove that under some relevant requirements (but not convexity) on the energy functional, weak solutions (and in particular energetic solutions, and BV solutions) are of class SBV. To our knowledge, there is no available results on the regularity of *weak solutions* for general, non-convex energy functionals.

Moreover, in the case that the solutions have only finitely many jumps, a kind of piecewise C^1-smoothness can be obtained. We also give some condition ensuring that weak solutions have only finitely many jumps. A regularity result for the vector-valued cases is provided in the end of this chapter.

1 Regularity results

Let $x(\cdot)$ be a weak solution to the system described by the energy functional $\mathscr{E}(t,x)$ and the dissipation functional $\Psi(x,y) := |x-y|$. More precisely, $x : [0,T] \mapsto \mathbb{R}$ is a BV function which satisfies two conditions:

(w-LS) (Weak local stability) For all $t \in (0,T)$, if $x(\cdot)$ is continuous at t, then

$$|\partial_x \mathscr{E}(t, x(t))| \le 1.$$

(ED-upper) (Energy-dissipation upper bound) For all $0 \le t_1 \le t_2 \le T$,

$$\mathscr{E}(t_2, x(t_2)) - \mathscr{E}(t_1, x(t_1)) \le \int_{t_1}^{t_2} \partial_t \mathscr{E}(s, x(s)) ds - \mathscr{D}iss(x(\cdot);[t_1,t_2]).$$

Remark. Energetic solutions, *BV solutions* and *epsilon-neighborhood solutions* satisfy (w-LS) and (ED-upper).

We shall use the following assumptions.

(H1) $\mathscr{E}(t,x)$ is of class C^3 and the set

$$\{(t,x) \in (0,T) \times \mathbb{R} \mid \partial_x \mathscr{E}(t,x) \in \{-1,1\}, \partial_{xx}\mathscr{E}(t,x) = \partial_{xxx}\mathscr{E}(t,x) = 0\}$$

has only finitely many elements.

(H2) The set

$$\{(t,x) \in (0,T) \times \mathbb{R} \mid \partial_x \mathscr{E}(t,x) \in \{-1,1\}, \partial_{xx}\mathscr{E}(t,x) = \partial_{xt}\mathscr{E}(t,x) = 0, \\ [\partial_{xxt}\mathscr{E}(t,x)]^2 = \partial_{xtt}\mathscr{E}(t,x) \cdot \partial_{xxx}\mathscr{E}(t,x)\}$$

has only finitely many elements.

(H3) The set

$$\{(t,x) \in (0,T) \times \mathbb{R} \mid \partial_x \mathscr{E}(t,x) \in \{-1,1\}, \partial_{xx}\mathscr{E}(t,x) = \partial_{xt}\mathscr{E}(t,x) = 0\}$$

has only finitely many elements.

(H4) The set

$$\{(t,x) \in [0,T] \times \mathbb{R} \mid \partial_x \mathscr{E}(t,x) \in \{-1,1\}, \partial_{xt}\mathscr{E}(t,x) = \partial_{xtt}\mathscr{E}(t,x) = 0\}$$

is empty.

(H5) The set

$$\{(t,x) \in [0,T] \times \mathbb{R} \mid \partial_x \mathscr{E}(t,x) \in \{-1,1\}, \partial_{xx}\mathscr{E}(t,x) = 0\}$$

is empty.

These conditions hold true for a dense class of energy functionals (we are not going to specify exactly the meaning of dense). Note that no convexity is imposed.

All our regularity results apply to the scalar case ($d = 1$). Only the last result works in an arbitrary dimension.

Theorem 3.3 (SBV regularity)**.** *Assume that the function $x(\cdot)$ has bounded variation and satisfies* (w-LS) *and* (ED-upper)*. If* (H1) *holds true, then $x(\cdot)$ is of class SBV.*

In the next theorem, we consider the differentiability of energetic solutions in the case that they have finitely many jumps.

Theorem 3.4 (Differentiability)**.** *Assume that $x(\cdot)$ is a BV function satisfying* (S) *and* (ED-upper) *and $x(\cdot)$ has only finitely many jump points. If* (H1) *holds true, then we can decompose $[0,T]$ into four disjoint sets I_1, I_2, I_3 and J such that the following holds true.*

(i) For every $t \in I_1$, $x'(t)$ does not exist and either $x'_-(t) = 0$ or $x'_+(t) = 0$.

(ii) For every $t \in I_2$, $x'_-(t)$ and $x'_+(t)$ do exist, but they are different. Moreover, $x(\cdot)$ is differentiable in a neighborhood of t (except the point t itself) and

$$x'_+(t) = \lim_{s \downarrow t} x'(s), \quad x'_-(t) = \lim_{s \uparrow t} x'(s).$$

(iii) For every $t \in I_3$, $x(\cdot)$ is differentiable at t, namely $x'(t)$ exists.

(iv) J is the jump set of $x(\cdot)$.

Notice that both I_1 and I_2 are discrete sets. Moreover, if (H2) *also holds true, then $I_1 \cup I_2$ is also a discrete set.*

Here the right and left derivatives $x'_+(t)$, $x'_-(t)$ are defined by

$$x'_+(t) := \lim_{s \downarrow t} \frac{x(s) - x(t)}{s - t}, \quad x'_-(t) := \lim_{s \uparrow t} \frac{x(s) - x(t)}{s - t}.$$

In the next two theorems, we consider the piecewise C^1-smoothness of the energetic solutions.

Theorem 3.5 (Weak C^1 regularity)**.** *Assume that $x(\cdot)$ is a BV function satisfying* (S) *and* (ED-upper) *and $x(\cdot)$ has only finitely many jump points. If* (H1) *and* (H3) *hold true, then there exists a set I of isolated points such that for any $t \in (0,T)\backslash I$, the classical derivative $x'(t)$ exists. Moreover, the function $x'(\cdot)$ is continuous on $(0,T)\backslash I$.*

Here we say that t is an isolated point of I if there exists $\varepsilon > 0$ such that

$$(t - \varepsilon, t + \varepsilon) \cap I = \{t\},$$

Theorem 3.6 (C^1 regularity)**.** *Assume that $x(\cdot)$ is a BV function satisfying* (S) *and* (ED-upper) *and $x(\cdot)$ has only finitely many jump points. If* (H1), (H3) *and* (H4) *hold true, then there exist* finite *disjoint open intervals $\{I_n\}_{n\geq 1}^M$ such that $[0,T] = \cup_{n\geq 1}^M \overline{I_n}$, and $x(\cdot)$ is C^1 on any interval I_n.*

In Theorem 3.4, 3.5, and 3.6, we require that the energetic solution has finitely many jump points. In the following theorem, we give a *simple condition* to make sure a weak solution has only finitely many jumps in one dimension.

Theorem 3.7 (Finite jumps)**.** *Assume that $x(\cdot)$ is a BV function satisfying* (w-LS) *and* (ED-upper)*. If* (H5) *holds true, then $x(\cdot)$ has only finitely many jumps.*

So far, all of the regularity theorems are stated in one dimension, since our techniques seem rather specific for one dimension. Now we give one generalization for SBV regularity in higher dimensions.

Theorem 3.8 (SBV regularity for higher dimensions)**.** *Let $x : [0,T] \to \mathbb{R}^d$ $(d \geq 1)$ be a BV function satisfying* (w-LS) *and* (ED-upper)*. Moreover, we assume that*

(H6) *$\mathscr{E}(t,x)$ is of class $C^3(\mathbb{R}^{d+1})$ and the set*

$$\{(t,x) \in (0,T) \times \mathbb{R}^d \mid |\nabla_x \mathscr{E}(t,x)| = 1, F(t,x) = (\nabla_x \mathscr{E}(t,x)) \cdot (\nabla_x F(t,x)) = 0\}$$

is countable, where the function $F(t,x)$ is defined by

$$F(t,x) := (\nabla_x \mathscr{E}(t,x)) \cdot H(t,x) \cdot (\nabla_x \mathscr{E}(t,x))^T$$

and the Hessian matrix $H(t,x)$ is defined by

$$[H(t,x)]_{ij} := (\partial_{x_i} \partial_{x_j} \mathscr{E})(t,x).$$

Then $x(\cdot)$ is of class SBV.

The previous theorems are proved in the next sections.

2 Any increasing function is an energetic solution

To prove Theorem 3.2, we start with some classical results.

Lemma 3.9. *For any closed set C in $\mathbb{R}^d$, there exists a smooth function φ such that $\varphi : \mathbb{R}^d \to [0,1]$ and $\varphi^{-1}(0) = C$.*

Proof. Since the set $\mathbb{R}^d \backslash C$ is open, we can find a family of open balls $\{B_n\}$ such that

$$\mathbb{R}^d \backslash C = \bigcup_{n\in\mathbb{N}} B_n.$$

Moreover, a classical result tells us that, for any $n \in \mathbb{N}$, there exist $\varphi_n : \mathbb{R}^d \to [0,1]$ such that φ_n is of class C^∞ and $\varphi_n^{-1}(0) = \mathbb{R}^d \backslash B_n$.

Take $\varphi := \sum_{n\in\mathbb{N}} \alpha_n \varphi_n$ with $\alpha_n > 0$ for all n. This implies $\varphi^{-1}(0) = C$.

Now for every $n \in \mathbb{N}$, we choose α_n such that $\|D^k \varphi_n\|_\infty \cdot \alpha_n \leq 2^{-n}$ for all $k = 0, 1, \dots, n$. It is easy to check that $\varphi(\mathbb{R}^d) \in [0,1]$ and φ is of class C^∞. This completes the proof of Lemma 3.9. □

Lemma 3.10. *If $u : [0,T] \mapsto \mathbb{R}$ is an increasing function, then there exists a smooth function $g : [0,T] \times \mathbb{R} \to \mathbb{R}$ such that*

$$g(t,x) \in [-1,0) \text{ for all } t \in [0,T] \text{ and for all } x < u(t),$$

$$g(t,x) \in (0,1] \text{ for all } t \in [0,T] \text{ and for all } x > u(t),$$

$$g(t,x) = 0 \text{ for all } t \in [0,T] \text{ and for all } x = u(t).$$

Proof. Define

$$C_1 := \{(t,x) \mid x \geq u(t^-)\}, \quad C_2 := \{(t,x) \mid x \leq u(t^+)\}.$$

We show that C_1 and C_2 are closed sets in $\mathbb{R}^2$. For example, to prove that C_1 is closed, we need to show that if a sequence $\{(t_n, x_n)\}_{n\geq 1} \subset C_1$ converges to (t_0, x_0), then $(t_0, x_0) \in C_1$, namely $x_0 \geq u(t_0^-)$. Indeed, if $s < t_0$, then for n large enough we have $t_n > s$, and hence $x_n \geq u(t_n^-) \geq u(s)$. Thus $x_0 = \lim x_n \geq u(s)$ for all $s < t_0$, which implies that $x_0 \geq \lim_{s\uparrow t_0} u(s) = u(t_0^-)$. Thus C_1 is closed. Similarly, we have C_2 is closed.

Applying Lemma 3.9 we can choose two smooth functions $g_1 : \mathbb{R}^2 \to [0,1]$ and $g_2 : \mathbb{R}^2 \to [0,1]$ such that

$$g_1^{-1}(0) = C_1 \text{ and } g_2^{-1}(0) = C_2.$$

We define

$$g(t,x) := g_2(t,x) - g_1(t,x) \text{ for all } (t,x) \in [0,T] \times \mathbb{R}.$$

It is straight-forward to see that the function g has all desired properties. □

Now we are able to show

Proof of Theorem 3.2. We choose the energy functional $\mathscr{E}(t,x)$ such that $g(t,x) = \partial_x \mathscr{E}(t,x) + 1$ (here g is from Lemma 3.10), the dissipation function $\Psi(x) := |x|$, and the initial value $x_0 := u(0)$. We shall prove that u is an energetic solution of the system $(\mathscr{E}, |.|, u(0))$.

By Proposition 1.4, $x(\cdot)$ is an energetic solution to the system $(\mathscr{E}, |.|, x_0)$ if the following three conditions hold.

(i) $x(\cdot)$ is left-continuous.

(ii) $\mathscr{D}iss(x(\cdot); [0,T]) = |x(T) - x_0|$.

(iii) For all $t \in [0,T]$, $x(t)$ minimizes the functional $x \mapsto \mathscr{E}(t,x) + |x - x_0|$ for $x \in \mathbb{R}$.

Thus it remains to check that u satisfies the condition (iii). We shall use the fact that for all t, $\partial_x \mathscr{E}(t, u(t)) = -1$ and $\partial_x \mathscr{E}(t,x) \in (-1, 0]$ if $x > u(t)$, and $\partial_x \mathscr{E}(t,x) \in [-2, -1)$ if $x < u(t)$. We distinguish two cases.

Case 1: $x > u(t)$. By the smoothness of $\mathscr{E}$, we can write

$$\begin{aligned} \mathscr{E}(t,x) &= \mathscr{E}(t,u(t)) + \int_{u(t)}^{x} \partial_x \mathscr{E}(t,z) dz \\ &> \mathscr{E}(t,u(t)) + \int_{u(t)}^{x} (-1) = \mathscr{E}(t,u(t)) + u(t) - x. \end{aligned}$$

Thus,

$$\begin{aligned} \mathscr{E}(t,x) + |x - u(0)| &> [\mathscr{E}(t,u(t)) + u(t) - x] + [x - u(0)] \\ &= \mathscr{E}(t,u(t)) + u(t) - u(0). \end{aligned}$$

Case 2: $x < u(t)$. Similarly to Case 1, we write

$$\begin{aligned} \mathscr{E}(t,x) &= \mathscr{E}(t,u(t)) - \int_{x}^{u(t)} \partial_x \mathscr{E}(t,z) dz \\ &> \mathscr{E}(t,u(t)) - \int_{x}^{u(t)} (-1) = \mathscr{E}(t,u(t)) + u(t) - x. \end{aligned}$$

Hence,

$$\begin{aligned} \mathscr{E}(t,x) + |x - u(0)| &> [\mathscr{E}(t,u(t)) + u(t) - x] + [x - u(0)] \\ &= \mathscr{E}(t,u(t)) + u(t) - u(0). \end{aligned}$$

In summary, $u(t)$ is the unique minimizer for the functional $x \mapsto \mathscr{E}(t,x) + |x - x(0)|$ over $x \in \mathbb{R}$. This completes the proof. □

3 SBV regularity

Now we prove Theorem 3.3.

Proof of Theorem 3.3. **Step 1.** Thanks to Proposition 1.5, we can assume that $x(\cdot)$ is right-continuous.

By dividing $(0,T)$ into smaller intervals if necessary, we can assume that the set

$$\{(t,x) \in (0,T) \times \mathbb{R} \mid \partial_x \mathscr{E}(t,x) \in \{-1,1\}, \partial_{xx}\mathscr{E}(t,x) = \partial_{xxx}\mathscr{E}(t,x) = 0\}$$

is empty.

Step 2. Since $x(\cdot)$ is a BV function in 1-dim which is right-continuous, there is a real-valued Radon measure μ such that

$$x(t) = \text{const} + \mu((0,t]) \quad \text{for all } t \in [0,T].$$

By Lebesgue Decomposition Theorem we can write

$$\mu = f dx + \mu_s$$

where $f \in L^1$ and $\mu_s = \mu|_S$ with

$$S = \left\{ t \in (0,T) \mid \lim_{h \downarrow 0} \frac{|\mu|(t-h,t+h)}{h} = \infty \right\}.$$

Let J be the jump set of $x(\cdot)$. We split μ_s into the Cantor part $\mu_c := \mu|_{S \setminus J}$ and the jump part $\mu_J := \mu|_J$. To show that $x(\cdot)$ is of SBV, we need to prove that $\mu_c = 0$.

Step 3. Next, we shall use the following lemmas:

Lemma 3.11. *For any BV function $x : [0,T] \to \mathbb{R}$ which is right-continuous, the set*

$$A := \left\{ t \in (0,T) \setminus J \mid \liminf_{h \to 0} \left| \frac{x(t+h) - x(t)}{h} \right| < \infty \right\}$$

has $|\mu_s|$-measure 0.

Lemma 3.12. *Assume that $x : [0,T] \to \mathbb{R}$ has bounded variation and satisfies (w-LS) and (ED-upper). If (H1) holds true, then the set*

$$B := \left\{ t \in (0,T) \setminus J \mid \lim_{h \to 0} \left| \frac{x(t+h) - x(t)}{h} \right| = \infty \right\}$$

is at most countable. Therefore, $|\mu_s|(B) = 0$.

Step 4. Since μ_c is the restriction of μ_s on $(0,T)\setminus J$, $\mu_c = 0$ if $|\mu_s|((0,T)\setminus J) = 0$. Because we can write $(0,T)\setminus J = A \cup B$, Lemma 3.11 and 3.12 ensure that $|\mu_s|((0,T)\setminus J) = 0$. This completes the proof of Theorem 3.3. □

It remains to show Lemma 3.11 and 3.12.

Proof of Lemma 3.11. **Step 1.** For any $h > 0$, we have

$$x(t+h) - x(t) = \mu((0, t+h]) - \mu((0, t]) = \mu((t, t+h])$$

Thus, we can rewrite

$$A = \left\{ t \in (0, T) \backslash J \mid \liminf_{h \to 0} \left| \frac{\mu((t, t+h])}{h} \right| < \infty \right\}.$$

If we define

$$A_k := \left\{ t \in (0, T) \backslash J \mid \left| \frac{\mu((t, t+h])}{h} \right| < k \quad \text{for some } h > 0 \text{ arbitrary close to } 0 \right\},$$

then $A \subset \cup_{k=1}^{\infty} A_k$. We can prove that A_k is Borel, and hence it is $|\mu_s|$-measurable. We will obtain $|\mu_s|(A) = 0$ if we can check $|\mu_s|(A_k) = 0$ for all k.

Step 2. Since $\mu_s \perp \mathcal{L}^1$, there exists a Borel set S_p such that $|\mu_s|(S_p^c) = 0$ and $\mathcal{L}^1(S_p) = 0$. For any $\varepsilon > 0$ there exists an open set $U_\varepsilon \supset S_p$ such that $\mathcal{L}^1(U_\varepsilon) < \varepsilon$. We have

$$\mu_s(A_k) = \mu_s(A_k'),$$

where $A_k' = A_k \cap U_\varepsilon$.

Step 3. Next, we consider the following family $\mathcal{F}$ covering A_k' where

$$\mathcal{F} = \{[t, t+h] : t \in A_k' \text{ and } h \text{ is chosen such that } [t, t+h] \subset U_\varepsilon \text{ and } |\mu|((t, t+h]) < kh\}.$$

Notice that since t does not belong to J, $\mu([t, t+h]) = \mu((t, t+h])$. We refine $\mathcal{F}$ to $\mathcal{F}'$ such that $\mathcal{F}'$ still covers A_k' as follows: if $I \in \mathcal{F}$ is a subset of another interval of $\mathcal{F}$, or I is a subset of the union of two other intervals of $\mathcal{F}$, then we omit I.

After refining $\mathcal{F}$, we obtain the family $\mathcal{F}'$ with the following properties.

P1. No interval of $\mathcal{F}'$ is a subset of another interval of $\mathcal{F}'$.

P2. Three different intervals of $\mathcal{F}'$ always have no common element (otherwise, two of them cover the remaining one). As a consequence, any $t \in A_k'$ is covered by at most 2 intervals of $\mathcal{F}'$.

P3. Any interval of $\mathcal{F}'$ is disjoint from all of the others except at most 2 intervals. In fact, if $J \cap I \neq \emptyset$, $J \not\subset I$ and $I \not\subset J$, then J must contain precisely one of the two boundary points of I. Therefore, by property P2, there are at most 2 intervals of $\mathcal{F}'$, which are different from I and have nontrivial intersections with I.

P4. For any $I \in \mathcal{F}'$, there is a nontrivial interval $(a, b) \subset I$ such that

$$(a, b) \cap \left(\bigcup_{I' \in \mathcal{F}' \backslash \{\mathrm{I}\}} I' \right) = \emptyset.$$

In fact, assume that for all $(a, b) \subset I$, there exists $J \in \mathcal{F}'$, $J \neq I$, such that $(a, b) \cap J \neq \emptyset$. By property P3, there are (at most) two intervals $J_1, J_2 \in \mathcal{F}'$ such that for all $(a, b) \subset I$, then either $(a, b) \cap J_1 \neq \emptyset$ or $(a, b) \cap J_2 \neq \emptyset$. It implies that $I \subset J_1 \cup J_2$, which is a contradiction.

P5. $\mathcal{F}'$ is at most countable. In fact, by property P4, each $I \in \mathcal{F}'$ contains a nontrivial interval (a, b) which has empty intersection with all of the other intervals of $\mathcal{F}'$. In this interval (a, b) we can choose a rational number c_I. Since $I \mapsto c_I$ is injective from $\mathcal{F}'$ into $\mathbb{Q}$, we conclude that $\mathcal{F}'$ is at most countable.

P6. We can divide $\mathcal{F}'$ into three subfamilies $\mathcal{F}'_1$, $\mathcal{F}'_2$, $\mathcal{F}'_3$, such that each subfamily is disjoint. This can be done by induction using property P2.

Step 4. Now we have

$$|\mu_s|(A_k) = |\mu_s|(A'_k) \le |\mu_s| \left(\bigcup_{I \in \mathcal{F}'} I \right) \le \sum_{I \in \mathcal{F}'} |\mu_s|(I) \le \sum_{j=1}^{3} \left(\sum_{I \in \mathcal{F}'_j} |\mu_s|(I) \right).$$

Recall that any $I \in \mathcal{F}'$ satisfies $|\mu_s|(I) \le k\mathcal{L}^1(I)$. Moreover, for any $j = 1, 2, 3$, the family $\mathcal{F}'_j$ contains disjoint intervals $I \subset U_\varepsilon$. Therefore,

$$\sum_{I \in \mathcal{F}'_j} |\mu_s|(I) \le k \sum_{I \in \mathcal{F}'_j} \mathcal{L}^1(I) = k\mathcal{L}^1(U_\varepsilon) \le k\varepsilon \quad \text{for all } j = 1, 2, 3.$$

Thus

$$|\mu_s|(A_k) \le 3k\varepsilon.$$

Because it holds true for any $\varepsilon > 0$, we conclude that $|\mu_s|(A_k) = 0$. This completes the proof of Lemma 3.11. □

Before proving Lemma 3.12, we need some preliminary results.

Lemma 3.13. *Assume that $x : [0, T] \to \mathbb{R}$ has bounded variation and satisfies the weak local stability (w-LS) and the energy-dissipation upper bound (ED-upper)*

$$\mathscr{E}(t_2, x(t_2)) - \mathscr{E}(t_1, x(t_1)) \le \int_{t_1}^{t_2} \partial_t \mathscr{E}(s, x(s)) ds - \mathscr{D}iss(x(\cdot); [t_1, t_2])$$

for all $0 \le t_1 \le t_2 \le T$, then we have

$$\partial_x \mathscr{E}(t, x(t)) \in \{-1, 1\}$$

for all $t \notin J$ and $t \notin \operatorname{int}(N \cup J)$. Here we denote by J the jump set of $x(\cdot)$ and

$$N := \{t \mid x'(t) \text{ exists and } x'(t) = 0\}. \tag{3.1}$$

Moreover, if we assume furthermore that $x(\cdot)$ is right-continuous and satisfies the global stability (S), then $\partial_x \mathscr{E}(t, x(t)) \in \{-1, 1\}$ for all $t \notin \operatorname{int}(N)$.

Remark. Recall that the global stability, namely $x(t)$ is a global minimizer for the functional

$$z \in \mathbb{R} \mapsto \mathscr{E}(t,z) + |z - x(t)|,$$

implies that

$$0 \in \partial_x[\mathscr{E}(t,x) + |x - x(t)|]_{x=x(t)}.$$

This yields the weak local stability,

$$\partial_x \mathscr{E}(t, x(t)) \in [-1, 1].$$

Proof. **Step 1.** First, we show that if $t \notin N \cup J$, then $\partial_x \mathscr{E}(t, x(t)) \in \{-1, 1\}$.

Since $t \notin N$, we can find a sequence $t_n \to t$ and $t_n \neq t$ such that

$$\liminf_{n\to\infty} \left| \frac{x(t_n) - x(t)}{t_n - t} \right| > 0. \tag{3.2}$$

Case 1. Assume that $t_n \downarrow t$. From the energy-dissipation upper bound, one has

$$\mathscr{E}(t_n, x(t_n)) - \mathscr{E}(t, x(t)) \leq \int_t^{t_n} \partial_t \mathscr{E}(s, x(s)) ds - \mathscr{D}iss(x(\cdot); [t, t_n]).$$

Using Taylor's expansion on the left-hand side and the continuity of $s \mapsto \partial_t \mathscr{E}(s, x(s))$ on the right-hand side, we obtain

$$\begin{aligned} & \partial_t \mathscr{E}(t, x(t)) \cdot (t_n - t) + \partial_x \mathscr{E}(t, x(t)) \cdot (x(t_n) - x(t)) + o(x(t_n) - x(t)) + o(t_n - t) \\ \leq \; & (t_n - t) \cdot \partial_t \mathscr{E}(t, x(t)) - \mathscr{D}iss(x(\cdot); [t, t_n]) + o(t_n - t). \end{aligned}$$

Dividing this inequality by $|x(t_n) - x(t)|$ and using (3.2), we obtain

$$\partial_x \mathscr{E}(t, x(t)) \cdot \frac{x(t_n) - x(t)}{|x(t_n) - x(t)|} \leq -\frac{\mathscr{D}iss(x(\cdot); [t, t_n])}{|x(t_n) - x(t)|} + o(1) \leq -1 + o(1). \tag{3.3}$$

Consequently, $|\partial_x \mathscr{E}(t, x(t))| \geq 1$. On the other hand, $|\partial_x \mathscr{E}(t, x(t))| \leq 1$ by (w-LS). Thus $|\partial_x \mathscr{E}(t, x(t))| = 1$.

Case 2. Assume that $t_n \uparrow t$. From the energy-dissipation upper bound, one has

$$\mathscr{E}(t_n, x(t_n)) - \mathscr{E}(t, x(t)) \geq \int_t^{t_n} \partial_t \mathscr{E}(s, x(s)) ds + \mathscr{D}iss(x(\cdot); [t_n, t]).$$

Following the above proof, we obtain

$$\partial_x \mathscr{E}(t, x(t)) \cdot \frac{x(t_n) - x(t)}{|x(t_n) - x(t)|} \geq \frac{\mathscr{D}iss(x(\cdot); [t_n, t])}{|x(t_n) - x(t)|} + o(1) \geq 1 + o(1).$$

This also implies that $|\partial_x \mathscr{E}(t, x(t))| = 1$.

Step 2. We show that if $t \notin J$ and $t \notin \text{int}(N \cup J)$, then $\partial_x \mathscr{E}(t, x(t)) \in \{-1, 1\}$.

Since $t \notin \operatorname{int}(N \cup J)$, there exists a sequence $t_n \to t$ such that $t_n \notin N \cup J$ for all $n \geq 1$. By the previous step, $\partial_x \mathscr{E}(t_n, x(t_n)) \in \{-1, 1\}$ for all $n \geq 1$. Moreover, since $x(\cdot)$ is continuous at t, we get

$$\partial_x \mathscr{E}(t_n, x(t_n)) \to \partial_x \mathscr{E}(t, x(t)).$$

Therefore, $\partial_x \mathscr{E}(t, x(t)) \in \{-1, 1\}$.

Step 3. Now assume furthermore that $x(\cdot)$ is right-continuous and satisfies the global stability. We shall show that $\partial_x \mathscr{E}(t, x(t)) \in \{-1, 1\}$ for all $t \in J$. Then by the same continuity argument as above, we obtain $\partial_x \mathscr{E}(t, x(t)) \in \{-1, 1\}$ for all $t \notin \operatorname{int}(N)$.

In fact, since $x(t)$ is right-continuous and $t \in J$, we have

$$x_0 := x(t) = x(t^+) \neq x(t^-) =: x_-.$$

From the energy-dissipation upper bound, we get

$$\mathscr{E}(t, x_-) \geq \mathscr{E}(t, x_0) + |x_0 - x_-|.$$

On the other hand, from the stability of $x(t)$ we obtain

$$\mathscr{E}(t, x) \geq \mathscr{E}(t, x_-) - |x - x_-| \quad \text{for all } x \in \mathbb{R}.$$

Combining them together, we arrive at

$$\mathscr{E}(t, x) \geq \mathscr{E}(t, x_0) + |x_0 - x_-| - |x - x_-| \quad \text{for all } x \in \mathbb{R}. \tag{3.4}$$

We distinguish two cases: $x_0 < x_-$ and $x_0 > x_-$.

Case 1. $x_0 < x_-$. In this case, (3.4) implies that

$$\partial_x \mathscr{E}(t, x_0) = \lim_{x \downarrow x_0} \frac{\mathscr{E}(t, x) - \mathscr{E}(t, x_0)}{x - x_0} \geq \lim_{x \downarrow x_0} \frac{|x_- - x_0| - |x - x_-|}{x - x_0} = 1.$$

Case 2. $x_0 > x_-$. Using (3.4) again we get

$$\partial_x \mathscr{E}(t, x_0) = \lim_{x \uparrow x_0} \frac{\mathscr{E}(t, x) - \mathscr{E}(t, x_0)}{x - x_0} \leq \lim_{x \downarrow x_0} \frac{|x_- - x_0| - |x - x_-|}{x - x_0} = -1.$$

On the other hand, by continuity, we have $\partial_x \mathscr{E}(t, x_0) \in [-1, 1]$. Therefore, in both cases, we have $\partial_x \mathscr{E}(t, x(t)) = \partial_x \mathscr{E}(t, x_0) \in \{-1, 1\}$ when $t \in J$. □

Lemma 3.14. *Assume that $x : [0, T] \to \mathbb{R}$ has bounded variation and satisfies the weak local stability (w-LS) and the energy-dissipation upper bound (ED-upper). If $t \notin J \cup \operatorname{int}(N \cup J)$ and $\partial_{xx} \mathscr{E}(t, x(t)) \neq 0$, then for any sequence $t_n \to t$ such that $t_n \notin J$ and $t_n \notin \operatorname{int}(N \cup J)$ and $t_n \neq t$ for all $n \geq 1$, one has*

$$\lim_{n \to \infty} \frac{x(t_n) - x(t)}{t_n - t} = -\frac{\partial_{xt} \mathscr{E}(t, x(t))}{\partial_{xx} \mathscr{E}(t, x(t))}.$$

Moreover, if $x(\cdot)$ is right-continuous and satisfies (S) *and* (ED-upper)*, then if $t \notin J \cup \operatorname{int}(N)$ and $\partial_{xx} \mathscr{E}(t, x(t)) \neq 0$, then for any sequence $t_n \to t$ such that $t_n \notin \operatorname{int}(N), t_n \neq t$, one has*

$$\lim_{n \to \infty} \frac{x(t_n) - x(t)}{t_n - t} = -\frac{\partial_{xt} \mathscr{E}(t, x(t))}{\partial_{xx} \mathscr{E}(t, x(t))}.$$

Here the set N is defined in (3.1).

Proof. By Lemma 3.13 we have $\partial_x \mathscr{E}(t, x(t)) \in \{-1, 1\}$ and $\partial_x \mathscr{E}(t_n, x(t_n)) \in \{-1, 1\}$ for all $n \geq 1$. Due to the continuity of $s \mapsto \partial_x \mathscr{E}(s, x(s))$ at $s = t$, we obtain

$$\partial_x \mathscr{E}(t_n, x(t_n)) = \partial_x \mathscr{E}(t, x(t))$$

for n large enough. Therefore, by Taylor's expansion

$$\begin{aligned} 0 &= \partial_x \mathscr{E}(t_n, x(t_n)) - \partial_x \mathscr{E}(t, x(t)) \\ &= \partial_{xt} \mathscr{E}(t, x(t)) \cdot (t_n - t) + \partial_{xx} \mathscr{E}(t, x(t)) \cdot (x(t_n) - x(t)) + o(t_n - t) + o(x(t_n) - x(t)). \end{aligned}$$

This implies that

$$\lim_{n \to \infty} \frac{x(t_n) - x(t)}{t_n - t} = -\frac{\partial_{xt} \mathscr{E}(t, x(t))}{\partial_{xx} \mathscr{E}(t, x(t))}$$

when $\partial_{xx} \mathscr{E}(t, x(t)) \neq 0$. □

Now we are able to show

Proof of Lemma 3.12. Consider the set

$$E := \{t \in (0, T) \mid \partial_x \mathscr{E}(t, x(t)) \in \{-1, 1\}, \partial_{xx} \mathscr{E}(t, x(t)) = 0\}. \tag{3.5}$$

Recall that we are assuming that $\partial_{xxx} \mathscr{E}(t, x(t)) \neq 0$ for any $t \in E$. Consider an arbitrary t.

Case 1. If $t \in N \cup J$, then $t \notin B$ by definition of B and N.

Case 2. If t is an accumulation point of $(0, T) \backslash (N \cup J)$ and $t \notin E$, then we can find a sequence $t_n \to t$ such that $t_n \notin N \cup J$ and $t_n \neq t$ for all $n \geq 1$. By Lemma 3.14,

$$\lim_{n \to \infty} \frac{x(t_n) - x(t)}{t_n - t} = -\frac{\partial_{xt} \mathscr{E}(t, x(t))}{\partial_{xx} \mathscr{E}(t, x(t))}.$$

Thus in this case, $t \notin B$.

Case 3. If $t \notin J$ and t is an accumulation point of E, then we can find a sequence $s_n \in E$, $s_n \to t$. Using Taylor's expansion again, we get

$$\begin{aligned} 0 &= \partial_{xx} \mathscr{E}(s_n, x(s_n)) - \partial_{xx} \mathscr{E}(t, x(t)) \\ &= \partial_{xxt} \mathscr{E}(t, x(t)) \cdot (s_n - t) + \partial_{xxx} \mathscr{E}(t, x(t)) \cdot (x(s_n) - x(t)) \\ &\quad + o(s_n - t) + o(x(s_n) - x(t)). \end{aligned}$$

Since $\partial_{xxx} \mathscr{E}(t, x(t)) \neq 0$, we arrive at

$$\lim_{n \to \infty} \frac{x(s_n) - x(t)}{s_n - t} = -\frac{\partial_{xxt} \mathscr{E}(t, x(t))}{\partial_{xxx} \mathscr{E}(t, x(t))},$$

which is a finite number. Thus $t \notin B$.

Conclusion. In summary, if $t \in B$, then either t is an isolated point of $(0, T) \backslash (N \cup J)$, or t is an isolated point of E. Therefore, B is at most countable. Since $\mu_s(\{t\}) = 0$ for any $t \in B \subset (0, T) \backslash J$, we have $|\mu_s|(B) = 0$. This ends the proof of Lemma 3.12. □

Remark. The proof of Theorem 3.3 is unchanged if the set

$$\{(t, x) \in (0, T) \times \mathbb{R} \mid \partial_x \mathscr{E}(t, x) \in \{-1, 1\}, \partial_{xx} \mathscr{E}(t, x) = \partial_{xxx} \mathscr{E}(t, x) = 0\}$$

in condition (H1) is assumed to be countable.

4 Weak C^1 regularity

In this section we shall prove Theorem 3.5. Beside Lemma 3.13 and Lemma 3.14, we have some other useful preliminaries.

Lemma 3.15. *Assume that $x : [0,T] \to \mathbb{R}$ is right-continuous, has bounded variation and satisfies the global stability (S) and the energy-dissipation upper bound (ED-upper). If $t \notin \mathrm{int}(N)$ and $t \notin J$, then $\partial_{xx}\mathscr{E}(t,x(t)) \ge 0$. Moreover, if $t \in E$, then $\partial_x\mathscr{E}(t,x(t)) \cdot \partial_{xxx}\mathscr{E}(t,x(t)) \le 0$. Here the set N is defined in (3.1) and the set E is defined in (3.5).*

Proof. 1. From the global stability (S)

$$\mathscr{E}(t,x(t)) \le \mathscr{E}(t,x) + |x - x(t)| \quad \text{for all } x \in \mathbb{R} \text{ and for all } t \in [0,T],$$

using Taylor's expansion for $\mathscr{E}(t,\cdot)$ up to second order, we have

$$\begin{aligned}\mathscr{E}(t,x(t)) \quad &\le \quad \mathscr{E}(t,x(t)) + |x - x(t)| + \partial_x\mathscr{E}(t,x(t)) \cdot (x - x(t)) \\ &\quad + \partial_{xx}\mathscr{E}(t,x(t)) \cdot \frac{(x-x(t))^2}{2} + o(|x-x(t)|^2). \qquad (3.6)\end{aligned}$$

By Lemma 3.13, $\partial_x\mathscr{E}(t,x(t)) \in \{-1,1\}$ for all $t \notin \mathrm{int}(N)$. If $\partial_x\mathscr{E}(t,x(t)) = -1$, then taking a sequence $x \downarrow x(t)$ in (3.6) we get $\partial_{xx}\mathscr{E}(t,x(t)) \ge 0$. If $\partial_x\mathscr{E}(t,x(t)) = 1$, then taking a sequence $x \uparrow x(t)$ in (3.6), we also get $\partial_{xx}\mathscr{E}(t,x(t)) \ge 0$.

2. Now assuming $\partial_{xx}\mathscr{E}(t,x(t)) = 0$, we shall prove that $\partial_x\mathscr{E}(t,x(t)) \cdot \partial_{xxx}\mathscr{E}(t,x(t)) \le 0$. Using the above stability and Taylor's expansion for $\mathscr{E}(t,\cdot)$ up to third order, we get

$$\begin{aligned}\mathscr{E}(t,x(t)) \quad &\le \quad \mathscr{E}(t,x(t)) + |x - x(t)| + \partial_x\mathscr{E}(t,x(t)) \cdot (x - x(t)) \\ &\quad + \partial_{xxx}\mathscr{E}(t,x(t)) \cdot \frac{(x-x(t))^3}{6} + o(|x-x(t)|^3). \qquad (3.7)\end{aligned}$$

If $\partial_x\mathscr{E}(t,x(t)) = -1$, then taking a sequence $x_n \downarrow x(t)$ in (3.7) we get $\partial_{xxx}\mathscr{E}(t,x(t)) \ge 0$. If $\partial_x\mathscr{E}(t,x(t)) = 1$, then taking a sequence $x_n \uparrow x(t)$ in (3.6), we get $\partial_{xxx}\mathscr{E}(t,x(t)) \le 0$. Thus in both cases,

$$\partial_x\mathscr{E}(t,x(t)) \cdot \partial_{xxx}\mathscr{E}(t,x(t)) \le 0.$$

□

Lemma 3.16. *Assume that $x : [0,T] \to \mathbb{R}$ has bounded variation and satisfies the weak local stability (w-LS) and the energy-dissipation upper bound (ED-upper). Then for all $t \in (0,T)\backslash J$, one has*

$$\limsup_{s\to t} \left\{ \partial_x\mathscr{E}(t,x(t)) \cdot \frac{x(s)-x(t)}{s-t} \right\} \le 0.$$

Proof. We shall show that for any sequence $t_n \to t$ and $t_n \ne t$ then

$$\limsup_{n\to\infty} \left\{ \partial_x\mathscr{E}(t,x(t)) \cdot \frac{x(t_n)-x(t)}{t_n-t} \right\} \le 0.$$

Of course, we may assume that

$$\liminf_{n\to\infty} \left| \frac{x(t_n)-x(t)}{t_n-t} \right| > 0$$

and either $t_n \downarrow t$ or $t_n \uparrow t$.

Case 1. If $t_n \downarrow t$, then due to the inequality (3.3) in the proof of Lemma 3.13 one has

$$\lim_{n\to\infty} \partial_x \mathscr{E}(t, x(t)) \cdot \frac{x(t_n) - x(t)}{|x(t_n) - x(t)|} = -1.$$

This implies that

$$\lim_{n\to\infty} \partial_x \mathscr{E}(t, x(t)) \cdot \operatorname{sign}\left(\frac{x(t_n) - x(t)}{t_n - t}\right) = -1.$$

Case 2. If $t_n \uparrow t$, then similarly, one has

$$\lim_{n\to\infty} \partial_x \mathscr{E}(t, x(t)) \cdot \frac{x(t_n) - x(t)}{|x(t_n) - x(t)|} = 1,$$

and hence

$$\lim_{n\to\infty} \partial_x \mathscr{E}(t, x(t)) \cdot \operatorname{sign}\left(\frac{x(t_n) - x(t)}{t_n - t}\right) = -1.$$

Thus in all cases, we have

$$\lim_{n\to\infty} \partial_x \mathscr{E}(t, x(t)) \cdot \operatorname{sign}\left(\frac{x(t_n) - x(t)}{t_n - t}\right) = -1.$$

and the conclusion follows. □

Lemma 3.17. *Assume that* $x : [0, T] \to \mathbb{R}$ *is right-continuous, has bounded variation and satisfies the global stability (S) and the energy-dissipation upper bound (ED-upper). Let* $t \in \operatorname{int}[(0, T)\backslash\operatorname{int}(N)]$, $t \in E$, $t \notin J$ *and* $\partial_{xxx}\mathscr{E}(t, x(t)) \neq 0$, *then we have* $\partial_{xt}\mathscr{E}(t, x(t)) = 0$. *Here we denote* N *as in (3.1) and* E *as in (3.5).*

Proof. **Step 1.** Take an arbitrary sequence $t_n \to t$, $t_n \neq t$, $t_n \in \operatorname{int}[(0, T)\backslash\operatorname{int}(N)]$. By Lemma 3.13 and the continuity of the function $s \mapsto \partial_x\mathscr{E}(s, x(s))$ at $s = t$, we have $\partial_x\mathscr{E}(t_n, x(t_n)) = \partial_x\mathscr{E}(t, x(t))$. Using Taylor's expansion and $\partial_{xx}\mathscr{E}(t, x(t)) = 0$, we have

$$\begin{aligned} 0 &= \partial_x\mathscr{E}(t_n, x(t_n)) - \partial_x\mathscr{E}(t, x(t)) \\ &= \partial_{xt}\mathscr{E}(t, x(t)) \cdot (t_n - t) + o(x(t_n) - x(t))) + o(t_n - t). \end{aligned} \tag{3.8}$$

Thus we can conclude that $\partial_{xt}\mathscr{E}(t, x(t)) = 0$ if we can find a sequence $t_n \to t$ such that

$$\limsup_{n\to\infty} \frac{|x(t_n) - x(t)|}{|t_n - t|} < \infty.$$

Step 2. Since $x(\cdot)$ has bounded variation, it has at most countably many jumps. Moreover, since $t \in \operatorname{int}[(0, T)\backslash\operatorname{int}(N)]$, we can choose an arbitrary sequence $s_n \uparrow t$, $s_n \neq t$, $s_n \in \operatorname{int}[(0, T)\backslash\operatorname{int}(N)]$, and $s_n \notin J$. Therefore, by Lemma 3.15 we have $\partial_{xx}\mathscr{E}(s_n, x(s_n)) \geq 0 = \partial_{xx}\mathscr{E}(t, x(t))$. Using Taylor's expansion we have

$$\begin{aligned} 0 &\leq \partial_{xx}\mathscr{E}(s_n, x(s_n)) - \partial_{xx}\mathscr{E}(t, x(t)) \\ &= \partial_{xxt}\mathscr{E}(t, x(t)) \cdot (s_n - t) + \partial_{xxx}\mathscr{E}(t, x(t)) \cdot (x(s_n) - x(t)) \\ &\quad + o(x(s_n) - x(t)) + o(s_n - t). \end{aligned}$$

Dividing the above inequality by $(s_n - t) < 0$, we have

$$0 \geq \partial_{xxt}\mathscr{E}(t, x(t)) + (\partial_{xxx}\mathscr{E}(t, x(t)) + o(1)) \cdot \frac{x(s_n) - x(t)}{s_n - t} + o(1). \tag{3.9}$$

Note that $\partial_{xxx}\mathscr{E}(t, x(t)) \neq 0$ by our assumption.

Step 3. We distinguish two cases.

Case 1. Assume $\partial_x\mathscr{E}(t, x(t)) = -1$. Then $\partial_{xxx}\mathscr{E}(t, x(t)) > 0$ by Lemma 3.15. Therefore, (3.9) implies that

$$\limsup_{n\to\infty} \frac{x(s_n) - x(t)}{s_n - t} \leq -\frac{\partial_{xxt}\mathscr{E}(t, x(t))}{\partial_{xxx}\mathscr{E}(t, x(t))} < \infty.$$

On the other hand, by Lemma 3.16,

$$\liminf_{n\to\infty} \left\{ \frac{x(s_n) - x(t)}{s_n - t} \right\} \geq 0.$$

Therefore,

$$\limsup_{n\to\infty} \frac{|x(s_n) - x(t)|}{|s_n - t|} < \infty.$$

Case 2. Assume $\partial_x\mathscr{E}(t, x(t)) = 1$. Similarly, we have $\partial_{xxx}\mathscr{E}(t, x(t)) < 0$ by Lemma 3.15, and hence

$$\liminf_{n\to\infty} \frac{x(s_n) - x(t)}{s_n - t} \geq -\frac{\partial_{xxt}\mathscr{E}(t, x(t))}{\partial_{xxx}\mathscr{E}(t, x(t))} > -\infty.$$

Moreover, by Lemma 3.16,

$$\limsup_{n\to\infty} \left\{ \frac{x(s_n) - x(t)}{s_n - t} \right\} \leq 0.$$

Thus

$$\limsup_{n\to\infty} \frac{|x(s_n) - x(t)|}{|s_n - t|} < \infty.$$

Step 4. In summary, if $s_n \uparrow t$, then we always have

$$\limsup_{n\to\infty} \frac{|x(s_n) - x(t)|}{|s_n - t|} < \infty.$$

Therefore, choosing $t_n = s_n$ in (3.8), we conclude that $\partial_{xt}\mathscr{E}(t, x(t)) = 0$. □

Lemma 3.18. *Assume that $x : [0, T] \to \mathbb{R}$ is right-continuous, has bounded variation and satisfies the global stability* (S) *and the energy-dissipation upper bound* (ED-upper)*. If t is an accumulation point of $\partial \overset{\circ}{N}$ and $t \notin J$, we have $\partial_{xt}\mathscr{E}(t, x(t)) = 0$. Moreover, if $t \notin E$, then $x'(t) = 0$ and $\partial_{xtt}\mathscr{E}(t, x(t)) = 0$.*

Here we denote by $\partial \overset{\circ}{N}$ the boundary of the interior of N, N is defined as in (3.1) and E is defined as in (3.5).

Proof. **Step 1.** Since t is an accumulation point of $\partial \overset{\circ}{N}$, we can find $a_n \to t$, $b_n \to t$ such that $(a_n, b_n) \subset \text{int}(N)$ and $a_n, b_n \in \partial \overset{\circ}{N}$. By Lemma 3.13, and the continuity of $s \mapsto \partial_x \mathscr{E}(s, x(s))$ at $s = t$, one has, for n large enough,

$$\partial_x \mathscr{E}(a_n, x(a_n)) = \partial_x \mathscr{E}(t, x(t)) = \partial_x \mathscr{E}(b_n, x(b_n)) \in \{-1, 1\}.$$

Note that for all $s \in [a_n, b_n]$, $x(s) = c_n$, a constant independent of s. Consider the one-variable function

$$s \mapsto f_n(s) := \partial_x \mathscr{E}(s, c_n).$$

Since $f_n(a_n) = f_n(b_n)$, by Rolle's Theorem, we can find a number $s_n \in (a_n, b_n)$ such that $f_n'(s_n) = 0$. This means $\partial_{xt} \mathscr{E}(s_n, x(s_n)) = 0$. Since $s_n \to t$, one has

$$0 = \partial_{xt} \mathscr{E}(s_n, x(s_n)) \to \partial_{xt} \mathscr{E}(t, x(t)).$$

Step 2. Now we assume that $t \notin E$. We distinguish two cases.

Case 1. Let $t_n \notin \text{int}(N)$, $t_n \neq t$ and $t_n \to t$. Then by Lemma 3.14 we have

$$\lim_{n\to\infty} \frac{x(t_n) - x(t)}{t_n - t} = -\frac{\partial_{xt} \mathscr{E}(t, x(t))}{\partial_{xx} \mathscr{E}(t, x(t))} = 0.$$

Case 2. Let $s_n \in \text{int}(N)$ and $s_n \to t$. Since t is an accumulation point of $\partial \overset{\circ}{N}$, we can assume that $s_n \in (a_n, b_n) \subset \text{int}(N)$ with $a_n, b_n \in \partial \overset{\circ}{N}$. Using Case 1, one has

$$\lim_{n\to\infty} \frac{x(a_n) - x(t)}{a_n - t} = \lim_{n\to\infty} \frac{x(b_n) - x(t)}{b_n - t} = 0.$$

On the other hand, since $x'(s) = 0$ when $s \in (a_n, b_n)$, we have $x(s_n) = x(a_n) = x(b_n)$. Therefore,

$$\left| \frac{x(s_n) - x(t)}{s_n - t} \right| \le \max \left\{ \left| \frac{x(a_n) - x(t)}{a_n - t} \right|, \left| \frac{x(b_n) - x(t)}{b_n - t} \right| \right\} \to 0$$

as $n \to \infty$.

Thus in summary, for any sequence $t_n \to t$ and $t_n \neq t$ we always have

$$\lim_{n\to\infty} \frac{x(t_n) - x(t)}{t_n - t} \to -\frac{\partial_{xt} \mathscr{E}(t, x(t))}{\partial_{xx} \mathscr{E}(t, x(t))} = 0.$$

This means $x'(t) = 0$.

Step 3. Now we show that if we assume furthermore that $t \notin E$, then $\partial_{xtt} \mathscr{E}(t, x(t)) = 0$.

Since $t \notin E$ and the functions $s \mapsto \partial_x \mathscr{E}(s, x(s)), s \mapsto \partial_{xx} \mathscr{E}(s, x(s))$ are continuous at $s = t$, we have $s \notin E$ if s is in a neighborhood of t. In particular, if s is in a neighborhood of t, $s \in \text{int}[(0, T) \backslash \text{int}(N)]$ and $s \notin J$, then $\partial_{xx} \mathscr{E}(s, x(s)) > 0$ by Lemma 3.15 and

$$x'(s) = -\frac{\partial_{xt} \mathscr{E}(s, x(s))}{\partial_{xx} \mathscr{E}(s, x(s))}$$

by Lemma 3.14. Using Lemma 3.16, we conclude that

$$\partial_{xt}\mathscr{E}(s,x(s))\cdot\partial_x\mathscr{E}(s,x(s))\geq 0. \tag{3.10}$$

Let us assume that $\partial_x\mathscr{E}(t,x(t))=-1$ (the other case, $\partial_x\mathscr{E}(t,x(t))=1$, can be treated by the same way). If s is in a neighborhood of t, $s\notin J$ and $s\in \mathrm{int}[(0,T)\backslash\mathrm{int}(N)]$, then $\partial_x\mathscr{E}(s,x(s))<0$, and hence $\partial_{xt}\mathscr{E}(s,x(s))\leq 0$ by (3.10). In particular, we have

$$\partial_{xt}\mathscr{E}(a_n,x(a_n))\leq 0 \text{ and } \partial_{xt}\mathscr{E}(b_n,x(b_n))\leq 0$$

for n large enough, where $\{a_n\},\{b_n\}$ are taken as in Step 1.

On the other hand, it was already shown in Step 1 that there exists $t_n\in(a_n,b_n)$ such that $\partial_{xt}\mathscr{E}(t_n,x(t_n))=0$. Therefore, the function $g(s):=\partial_{xt}\mathscr{E}(s,x(s))$ has a local maximizer $s_n\in(a_n,b_n)$. Hence, for n large enough,

$$\partial_{xtt}\mathscr{E}(s_n,x(s_n))=g'(s_n)=0.$$

Since $s_n\to t$, by taking the limit as $n\to\infty$ we obtain $\partial_{xtt}\mathscr{E}(t,x(t))=0$. □

Now we are able to give

Proof of Theorem 3.5. Since $x(\cdot)$ has finite jump points and (H1), (H3) hold true, by dividing $(0,T)$ into subintervals if necessary, we may further assume that $x(\cdot)$ has no jumps on $(0,T)$ and

$$\{(t,x)\in(0,T)\times\mathbb{R}\mid \partial_x\mathscr{E}(t,x)\in\{-1,1\},\partial_{xx}\mathscr{E}(t,x)=\partial_{xxx}\mathscr{E}(t,x)=0\}=\emptyset,$$

$$\{(t,x)\in(0,T)\times\mathbb{R}\mid \partial_x\mathscr{E}(t,x)\in\{-1,1\},\partial_{xx}\mathscr{E}(t,x)=\partial_{xt}\mathscr{E}(t,x)=0\}=\emptyset.$$

We denote by I_1 the set of isolated points of $\partial\overset{\circ}{N}$. It remains to consider the case $t\notin I_1$. We distinguish the following cases.

Case 1. If $t\in\mathrm{int}(N)$, then $x'(t)=0$. Moreover, if s is in a neighborhood of t then $x'(s)=0$. Therefore, $x'(\cdot)$ is continuous at t.

Case 2. If $t\in\mathrm{int}[(0,T)\backslash\mathrm{int}(N)]$, then by Lemma 3.17 we have $t\notin E$. Therefore, by Lemma 3.14,

$$x'(t)=-\frac{\partial_{xt}\mathscr{E}(t,x(t))}{\partial_{xx}\mathscr{E}(t,x(t))}.$$

Since the same formula also holds true for any s in a neighborhood of t, we have that $x'(\cdot)$ is continuous at t.

Case 3. If t is an accumulation point of $\partial\overset{\circ}{N}$, then $\partial_{xt}\mathscr{E}(t,x(t))=0$ by Lemma 3.18. Therefore, $t\notin E$. By Lemma 3.18 one has $x'(t)=0$. Next, we shall show that if $t_n\to t$ and $t_n\notin I_1$, then $x'(t_n)\to x'(t)=0$. Indeed, if $t_n\in\mathrm{int}[(0,T)\backslash\mathrm{int}(N)]$, then

$$x'(t_n)=-\frac{\partial_{xt}\mathscr{E}(t_n,x(t_n))}{\partial_{xx}\mathscr{E}(t_n,x(t_n))}\to-\frac{\partial_{xt}\mathscr{E}(t,x(t))}{\partial_{xx}\mathscr{E}(t,x(t))}=0.$$

Otherwise, if $t_n\in\mathrm{int}(N)$ or t_n is an accumulation point of $\partial\overset{\circ}{N}$, then we already have $x'(t_n)=0$.

In summary, if $t \in (0,T)\backslash I_1$ one has

$$x'(t) = \begin{cases} -\dfrac{\partial_{xt}\mathscr{E}(t,x(t))}{\partial_{xx}\mathscr{E}(t,x(t))} & \text{if } t \in \text{int}[(0,T)\backslash\text{int}(N)], \\ 0 & \text{otherwise,} \end{cases}$$

and $x'(\cdot)$ is continuous on $(0,T)\backslash I_1$. This completes the proof of Theorem 3.5. □

Remark. In general, the set I in the statement of Theorem 3.5 contains the following points: the isolated points of $\partial \overset{\circ}{N}$ (namely the set I_1 in the above proof), the jump points, and the points t such that $\partial_x\mathscr{E}(t,x(t)) \in \{-1,1\}$, $\partial_{xx}\mathscr{E}(t,x(t)) = 0$ and either $\partial_{xxx}\mathscr{E}(t,x(t)) = 0$ or $\partial_{xt}\mathscr{E}(t,x(t)) = 0$.

5 C^1 regularity

Now we prove Theorem 3.6. First, we need the following lemmas.

Lemma 3.19. *Assume that $x : [0,T] \to \mathbb{R}$ is right-continuous, has bounded variation and satisfies the global stability (S) and the energy-dissipation upper bound (ED-upper). Assume furthermore that $t \notin J, t \notin \text{int}(N)$, $\partial_{xx}\mathscr{E}(t,x(t)) = \partial_{xt}\mathscr{E}(t,x(t)) = 0$ and $\partial_{xxx}\mathscr{E}(t,x(t)) \neq 0$. If there exists a sequence $s_n \notin J \cup \text{int}(N)$ such that $s_n \downarrow t$ (or $s_n \uparrow t$), then the limit*

$$\lim_{s\notin J\cup\text{int}(N), s\downarrow t} \frac{x(s)-x(t)}{s-t} \quad \left(or \lim_{s\notin J\cup\text{int}(N), s\uparrow t} \frac{x(s)-x(t)}{s-t}, respectively \right)$$

exists and it is a solution of the equation

$$\partial_{xtt}\mathscr{E}(t,x(t)) + 2\partial_{xxt}\mathscr{E}(t,x(t)) \cdot X + \partial_{xxx}\mathscr{E}(t,x(t)) \cdot X^2 = 0. \tag{3.11}$$

Moreover, if there is a sequence $t_n \to t$ such that $t_n \neq t$, $t_n \notin \text{int}(N)$ and $\partial_{xx}\mathscr{E}(t_n,x(t_n)) = 0$ for all $n \geq 1$, then (3.11) has the unique solution $-\partial_{xxt}\mathscr{E}(t,x(t))/\partial_{xxx}\mathscr{E}(t,x(t))$. Here the set N is defined as in (3.1).

Proof. **Step 1.** Let $t_n \to t$ and $t_n \notin J \cup \text{int}(N)$. We have $\partial_x\mathscr{E}(t_n,x(t_n)) = \partial_x\mathscr{E}(t,x(t))$ by Lemma 3.13 and the continuity of $s \mapsto \partial_x\mathscr{E}(s,x(s))$ at $s=t$. Using Taylor's expansion we obtain

$$\begin{aligned} 0 &= \partial_x\mathscr{E}(t_n,x(t_n)) - \partial_x\mathscr{E}(t,x(t)) \\ &= \partial_{xtt}\mathscr{E}(t,x(t)) \cdot (t_n-t)^2 + 2\partial_{xxt}\mathscr{E}(t,x(t)) \cdot (x(t_n)-x(t)) \cdot (t_n-t) \\ &\quad +\partial_{xxx}\mathscr{E}(t,x(t)) \cdot (x(t_n)-x(t))^2 + o(|x(t_n)-x(t)|^2) + o(|t_n-t|^2). \end{aligned}$$

Dividing this equality by $(t_n-t)^2$ and taking the limit as $n \to \infty$ we get

$$\partial_{xtt}\mathscr{E}(t,x(t)) + 2\partial_{xxt}\mathscr{E}(t,x(t)) \cdot \frac{x(t_n)-x(t)}{t_n-t} + \partial_{xxx}\mathscr{E}(t,x(t)) \cdot \left(\frac{x(t_n)-x(t)}{t_n-t}\right)^2 \to 0. \tag{3.12}$$

Notice that, (3.12) also shows that the solutions of (3.11) are real. Moreover, if we denote by X_1 and X_2 the two solutions of the equation (3.11), then

$$\min\left\{ \left| \frac{x(t_n)-x(t)}{t_n-t} - X_1 \right|, \left| \frac{x(t_n)-x(t)}{t_n-t} - X_2 \right| \right\} \to 0$$

as $n \to \infty$.

Step 2. Using Lemma 3.15 and Taylor's expansion one has

$$\begin{aligned} 0 &\leq \partial_{xx}\mathscr{E}(t_n, x(t_n)) - \partial_{xx}\mathscr{E}(t, x(t)) \\ &= \partial_{xxt}\mathscr{E}(t, x(t)) \cdot (t_n - t) + \partial_{xxx}\mathscr{E}(t, x(t)) \cdot (x(t_n) - x(t)) \\ &\quad + o(t_n - t) + o(x(t_n) - x(t)). \end{aligned} \tag{3.13}$$

By Lemma 3.13, $\partial_x\mathscr{E}(t, x(t)) \in \{-1, 1\}$. We distinguish two cases.

Case 1. $\partial_x\mathscr{E}(t, x(t)) = -1$. In this case, by Lemma 3.15 we have $\partial_{xxx}\mathscr{E}(t, x(t)) > 0$. Therefore, from the inequality (3.13), if $t_n \downarrow t$, then

$$\liminf_{n\to\infty} \frac{x(t_n) - x(t)}{t_n - t} \geq -\frac{\partial_{xxt}\mathscr{E}(t, x(t))}{\partial_{xxx}\mathscr{E}(t, x(t))}; \tag{3.14}$$

while if $t_n \uparrow t$, then

$$\limsup_{n\to\infty} \frac{x(t_n) - x(t)}{t_n - t} \leq -\frac{\partial_{xxt}\mathscr{E}(t, x(t))}{\partial_{xxx}\mathscr{E}(t, x(t))}. \tag{3.15}$$

Since

$$X_1 + X_2 = -\frac{\partial_{xxt}\mathscr{E}(t, x(t))}{\partial_{xxx}\mathscr{E}(t, x(t))},$$

we have

$$\max\{X_1, X_2\} \geq -\frac{\partial_{xxt}\mathscr{E}(t, x(t))}{\partial_{xxx}\mathscr{E}(t, x(t))} \geq \min\{X_1, X_2\},$$

and then together with (3.14) and (3.15), the convergence in (3.12) reduces to

$$\lim_{n\to\infty} \frac{x(t_n) - x(t)}{t_n - t} = \max\{X_1, X_2\} \quad \text{if } t_n \downarrow t,$$

and

$$\lim_{n\to\infty} \frac{x(t_n) - x(t)}{t_n - t} = \min\{X_1, X_2\} \quad \text{if } t_n \uparrow t.$$

Case 2. If $\partial_x\mathscr{E}(t, x(t)) = 1$, then similarly,

$$\lim_{n\to\infty} \frac{x(t_n) - x(t)}{t_n - t} = \min\{X_1, X_2\} \quad \text{if } t_n \downarrow t,$$

and

$$\lim_{n\to\infty} \frac{x(t_n) - x(t)}{t_n - t} = \max\{X_1, X_2\} \quad \text{if } t_n \uparrow t.$$

In both cases, the first conclusion of Lemma 3.19 follows.

Step 3. Now assume that there is a sequence $t_n \to t$ such that $t_n \neq t$, $t_n \notin \text{int}(N)$ and $\partial_{xx}\mathscr{E}(t_n, x(t_n)) = 0$ for all $n \geq 1$. Using Taylor's expansion,

$$\begin{aligned} 0 &= \partial_{xx}\mathscr{E}(t_n, x(t_n)) - \partial_{xx}\mathscr{E}(t, x(t)) \\ &= \partial_{xxt}\mathscr{E}(t, x(t)) \cdot (t_n - t) + \partial_{xxx}\mathscr{E}(t, x(t)) \cdot (x(t_n) - x(t)) \\ &\quad + o(t_n - t) + o(x(t_n) - x(t)), \end{aligned}$$

we find that

$$\lim_{n\to\infty} \frac{x(t_n) - x(t)}{t_n - t} = -\frac{\partial_{xxt}\mathscr{E}(t, x(t))}{\partial_{xxx}\mathscr{E}(t, x(t))}.$$

Thus $-\partial_{xxt}\mathscr{E}(t, x(t))/\partial_{xxx}\mathscr{E}(t, x(t))$ is a solution to (3.11). Substituting this solution into (3.11) we find that

$$[\partial_{xxt}\mathscr{E}(t, x(t))]^2 = \partial_{xtt}\mathscr{E}(t, x(t)) \cdot \partial_{xxx}\mathscr{E}(t, x(t)),$$

which implies that (3.11) has a unique solution. □

Lemma 3.20. *Assume that $x : [0, T] \to \mathbb{R}$ is right-continuous, has bounded variation and satisfies the global stability (S) and the energy-dissipation upper bound (ED-upper). Let t be an accumulation point of $\partial \overset{\circ}{N}$, $t \notin J$ and either $t \notin E$, or $t \in E$ and $\partial_{xxx}\mathscr{E}(t, x(t)) \neq 0$. Then $x'(t) = 0$, and $\partial_{xtt}\mathscr{E}(t, x(t)) = 0$. Here we denote E as in (3.5), N as in (3.1) and $\partial \overset{\circ}{N}$ the boundary of the interior of N.*

Proof. Since t is an accumulation point of $\partial \overset{\circ}{N}$, Lemma 3.18 ensures that $\partial_{xt}\mathscr{E}(t, x(t)) = 0$. If $t \notin E$, then Lemma 3.18 also implies that $x'(t) = 0$ and $\partial_{xtt}\mathscr{E}(t, x(t)) = 0$. Therefore, it remains to consider the case when $t \in E$ and $\partial_{xxx}\mathscr{E}(t, x(t)) \neq 0$.

Step 1. Since t is an accumulation point of $\partial \overset{\circ}{N}$, there exists a sequence $\{(a_n, b_n)\}$ such that $(a_n, b_n) \subset \mathrm{int}(N)$, $a_n, b_n \in \partial \overset{\circ}{N}$ for all $n \geq 1$, and $a_n, b_n \downarrow t$ (or $a_n, b_n \uparrow t$). By Lemma 3.19, we have

$$\lim_{n\to\infty} \frac{x(a_n) - x(t)}{a_n - t} = \lim_{n\to\infty} \frac{x(b_n) - x(t)}{b_n - t} = X_1,$$

where X_1 is a solution to (3.11). Note that $x(s) = c_n$ is a constant on $[a_n, b_n]$. Therefore, if $t_n \in [a_n, b_n]$ for all $n \geq 1$, then using the fact that $x(\cdot)$ is a constant in $[a_n, b_n]$, one has

$$\left| \frac{x(t_n) - x(t)}{t_n - t} - X_1 \right| \leq \max \left\{ \left| \frac{x(b_n) - x(t)}{b_n - t} - X_1 \right|, \left| \frac{x(a_n) - x(t)}{a_n - t} - X_1 \right| \right\} \to 0.$$

Thus if $t_n \in [a_n, b_n]$, then

$$\lim_{n\to\infty} \frac{x(t_n) - x(t)}{t_n - t} = X_1.$$

Step 2. On the other hand, by Lemma 3.13, the fact that $a_n, b_n, t \notin \mathrm{int}(N)$, we have

$$\partial_x\mathscr{E}(a_n, x(a_n)), \partial_x\mathscr{E}(t, x(t)), \partial_x\mathscr{E}(b_n, x(b_n)) \in \{-1, 1\} \text{ for all } n \in \mathbb{N}.$$

Moreover, by the continuity of $s \mapsto \partial_x\mathscr{E}(s, x(s))$ at $s = t$, we get

$$\partial_x\mathscr{E}(a_n, x(a_n)) = \partial_x\mathscr{E}(t, x(t)) = \partial_x\mathscr{E}(b_n, x(b_n)) \in \{-1, 1\}. \tag{3.16}$$

for n large enough. Consider the one-variable function

$$f_n(s) := \partial_x\mathscr{E}(s, c_n) \quad \text{on } s \in [a_n, b_n], \tag{3.17}$$

where recall that $x(s) = c_n$ for all $s \in [a_n, b_n]$. Since $f_n(a_n) = f_n(b_n)$, by applying Rolle's Theorem, we can find $t_n \in (a_n, b_n)$ such that

$$\partial_{xt}\mathscr{E}(t_n, x(t_n)) = f_n'(t_n) = 0.$$

Using Taylor's expansion we have

$$\begin{aligned} 0 &= \partial_{xt}\mathscr{E}(t_n, x(t_n)) - \partial_{xt}\mathscr{E}(t, x(t)) \\ &= \partial_{xtt}\mathscr{E}(t, x(t)) \cdot (t_n - t) + \partial_{xxt}\mathscr{E}(t, x(t)) \cdot (x(t_n) - x(t)) + o(t_n - t) + o(x(t_n) - x(t)). \end{aligned}$$

Dividing this equality by $t_n - t$ and taking the limit as $n \to \infty$ we obtain

$$\partial_{xtt}\mathscr{E}(t, x(t)) + \partial_{xxt}\mathscr{E}(t, x(t)) \cdot X_1 = 0. \tag{3.18}$$

Step 3. We show that $\partial_{xtt}\mathscr{E}(t, x(t)) = 0$. Assume by contradiction that $\partial_{xtt}\mathscr{E}(t, x(t)) \neq 0$. Then from (3.18), we must have $\partial_{xxt}\mathscr{E}(t, x(t)) \neq 0$ and

$$X_1 = -\frac{\partial_{xtt}\mathscr{E}(t, x(t))}{\partial_{xxt}\mathscr{E}(t, x(t))} \neq 0.$$

Since X_1 is a solution to (3.11), we obtain

$$[\partial_{xxt}\mathscr{E}(t, x(t))]^2 = \partial_{xtt}\mathscr{E}(t, x(t)) \cdot \partial_{xxx}\mathscr{E}(t, x(t)), \tag{3.19}$$

which in particular implies that X_1 is the unique solution to (3.11).

In view of (3.16), we may assume that

$$\partial_x\mathscr{E}(a_n, x(a_n)) = \partial_x\mathscr{E}(t, x(t)) = \partial_x\mathscr{E}(b_n, x(b_n)) = -1$$

for n large enough (the other case can be treated by the same way).

By Lemma 3.15 one has $\partial_{xxx}\mathscr{E}(t, x(t)) > 0$. From (3.19) one has $\partial_{xtt}\mathscr{E}(t, x(t)) > 0$. By the continuity of $s \mapsto \partial_{xtt}\mathscr{E}(s, x(s))$ at $s = t$, we have $\partial_{xtt}\mathscr{E}(s, x(s)) > 0$ when s is in a neighborhood of t. In particular, the function $f_n(s)$ defined by (3.17) satisfies

$$f_n''(s) = \partial_{xtt}\mathscr{E}(s, x(s)) > 0 \quad \text{for all } s \in (a_n, b_n)$$

for n large enough.

Thus f_n is strictly convex on $[a_n, b_n]$. Consequently, if we choose $s := (a_n + b_n)/2$, then

$$\partial_x\mathscr{E}(s, x(s)) = f_n(s) < \frac{f(a_n) + f(b_n)}{2} = -1.$$

However, this contradicts the fact that $\partial_x\mathscr{E}(s, x(s)) \geq -1$ for all $s \notin \mathrm{int}(N)$ by Lemma 3.13. Thus we must have $\partial_{xtt}\mathscr{E}(t, x(t)) = 0$.

Step 4. Now we show that $X_1 = 0$. In fact, if $\partial_{xxt}\mathscr{E}(t, x(t)) \neq 0$, then from $\partial_{xtt}\mathscr{E}(t, x(t)) = 0$ and (3.18) we must have $X_1 = 0$. Otherwise, if $\partial_{xxt}\mathscr{E}(t, x(t)) = 0$, then 0 is the unique solution to the equation (3.11), and hence we also have $X_1 = 0$.

Step 5. Now we show that $x'(t) = 0$. We distinguish three cases.

Case 1. Assume that there exists $a < t$ such that $(a, t) \subset \text{int}(N)$. It is obvious that $x'_-(t) = 0 = \lim_{s \uparrow t} x'(s)$. It remains to show that $x'_+(t) = 0$, namely to show that

$$\lim_{n\to\infty} \frac{x(t_n) - x(t)}{t_n - t} = 0$$

provided that $t_n \downarrow t$.

First, we assume that $t_n \in \text{int}(N)$ and $t_n \downarrow t$. Note that $(t, b) \not\subset \text{int}(N)$ for all $b > t$ (otherwise, by the continuity we have $x(a) = x(t) = x(b)$ and $t \in (a, b) \subset \text{int}(N)$, which is a contradiction). Therefore, as in Step 1, we can choose the sequence $\{(a_n, b_n)\}$ such that $(a_n, b_n) \subset \text{int}(N)$, $a_n, b_n \in \partial \overset{\circ}{N}$ for all $n \geq 1$, and $a_n, b_n \downarrow t$. Therefore, it follows from Step 1 and the fact that $X_1 = 0$

$$\lim_{n\to\infty} \frac{x(t_n) - x(t)}{t_n - t} = \lim_{n\to\infty} \frac{x(a_n) - x(t)}{a_n - t} = 0.$$

Next, assume that $t_n \notin \text{int}(N)$ and $t_n \downarrow t$. Then by Lemma 3.19 we have

$$\lim_{n\to\infty} \frac{x(t_n) - x(t)}{t_n - t} = \lim_{n\to\infty} \frac{x(a_n) - x(t)}{a_n - t} = 0.$$

Thus for any sequence $t_n \downarrow t$ we obtain

$$\lim_{n\to\infty} \frac{x(t_n) - x(t)}{t_n - t} = 0.$$

Therefore, $x'_+(t) = 0$. Thus $x'(t) = 0$.

Case 2. If there exists $b > t$ such that $(t, b) \subset \text{int}(N)$, then similarly to Case 1 we have $x'(t) = 0$.

Case 3. Finally, assume that $(a, t) \not\subset \text{int}(N)$ for all $a < t$, and $(t, b) \not\subset \text{int}(N)$ for all $b > t$. Then by the same proof in Case 1, using the fact that $(t, b) \not\subset \text{int}(N)$ for all $b > t$, we have $x'_+(t) = 0$. Similarly, using the fact that $(a, t) \not\subset \text{int}(N)$ for all $a < t$, we obtain $x'_-(t) = 0$. Thus $x'(t) = 0$. This completes our proof. □

Now we are able to prove the main result of this section.

Proof of Theorem 3.6. **Step 1.** Since $x(\cdot)$ has only finitely many jumps and (H1), (H3), (H4) hold true, by dividing $(0, T)$ into subintervals if necessary, we may assume that $x(\cdot)$ has no jumps and

$$\{(t, x) \in (0, T) \times \mathbb{R} \mid \partial_x \mathscr{E}(t, x) \in \{-1, 1\}, \partial_{xx} \mathscr{E}(t, x) = \partial_{xxx} \mathscr{E}(t, x) = 0\} = \emptyset,$$
$$\{(t, x) \in (0, T) \times \mathbb{R} \mid \partial_x \mathscr{E}(t, x) \in \{-1, 1\}, \partial_{xx} \mathscr{E}(t, x) = \partial_{xt} \mathscr{E}(t, x) = 0\} = \emptyset,$$
$$\{(t, x) \in [0, T] \times \mathbb{R} \mid \partial_x \mathscr{E}(t, x) \in \{-1, 1\}, \partial_{xt} \mathscr{E}(t, x) = \partial_{xtt} \mathscr{E}(t, x) = 0\} = \emptyset.$$

Step 2. Assume that $\partial \overset{\circ}{N}$ has an accumulation point t. Then we have $\partial_x \mathscr{E}(t, x(t)) \in \{-1, 1\}$ by Lemma 3.13. Note that if $t = 0$ or $t = T$, then Lemma 3.13 is not applicable directly to

t, but because t is an accumulation point of $\partial \overset{\circ}{N}$, we can apply Lemma 3.13 to the points in $\partial \overset{\circ}{N} \cap (0, T)$ first, and then take the limit to get the conclusion at t.

Next, we have $\partial_{xt}\mathscr{E}(t, x(t)) = 0$ by Lemma 3.18, and $\partial_{xtt}\mathscr{E}(t, x(t)) = 0$ by Lemma 3.18 (when $t \notin E$) and Lemma 3.20 (when $t \in E$ and $\partial_{xxx}\mathscr{E}(t, x(t)) \neq 0$). Note that these lemmas apply even if $t = 0$ or $t = T$.

Thus

$$\partial_x\mathscr{E}(t, x(t)) \in \{-1, 1\}, \partial_{xt}\mathscr{E}(t, x(t)) = 0, \partial_{xtt}\mathscr{E}(t, x(t)) = 0.$$

By condition (H3), this case cannot happen. Therefore, $\partial \overset{\circ}{N}$ has no accumulation point. Thus $\partial \overset{\circ}{N}$ has finitely many points, and hence $\text{int}(N) \cup \text{int}[(0, T)\backslash\text{int}(N)]$ is the union of finitely many open intervals.

Step 3. Finally, if $t \in \text{int}(N)$, then $x'(t) = 0$. On the other hand, if $t \in \text{int}[(0, T)\backslash\text{int}(N)]$, then by Lemma 3.17 we have $t \notin E$, and hence

$$x'(t) = -\frac{\partial_{xt}\mathscr{E}(t, x(t))}{\partial_{xx}\mathscr{E}(t, x(t))}$$

by Lemma 3.14. Thus we can conclude that $x(\cdot)$ is of class C^1 in $\text{int}(N) \cup \text{int}[(0, T)\backslash\text{int}(N)]$. The proof is completed. □

6 Differentiability

Now we prove Theorem 3.4. First we need the following lemma.

Lemma 3.21. *Assume that $x(\cdot)$ is continuous, has bounded variation and satisfies the global stability (S) and the energy-dissipation upper bound (ED-upper). If $t \in \text{int}[(0, T)\backslash\text{int}(N)]$, $t \in E$ and $\partial_{xxx}\mathscr{E}(t, x(t)) \neq 0$, then the right and left derivatives*

$$x'_+(t) := \lim_{s \downarrow t} \frac{x(s) - x(t)}{s - t}, \quad x'_-(t) := \lim_{s \uparrow t} \frac{x(s) - x(t)}{s - t},$$

exist and they are two solutions of the equation (3.11).

Moreover, if t is an accumulation point of E, then

$$x'(t) = -\frac{\partial_{xxt}\mathscr{E}(t, x(t))}{\partial_{xxx}\mathscr{E}(t, x(t))}$$

and it is the unique solution to the equation (3.11).

On the other hand, if t is an isolated point of E, then either

$$x'(t) = -\frac{\partial_{xxt}\mathscr{E}(t, x(t))}{\partial_{xxx}\mathscr{E}(t, x(t))},$$

or

$$x'_+(t) = \lim_{s \downarrow t} x'(s), \quad x'_-(t) = \lim_{s \uparrow t} x'(s).$$

Here the set E is defined as in (3.5) and the set N is defined as in (3.1).

Proof. **Step 1.** Since $t \in \operatorname{int}[(0,T)\backslash \operatorname{int}(N)]$ and $t \in E$, Lemma 3.17 ensures that $\partial_{xt}\mathscr{E}(t,x(t)) = 0$. Therefore, by Lemma 3.19, we get that $x'_+(t)$, $x'_-(t)$ exist and they are two solutions of the equation (3.11).

Step 2. If t is an accumulation point of E, then by Lemma 3.19 again, the equation (3.11) has a unique solution $-\partial_{xxt}\mathscr{E}(t,x(t))/\partial_{xxx}\mathscr{E}(t,x(t))$. Therefore,

$$x'(t) = -\frac{\partial_{xxt}\mathscr{E}(t,x(t))}{\partial_{xxx}\mathscr{E}(t,x(t))}.$$

Step 3. Now we assume that t is an isolated point of E. If the equation (3.11) has a unique solution, then it must be $-\partial_{xxt}\mathscr{E}(t,x(t))/\partial_{xxx}\mathscr{E}(t,x(t))$, and hence

$$x'(t) = -\frac{\partial_{xxt}\mathscr{E}(t,x(t))}{\partial_{xxx}\mathscr{E}(t,x(t))}.$$

Otherwise, if the equation (3.11) has two distinct solutions, then we shall show that

$$x'_+(t) = \lim_{s\downarrow t} x'(s), \quad x'_-(t) = \lim_{s\uparrow t} x'(s).$$

In fact, since t is an isolated point of E, when s is in a neighborhood of t we have $s \notin E$. Therefore, using Lemma 3.14 and de L'Hôpital's rule, we have, as $s \downarrow t$,

$$\begin{aligned} x'(s) &= -\frac{\partial_{xt}\mathscr{E}(s,x(s))}{\partial_{xx}\mathscr{E}(s,x(s))} = -\frac{\left(\frac{\partial_{xt}\mathscr{E}(s,x(s))-\partial_{xt}\mathscr{E}(t,x(t))}{s-t}\right)}{\left(\frac{\partial_{xx}\mathscr{E}(s,x(s))-\partial_{xx}\mathscr{E}(t,x(t))}{s-t}\right)} \\ &\to -\frac{\partial_{xtt}\mathscr{E}(t,x(t)) + \partial_{xxt}\mathscr{E}(t,x(t))\, x'_+(t)}{\partial_{xxt}\mathscr{E}(t,x(t)) + \partial_{xxx}\mathscr{E}(t,x(t))\, x'_+(t)} = x'_+(t). \end{aligned}$$

Here in the last identity we have used that $x'_+(t)$ solves the equation (3.11). Note that $\partial_{xxt}\mathscr{E}(t,x(t)) + \partial_{xxx}\mathscr{E}(t,x(t))\, x'_+(t) \neq 0$ because the equation (3.11) has two distinct solutions.

Similarly, as $s \uparrow t$,

$$x'(s) = -\frac{\partial_{xt}E(s,x(s))}{\partial_{xx}E(s,x(s))} \to x'_-(t).$$

The proof is completed. □

Now we are able to prove Theorem 3.4.

Proof of Theorem 3.4. **Step 1.** Assume that $x(\cdot)$ has only finitely many jumps and (H1) holds. By dividing $(0,T)$ into subintervals if necessary, we may further assume that $x(\cdot)$ has no jumps and

$$\{(t,x) \in (0,T)\times\mathbb{R} \mid \partial_x\mathscr{E}(t,x) \in \{-1,1\}, \partial_{xx}\mathscr{E}(t,x) = \partial_{xxx}\mathscr{E}(t,x) = 0\} = \emptyset.$$

Thus either $t \notin E$, or $t \in E$ and $\partial_{xxx}\mathscr{E}(t,x(t)) \neq 0$. Choose I_3 and I_1 as follows

$$\begin{aligned} I_3 &:= \{t \in [0,T] \mid x(\cdot) \text{ is differentiable at } t\}; \\ I_1 &:= \{t \in [0,T]\backslash I_3 \mid t \text{ is an isolated point of } \partial \mathring{N}\}. \end{aligned}$$

Now we consider the case t is not an isolated point of $\partial \overset{\circ}{N}$. We have the following cases.

Case 1. If $t \in \text{int}(N)$, then $x'(t) = 0$ by definition.

Case 2. If t is an accumulation point of $\partial \overset{\circ}{N}$, then $x'(t) = 0$ by Lemma 3.20.

Case 3. If $t \in \text{int}[(0,T)\backslash\text{int}(N)]$ and $t \notin E$, then by Lemma 3.14,

$$x'(t) = -\frac{\partial_{xt}\mathscr{E}(t, x(t))}{\partial_{xx}\mathscr{E}(t, x(t))}.$$

Case 4. If $t \in \text{int}[(0,T)\backslash\text{int}(N)]$ and t is an accumulation point of E, then by Lemma 3.21,

$$x'(t) = -\frac{\partial_{xxt}\mathscr{E}(t, x(t))}{\partial_{xxx}\mathscr{E}(t, x(t))}.$$

Case 5. If $t \in \text{int}[(0,T)\backslash\text{int}(N)]$ and t is an isolated point of E, then by Lemma 3.21, we have either

$$x'(t) = -\frac{\partial_{xxt}\mathscr{E}(t, x(t))}{\partial_{xxx}\mathscr{E}(t, x(t))},$$

or there exist $x'_+(t)$, $x'_-(t)$ and

$$x'_+(t) = \lim_{s\downarrow t} x'(s),\ x'_-(t) = \lim_{s\uparrow t} x'(s).$$

Thus we can choose I_2 as follows

$$I_2 := \{t \in [0,T]\backslash(I_1 \cup I_3) \mid t \text{ is an isolated point of } E \text{ in } \text{int}[(0,T)\backslash\text{int}(N)] \text{ and } x'_-(t) \neq x'_+(t)\}.$$

Step 2. Assume that (H2) also holds. Then by dividing $(0,T)$ into subintervals again we may assume further that

$$\begin{aligned}\{(t,x) \in (0,T)\times\mathbb{R} \mid \partial_x\mathscr{E}(t,x) \in \{-1,1\}, \partial_{xx}\mathscr{E}(t,x) = \partial_{xt}\mathscr{E}(t,x) = 0,\\ [\partial_{xxt}\mathscr{E}(t,x)]^2 = \partial_{xtt}\mathscr{E}(t,x)\cdot\partial_{xxx}\mathscr{E}(t,x)\} \quad = \quad \emptyset. \quad (3.20)\end{aligned}$$

We show that in this case the set $I := I_1 \cup I_2$ only contains isolated points. Assume by contradiction that t is an accumulation point of I. Thus we must have a sequence $t_n \to t \in I_1$ with $t_n \in I_2$ for all $n \geq 1$. By Lemma 3.17 we have $\partial_{xt}\mathscr{E}(t_n, x(t_n)) = 0$ for all n. Since $\partial_{xx}\mathscr{E}(t_n, x(t_n)) = \partial_{xt}\mathscr{E}(t_n, x(t_n)) = 0$, taking the limit as $n \to \infty$ we get $\partial_{xx}\mathscr{E}(t, x(t)) = \partial_{xt}\mathscr{E}(t, x(t)) = 0$. Therefore, by the second statement of Lemma 3.19, the equation (3.11) has a unique solution $-\partial_{xxt}\mathscr{E}(t, x(t))/\partial_{xxx}\mathscr{E}(t, x(t))$. This implies that

$$[\partial_{xxt}\mathscr{E}(t,x)]^2 = \partial_{xtt}\mathscr{E}(t,x)\cdot\partial_{xxx}\mathscr{E}(t,x).$$

However, since $\partial_{xx}\mathscr{E}(t, x(t)) = \partial_{xt}\mathscr{E}(t, x(t)) = 0$ and $\partial_x\mathscr{E}(t, x(t)) \in \{-1, 1\}$ (by Lemma 3.13), we obtain a contradiction to the assumption (3.20). The proof is completed. □

7 Condition for finite jump set

Now we prove Theorem 3.7.

Proof. **Step 1.** Since $x(\cdot)$ is a BV function, we have $L := \sup_{0\le t\le T}|x(t)| < \infty$. For any $t \in [0,T]$, define

$$\mathscr{F}(t) := \{x \in [-L, L] : |\partial_x \mathscr{E}(t,x)| = 1\}.$$

We shall show that there exists $\varepsilon > 0$ independent of t such that if $x, y \in \mathscr{F}(t)$ and $x \neq y$, then $|x - y| \geq \varepsilon$.

Indeed, we assume by contradiction that there exists a sequence $\{t_n\}_{n=1}^{\infty} \subset [0,T]$ and $x_n, y_n \in \mathscr{F}(t_n)$ such that $x_n < y_n$ and $|x_n - y_n| \to 0$. By compactness, after passing to subsequences if necessary, we may assume that $t_n \to t_0$, $x_n \to x_0$ and $y_n \to x_0$. Using the continuity of $\partial_x\mathscr{E}$, we have $|\partial_x\mathscr{E}(t_0, x_0)| = 1$.

On the other hand, since $|\partial_x\mathscr{E}(t_n, x_n)|^2 = 1 = |\partial_x\mathscr{E}(t_n, y_n)|^2$, by applying Rolle's Theorem for the function $z \mapsto |\partial_x\mathscr{E}(t_n, z)|^2$, we can find an element $z_n \in (x_n, y_n)$ such that $\partial_{xx}\mathscr{E}(t_n, z_n) = 0$. Taking $n \to \infty$, we obtain $\partial_{xx}\mathscr{E}(t_0, x_0) = 0$.

Thus $|\partial_x\mathscr{E}(t_0, x_0)| = 1$ and $\partial_{xx}\mathscr{E}(t_0, x_0) = 0$, which contradict (H5). Therefore, there exists $\varepsilon > 0$ independent of t, such that $|x - y| \geq \varepsilon$ for all $x, y \in \mathscr{F}(t)$ and $x \neq y$.

Step 2. We assume that $x(\cdot)$ jumps at t, namely $x(t^-) \neq x(t^+)$, here

$$x(t^-) := \lim_{s\uparrow t} x(s) \quad \text{and} \quad x(t^+) := \lim_{s\downarrow t} x(s).$$

We shall show that $|x(t^-) - x(t^+)| \geq \varepsilon$.

From the weak local stability of $x(\cdot)$, we have $|\partial_x\mathscr{E}(t, x(t^-))| \leq 1$ and $|\partial_x\mathscr{E}(t, x(t^+))| \leq 1$. If $|\partial_x\mathscr{E}(t, x(t^-))| = 1 = |\partial_x\mathscr{E}(t, x(t^-))|$, then by Step 1 we already get $|x(t^-) - x(t^+)| \geq \varepsilon$. Hence, let us assume that

$$\min\{|\partial_x\mathscr{E}(t, x(t^-))|, |\partial_x\mathscr{E}(t, x(t^+))|\} < 1. \tag{3.21}$$

Using the energy-dissipation upper bound, we get

$$|x(t^+) - x(t^-)| \leq \mathscr{E}(t, x(t^-)) - \mathscr{E}(t, x(t^+)) = \left|\int_{x(t^+)}^{x(t^-)} \partial_x\mathscr{E}(t,z)dz\right| \leq \int_I |\partial_x\mathscr{E}(t,z)| \tag{3.22}$$

where I is the closed interval between $x(t^-)$ and $x(t^+)$.

From (3.21) and (3.22), we conclude that there exists y between $x(t^-)$ and $x(t^+)$ such that $|\partial_x\mathscr{E}(t,y)| > 1$. Since $|\partial_x\mathscr{E}(t, x(t^-))| \leq 1 < |\partial_x\mathscr{E}(t,y)|$, there exists z_- between $x(t^-)$ and y such that $|\partial_x\mathscr{E}(t, z_-)| = 1$ (here z_- may be equal to $x(t^-)$). Similarly, there exists z_+ between $x(t^+)$ and y such that $|\partial_x\mathscr{E}(t, z_+)| = 1$ (here z_+ may be equal to $x(t^+)$). Since $z_+ \neq z_-$, we have $|z_+ - z_-| \geq \varepsilon$ by Step 1. Thus $|x(t^+) - x(t^-)| \geq |z_+ - z_-| \geq \varepsilon$.

Step 3. Thus by Step 2, any jump step is not less than ε. Since $x(\cdot)$ is a BV function, it can only have finitely many jumps. □

8 SBV regularity in the vector-valued case

Now we prove Theorem 3.8. The proof of this Theorem is quite similar to Theorem 3.3.

First, we can assume that $x(\cdot)$ is right-continuous thanks to Proposition 1.5. Since $x(\cdot)$ is a BV function on $[0,T]$ which is right-continuous, there is a vector-valued Radon measure μ such that

$$x(t) = \mathrm{c} + \mu((0,t]) \quad \text{for all } t \in [0,T] \text{ and for some vector } c \text{ in } \mathbb{R}^d.$$

By the Lebesgue Decomposition Theorem we can write

$$\mu = f dx + \mu_s$$

where $f \in L^1$ and $\mu_s = \mu|_S$ with

$$S = \left\{ t \in (0,T) \mid \lim_{h \downarrow 0} \frac{|\mu|(t-h,t+h)}{h} = \infty \right\}.$$

Let J be the jump set of $x(\cdot)$. We split μ_s into the Cantor part $\mu|_{S\setminus J}$ and the jump part $\mu|_J$. To show that $x(\cdot)$ is of SBV, we need to prove that $\mu_c = 0$. This fact follows from the following two lemmas.

Lemma 3.22. *For any BV function $x : [0,T] \to \mathbb{R}^d$ which is right-continuous, the set*

$$A := \left\{ t \in (0,T)\setminus J \mid \liminf_{h \to 0} \left| \frac{x(t+h) - x(t)}{h} \right| < \infty \right\}$$

has $|\mu_s|$-measure 0.

Lemma 3.23. *Assume that $x : [0,T] \to \mathbb{R}^d$ has bounded variation and satisfies* (w-LS) *and* (ED-upper)*. If* (H6) *holds, then the set*

$$B := \left\{ t \in (0,T)\setminus J \mid \lim_{h \to 0} \left| \frac{x(t+h) - x(t)}{h} \right| = \infty \right\}$$

is at most countable. Therefore $|\mu_s|(B) = 0$.

Since we can write $(0,T) = A \cup B \cup J$, Lemma 3.22 and 3.23 ensure that $|\mu_s|((0,T)\setminus J) = 0$. This implies $\mu_c = 0$. And the proof of Theorem 3.8 is complete.

The proof of Lemma 3.22 is the same as in the one-dimensional case, cf. Lemma 3.11. Hence, we shall only concentrate on the proof of Lemma 3.23. Our key observation is the following.

Lemma 3.24. *Assume that $x(\cdot)$ satisfies the weak local stability and the energy-dissipation upper bound. Then for all $t \in (0,T)\setminus(N \cup J)$, we have*

$$|\nabla_x \mathscr{E}(t, x(t))| = 1,$$

where N denotes the null set of the derivative of $x(\cdot)$,

$$N := \{ t \in (0,T) \mid x'(t) \text{ exists and } x'(t) = 0 \}.$$

Proof. Since $t \notin N$, we can find a sequence $t_n \to t$ and $t_n \neq t$ such that

$$\liminf_{n\to\infty} \left| \frac{x(t_n) - x(t)}{t_n - t} \right| > 0. \tag{3.23}$$

By passing to a subsequence if necessary, we can further assume that

$$\lim_{n\to\infty} \frac{x(t_n) - x(t)}{|x(t_n) - x(t)|} = b \text{ in } \mathbb{R}^d \text{ with } |b| = 1. \tag{3.24}$$

Case 1. Assume that $t_n \downarrow t$. From the energy-dissipation upper bound, one has

$$\mathscr{E}(t_n, x(t_n)) - \mathscr{E}(t, x(t)) \le \int_t^{t_n} \partial_t \mathscr{E}(s, x(s))ds - \mathscr{D}iss(x(\cdot); [t, t_n]).$$

Using Taylor's expansion on the left-hand side, the continuity of $s \mapsto \partial_x \mathscr{E}(s, x(s))$ at $s = t$ on the right-hand side, we obtain

$$\begin{aligned} & \partial_t \mathscr{E}(t, x(t)) \cdot (t_n - t) + \nabla_x \mathscr{E}(t, x(t)) \cdot (x(t_n) - x(t)) + o(|x(t_n) - x(t)|) + o(t_n - t) \\ \le \; & \partial_t \mathscr{E}(t, x(t)) \cdot (t_n - t) + o(t_n - t) - \mathscr{D}iss(x(\cdot); [t, t_n]). \end{aligned}$$

Dividing this inequality by $|x(t_n) - x(t)|$ and using (3.23) and (3.24), we obtain

$$\nabla_x \mathscr{E}(t, x(t)) \cdot \frac{x(t_n) - x(t)}{|x(t_n) - x(t)|} \le -\frac{\mathscr{D}iss(x(\cdot); [t, t_n])}{|x(t_n) - x(t)|} + o(1) \le -1 + o(1).$$

Hence

$$\nabla_x \mathscr{E}(t, x(t)) \cdot b \le -1. \tag{3.25}$$

On the other hand, due to the Schwarz inequality and the weak local stability of $x(\cdot)$, we have

$$-\nabla_x \mathscr{E}(t, x(t)) \cdot b \le |\nabla_x \mathscr{E}(t, x(t)) \cdot b| \le |\nabla_x \mathscr{E}(t, x(t))| \cdot |b| \le 1.$$

Or equivalently,

$$\nabla_x \mathscr{E}(t, x(t)) \cdot b \ge -1. \tag{3.26}$$

Thus from the inequalities (3.25) and (3.26), we conclude that $\nabla_x \mathscr{E}(t, x(t)) = -b$. In particular, $|\nabla_x \mathscr{E}(t, x(t))| = 1$.

Case 2. Assume that $t_n \uparrow t$. From the energy-dissipation upper bound, one has

$$\mathscr{E}(t_n, x(t_n)) - \mathscr{E}(t, x(t)) \ge \int_t^{t_n} \partial_t \mathscr{E}(s, x(s))ds + \mathscr{D}iss(x(\cdot); [t_n, t]).$$

Following the above proof, we obtain

$$\nabla_x \mathscr{E}(t, x(t)) \cdot \frac{x(t_n) - x(t)}{|x(t_n) - x(t)|} \ge \frac{\mathscr{D}iss(x(\cdot); [t_n, t])}{|x(t_n) - x(t)|} + o(1) \ge 1 + o(1).$$

This also implies that $\nabla_x \mathscr{E}(t, x(t)) = b$. In particular, $|\nabla_x \mathscr{E}(t, x(t))| = 1$. □

Remark. From the proof, we can see that if $t \notin N \cup J$, then

$$\lim_{s \downarrow t} \frac{x(s) - x(t)}{|x(s) - x(t)|} = -\nabla_x \mathscr{E}(t, x(t)).$$

and

$$\lim_{s \uparrow t} \frac{x(s) - x(t)}{|x(s) - x(t)|} = \nabla_x \mathscr{E}(t, x(t)).$$

Moreover, if the limit

$$\lim_{n \to \infty} \frac{x(t_n) - x(t)}{t_n - t} = v \quad \text{in } \mathbb{R}^d$$

exists then $v = -c\nabla_x \mathscr{E}(t, x(t))$ for some number $c > 0$.

Proof of Lemma 3.23. Denote the "bad set"

$$E := \{t \in (0, T) \mid |\nabla_x \mathscr{E}(t, x(t))| = 1, F(t, x(t)) = 0\},$$

where

$$F(t, x) := (\nabla_x \mathscr{E}(t, x)) \cdot H(t, x) \cdot (\nabla_x \mathscr{E}(t, x))^T$$

and

$$[H(t, x)]_{ij} := (\partial_{x_i} \partial_{x_j} \mathscr{E})(t, x).$$

We distinguish the following cases.

1. If $t \in N \cup J$, then $t \notin B$.

2. If $t \in (0, T) \backslash (N \cup J)$ and t is an isolated point of $(0, T) \backslash (N \cup J)$, then we can ignore it, since the set of the isolated points is at most countable.

3. Thus it remains to consider the case when t is an accumulation point of $(0, T) \backslash (N \cup J)$.

We show that in this case if $t \in B$, then $t \in E$. In fact, since t is an accumulation point of $(0, T) \backslash (N \cup J)$ we can find a sequence t_n in $(0, T) \backslash (N \cup J)$, $t_n \neq t$, and $t_n \to t$. Because $|\nabla_x \mathscr{E}(s, x(s))|^2 = 1$ for all $s \notin N \cup J$, using Taylor's expansion we can write

$$\begin{aligned} 0 &= |\nabla_x \mathscr{E}(t_n, x(t_n))|^2 - |\nabla_x \mathscr{E}(t, x(t))|^2 \\ &= 2\nabla_x \mathscr{E}(t, x(t)) \cdot \nabla_x \partial_t \mathscr{E}(t, x(t)) \cdot (t_n - t) + 2\nabla_x \mathscr{E}(t, x(t)) \cdot H(t, x(t)) \cdot (x(t_n) - x(t))^T \\ &\quad + o(|x(t_n) - x(t)|) + o(|t_n - t|). \end{aligned}$$

Next, we divide the latter equation by $|x(t_n) - x(t)|$ and take the limit as $t_n \to t$. Since

$$\left| \frac{x(t_n) - x(t)}{t_n - t} \right| \to \infty \quad \text{and} \quad \frac{x(t_n) - x(t)}{|x(t_n) - x(t)|} \to \pm \nabla_x \mathscr{E}(t, x(t)),$$

we obtain

$$\nabla_x \mathscr{E}(t, x(t)) \cdot H(t, x(t)) \cdot (\nabla_x \mathscr{E}(t, x(t)))^T = 0,$$

which implies $t \in E$.

4. If t is an accumulation point of $(0, T) \backslash (N \cup J)$ and t is an isolated point of E, then this point can be ignored since there are at most countably many such points.

5. If t is an accumulation point of $(0,T)\backslash(N\cup J)$, t is an accumulation point of E and

$$(\nabla_x \mathscr{E}(t,x(t)))\cdot(\nabla_x F(t,x(t)))=0,$$

then by (H6) we know that there are at most countably many such points.

6. If t is an accumulation point of $(0,T)\backslash(N\cup J)$, t is an accumulation point of E and

$$(\nabla_x \mathscr{E}(t,x(t)))\cdot(\nabla_x F(t,x(t)))\neq 0.$$

In this case, there exists a sequence $\{t_n\}$ in E converging to t. We prove that $t\notin B$.

Assume by contradiction that $t\in B$. Since both t_n and t belong to E, we have

$$\nabla_x\mathscr{E}(t,x(t))\cdot H(t,x(t))\cdot(\nabla_x\mathscr{E}(t,x(t)))^T=0=\nabla_x\mathscr{E}(t_n,x(t_n))\cdot H(t_n,x(t_n))\cdot(\nabla_x\mathscr{E}(t_n,x(t_n)))^T.$$

Or equivalently

$$\begin{aligned}
&\sum_{i=1}^d(\partial_{x_i}\mathscr{E}(t,x(t)))^2\cdot(\partial_{x_ix_i}\mathscr{E}(t,x(t)))+2\sum_{i,j=1}^d\partial_{x_i}\mathscr{E}(t,x(t))\cdot\partial_{x_j}\mathscr{E}(t,x(t))\cdot\partial_{x_ix_j}\mathscr{E}(t,x(t))\\
=\;&\sum_{i=1}^d(\partial_{x_i}\mathscr{E}(t_n,x(t_n)))^2\cdot(\partial_{x_ix_i}\mathscr{E}(t_n,x(t_n)))+2\sum_{i,j=1}^d\partial_{x_i}\mathscr{E}(t_n,x(t_n))\cdot\partial_{x_j}\mathscr{E}(t_n,x(t_n))\cdot\partial_{x_ix_j}\mathscr{E}(t_n,x(t_n))\\
=\;&0.
\end{aligned}$$

Substracting the right-hand side to the left-hand side, and then using Taylor's expansion we get

$$\begin{aligned}
&\sum_{i=1}^d(\partial_{x_i}(t_n)+\partial_{x_i})\cdot\left((t_n-t)\cdot\partial_{x_it}+\sum_{k=1}^d(x_k(t_n)-x_k(t))\cdot\partial_{x_ix_k}\right)\cdot\partial_{x_ix_i}(t_n)\\
+\;&\sum_{i=1}^d(\partial_{x_i})^2\cdot\left((t_n-t)\cdot\partial_{x_ix_it}+\sum_{k=1}^d(x_k(t_n)-x_k(t))\cdot\partial_{x_ix_ix_k}\right)\\
+\;&2\sum_{i,j=1}^d\left((t_n-t)\cdot\partial_{x_it}+\sum_{k=1}^d(x_k(t_n)-x_k(t))\cdot\partial_{x_ix_k}\right)\cdot\partial_{x_j}(t_n)\cdot\partial_{x_ix_j}(t_n)\\
+\;&2\sum_{i,j=1}^d\partial_{x_i}\cdot\left((t_n-t)\cdot\partial_{x_jt}+\sum_{k=1}^d(x_k(t_n)-x_k(t))\cdot\partial_{x_jx_k}\right)\cdot\partial_{x_ix_j}(t_n)\\
+\;&2\sum_{i,j=1}^d\partial_{x_i}\cdot\partial_{x_j}\cdot\left((t_n-t)\cdot\partial_{x_ix_jt}+\sum_{k=1}^d(x_k(t_n)-x_k(t))\cdot\partial_{x_ix_jx_k}\right)\\
=\;&0
\end{aligned}$$

where ∂_{x_i} means $\partial_{x_i}\mathscr{E}(t,x(t))$ and $\partial_{x_i}(t_n)$ means $\partial_{x_i}\mathscr{E}(t_n,x(t_n))$. Dividing this equality by $|x(t_n)-x(t)|$, taking $t_n\to t$, and using

$$\frac{x(t_n)-x(t)}{|x(t_n)-x(t)|}\to\pm\nabla_x\mathscr{E}(t,x(t)),$$

and

$$\left|\frac{x(t_n)-x(t)}{t_n-t}\right| \to \infty$$

we arrive at

$$\begin{aligned}
&\sum_{k=1}^{d}\partial_{x_k}\cdot\left(2\sum_{i=1}^{d}\partial_{x_ix_k}\cdot\partial_{x_ix_i}\cdot\partial_{x_i}\right)+\sum_{k=1}^{d}\partial_{x_k}\cdot\left(\sum_{i=1}^{d}\partial_{x_ix_ix_k}\cdot(\partial_{x_i})^2\right)\\
+\;&\sum_{k=1}^{d}\partial_{x_k}\cdot\left(2\sum_{i,j=1}^{d}\partial_{x_ix_k}\cdot\partial_{x_j}\cdot\partial_{x_ix_j}\right)+\sum_{k=1}^{d}\partial_{x_k}\cdot\left(2\sum_{i,j=1}^{d}\partial_{x_i}\cdot\partial_{x_jx_k}\cdot\partial_{x_ix_j}\right)\\
+\;&\sum_{k=1}^{d}\partial_{x_k}\cdot\left(2\sum_{i,j=1}^{d}\partial_{x_i}\cdot\partial_{x_j}\cdot\partial_{x_ix_jx_k}\right)=0.
\end{aligned}$$

This is equivalent to

$$\nabla_x\mathscr{E}(t,x(t))\cdot(\nabla_xF(t,x(t)))=0.$$

Thus $t \in E$ and $\nabla_x\mathscr{E}(t,x(t))\cdot(\nabla_xF(t,x(t)))=0$, which is a contradiction. Therefore, we must have $t \notin B$.

Conclusion: The set B is a subset of the union of the following three sets: isolated points of $[(0,T)\backslash(N\cup J)]$, isolated points of E, and the set

$$G := \{t\in(0,T)\mid |\nabla_x\mathscr{E}(t,x(t))|=1, F(t,x(t))=(\nabla_x\mathscr{E}(t,x(t)))\cdot(\nabla_xF(t,x(t)))=0\}.$$

Consequently, B is at most countable. This completes the proof of Lemma 3.23 in the d-dimensional case. □

Chapter 4

Examples

1 Example 4.1

Example 4.1. Consider the system defined by the energy functional $\mathscr{E}(t,x) := t\,(x^6 - x^4) - |x|$, where $t \in [0,1]$ and $x \in \mathbb{R}$, the dissipation function $\Psi(x) := |x|$, and the initial value $x_0 := 0$. We have the following claims.

(i) The energetic solutions constructed by time-discretization are

$$x(0) = 0,\, x(t) = \sqrt{2/3} \text{ for all } t \in (0,1]$$

and

$$x(0) = 0,\, x(t) = -\sqrt{2/3} \text{ for all } t \in (0,1].$$

These energetic solutions have "reasonable" jumps and they also satisfy the definition of BV solutions.

(ii) The BV solution corresponding to the viscous dissipation $\Psi_\varepsilon(x) = |x| + \frac{\varepsilon}{2}x^2$ is

$$x(t) = 0 \;\text{ for all } t \in [0,1].$$

This solution does not satisfy the strong local stability (s-LS).

Recall that $t \mapsto x(t)$ satisfies strong local stability (s-LS) if

$$x(t) \text{ is local minimizer of } x \mapsto \mathscr{E}(t,x) + |x - x(t)| \text{ at every continuity point.}$$

(iii) The BV solutions corresponding to the viscous dissipation $\Psi_\varepsilon(x) = |x| + \varepsilon^5 x^6$ with $\varepsilon^{-25/18}\tau \to \infty$ (where τ is the time step in the discretization) are precisely the energetic solutions. Thus the BV solutions obtained by vanishing viscosity depend on the choice of the viscosity.

(iv) The epsilon-neighborhood solutions x^ε are independent of ε when $\varepsilon \in (0, \sqrt{2/3}]$. The BV solutions constructed by epsilon-neighborhood are precisely the energetic solutions.

Proof. **1. Energetic solutions.** We show that the energetic solutions constructed by time-discretization are

$$x(0) = 0,\, x(t) = \sqrt{2/3} \text{ for all } t \in (0,1]$$

and

$$x(0) = 0,\ x(t) = -\sqrt{2/3} \text{ for all } t \in (0,1].$$

Fix a small time step $\tau > 0$ and consider a partition $\{t_i\}_{i=0}^N$ of $[0,1]$ such that $0 = t_0 < \cdots < t_N \leq 1$ and $t_i - t_{i-1} = \tau$ for all $i = 1, 2, \ldots, N$, here $N \in \mathbb{N}$ satisfies $1 \in [\tau N, \tau(N+1))$. To find the discretized solution $x^\tau(t)$ when $t \in [0,1]$, it suffices to compute $x_i := x^\tau(t_i)$.

Step 1: Given $x_0 = 0$, we calculate x_1. By definition, $x_1 := x^\tau(t_1)$ minimizes the functional

$$F_1(x) := \mathscr{E}(t_1, x) + |x| = t_1(x^6 - x^4)$$

over $x \in \mathbb{R}$.

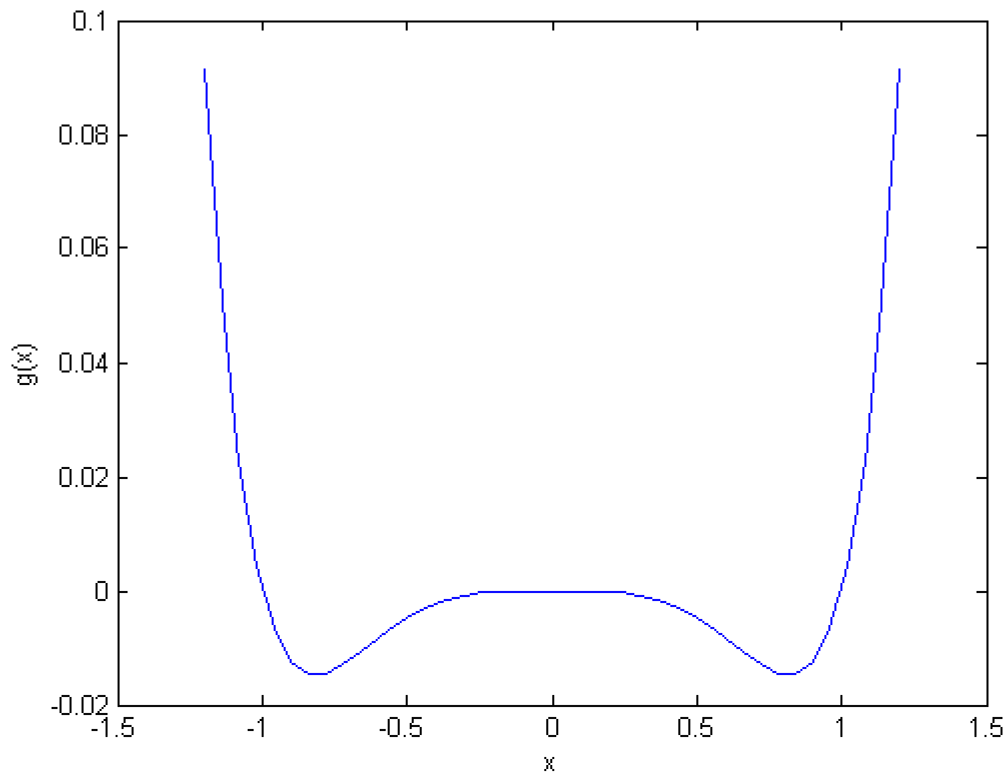

Figure 1. $\mathscr{E}(t, x) + |x|$ with $t = 0.1$.

A direct computation shows that F_1 attains its minimum at $x = \pm\sqrt{2/3}$. Thus, we can choose either $x_1 = \sqrt{2/3}$ or $x_1 = -\sqrt{2/3}$.

Step 2: Given $x_1 = \pm\sqrt{2/3}$, we calculate x_2. Let us consider the case $x_1 = \sqrt{2/3}$ since the other case can be treated in the same way. By definition, x_2 minimizes the functional

$$F_2(x) := \mathscr{E}(t_2, x) + \left|x - \sqrt{2/3}\right| = t_2(x^6 - x^4) - |x| + \left|x - \sqrt{2/3}\right|$$

over $x \in \mathbb{R}$. If $x < 0$, then $F_2(x) > F_2(-x)$. Therefore, it suffices to consider when $x \geq 0$. We distinguish two cases.

Case 1: If $x \geq \sqrt{2/3}$ then

$$F_2(x) = t_2(x^6 - x^4) - \sqrt{2/3}.$$

Thus the unique minimizer for F_2 when $x \geq \sqrt{2/3}$ is $\sqrt{2/3}$.

Case 2: If $0 \leq x < \sqrt{2/3}$ then

$$\begin{aligned} F_2(x) &= t_2(x^6 - x^4) - x + \sqrt{2/3} - x \\ &> t_2(x^6 - x^4) - \sqrt{2/3} \geq F_2\left(\sqrt{2/3}\right). \end{aligned}$$

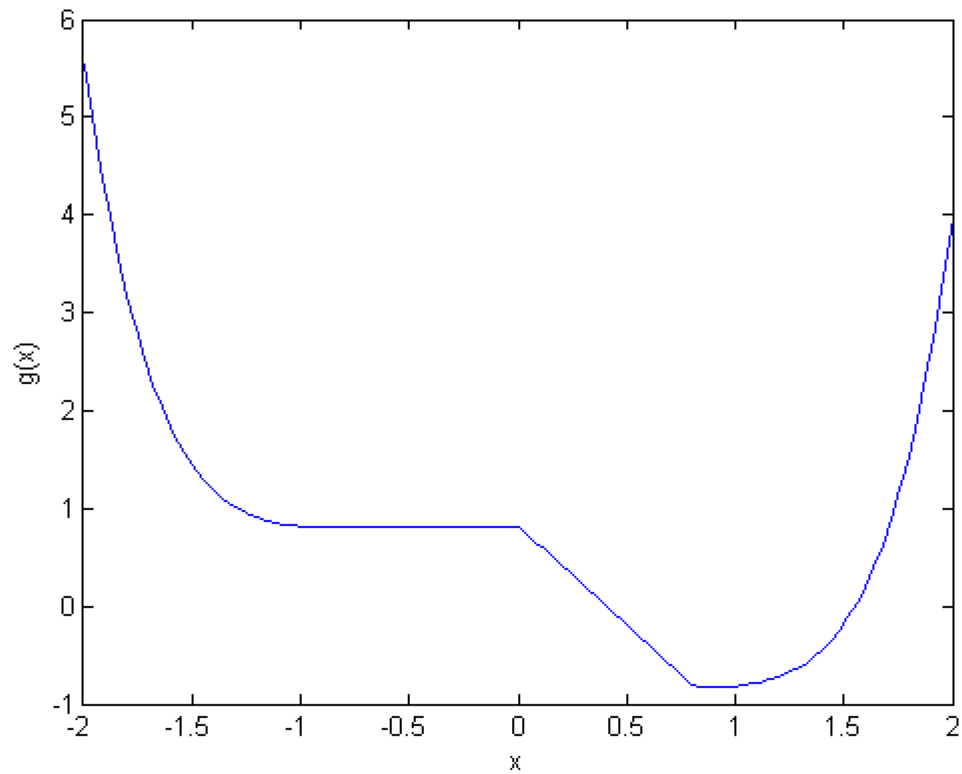

Figure 2. $\mathscr{E}(t,x) + |x - \sqrt{2/3}|$ with $t = 0.1$.

We can conclude that the unique minimizer for F_2 is $x_2 = \sqrt{2/3}$. Thus if $x_1 = \sqrt{2/3}$, then $x_2 = x_1$. Similarly, if $x_1 = -\sqrt{2/3}$, we also have $x_2 = x_1$.

Step 3. By the same way, we have $x_i = x_1$ for all $i = 1, 2, ..., N$. Thus the discretized solution x^τ is either

$$x^\tau(t) = 0 \text{ when } t < \tau \text{ and } x^\tau(t) = \sqrt{2/3} \text{ when } t \in [\tau, 1],$$

or

$$x^\tau(t) = 0 \text{ when } t < \tau \text{ and } x^\tau(t) = -\sqrt{2/3} \text{ when } t \in [\tau, 1].$$

Step 4. Taking the limit of the sequence $x^\tau(t)$ when $\tau \to 0$, we see that the energetic solution is either

$$x(0) = 0,\ x(t) = \sqrt{2/3} \text{ for all } t \in (0,1],$$

or

$$x(0) = 0,\ x(t) = -\sqrt{2/3} \text{ for all } t \in (0,1].$$

2. The energetic solutions are BV solutions. We need to verify that the energetic solutions satisfy the new energy dissipation balance

$$\mathscr{E}(t, x(t)) - \mathscr{E}(0, x(0)) = \int_0^t \partial_t \mathscr{E}(s, x(s))ds - \mathscr{D}iss_{new}(x; [0,t]) \tag{4.1}$$

for all $t \in (0,1]$. For example, we consider the solution

$$x(0) = 0,\ x(t) = \sqrt{\frac{2}{3}} \text{ for all } t \in (0,1].$$

A direct computation gives us

$$\mathscr{E}(t, x(t)) - \mathscr{E}(0, x(0)) = -\frac{4}{27}t - \sqrt{\frac{2}{3}}$$

and

$$\int_0^t \partial_t \mathscr{E}(s, x(s))ds = \int_0^t (x(s)^6 - x(s)^4)ds = \left(\left(\sqrt{\frac{2}{3}}\right)^6 - \left(\sqrt{\frac{2}{3}}\right)^4\right)t = -\frac{4}{27}t.$$

Moreover,

$$\begin{aligned}\mathscr{D}iss_{new}(x; [0,t]) &= \mathscr{D}iss(x; [0,t]) - \Delta(0, x(0), x(0^+)) + \Delta_{new}(0, x(0), x(0^+)) \\ &= \sqrt{2/3} - \sqrt{2/3} + \int_0^{\sqrt{2/3}} \max\{1, |-1|\}dy = \sqrt{2/3}.\end{aligned}$$

Thus (4.1) holds true.

3. The BV solution for $\Psi_\varepsilon(x) = |x| + \frac{\varepsilon}{2}x^2$ is $x(t) = 0$ for all $t \in [0,1]$.

First we find the discretized solutions. Fix $\varepsilon > 0$ and $\tau > 0$ such that $\varepsilon/\tau > 2$. To compute the discretized solution $x^{\tau,\varepsilon}(t)$, we shall calculate $x_i := x^{\tau,\varepsilon}(t_i)$ with $t_0 = 0$ and $t_i = i/N$ for $i = 1, \dots, N$. Here $N \in \mathbb{N}$ is such that $1 \in [\tau N, \tau(N+1))$. We will show that the incremental problem

$$x_i \in \operatorname{argmin}_{x \in \mathbb{R}} \{\mathscr{E}(t_i, x) + |x - x_{i-1}| + \frac{\varepsilon}{2\tau}|x - x_{i-1}|^2\}$$

admits the unique minimizer $x_i = 0$ for every $i = 1, \dots, N$.

Let us start by considering

$$x_1 \in \text{argmin}_{x\in\mathbb{R}} F(x)$$

where

$$F(x) := \mathscr{E}(t_1, x) + |x| + \frac{\varepsilon}{2\tau}|x|^2 = t_1(x^6 - x^4) + ex^2$$

and $e := \frac{\varepsilon}{2\tau}$. Since $e > 1 \geq t_1$, we have

$$F(x) = t_1(x^6 - x^4 + x^2) + (e - t_1)x^2 \geq 0$$

for all $x \in \mathbb{R}$ and $x = 0$ is the unique minimizer for F. Thus, $x_1 = 0$.

In the above argument, we can replace t_1 by any $t_i \leq 1 < e$. Therefore, by induction, we obtain easily that $x_i = 0$ for all $i = 2, \ldots, N$. Thus, for every $\varepsilon > 0$, for every $\tau > 0$ such that $\varepsilon/\tau > 2$, the unique discretized solution is defined by $x^{\tau,\varepsilon}(t) = 0$ for all $t \in [0, 1]$.

By taking the limit when $\tau \to 0$ and $\varepsilon/\tau \to \infty$, we get the BV solution $x(t) = 0$ for all $t \in [0, 1]$.

4. The BV solution $x(t) = 0$ does not satisfy the strong local stability (s-LS). Since the BV solution $x(t) = 0$ is not a local minimizer for the functional $\mathscr{E}(t, x) + |x| = t(x^6 - x^4)$ for $t > 0$, we see that it does not satisfy the strong local stability (s-LS) for any $t > 0$. In fact, $x = 0$ is a local maximizer for the functional $\mathscr{E}(t, x) + |x| = t(x^6 - x^4)$ when $t > 0$ (see Figure 1).

5. The BV solutions corresponding to $\Psi_\varepsilon(x) = |x| + \varepsilon^5 x^6$, with $\varepsilon^{-25/18}\tau \to \infty$, are energetic solutions.

Step 1. We start with the discretized solutions. Let $\varepsilon > 0$ and $\tau > 0$ such that $\tau \to 0$ and $\varepsilon^{-25/18}\tau \to \infty$. To compute the discretized solution $x^{\tau,\varepsilon}(t)$, it suffices to calculate $x_i := x^{\tau,\varepsilon}(t_i)$ with $t_0 = 0$ and $t_i = i/N$ for $i = 1, \ldots, N$. Here $N \in \mathbb{N}$ satisfies $1 \in [\tau N, \tau(N + 1))$.

The sequence $\{x_i\}_{i=0}^N$ solves the incremental problem

$$x_i \in \text{argmin}_{x\in\mathbb{R}}\{\mathscr{E}(t_i, x) + |x - x_{i-1}| + \frac{\varepsilon^5}{\tau^5}|x - x_{i-1}|^6\}$$

when $i = 1, \ldots, N$.

When $i = 1$, since $x_0 = 0$, x_1 is a minimizer for

$$F_1(x) \quad := \mathscr{E}(t_1, x) + |x| + \tfrac{\varepsilon^5}{\tau^5}x^6 = (\tau + e^5)x^6 - \tau x^4.$$

with $e := \varepsilon/\tau$. A direct computation shows that

$$x_1 = \pm\sqrt{\frac{2\tau}{3(\tau + e^5)}}.$$

In the following, we shall consider the case $x_1 = \sqrt{\frac{2\tau}{3(\tau+e^5)}}$. The other case can be treated in the same way.

Step 2. We show that if $0 < x_1 \le \sqrt{2/3}$, then for all $i = 1, 2, ..., N$

$$0 \le x_{i-1} \le x_i \le \sqrt{2/3}$$

and

$$t_i(6x_i^5 - 4x_i^3) + 6e^5(x_i - x_{i-1})^5 = 0. \tag{4.2}$$

We assume by induction that $x_{i-1} \le \sqrt{2/3}$. Recall that, for any $i = 2, ..., N$, x_i is a minimizer for

$$\begin{aligned} F_2(x) &:= \mathscr{E}(t_i, x) + |x - x_{i-1}| + e^5|x - x_{i-1}|^6 \\ &= t_i(x^6 - x^4) - |x| + |x - x_{i-1}| + e^5|x - x_{i-1}|^6. \end{aligned}$$

We shall show that $x_i \ge x_{i-1}$. If $x_i < 0$, then $F_2(x_i) > F_2(-x_i)$ because $x_{i-1} > 0$, which contradicts the fact that x_i is a minimizer for F_2. Thus $x_i \ge 0$.

Now using $-|x| + |x_i - x_{i-1}| \ge -x_{i-1}$, we obtain

$$F_2(x_i) \ge g(x_i) \quad \text{with } g(x) := t_i(x^6 - x^4) - x_{i-1} + e^5(x - x_{i-1})^6$$

The minimization problem $\inf_{x \ge 0} g(x)$ has a minimizer $y \ge 0$ which satisfies the equation

$$t_i(6y^5 - 4y^3) + 6e^5(y - x_{i-1})^5 = 0.$$

It is easy to see that $x_{i-1} \le y \le \sqrt{2/3}$. In fact, if $y < x_{i-1} \le \sqrt{2/3}$, then both $6y^5 - 4y^3$ and $(y - x_{i-1})^5$ are strictly negative, which is a contradiction. Similarly, if $y > \sqrt{2/3} > x_{i-1}$, then both $6y^5 - 4y^3$ and $(y - x_{i-1})^5$ are strictly positive, which is also a contradiction. Thus $x_{i-1} \le y \le \sqrt{2/3}$.

Therefore, $F_2(x_i) \ge g(x_i) \ge g(y) = F_2(y)$. Therefore, we must have $x_i = y \in [x_{i-1}, \sqrt{2/3}]$ and x_i satisfies the equation (4.2).

Step 3. Fix $\delta > 0$. We shall show that for ε and τ small and $\varepsilon^{-25/18}\tau$ large, we have

$$x_L + \delta \ge \sqrt{2/3},$$

for all $L \ge \delta N$.

We assume by contradiction that $x_L \le \sqrt{2/3} - \delta$. From (4.2) we have

$$x_i - x_{i-1} = \left(\frac{t_i}{6e^5} |6x_i^5 - 4x_i^3| \right)^{1/5}.$$

For any $1 \le i \le L$, one has $x_1 \le x_i \le \sqrt{2/3} - \delta$. Therefore,

$$|6x_i^5 - 4x_i^3| = x_i^3(4 - 6x_i^2) \ge \delta x_1^3.$$

Thus

$$x_i - x_{i-1} \ge \left(\frac{\delta x_1^3 t_i}{6e^5} \right)^{1/5}$$

for all $i = 1, 2, ..., L$. Taking the sum we obtain

$$\begin{aligned} x_L &\geq \sum_{i=1}^{L} \left(\frac{\delta x_1^3 t_i}{6e^5} \right)^{1/5} = \left(\frac{\delta x_1^3 \tau}{6e^5} \right)^{1/5} L^{6/5} L^{-1} \sum_{i=1}^{L} \left(\frac{i}{L} \right)^{1/5} \\ &\geq \frac{1}{2} \left(\frac{\delta x_1^3 \tau}{6e^5} \right)^{1/5} L^{6/5} \int_0^1 s^{1/5} ds \\ &= \frac{1}{2} \cdot \frac{5}{6} \left(\frac{\delta x_1^3 \tau}{6e^5} \right)^{1/5} L^{6/5}. \end{aligned}$$

Here we have used that

$$L^{-1} \sum_{i=1}^{L} \left(\frac{i}{L} \right)^{1/5} \geq \frac{1}{2} \int_0^1 s^{1/5} ds$$

when L is large enough, since

$$\lim_{L \to \infty} L^{-1} \sum_{i=1}^{L} \left(\frac{i}{L} \right)^{1/5} = \int_0^1 s^{1/5} ds.$$

Finally, replacing $L \geq \delta N = \delta/\tau$, $x_1 \geq \frac{1}{2}\sqrt{\tau/e^5}$ and $e = \varepsilon/\tau$, we obtain

$$\sqrt{2/3} \geq x_L \geq C_\delta (\varepsilon^{-25/18} \tau)^{9/5}$$

for a constant $C_\delta > 0$ independent of ε and τ. However, this inequality contradicts the assumption that $\varepsilon^{-25/18}\tau \to \infty$. Thus we must have $x_L + \delta \geq \sqrt{2/3}$ for all $L \geq \delta N$.

Step 4. After passing to the limit as $\tau \to 0$ and $\varepsilon^{-25/18}\tau \to \infty$, we obtain a BV solution $x(\cdot)$ satisfying

$$\sqrt{2/3} - \delta \leq x(t) \leq \sqrt{2/3}$$

for all $1 \geq t > \delta > 0$. Therefore, we get $x(t) = \sqrt{2/3}$ for all $t \in (0, 1]$.

Similarly, if we have $x_1 < 0$ in Step 2, then we get the BV solution $x(t) = -\sqrt{2/3}$ for all $t \in (0, 1]$. The two BV solutions are precisely the energetic solutions obtained before.

Numerical computation. To verify the above theoretical argument, we can compute the discretized solution $x^{\tau,\varepsilon}(t)$ by using Maple-software as follows.

Program.

```
tau := 0.0001; e = 20; N := 1/tau;
f := s * (z^6 - z^4) + (z - a)^6 * e^5;

for i from 1 to N do
t[i] := i * tau;
end do;

x[1] := fsolve(subs(s = t[1], a = 0, diff(f, z)), z = exp(-20)..2);
for i from 2 to N do
x[i] := fsolve(subs(s = t[i], a = x[i - 1], diff(f, z)), z = exp(-20)..2);
end do;

plot([seq([t[i], x[i]], i = 1..N)]);
```

Result.

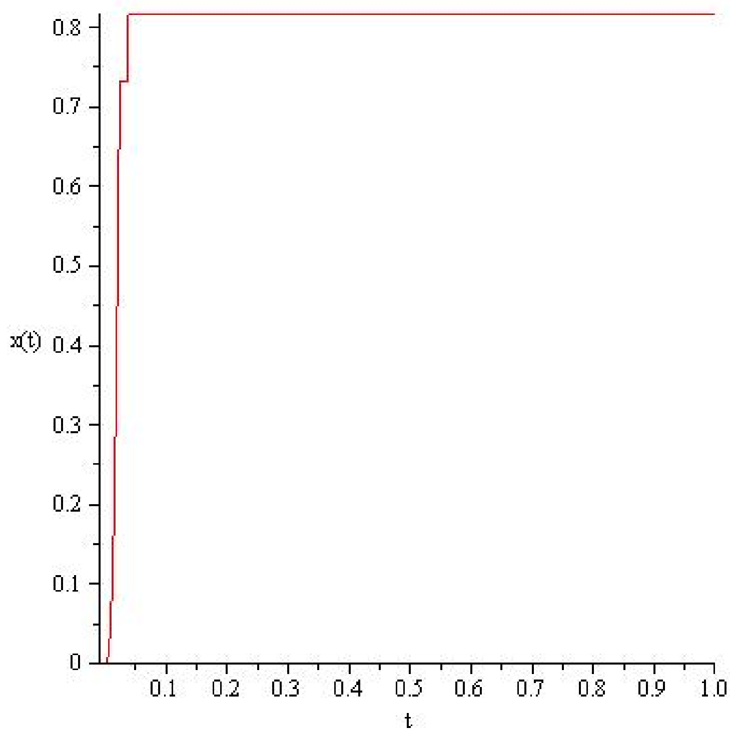

Figure 3. The discretized solution $x^{\tau,\varepsilon}(t)$ solution with viscosity $\varepsilon^5 x^6$ when $\tau = 0.0001$ and $\varepsilon = 0.002$.

6. The BV solutions constructed by epsilon-neighborhood are energetic solutions.

Step 1. Let $\varepsilon > 0$ and $\tau > 0$ be small. Let us compute the discretized solution $x^{\varepsilon,\tau}(t)$.

Denote $x_0 := 0$, it suffices to calculate $x_i := x^{\varepsilon,\tau}(t_i)$ with $t_i = i/N$ for $i = 1, \ldots, N$. Here $N \in \mathbb{N}$ is such that $1 \in [\tau N, \tau(N+1))$.

By definition, for all $i = 1, 2, \ldots, N$, the value x_i is a minimizer for the functional

$$\mathscr{E}(t_i, x) + |x - x_{i-1}|$$

over $x \in [x_{i-1} - \varepsilon, x_{i-1} + \varepsilon]$.

Since $x_0 = 0$, x_1 is a minimizer for the functional

$$F_1(x) = \mathscr{E}(t_1, x) + |x| = t_1(x^6 - x^4)$$

over $x \in [-\varepsilon, \varepsilon]$. It is easy to verify (see Figure 1), if $\varepsilon \le \sqrt{2/3}$, then $x_1 = \pm\varepsilon$.

In the following, we shall concentrate on the case $x_1 = \varepsilon$. The other case can be treated in the same way.

Step 2. We show that if $x_1 > 0$, then

$$x_i = \min\{x_{i-1} + \varepsilon, \sqrt{2/3}\}$$

for all $i = 1, 2, ..., N$. By induction, we can assume that $\varepsilon \le x_{i-1} \le \sqrt{2/3}$. Recall that x_i is a minimizer for

$$F_2(x) := \mathscr{E}(t_i, x) + |x - x_{i-1}| = t_i(x^6 - x^4) - |x| + |x - x_{i-1}|$$

over $[x_{i-1} - \varepsilon, x_{i-1} + \varepsilon]$. Using the inequality $-|x| + |x - x_{i-1}| \ge -x_{i-1}$ and the properties of the function $x^6 - x^4$ (see Figure 1), we have

$$F_2(x_i) \ge t_i(x^6 - x^4) - x_{i-1} \ge t_i(y^6 - y^4) - x_{i-1} = F(y)$$

with $y := \min\{x_{i-1} + \varepsilon, \sqrt{2/3}\}$, for all $x \in [x_{i-1} - \varepsilon, x_{i-1} + \varepsilon]$. Therefore, we must have $x_i = y = \min\{x_{i-1} + \varepsilon, \sqrt{2/3}\}$.

Step 3. Now we fix $\varepsilon \in (0, \sqrt{2/3}]$ and passing to the limit when $\tau \to \infty$, we see that the limit x^ε of $x^{\varepsilon,\tau}$ is

$$x^\varepsilon(0) = 0,\ x^\varepsilon(t) = \sqrt{2/3} \text{ for all } t \in (0, 1].$$

Similarly, if in Step 2 we assume $x_1 < 0$, then we get the epsilon-neighborhood solution

$$x^\varepsilon(0) = 0,\ x^\varepsilon(t) = -\sqrt{2/3} \text{ for all } t \in (0, 1].$$

In fact, these limits are independent of ε and they are precisely the energetic solutions. Hence, when we take the limit when $\varepsilon \to 0$, we get the same functions. □

2 Example 4.2

Example 4.2. Consider the system defined by the energy functional

$$\mathscr{E}(t, x) := x^2 - x^4 + 0.3\,x^6 + t\,(1 - x^2) - |x|, \quad t \in [0, 2],\ x \in \mathbb{R},$$

the dissipation function $\Psi(x) := |x|$, and the initial value $x_0 := 0$. We have

(i) The energetic solutions constructed by time-discretization satisfy either

$$x(t) = 0 \quad \text{if} \;\; t < 1/6,\; x(1/6) \in \{0, \sqrt{5/3}\},\; x(t) = \frac{\sqrt{10 + \sqrt{10 + 90t}}}{3} \text{ if } t > 1/6$$

or

$$x(t) = 0 \quad \text{if} \;\; t < 1/6,\; x(1/6) \in \{0, -\sqrt{5/3}\},\; x(t) = -\frac{\sqrt{10 + \sqrt{10 + 90t}}}{3} \text{ if } t > 1/6.$$

These solutions jump at $t = 1/6$. This jump point is not reasonable (the reasonable jump is at $t = 1$). These energetic solutions are not BV solutions.

(ii) The BV solution corresponding to the viscous dissipation $\Psi_\varepsilon(x) = |x| + \varepsilon x^2$ is

$$x(t) = 0 \text{ for all } t \in [0, 2].$$

(iii) The BV solutions constructed by epsilon-neighborhood satisfy either

$$x(t) = 0 \quad \text{if} \;\; t < 1,\; x(t) = \frac{\sqrt{10 + \sqrt{10 + 90t}}}{3} \text{ if } t > 1$$

or

$$x(t) = 0 \quad \text{if} \;\; t < 1,\; x(t) = -\frac{\sqrt{10 + \sqrt{10 + 90t}}}{3} \text{ if } t > 1.$$

The jump point at $t = 1$ is reasonable. These BV solutions do not satisfy the energy-dissipation balance (ED).

Proof. **1. Energetic solutions via time-discretization.**
Step 1. Fix a time step $\tau > 0$. To find the discretized solution $x^\tau(t)$, it suffices to calculate $x_i := x^\tau(t_i)$ where $0 = t_0 < \cdots < t_N \le 1$ and $t_i - t_{i-1} = \tau$ for all $i = 1, 2, \ldots, N$. Here $N \in \mathbb{N}$ satisfies $1 \in [\tau N, \tau(N+1))$.

We have $x_0 = 0$ and for all $i = 1, 2, \ldots, N$, x_i is a minimizer of the functional

$$F_i(x) := \mathscr{E}(t_i, x) + |x - x_{i-1}|$$

over $x \in \mathbb{R}$.

Step 2. Let us fix $t \in (0, 2]$ and consider the functional

$$F(x) := \mathscr{E}(t, x) + |x| = x^2 - x^4 + 0.3\, x^6 + t(1 - x^2), \quad x \in \mathbb{R}.$$

We have

$$F'(x) = x(2 - 2t + 4x^2 + 1.8\, x^4).$$

When $t < 1$, $F(x)$ has five critical points

$$x = 0 \;\; \text{and} \;\; x = \pm\frac{\sqrt{10 \pm \sqrt{10 + 90t}}}{3}.$$

Among these five critical points, there are three local minimizers

$$x = 0 \quad \text{and} \quad x = \pm y(t),$$

where

$$y(t) := \frac{\sqrt{10 + \sqrt{10 + 90t}}}{3}.$$

On the other hand, when $t \geq 1$, $F(x)$ has three critical points $x = 0$ and $x = \pm y(t)$, among those critical points there are two local minimizers $x = \pm y(t)$.

Note that

$$F(\pm y(t)) - F(0) = \frac{1}{243}(10 + \sqrt{10 + 90t})(8 - 18t - \sqrt{10 + 90t}),$$

which is positive if $t < 1/6$ and negative if $t > 1/6$.

Thus we can conclude that if $t < 1/6$, then F has the unique minimizer $x = 0$, and if $1/6 < t \leq 2$, then F has two minimizers at $\pm y(t)$. Moreover, if $t < 1$, then $x = 0$ is a local minimizer for F; and if $t > 1$, then $x = 0$ is a local maximizer for F.

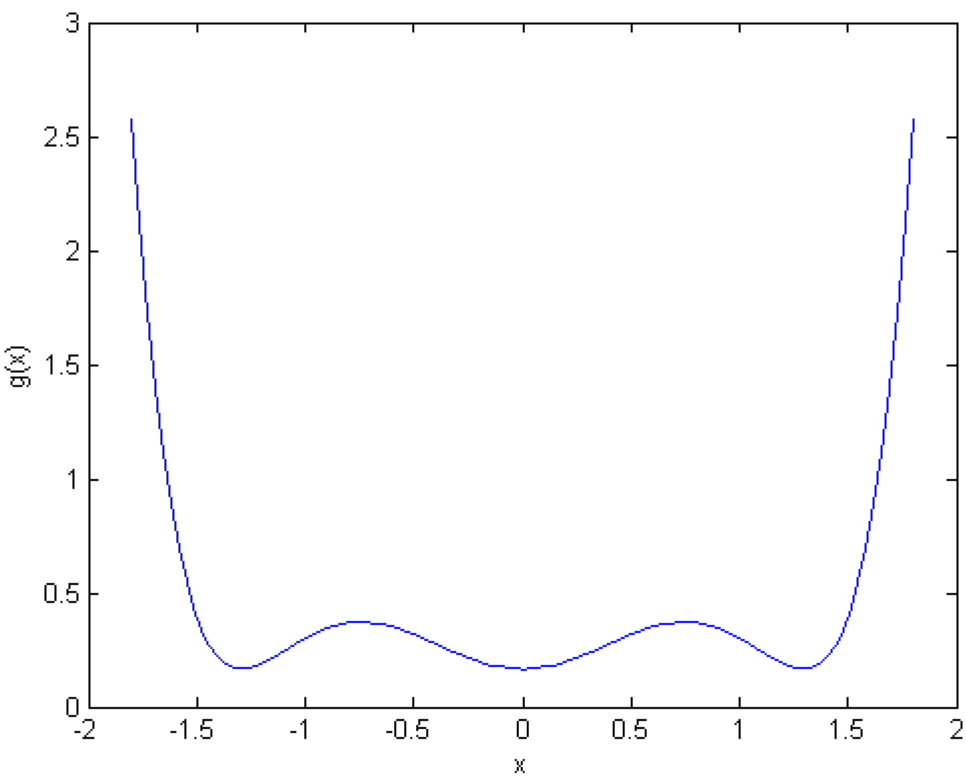

Figure 4. The function $F(x)$ with $t = 1/6$.

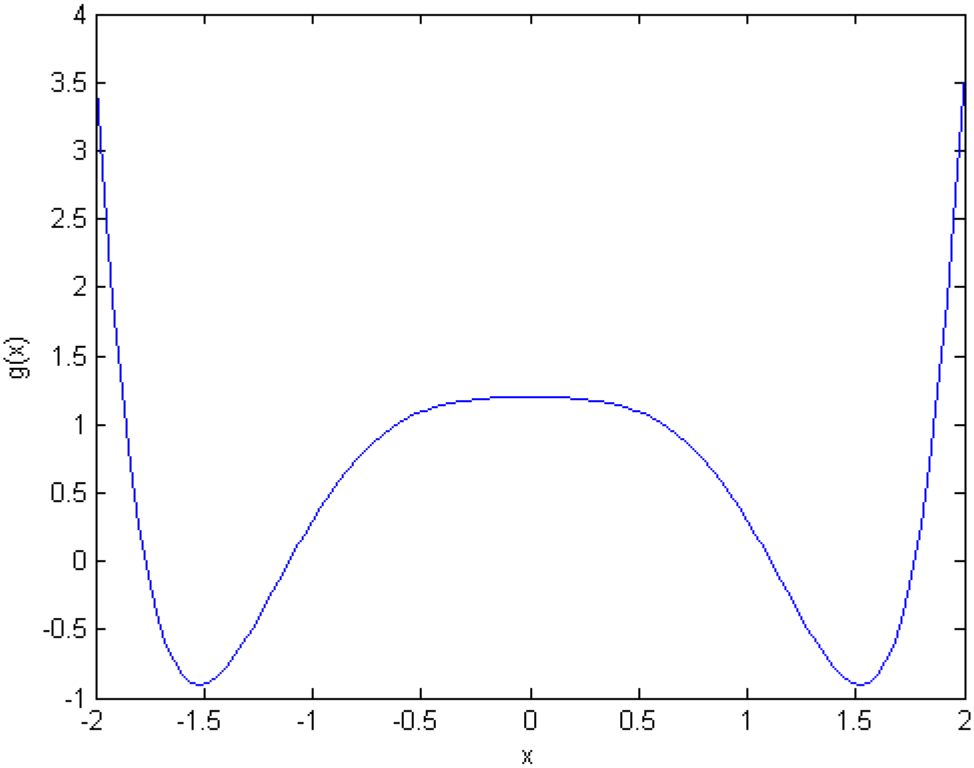

Figure 5. The function $F(x)$ with $t = 1.2$.

Step 3. Recall that x_1 minimizes the functional

$$\mathscr{E}(t_1, x) + |x| = x^2 - x^4 + 0.3\,x^6 + t_1(1 - x^2), \quad x \in \mathbb{R}.$$

From the analysis of $F(x)$ above, we have $x_1 = 0$.

More generally, if $x_{i-1} = 0$, then x_i is a minimizer for the functional

$$\mathscr{E}(t_i, x) + |x| = x^2 - x^4 + 0.3\,x^6 + t_i(1 - x^2), \quad x \in \mathbb{R}.$$

Thus we have $x_i = 0$ if $t_i < 1/6$; $x_i \in \{\pm y(t_i)\}$ if $t_i > 1/6$; and $x_i \in \{0, \pm y(t_i)\} = \left\{0, \pm\sqrt{5/3}\right\}$ if $t_i = 1/6$.

Step 4. Next, we show that if $x_{i-1} = y(t_{i-1}) > 0$, then $x_i = y(t_i)$ (similarly, if $x_{i-1} = -y(t_{i-1}) < 0$, then $x_i = -y(t_i)$). Recall that x_i is a minimizer for the functional

$$F_i(x) := \mathscr{E}(t_i, x) + |x - x_{i-1}| = x^2 - x^4 + 0.3\,x^6 + t_i(1 - x^2) - |x| + |x - x_{i-1}|$$

over $\mathbb{R}$. If $x < 0$, then $F_i(x) > F_i(-x)$ since $x_{i-1} > 0$. Hence $x_i \geq 0$. Using the inequality $-|x| + |x - x_{i-1}| \geq -x_{i-1}$ we have

$$F_i(x) \geq g(x) := x^2 - x^4 + 0.3\,x^6 + t_i(1 - x^2) - x_{i-1}.$$

By the same analysis of F and the fact that $t_i > t_{i-1} \geq 1/6$ (since $x_{i-1} \neq 0$), we can conclude that the minimization problem $\inf_{x \geq 0} g(x)$ has the unique minimizer $y(t_i)$. Moreover,

$g(y(t_i)) = F_i(y(t_i))$. Therefore, we must have $x_i = y(t_i)$.

Step 5. Taking the limit of the sequence $x^\tau(t)$ when $\tau \to 0$, we can see that the energetic solution satisfies either

$$x(t) = 0 \text{ if } t \in [0, 1/6),\ x(1/6) \in \{0, \sqrt{5/3}\},\ x(t) = y(t) \text{ if } t \in [1/6, 2],$$

or

$$x(t) = 0 \text{ if } t \in [0, 1/6),\ x(1/6) \in \{0, -\sqrt{5/3}\},\ x(t) = -y(t) \text{ if } t \in [1/6, 2].$$

Remark. These solutions jump at $t = 1/6$, from $x = 0$ to $x = \pm\sqrt{5/3}$. However, according to Figure 4, we can see that the jump at $t = 1/6$ is not reasonable, since along the jump step there is some moment, the energy plus dissipation is increased.

2. The energetic solutions constructed above are not BV solutions.

Let us consider the jump point $t = 1/6$. We shall show that if $x(\cdot)$ is an energetic solution constructed in Step 1, then at $t = 1/6$,

$$\mathscr{E}(t, x(t^+)) - \mathscr{E}(t, x(t^-)) > -\Delta_{new}(t, x(t^-), x(t^+)).$$

In fact, a direct computation gives us at $t = 1/6$,

$$\mathscr{E}(t, x(t^+)) - \mathscr{E}(t, x(t^-)) = \mathscr{E}(1/6, \sqrt{5/3}) - \mathscr{E}(1/6, 0) = -\sqrt{5/3}.$$

On the other hand, at $t = 1/6$ we have

$$\begin{aligned}\Delta_{new}(t, x(t^-), x(t^+)) &= \int_0^{\sqrt{15}/3} \max\left\{1, \left|\frac{2}{3}y - 4y^3 + 1.8y^5 - 1\right|\right\} dy \\ &= \int_0^{\sqrt{5}/3} 1dy + \int_{\frac{\sqrt{5}}{3}}^{\frac{\sqrt{15}}{3}} \left(-\frac{2}{3}y + 4y^3 - 1.8y^5 + 1\right)dy \\ &= \frac{\sqrt{5}}{3} + \frac{185}{486} + \frac{\sqrt{15}}{3} - \frac{\sqrt{5}}{3} = \frac{185}{486} + \sqrt{5/3}.\end{aligned}$$

Thus, $\mathscr{E}(t, x(t^+)) - \mathscr{E}(t, x(t^-)) > -\Delta_{new}(t, x(t^-), x(t^+))$ at $t = 1/6$.

3. BV solutions corresponding to the viscous dissipation $\Psi_\varepsilon(x) = |x| + \varepsilon x^2$.

We construct the BV solutions via vanishing viscosity with the viscous term εx^2. Let $\varepsilon > 0$ and $\tau > 0$ satisfy $e := \varepsilon/\tau > 2$. For $t \in (0, 2]$, we consider the function

$$F(x) := \mathscr{E}(t, x) + |x| + e|x|^2 = t + (1 + e - t)\,x^2 - x^4 + 0.3\,x^6, \quad x \in \mathbb{R}.$$

Since $1 + e - t \geq 1$, one has

$$F(x) \geq t + x^2 - x^4 + 0.3\,x^6 = t + \frac{1}{6}x^2 + \left(\sqrt{\frac{5}{6}}x - \sqrt{\frac{3}{10}}x^3\right)^2 \geq t = F(0).$$

Thus F has the unique minimizer $x = 0$.

Consequently, the discretized solution $x^{\tau,\varepsilon}$ is identically equal to 0. Therefore, after passing to the limit, this BV solution is also identically equal to 0.

4. BV solutions by epsilon-neighborhood.
Step 1. Let $\varepsilon > 0$ and $\tau > 0$ be small. To compute the discretized solution $x^{\varepsilon,\tau}(t)$, it suffices to calculate $x_i := x^{\varepsilon,\tau}(t_i)$ with $t_0 = 0$ and $t_i = i/N$ for $i = 1, \dots, N$. Here $N \in \mathbb{N}$ is such that $1 \in [\tau N, \tau(N+1))$.

By definition, for all $i = 1, 2, \dots, N$, the value x_i is a minimizer for the functional

$$F_i(x) := \mathscr{E}(t_i, x) + |x - x_{i-1}| = x^2 - x^4 + 0.3\, x^6 + t_i(1 - x^2) - |x| + |x - x_{i-1}|$$

over $x \in [x_{i-1} - \varepsilon, x_{i-1} + \varepsilon]$.

Step 2. In particular, if $x_{i-1} = 0$, then x_i is a minimizer for

$$F_i(x) := x^2 - x^4 + 0.3\, x^6 + t_i(1 - x^2)$$

over $x \in [-\varepsilon, \varepsilon]$.

If $t_i < 1$, then $F_i(x)$ has a local minimizer at $x = 0$ (see Figure 4) and the distance from 0 to the closest critical points of F_i is

$$\frac{\sqrt{10 - \sqrt{10 + 90t_i}}}{3} = \frac{1}{3}\sqrt{\frac{100 - (10 + 90t_i)}{10 + \sqrt{10 + 90t_i}}} \geq \sqrt{\frac{1 - t_i}{2}}.$$

Therefore, if $\varepsilon < \sqrt{(1 - t_i)/2}$, then $x = 0$ is the unique minimizer for $F_i(x)$ on $x \in [-\varepsilon, \varepsilon]$. Hence $x_i = 0$. By induction, we can conclude that if $t_i < 1 - 2\varepsilon^2$, then $x_i = 0$.

Step 3. We show that if $t_i \in [1 - 2\varepsilon^2, 1]$, then $x_i \in [-y(t_i), y(t_i)]$. By induction, we can assume that $x_{i-1} \in [-y(t_{i-1}), y(t_{i-1})]$.

We assume by contradiction that $x_i > y(t_i)$. Because $x_{i-1} \leq y(t_{i-1}) < y(t_i) < x_i \leq x_{i-1} + \varepsilon$, there exists $a \in (y(t_i), x_i) \cap [x_{i-1} - \varepsilon, x_{i-1} + \varepsilon]$. Then using the fact that the function $x \mapsto x^2 - x^4 + 0.3\, x^6 + t_i(1 - x^2)$ is strictly increasing on $[y(t_i), \infty)$ we have

$$\begin{aligned} F_i(x_i) &= x_i^2 - x_i^4 + 0.3x_i^6 + t_i(1 - x_i^2) - x_{i-1} \\ &> a^2 - a^4 + 0.3a^6 + t_i(1 - a^2) - x_{i-1} = F_i(a). \end{aligned}$$

This contradicts the assumption that x_i is a minimizer for $F_i(x)$ over $x \in [x_{i-1} - \varepsilon, x_{i-1} + \varepsilon]$. Thus we must have $x_i \leq y(t_i)$. In the same way we obtain $x_i \geq -y(t_i)$.

Step 4. Now assume that $t_i \in (1, 2]$ and $x_{i-1} \in [-y(t_{i-1}), y(t_{i-1})]$. We show that if $x_{i-1} = 0$, then $x_i \in \{\pm\varepsilon\}$; if $x_{i-1} \in (0, y(t_{i-1})]$, then $x_i = \min\{x_{i-1} + \varepsilon, y(t_i)\}$; and if $x_{i-1} \in [-y(t_{i-1}), 0)$, then $x_i = \max\{x_{i-1} - \varepsilon, -y(t_i)\}$.

Case 1. If $x_{i-1} = 0$, then x_i is a minimizer for

$$F_i(x) = x^2 - x^4 + 0.3\, x^6 + t_i(1 - x^2)$$

over $x \in [-\varepsilon, \varepsilon]$. Since $x = 0$ is a local minimizer for F_i and $x \mapsto F_i(x)$ is a strictly decreasing function on $x \in [-\varepsilon, \varepsilon]$ (see Figure 5), we have $x_i = \pm\varepsilon$.

Case 2. We consider the case when $x_{i-1} \in (0, y(t_{i-1})]$. Recall that x_i is a minimizer for

$$F_i(x) = x^2 - x^4 + 0.3\,x^6 + t_i(1 - x^2) - |x| + |x - x_{i-1}|$$

over $[x_{i-1} - \varepsilon, x_{i-1} + \varepsilon]$. Using the inequality $-|x| + |x - x_{i-1}| \geq -x_{i-1}$, we have

$$F_i(x) \geq g(x) \quad \text{with } g(x) := x^2 - x^4 + 0.3\,x^6 + t_i(1 - x^2) - x_{i-1}.$$

Since in the interval $x \in [x_{i-1} - \varepsilon, x_{i-1} + \varepsilon]$, $g(x)$ has the unique minimizer $\widetilde{x}_i := \min\{x_{i-1} + \varepsilon, y(t_i)\}$ (see Figure 5), then

$$F_i(x_i) \geq g(x_i) \geq g(\widetilde{x}_i) = F_i(\widetilde{x}_i).$$

Thus we can conclude that $x_i = \widetilde{x}_i = \min\{x_{i-1} + \varepsilon, y(t_i)\}$.

Case 3. If $x_{i-1} \in [-y(t_{i-1}), 0)$, then similarly to Case 2, we have $x_i = \max\{x_{i-1} - \varepsilon, -y(t_i)\}$.

Step 5. Taking the limit $\tau \to 0$, we obtain that the epsilon-neighborhood solution $x^\varepsilon(\cdot)$ satisfies $x^\varepsilon(t) = 0$ if $t < 1 - 2\varepsilon^2$ and either $x^\varepsilon(t) = y(t)$ or $x^\varepsilon(t) = -y(t)$ for all $t \in (1, 2]$.

Taking the limit $\varepsilon \to 0$, we obtain that the BV solution constructed by epsilon-neighborhood satisfies that $x(t) = 0$ if $t \in (0, 1)$ and either $x(t) = y(t)$ or $x(t) = -y(t)$ for $t \in (1, 2)$.

Remark. Thus the BV solutions constructed by epsilon-neighborhood jump at $t = 1$, from 0 to $\pm y(1) = \pm\sqrt{20}/3$. This jump is reasonable since $x = 0$ is a local minimizer for the corresponding functional if $t < 1$ (see Figure 4), and $x = 0$ is a local maximizer when $t > 1$ (see Figure 5).

5. The BV solutions constructed by epsilon-neighborhood do not satisfy the energy-dissipation balance (ED).

We consider the solution $x(t) = 0$ if $t \in (0, 1)$ and $x(t) = y(t)$ if $t \in (1, 2)$. The other solution can be treated in the same way.

Indeed, at the jump point $t = 1$, one has

$$\Delta(t, x(t^-), x(t^+)) = |x(t^-) - x(t^+)| = \frac{2\sqrt{5}}{3}.$$

On the other hand, a direct computation gives us at $t = 1$,

$$\mathscr{E}(t, x(t^+)) - \mathscr{E}(t, x(t^-)) = \mathscr{E}(1, \sqrt{20}/3) - \mathscr{E}(1, 0) = -\frac{400}{243} - \frac{\sqrt{20}}{3}.$$

Therefore,

$$\mathscr{E}(t, x(t^+)) - \mathscr{E}(t, x(t^-)) < -\Delta(t, x(t^-), x(t^+)).$$

Thus the solutions $x(\cdot)$ do not satisfy the energy-dissipation balance. $\square$

3 Example 4.3

Example 4.3. Consider the system defined by the energy functional

$$\mathscr{E}(t,x) := t\left(\frac{x}{4} - \frac{3}{4}|x| + |x+1| + |x-2|\right) - |x|,\ t \in [0,1],\ x \in \mathbb{R},$$

the dissipation function $\Psi(x) := |x|$, and the initial value $x_0 := 0$. We have

(i) The energetic solutions constructed by time-discretization are

$$x(0) = 0 \ \text{ and } x(t) = -1 \ \text{ for all } t \in (0,1],$$

and

$$x(0) = 0 \ \text{ and } x(t) = 2 \ \text{ for all } t \in (0,1].$$

The jump point $t = 0$ is reasonable. These energetic solutions are also BV solutions.

(ii) The BV solution corresponding to the viscous dissipation $\Psi_\varepsilon(x) = |x| + \varepsilon x^2$ is

$$x(0) = 0 \ \text{ and } x(t) = -1 \ \text{ for all } t \in (0,1].$$

The BV solution corresponding to

$$\Psi_\varepsilon(x) = \begin{cases} |x| + \varepsilon x^2 \text{ if } x \geq 0, \\ |x| + 4\varepsilon x^2 \text{ if } x \leq 0, \end{cases}$$

satisfies either $x(t) = -1$ for all $t \in (0,1]$, or $x(t) = 2$ for all $t \in (0,1]$.
The BV solution corresponding to

$$\Psi_\varepsilon(x) = \begin{cases} |x| + \varepsilon x^2 \text{ if } x \geq 0, \\ |x| + 5\varepsilon x^2 \text{ if } x \leq 0, \end{cases}$$

satisfies that $x(t) = 2$ for all $t \in (0,1]$.

(iii) The BV solution constructed by epsilon-neighborhood with the usual neighborhood $I_\varepsilon(a) = [a-\varepsilon, a+\varepsilon]$ is

$$x(0) = 0 \ \text{ and } x(t) = -1 \ \text{ for all } t \in (0,1].$$

The BV solution constructed by epsilon-neighborhood with the neighborhood $I_\varepsilon(a) = [a-\varepsilon, a+3\varepsilon]$ is

$$x(0) = 0 \ \text{ and } x(t) = 2 \ \text{ for all } t \in (0,1].$$

The BV solution constructed by epsilon-neighborhood with the neighborhood $I_\varepsilon(a) = [a-\varepsilon, a+2\varepsilon]$ coincide with the two above solutions.

Proof. **1. Energetic solutions.**

Step 1. Taking a small time step $\tau > 0$, we find the discretized solution $x^\tau(t)$. It suffices to calculate $x_i := x^\tau(t_i)$ where $0 = t_0 < \cdots < t_N \leq 1$ and $t_i - t_{i-1} = \tau$ for all $i = 1, 2, \ldots, N$. Here $N \in \mathbb{N}$ satisfies $1 \in [\tau N, \tau(N+1))$.

Recall that for all $i = 1, 2, \ldots, N$, x_i is a minimizer for the functional $\mathscr{E}(t_i, x) + |x - x_{i-1}|$ over $x \in \mathbb{R}$.

Step 2. Since $x_0 = 0$, x_1 is a minimizer for the functional

$$F_1(x) := \mathscr{E}(t_1, x) + |x| = t_1 \left(\frac{x}{4} - \frac{3}{4}|x| + |x+1| + |x-2| \right)$$

over $x \in \mathbb{R}$. A simple computation shows that $x_1 \in \{-1, 2\}$.

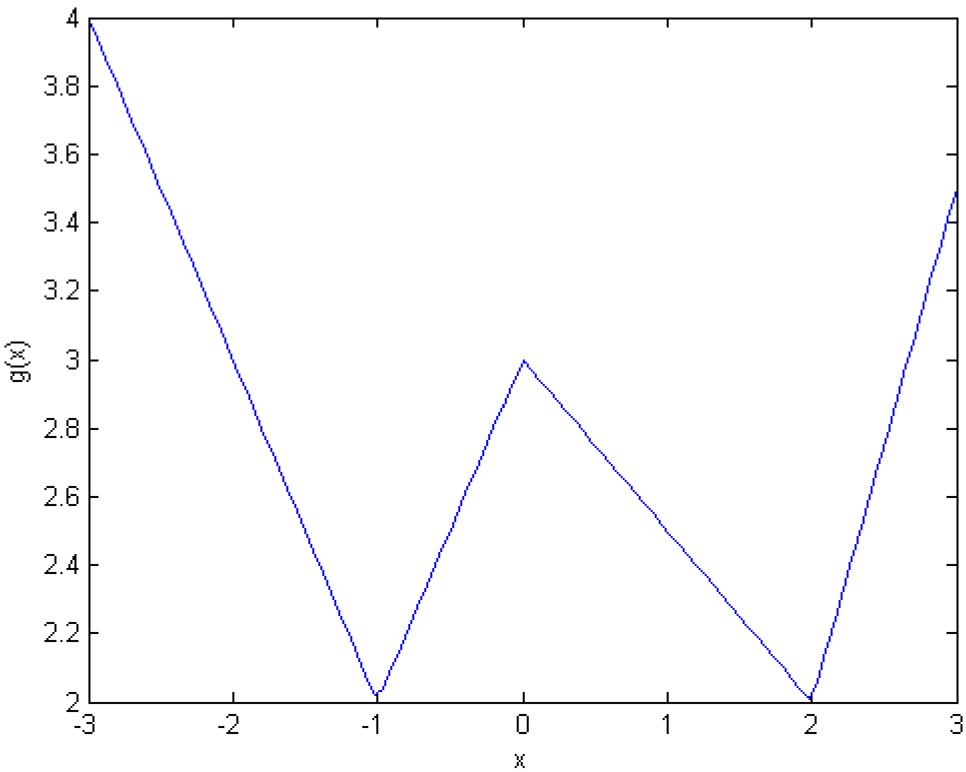

Figure 6. The function $F_1(x)/t_1$.

Step 2. We show that if $x_{i-1} = 2$, then $x_i = 2$. In fact, x_i is a minimizer for the functional

$$F_2(x) := \mathscr{E}(t_i, x) + |x - x_{i-1}| = t_i \left(\frac{x}{4} - \frac{3}{4}|x| + |x+1| + |x-2| \right) - |x| + |x-2|$$

over $x \in \mathbb{R}$. Using the inequality $-|x| + |x-2| \geq -2$ and the properties of the functional F_1 (see Figure 6) we have

$$F_2(x) = \frac{t_i}{t_1} F_1(x) - |x| + |x-2| \geq \frac{t_i}{t_1} F_1(2) - 2 = F_2(2)$$

for all $x \in \mathbb{R}$. Moreover, $x = 2$ is the unique minimizer for F_2 over $\mathbb{R}$. Thus $x_i = 2$.

Similarly, we can show that if $x_{i-1} = -1$, then $x_i = -1$.

Step 3. Taking the limit when $\tau \to \infty$, we see that the energetic solution is either

$$x(0) = 0 \text{ and } x(t) = -1 \quad \text{for all } t \in (0,1],$$

or

$$x(0) = 0 \text{ and } x(t) = 2 \quad \text{for all } t \in (0,1].$$

Remark. From Figure 6, we can see that the jump point $t = 0$ is reasonable.

2. Energetic solutions are also BV solutions.

We consider the case when $x(t) = 2$ for all $t \in (0,1]$, and the other case can be treated in the same way. We need to show that at the jump point $t = 0$ one has

$$\mathscr{E}(t, x(t^+)) - \mathscr{E}(t, x(t^-)) = -\Delta_{new}(t, x(t^-), x(t^+)).$$

By a direct computation at $t = 0$, we have

$$\mathscr{E}(t, x(t^+)) - \mathscr{E}(t, x(t^-)) = \mathscr{E}(0,2) - \mathscr{E}(0,0) = -2$$

and

$$\begin{aligned} \Delta_{new}(t, x(t^-), x(t^+)) &= \int_0^2 \max\{1, |\partial_x \mathscr{E}(t,y)|\} dy \\ &= \int_0^2 \max\{1, |-1|\} dy = 2. \end{aligned}$$

Thus, $\mathscr{E}(t, x(t^+)) - \mathscr{E}(t, x(t^-)) = -\Delta_{new}(t, x(t^-), x(t^+))$ at $t = 0$.

3. BV solutions constructed by vanishing viscosities Ψ_ε.

Let $\varepsilon > 0$ and $\tau > 0$ satisfy $e := \varepsilon/\tau > 1$. We choose $t_i = i/N$ for $i = 0, \dots, N$, where $N \in \mathbb{N}$is such that $1 \in [\tau N, \tau(N+1))$. To compute the discretized solution $x^{\tau,\varepsilon}(t)$, it suffices to calculate $x_i := x^{\tau,\varepsilon}(t_i)$, where $x_0 = 0$ and x_i is a minimizer for the functional

$$F_i(x) \quad := \quad \mathscr{E}(t_i, x) + \tau \Psi_\varepsilon\left(\frac{x - x_{i-1}}{\tau}\right)$$

over $x \in \mathbb{R}$. Then we take the limit when $\varepsilon \to 0$, $\tau \to 0$, and $e := \varepsilon/\tau \to +\infty$.

(a) We consider $\Psi_\varepsilon(x) = |x| + \varepsilon x^2$. We shall show that if $x_{i-1} \in [-1, 0]$, then

$$x_i = \max\left\{-1, x_{i-1} - \frac{t_i}{2e}\right\}$$

for all $i = 0, 1, ..., N$. We distinguish three cases.

Case 1. If $x \in [-1, 0]$ we have

$$\begin{aligned} F_i(x) &= t_i(x+3) - |x| + |x - x_{i-1}| + e(x - x_{i-1})^2 \\ &\geq t_i(x+3) + x_{i-1} + e(x - x_{i-1})^2 \\ &= e\left(x - x_{i-1} + \frac{t_i}{2e}\right)^2 + t_i(x_{i-1} + 3) + x_{i-1} - \frac{t_i^2}{4e} \\ &\geq e\left(\widetilde{x}_i - x_{i-1} + \frac{t_i}{2e}\right)^2 + t_i(x_{i-1} + 3) + x_{i-1} - \frac{t_i^2}{4e} = F_i(\widetilde{x}_i) \end{aligned}$$

where $\widetilde{x}_i := \max\{-1, x_{i-1} - t_i/(2e)\}$. Moreover, the equality holds if and only if $x = \widetilde{x}_i$.

Case 2. If $x < -1$, we have

$$\begin{aligned} F_i(x) &= t_i(1 - x) + x_{i-1} + e(x - x_{i-1})^2 \\ &> 2t_i + x_{i-1} + e(1 - x_{i-1})^2 = F_i(-1). \end{aligned}$$

Case 3. If $x \geq 0$, then

$$\begin{aligned} F_i(x) &= t_i\left(-\frac{x}{2} + |x+1| + |x-2|\right) - x_{i-1} + ex^2 \\ &\geq t_i\left(-\frac{x}{2} + 3\right) t_i - x_{i-1} + e(x - x_{i-1})^2 \\ &= e\left(x - x_{i-1} - \frac{t_i}{4e}\right)^2 + \left(\frac{t_i}{2} - 1\right) x_{i-1} + 3t_i - \frac{t_i^2}{16e} \\ &\geq \left(\frac{t_i}{2} - 1\right) x_{i-1} + 3t_i - \frac{t_i^2}{16e} > F_i(\widetilde{x}_i). \end{aligned}$$

Thus we can conclude that $F_i(x)$ has the unique minimizer $\widetilde{x}_i$. Consequently, $x_i = \widetilde{x}_i = \max\{-1, x_{i-1} - t_i/(2e)\}$.

After taking the limit when $\varepsilon \to 0, \tau \to 0$ such that $e = \varepsilon/\tau \to \infty$, we can see that the BV solution $x(\cdot)$ satisfies that $x(t) = -1$ for all $t \in (0, 1]$.

(b) Now we consider

$$\Psi_\varepsilon(x) = \begin{cases} |x| + \varepsilon x^2 \text{ if } x \geq 0, \\ |x| + 4\varepsilon x^2 \text{ if } x \leq 0, \end{cases}$$

Step 1. We show that $x_1 \in \{-t_1/(8e), t_1/(4e)\}$. Recall that x_1 is a minimizer for

$$F_1(x) = \begin{cases} t_1\left(\frac{x}{4} - \frac{3}{4}|x| + |x+1| + |x-2|\right) + ex^2 \text{ if } x \geq 0, \\ t_1\left(\frac{x}{4} - \frac{3}{4}|x| + |x+1| + |x-2|\right) + 4ex^2 \text{ if } x \leq 0. \end{cases}$$

Similarly to the above argument, we distinguish three cases.

Case 1. If $x \in [-1, 0]$ we have

$$F_1(x) = \left(x + \frac{t_1}{8e}\right)^2 + 3t_1 - \frac{t_1^2}{16e} \geq 3t_1 - \frac{t_1^2}{16e},$$

and we have the equality if and only if $x = -t_1/8e$.

Case 2. If $x < -1$, we have

$$F_1(x) = t_1(1-x) + 4ex^2 > 2t_1 + 5e.$$

Case 3. If $x \geq 0$, then

$$\begin{aligned} F_1(x) &= t_1\left(-\frac{x}{2} + |x+1| + |x-2|\right) + ex^2 \\ &\geq t_1\left(-\frac{x}{2} + 3\right) t_1 + ex^2 \\ &= e\left(x - \frac{t_1}{4e}\right)^2 + 3t_1 - \frac{t_1^2}{16e} \\ &\geq 3t_1 - \frac{t_1^2}{16e} \end{aligned}$$

and the equality occurs when $x = t_1/(4e)$.

Thus x_1 is either $-t_1/(8e)$ or $t_1/(4e)$.

Step 2. By distinguish three cases as above, we can show that if $x_{i-1} \in [-1, -t_1/(8e)]$, then

$$x_i = \max\left\{-1, x_{i-1} - \frac{t_i}{8e}\right\};$$

and if $x_{i-1} \in [t_1/(4e), 2]$, then

$$x_i = \min\left\{2, x_{i-1} + \frac{t_i}{4e}\right\}.$$

Step 3. Taking the limit when $\varepsilon \to 0, \tau \to 0$ such that $e = \varepsilon/\tau \to \infty$, we can see that the BV solution $x(\cdot)$ satisfies either $x(t) = -1$ for all $t \in (0,1]$, or $x(t) = 2$ for all $t \in (0,1]$.

(c) We consider

$$\Psi_\varepsilon(x) = \begin{cases} |x| + \varepsilon x^2 \text{ if } x \geq 0, \\ |x| + 5\varepsilon x^2 \text{ if } x \leq 0, \end{cases}$$

By similar computation, we can show that $x_1 = t_1/(4e)$ and

$$x_i = \min\left\{2, x_{i-1} + \frac{t_i}{4e}\right\}.$$

Hence, by taking the limit when $\varepsilon \to 0, \tau \to 0$ such that $e = \varepsilon/\tau \to \infty$, we can see that the BV solution $x(\cdot)$ satisfies that $x(t) = 2$ for all $t \in (0,1]$.

4. BV solutions constructed by epsilon-neighborhood.

Let $\varepsilon \in (0, 1/2)$ and let $\tau > 0$ be small. Let $t_i = i/N$ for $i = 0, \ldots, N$, where $N \in \mathbb{N}$ is such that $1 \in [\tau N, \tau(N+1))$.

To compute the discretized solution $x^{\varepsilon,\tau}(t)$, it suffices to calculate $x_i := x^{\varepsilon,\tau}(t_i)$. Recall that for all $i = 1, 2, \ldots, N$, the value x_i is a minimizer for the functional $\mathscr{E}(t_i, x) + |x - x_{i-1}|$

over $x \in I_\varepsilon(x_{i-1})$.

(a) Let $I_\varepsilon(a) := [a-\varepsilon, a+\varepsilon]$.
Step 1. Since $x_0 = 0$, x_1 is a minimizer for the functional

$$F_1(x) := \mathscr{E}(t_1, x) + |x| = t_1\left(\frac{x}{4} - \frac{3}{4}|x| + |x+1| + |x-2|\right)$$

over $x \in [-\varepsilon, \varepsilon]$. Note that when $x \in [-1, 2]$, we have $|x+1| + |x-2| = 3$. Therefore

$$F_1(x) = t_1\left(\frac{x}{4} - \frac{3}{4}|x| + 3\right)$$

Hence, a simple comparison shows that $x_1 = -\varepsilon$ (see Figure 6).

Step 2. Next, we shall show that if $x_{i-1} \in [-1, -\varepsilon]$, then

$$x_i = \max\{-1, x_{i-1} - \varepsilon\}.$$

In fact, x_i is a minimizer for the functional

$$F_2(x) := \mathscr{E}(t_i, x) + |x| = t_i\left(\frac{x}{4} - \frac{3}{4}|x| + |x+1| + |x-2|\right) - |x| + |x - x_{i-1}|$$

over $x \in [x_{i-1} - \varepsilon, x_{i-1} + \varepsilon]$. Using the inequality $-|x| + |x - x_{i-1}| \geq -|x_{i-1}| = x_{i-1}$, we obtain

$$F_2(x) \geq g(x) \quad \text{with } g(x) := t_i\left(\frac{x}{4} - \frac{3}{4}|x| + |x+1| + |x-2|\right) + x_{i-1}.$$

Note that when $x \in [x_{i-1} - \varepsilon, x_{i-1} + \varepsilon] \subset (-\infty, 0]$, $g(x)$ has the unique minimizer $\widetilde{x}_i := \max\{-1, x_{i-1} - \varepsilon\}$ (see Figure 6). Moreover,

$$F_2(x_i) \geq g(x_i) \geq g(\widetilde{x}_i) = F_2(\widetilde{x}_i).$$

Therefore, we must have $x_i = \widetilde{x}_i = \max\{-1, x_{i-1} - \varepsilon\}$.

Step 3. Taking the limit $\tau \to 0$, we obtain the epsilon-neighborhood solution

$$x^\varepsilon(0) = 0, \ x^\varepsilon(t) = -1 \ \text{ for all } t \in (0, 1].$$

Since the solution x^ε does not depend on $\varepsilon \in (0, 1/2)$, when taking the limit $\varepsilon \to 0$ we get the same solution.

(b) Let $I_\varepsilon(a) := [a-\varepsilon, a+3\varepsilon]$.
Since $x_0 = 0$, x_1 is a minimizer for the functional

$$F_1(x) := \mathscr{E}(t_1, x) + |x| = t_1\left(\frac{x}{4} - \frac{3}{4}|x| + |x+1| + |x-2|\right)$$

over $x \in [-\varepsilon, 3\varepsilon]$. A simple comparison shows that $x_1 = 3\varepsilon$ (see Figure 6).

By the same argument as in the above proof, we can show that if $x_{i-1} \in [3\varepsilon, 2]$, then

$$x_i = \min\{2, x_{i-1} + 3\varepsilon\}.$$

Therefore, after passing the limit when $\tau \to 0$, we obtain the epsilon-neighborhood solution

$$x^\varepsilon(0) = 0,\ x^\varepsilon(t) = 2 \text{ for all } t \in (0, 1].$$

Taking the limit when $\varepsilon \to 0$, we get the desired BV solution.

(c) Let $I_\varepsilon(a) := [a - \varepsilon, a + 2\varepsilon]$**.**

Since $x_0 = 0$, x_1 is a minimizer for the functional

$$F_1(x) := \mathscr{E}(t_1, x) + |x| = t_1 \left(\frac{x}{4} - \frac{3}{4}|x| + |x + 1| + |x - 2| \right)$$

over $x \in [-\varepsilon, 2\varepsilon]$. A simple comparison shows that $x_1 \in \{-\varepsilon, 2\varepsilon\}$ (see Figure 6).

By the same argument as in the above proof, we can show that if $x_{i-1} \in [-1, -\varepsilon]$, then $x_i = \max\{-1, x_{i-1} - \varepsilon\}$, and if $x_{i-1} \in [2\varepsilon, 2]$, then $x_i = \min\{2, x_{i-1} + 2\varepsilon\}$.

Taking the the limit when $\tau \to 0$, we see that the epsilon-neighborhood solution x^ε satisfies either

$$x^\varepsilon(0) = 0,\ x^\varepsilon(t) = -1 \text{ for all } t \in (0, 1],$$

or

$$x^\varepsilon(0) = 0,\ x^\varepsilon(t) = 2 \text{ for all } t \in (0, 1].$$

Taking the limit when $\varepsilon \to 0$, we get the BV solutions as desired. □

4 Example 4.4

Example 4.4. Consider the system defined by the energy functional

$$\mathscr{E}(t, x) := t \left(\frac{|x| - 1 - |x - 1|}{2} + |x - 2| - 1 \right) - |x|,\ t \in [0, 1],\ x \in \mathbb{R},$$

the dissipation function $\Psi(x) := |x|$, and the initial value $x_0 := 0$. We have

(i) The energetic solution constructed by time-discretization is

$$x(0) = 0 \text{ and } x(t) = 2 \text{ for all } t \in (0, 1].$$

(ii) All BV solutions constructed by epsilon-neighborhood (with the usual neighborhood $I_\varepsilon(a) = [a - \varepsilon, a + \varepsilon]$) satisfy that $x(\cdot)$ is increasing and

$$x(t) \in [0, 1) \text{ if } t \in [0, t_0) \text{ and } x(t) = 2 \text{ for all } t \in (t_0, 1].$$

for an arbitrary $t_0 \in [0, 1]$.

(iii) Now we consider the viscous dissipation of the form $\Psi_\varepsilon(x) := |x| + \varepsilon^{-1}\Psi_0(\varepsilon x)$, where Ψ_0 is convex and satisfies that

$$\lim_{x\to 0}\frac{\Psi_0(x)}{|x|} = 0 \quad \text{and} \quad \lim_{|x|\to\infty}\frac{\Psi_0(x)}{|x|} = \infty$$

If $\Psi_0(x) > 0$ when $x \neq 0$, then the BV solution corresponding to Ψ_ε is $x(t) = 0$ for all $t \in [0,1]$.

On the other hand, if $\Psi_0(x) = 0$ in a neighborhood of $x = 0$, then the BV solutions corresponding to Ψ_ε are precisely all BV solutions constructed by epsilon-neighborhood.

Proof. **1. Energetic solutions.** Take a small time step $\tau > 0$ and let $N \in \mathbb{N}$ satisfy $1 \in [\tau N, \tau(N+1))$. To find the discretized solution $x^\tau(t)$, it suffices to calculate $x_i := x^\tau(t_i)$ where $0 = t_0 < \cdots < t_N \leq 1$. Recall that for all $i = 1, 2, \ldots, N$, x_i is a minimizer for the functional $\mathscr{E}(t_i, x) + |x - x_{i-1}|$ over $x \in \mathbb{R}$.

In particular, since $x_0 = 0$, x_1 is a global minimizer for the functional

$$F_1(x) = t_1\, g(x) \quad \text{with } g(x) := \frac{|x| - 1 - |x-1|}{2} + |x-2| - 1.$$

A simple calculation (see Figure 7) shows that $x_1 = 2$.

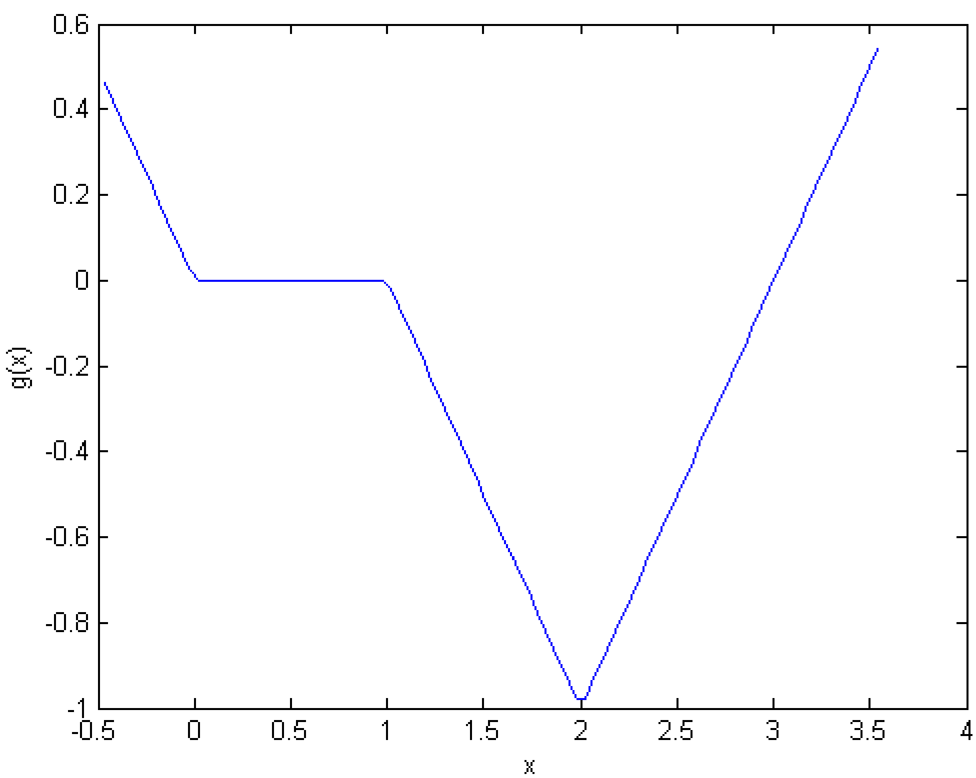

Figure 7. The function $g(x) = F_1(x)/t_1$.

Next, we shall show that if $x_{i-1} = 2$, then $x_i = 2$ for all $i \geq 1$. In fact, x_i is a minimizer for the functional

$$F_i(x) = t_i\, g(x) - |x| + |x-2|$$

over $x \in \mathbb{R}$. Using the inequality $-|x| + |x-2| \geq -2$ and the same analysis of F_1, we have

$$F_i(x) \geq t_i\, g(x) - 2 \geq t_i\, g(2) - 2 = F_i(2)$$

and the equality occurs if and only if $x = 2$. Therefore, $x_i = 2$.

Thus $x_i = 2$ for all $i \geq 1$. Hence, after passing the limit $\tau \to 0$, we get the energetic solution $x(t) = 2$ for all $t \in (0, 2]$.

2. BV solutions constructed by epsilon-neighborhood $I_\varepsilon(a) = a + I_\varepsilon(0)$. Let $\varepsilon > 0$ and $\tau > 0$ be small. Let $t_i = i/N$ for $i = 0, \ldots, N$, where $N \in \mathbb{N}$ satisfies $1 \in [\tau N, \tau(N+1))$. We need to compute the discretized solution $x^{\varepsilon,\tau}(t)$ by calculating $x_i := x^{\varepsilon,\tau}(t_i)$, where the value x_i is a minimizer for the function

$$F_i(x) := \mathscr{E}(t_i, x) + |x - x_{i-1}| = t_i\, g(x) - |x| + |x - x_{i-1}|$$

over $x \in I_\varepsilon(x_{i-1})$.

Step 1. We show that if $\sup I_\varepsilon(x_{i-1}) \leq 1$, then x_i can be chosen arbitrary in $[x_{i-1}, 1] \cap I_\varepsilon(x_{i-1})$.

First, notice that $-|x| + |x - x_{i-1}| \geq -x_{i-1}$ and we have the equality if and only if $x \geq x_{i-1}$. Hence,

$$F_i(x) \geq t_i\, g(x) - x_{i-1} \geq \inf_{a \in I_\varepsilon(x_{i-1})} g(a) - x_{i-1}$$

for all $x \in I_\varepsilon(x_{i-1})$. Due to the properties of $g(x)$ (see Figure 7), we have

$$\inf_{a \in I_\varepsilon(x_{i-1})} g(a) = g(y)$$

for any $y \in [x_{i-1}, 1] \cap I_\varepsilon(x_{i-1})$. Thus we can choose x_i arbitrarily in $[x_{i-1}, 1] \cap I_\varepsilon(x_{i-1})$.

Moreover, if $x < x_{i-1}$, then using the strict inequality $-|x| + |x - x_{i-1}| = x_{i-1} - 2x > -x_{i-1}$ we have

$$F_i(x) > t_i\, g(x) - x_{i-1} \geq t_i\, g(x_{i-1}) - x_{i-1} = F(x_{i-1}).$$

Therefore, we cannot choose $x_i < x_{i-1}$. Thus all possible choices of x_i are $x_i \in [x_{i-1}, 1] \cap I_\varepsilon(x_{i-1})$.

Step 2. We show that if $\sup I_\varepsilon(x_{i-1}) > 1$, then

$$x_i = \min\{2, \sup I_\varepsilon(x_{i-1})\}.$$

In fact, since the variational problem $\inf_{x \in I_\varepsilon(x_{i-1})} g(x)$ has the unique minimizer at $a_i := \min\{2, \sup I_\varepsilon(x_{i-1})\}$, we have

$$F_i(x) \geq t_i\, g(a_i) - x_{i-1} = F_i(a_i)$$

and the equality occurs if and only if $x = a_i$. Thus $x_i = a_i = \min\{2, \sup I_\varepsilon(x_{i-1})\}$.

Step 3. Note that with the assumption $I_\varepsilon(a) = a + I_\varepsilon(0)$, we have $\sup I_\varepsilon(x_{i-1}) = x_{i-1} + \delta_\varepsilon$ where $\delta_\varepsilon := \sup I_\varepsilon(0) > 0$. Hence, after passing to the limit $\tau \to 0$, and then $\varepsilon \to 0$, we obtain all BV solutions constructed by epsilon-neighborhood as desired.

3. BV solutions constructed by an arbitrary viscous dissipation $\Psi_\varepsilon(x) = |x| + \varepsilon^{-1}\Psi_0(\varepsilon x)$ with a convex function $\Psi_0 : \mathbb{R} \to [0,\infty)$ satisfying $\lim_{x\to 0}\Psi_0(x)/|x| = 0$ and $\lim_{|x|\to\infty}\Psi_0(x)/|x| = \infty$.
Let $\varepsilon > 0$ and $\tau > 0$. We choose $t_i = i/N$ for $i = 0,\dots,N$, where $N \in \mathbb{N}$ is such that $1 \in [\tau N, \tau(N+1))$. Let $\Psi_\varepsilon(x) := |x| + \varepsilon^{-1}\Psi_0(\varepsilon x)$ for some given convex function $\Psi_0 : \mathbb{R} \to [0,\infty)$ satisfying $\lim_{x\to 0}\Psi_0(x)/|x| = 0$ and $\lim_{|x|\to\infty}\Psi_0(x)/|x| = \infty$. To compute the discretized solution $x^{\tau,\varepsilon}(\cdot)$, it suffices to calculate $x_i := x^{\tau,\varepsilon}(t_i)$, where x_i is a minimizer for the functional

$$\begin{aligned} F_i(x) &:= \mathscr{E}(t_i, x) + \tau\Psi_\varepsilon\left(\frac{x - x_{i-1}}{\tau}\right) \\ &= t_i g(x) - |x| + |x - x_{i-1}| + e^{-1}\Psi_0(e(x - x_{i-1})) \end{aligned}$$

over $x \in \mathbb{R}$, where $e := \varepsilon/\tau$. BV solutions $x(\cdot)$ are obtained from the discretized solution $x^{\tau,\varepsilon}(\cdot)$ after taking the limit as $\tau \to 0$, $\varepsilon \to 0$ and $e \to \infty$.

(a) We consider the case that $\Psi_0(x) > 0$ if $x > 0$. We shall show that in this case the unique BV solution is $x(t) = 0$ for all $t \in [0,1]$.

It suffices to show that if $e := \varepsilon/\tau$ is large enough, then $x_i = 0$ for all $i \geq 0$. By induction, we can assume that $x_{i-1} = 0$ and x_i is a minimizer for the functional

$$F_i(x) = t_i g(x) + e^{-1}\Psi_0(ex)$$

over $x \in \mathbb{R}$. We have $F_i(0) = 0$. If $x \neq 0$, we distinguish two cases.

Case 1. If $0 < x \leq 1$, then using the inequality $g(x) \geq g(0)$ and $\Psi_0(ex) > 0$, we obtain $F_i(x) > 0 = F_i(0)$.

Case 2. If $x \geq 1$, then using the inequality $g(x) \geq g(2)$ we have

$$F_i(x) \geq g(2) + e^{-1}\Psi_0(ex).$$

Since $\lim_{|x|\to\infty}\Psi_0(x)/|x| = \infty$, for e large enough we have

$$\frac{\Psi_0(ex)}{ex} > |g(2)|.$$

Therefore, $F_i(x) > 0 = F_i(0)$ for all $x \geq 1$.

Thus the unique minimizer for F_i is $x = 0$. Hence $x_i = 0$. Consequently, the unique BV solution is $x(t) = 0$ for all $t \in [0,1]$.

(b) We consider the case when $\Psi_0(x) = 0$ in a neighborhood of 0. We shall show that the BV solutions in this case are the same with all BV solutions constructed by epsilon-neighborhood.

Let $\delta := \sup \Psi_0^{-1}(0)$, namely $\delta > 0$ is the largest number such that $\Psi_0(\delta) = 0$. Let $\delta' > 0$ be an arbitrary small number. Note that both δ and δ' are independent of ε and τ.

Step 1. We show that if $x_{i-1} \in [0,2]$, then $x_i \geq x_{i-1}$.

In fact, if $x_i < x_{i-1}$, then using the inequalities $\Psi_0 \geq 0$, $-|x_i| + |x_i - x_{i-1}| = x_{i-1} - 2x_i > -x_{i-1}$, and $g(x_i) \geq g(x_{i-1})$ (since $g(x)$ is decreasing in $(-\infty, 2]$, see Figure 7), we have

$$\begin{aligned} F_i(x_i) &= t_i\, g(x_i) - |x_i| + |x_i - x_{i-1}| + e^{-1}\Psi_0(e\,(x_i - x_{i-1})) \\ &> t_i\, g(x_{i-1}) - x_{i-1} = F_i(x_{i-1}). \end{aligned}$$

This contradicts the assumption that $F_i(x_i) = \inf_{x\in\mathbb{R}} F_i(x)$. Thus $x_i \geq x_{i-1}$.

Step 2. We show that if $x_{i-1} + \delta e^{-1} \in (1, 2]$, then $x_i \geq x_{i-1} + \delta e^{-1}$.

If $x_i \leq x_{i-1} + \delta e^{-1}$, then using the inequalities $-|x_i| + |x_i - x_{i-1}| \geq x_{i-1}$, $\Psi_0 \geq 0$, and $g(x_i) > g(x_{i-1} + \delta e^{-1})$ (since $g(x)$ is strictly decreasing in [1,2], see Figure 7), we obtain

$$F_i(x_i) > t_i\, g(x_{i-1} + \delta e^{-1}) - x_{i-1} = F_i(x_{i-1} + \delta e^{-1}).$$

This contradicts the assumption that $F_i(x_i) = \inf_{x\in\mathbb{R}} F_i(x)$. Thus $x_i \geq x_{i-1} + \delta e^{-1}$.

Remark. Note that although the number δe^{-1} is small, we have

$$N\delta e^{-1} = \frac{N\tau\delta}{\varepsilon} \geq \frac{\delta}{2\varepsilon} \to \infty$$

as $\varepsilon \to 0$ and $\tau \to 0$.

Step 3. We show that for any given δ', if e is large enough (namely e is larger than a constant dependent only on δ'), then $x_i \leq x_{i-1} + \delta'$ for all $i \geq 1$.

We assume by contradiction that $x_i > x_{i-1} + \delta'$. Then

$$F_i(x_i) - F_i(x_{i-1}) = t_i(g(x_i) - g(x_{i-1})) + e^{-1}\Psi_0(e(x_i - x_{i-1})).$$

Note that

$$g(y) - g(x) \geq -(y - x)$$

for all $y \geq x$. On the other hand, since $\lim_{|x|\to\infty} \Psi_0(x)/|x| = \infty$, there exists $L > 0$ (dependent only on Ψ_0) such that $\Psi_0(x) \geq 2x$ for all $x \geq L$. When $e \geq L/\delta'$, we have $e(x_i - x_{i-1}) > e\delta' \geq L$ and hence

$$e^{-1}\Psi_0(e(x_i - x_{i-1})) \geq 2(x_i - x_{i-1}).$$

Thus

$$F_i(x_i) - F_i(x_{i-1}) \geq (2 - t_i)(x_i - x_{i-1}) > 0.$$

This contradicts the assumption that $F_i(x_i) = \inf_{x\in\mathbb{R}} F_i(x)$. Thus $x_i \leq x_{i-1} + \delta'$.

Step 4. We show that if $x_{i-1} \geq 2$, then $x_i \leq x_{i-1}$.

We assume by contradiction that $x_i > x_{i-1}$. Since $g(x)$ is strictly increasing when $x \geq 2$, we obtain

$$\begin{aligned} F_i(x_i) &= t_i\, g(x_i) - x_{i-1} + e^{-1}\Psi_0(e(x_i - x_{i-1})) \\ &> t_i\, g(x_{i-1}) - x_{i-1} = F_i(x_{i-1}). \end{aligned}$$

This contradicts the assumption that $F_i(x_i) = \inf_{x\in\mathbb{R}} F_i(x)$. Thus $x_i \le x_{i-1}$.

Step 5. Now taking the limit as $\tau \to 0$, $\varepsilon \to 0$ and $e = \varepsilon/\tau \to \infty$, we obtain the BV solutions $x(\cdot)$. These solutions have the property that there exists $t_0 > 0$ (t_0 may be larger than or equal to 1) such that $x(t)$ is increasing on $[0, t_0)$ and $x(t) \in [0, 1)$ for all $t \in [0, t_0)$; moreover, $x(t) \in [2, 2 + \delta']$ for all $t \in (t_0, 1]$.

Since the latter property holds for an arbitrary $\delta' > 0$, letting $\delta' \to 0$ we conclude that the BV solutions thus obtained coincide with those constructed by ε-neighborhood. □

5 Example 4.5

Example 4.5. Consider the system defined by the energy functional $\mathscr{E}(t, x) := t\, g(x) - x$ with $g(x) := x^5 \sin(1/x), t \in [0, 1]$, the dissipation function $\Psi(x) := |x|$, and the initial value $x_0 := 0$. Note that $g(\cdot)$ has a unique global minimizer $z_1 = 0.2638367621...$. Moreover,

(i) The energetic solution constructed by time-discretization is

$$x(0) = 0 \text{ and } x(t) = z_1 \text{ for all } t \in (0, 1].$$

(ii) The BV solution constructed by epsilon-neighborhood is $x(t) = 0$ for all $t \in [0, 1]$. Here we can choose any neighborhood of the form $I_\varepsilon(a) = a + I_\varepsilon(0)$ where $I_\varepsilon(0)$ is a closed connected neighborhood of 0 with diameter of order $O(\varepsilon)$.

(iii) The BV solution constructed by vanishing viscosity is $x(t) = 0$ for all $t \in [0, 1]$. Here we can choose an arbitrary viscous dissipation of the form $\Psi_\varepsilon(x) = |x| + \varepsilon^{-1}\Psi_0(\varepsilon x)$ where $\Psi_0 : \mathbb{R} \to [0, \infty)$ is convex and satisfies that

$$\lim_{x\to 0} \frac{\Psi_0(x)}{|x|} = 0, \lim_{|x|\to\infty} \frac{\Psi_0(x)}{|x|} = \infty.$$

Proof. **1. Properties of g.** Here we list some key properties of the function g, which can be verified by a direct computation.

First, we have $g(x) = g(-x)$, $g(0) = 0$ and $\lim_{|x|\to\infty} g(x) = \infty$. In the region $(0, \infty)$, the set of zeroes of the function

$$g'(x) = 5x^4 \sin(1/x) - x^3 \cos(1/x)$$

is $\{z_i\}_{n=1}^\infty$, where each z_n has multiplicity 1 and

$$z_1 > z_2 > \cdots > \lim_{n\to\infty} z_n = 0.$$

Remark. In fact, the numbers z_n^{-1} are solutions to the equation $5 \sin z = z \cos z$, which can be computed numerically: $z_1 = 0.2638367621...$, $z_2 = 0.1379263106...$, $z_3 = 0.07251993503...$ etc.

As a consequence, for every $n = 1, 2, \ldots$, we see that z_{2n-1} is a local minimizer of g (in particular, z_1 is a global minimizer of g) and z_{2n} is a local maximizer of g (see Figure 8). Moreover, it is straightforward to check that $g(x) + |x| > 0$ for every $x \neq 0$ (see Figure 9).

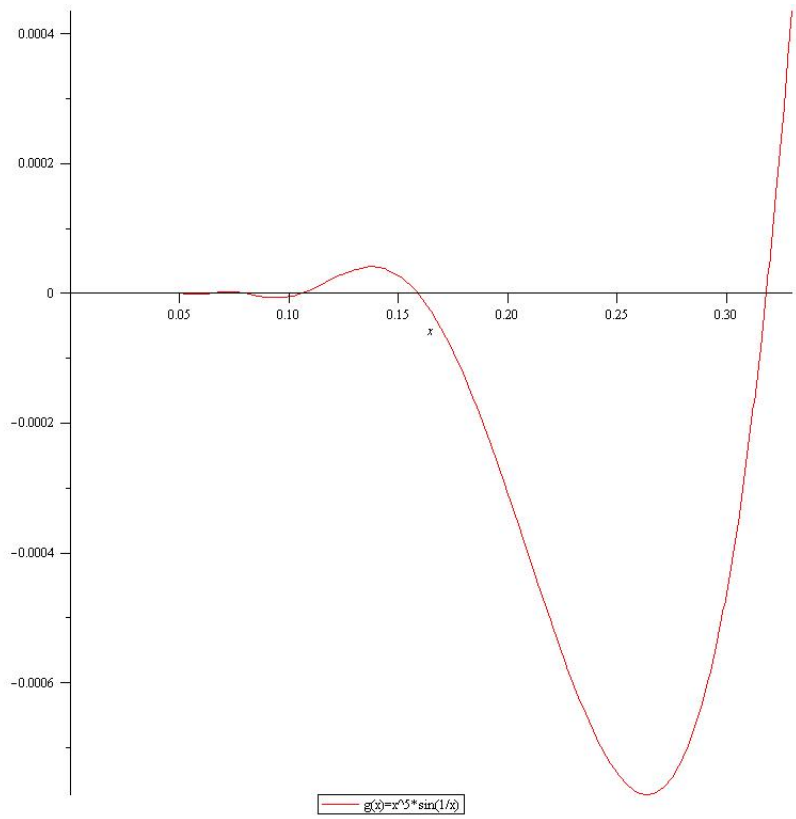

Figure 8. The function $g(x) = x^5 \sin(1/x)$.

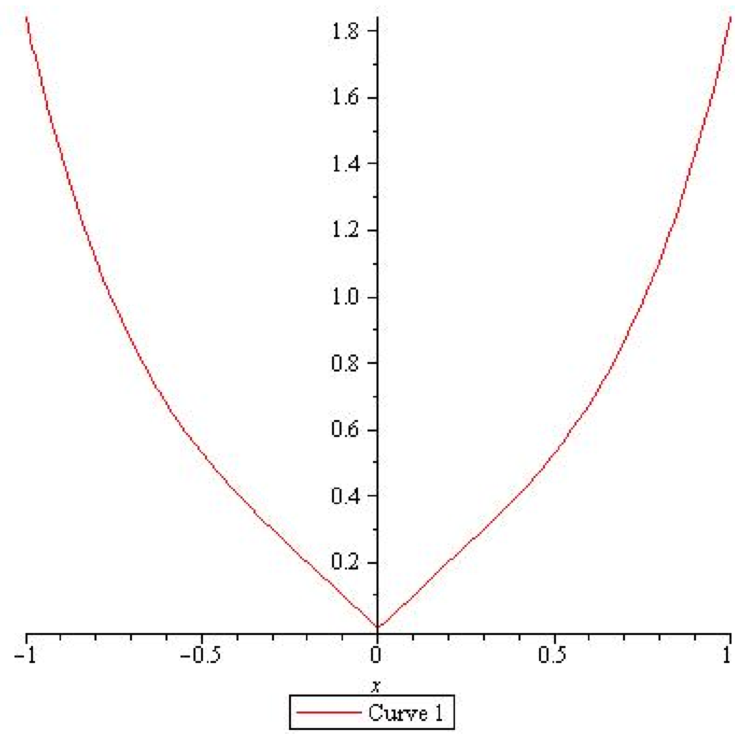

Figure 9. The function $g(x) + |x| = x^5 \sin(1/x) + |x|$.

2. Energetic solutions. Take a small time step $\tau > 0$ and let $N \in \mathbb{N}$ satisfy $1 \in$

$[\tau N, \tau(N+1))$. To find the discretized solution $x^\tau(t)$, it suffices to calculate $x_i := x^\tau(t_i)$ where $0 = t_0 < \cdots < t_N \leq 1$.

Recall that for all $i = 1, 2, \ldots, N$, x_i is a minimizer for the functional

$$\mathscr{E}(t_i, x) + |x - x_{i-1}| = t_i\, g(x) - x + |x - x_{i-1}|$$

over $x \in \mathbb{R}$.

First, since $x_0 = 0$, x_1 is a minimizer for $t_1\, g(x) - x + |x|$ over $x \in \mathbb{R}$. Note that $g(x) \geq g(z_1)$ (and the equality occurs if and only if $x = \pm z_1$) and $-x + |x| \geq 0$ (and the equality occurs if and only if $x \geq 0$). Therefore,

$$t_1\, g(x) - x + |x| \geq t_1\, g(z_1)$$

and the equality occurs if and only if $x = z_1$. Thus $x_1 = z_1$.

Next, since $x_1 = z_1$, x_2 is a minimizer for $t_2\, g(x) - x + |x - x_1|$ over $x \in \mathbb{R}$. Since $g(x) \geq g(z_1)$ and $-x + |x - x_1| \geq -x_1$, we have

$$t_2\, g(x) - x + |x - x_1| \geq t_2\, g(z_1)$$

and the equality occurs if and only if $x = z_1$. Thus $x_2 = z_1$.

In the same way, we get $x_i = z_1$ for all $i = 1, 2, \ldots, N$. Thus after passing to the limit as $\tau \to 0$, we obtain the energetic solution $x(t) = z_1$ for all $t \in (0, 1]$.

3. BV solutions by epsilon-neighborhood.

Step 1. Let $\varepsilon > 0$ and $\tau > 0$ be small. Let $t_i = i/N$ for $i = 0, \ldots, N$, where $N \in \mathbb{N}$ satisfies $1 \in [\tau N, \tau(N+1))$. We need to compute the discretized solution $x^{\varepsilon,\tau}(t)$ by calculating $x_i := x^{\varepsilon,\tau}(t_i)$, where the value x_i is a minimizer for the function

$$F_i(x) := \mathscr{E}(t_i, x) + |x - x_{i-1}| = t_i\, g(x) - |x| + |x - x_{i-1}|$$

over $x \in I_\varepsilon(x_{i-1})$.

For every $\delta > 0$ small, let $j_\delta \in \mathbb{N}$ be the largest number satisfying

$$\min\{|z_{2j_\delta - 1} - z_{2j_\delta}|, |z_{2j_\delta - 1} - z_{2j_\delta - 2}|\} \geq \delta.$$

Note that $j_\delta \to \infty$ as $\delta \to 0$.

Step 2. Now fix $\delta > 0$. For $\varepsilon > 0$ small, we have $I_\varepsilon(0) \subset [-\delta, \delta]$. We shall show that the discrete solution

$$0 \leq x^{\varepsilon,\tau}(t) \leq z_{2j_\delta - 1}$$

for all $t \in [0, 1]$. It suffices to show that $x_i = x^{\varepsilon,\tau}(t_i) \in [0, z_{2j_\delta - 1}]$ for all $i \in \{1, 2, \ldots, N\}$.

First, we show that $x_i \geq 0$ for all $i = 1, 2, \ldots, N$. Since $x_0 = 0$, x_1 is a minimizer for the function $x \mapsto t_1\, g(x) - x + |x|$ over $I_\varepsilon(0)$. If $x < 0$, then

$$t_1\, g(x) - x + |x| = t_1\, g(x) - 2x > t_1\, (g(x) - x) > 0$$

(see Figure 9). Therefore, we must have $x_1 \geq 0$. By the same argument, we get $x_i \geq 0$ for all $i = 1, 2, \ldots, N$.

Now we show that $x_k \leq z_{2j_\delta-1}$ for all $k = 1, 2, \ldots, N$. Assume by contradiction that $x_k > z_{2j_\delta-1}$ for some $k \in \{1, 2, \ldots, N\}$. Then we can choose the smallest number $i \in \{1, 2, \ldots, N\}$ such that $z_{2j_\delta-1} \in [x_{i-1}, x_i]$. Recall that x_i is a minimizer of the function

$$t_i\, g(x) - x + |x - x_{i-1}|$$

over $x \in I_\varepsilon(x_{i-1})$. Since x_{i-1} and x_i belong to $I_\varepsilon(x_{i-1})$, we have $z_{2j_\delta-1} \in [x_{i-1}, x_i] \subset I_\varepsilon(x_{i-1})$. Note that $I_\varepsilon(x_{i-1}) \subset [z_{2j_\delta}, z_{2j_\delta-2}]$ and the function g is strictly decreasing on $[z_{2j_\delta}, z_{2j_\delta-1}]$ and strictly increasing on $[z_{2j_\delta-1}, z_{2j_\delta-2}]$. Therefore, $g(x)$ has the unique minimizer $z_{2j_\delta-1}$ over $x \in I_\varepsilon(x_{i-1})$. Hence, there holds

$$t_i\, g(x) - x + |x - x_{i-1}| \geq t_i\, g(z_{2j_\delta-1}) - z_{2j_\delta-1} + |z_{2j_\delta-1} - x_{i-1}|$$

for all $x \in I_\varepsilon(x_{i-1})$ and the equality occurs if and only if $x = z_{2j_\delta-1}$. Thus $x_i = z_{2j_\delta-1}$.

Similarly, we have $x_k = z_{2j_\delta-1}$ for all $k = i, i+1, \ldots, N$. On the other hand, it is obviously that $x_k \leq z_{2j_\delta-1}$ for every $k \leq i-1$ (due to the choice of i). Hence, we can conclude that $x_k \leq z_{2j_\delta-1}$ for all $k = i, i+1, \ldots, N$.

Thus $0 \leq x^{\varepsilon,\tau}(t) \leq z_{2j_\delta-1}$ for all $t \in [0, 1]$.

Step 3. After passing to the limit as $\tau \to 0$, we see that $0 \leq x^\varepsilon(t) \leq z_{2j_\delta-1}$ for all $t \in [0, 1]$, provided $\varepsilon > 0$ is small enough such that $I_\varepsilon(0) \subset [-\delta, \delta]$. Then taking the limit as $\varepsilon \to 0$, we obtain the BV solution $0 \leq x(t) \leq z_{2j_\delta-1}$ for all $t \in [0, 1]$. Since $\lim_{\delta\to 0} j_\delta = \infty$ and $\lim_{n\to\infty} z_n = 0$, we can take $\delta \to 0$ to obtain that $x(t) = 0$ for all $t \in [0, 1]$.

4. BV solutions by vanishing viscosity.

Step 1. Let $\varepsilon > 0$ and $\tau > 0$. We choose $t_i = i/N$ for $i = 0, \ldots, N$, where $N \in \mathbb{N}$ is such that $1 \in [\tau N, \tau(N+1))$. Let $\Psi_0 : \mathbb{R} \to [0, \infty)$ be a convex function such that $\Psi_0(0) = 0$ and $\lim_{|x|\to\infty} \Psi_0(x)/|x| = \infty$. To compute the discretized solution $x^{\tau,\varepsilon}(t)$, it suffices to calculate $x_i := x^{\tau,\varepsilon}(t_i)$, where x_i is a minimizer for the function

$$F_i(x) = t_i\, g(x) - x + |x - x_{i-1}| + e^{-1}\Psi_0(e(x - x_{i-1}))$$

over $x \in \mathbb{R}$, where $e := \varepsilon/\tau$.

Step 2. We show that $x_i \in [0, z_1]$ for all $i = 1, 2, ..., N$.

Assume by contradiction that $x_i < 0$ for some i. If we take the smallest i such that $x_i < 0$, then $x_{i-1} \geq 0$. We have

$$F_i(x_i) - F_i(x_{i-1}) = t_i(g(x_i) - g(x_{i-1})) + 2|x_{i-1}| - 2|x_i| + e^{-1}\Psi_0(e(x - x_{i-1})) > 0$$

since $g(x) + |x| > 0$ for all $x \neq 0$. However, it is a contradiction to the assumption that x_i is a minimizer for F_i. Thus we must have $x_i \geq 0$ for all $i = 1, 2, \ldots, N$.

Assume by contradiction that $x_i > z_1$ for some i. If we take the smallest i such that $x_i > z_1$, then $x_{i-1} \leq z_1$. We have

$$F_i(x_i) - F_i(z_1) = t_i(g(x_i) - g(z_1)) + e^{-1}\left(\Psi_0(e(x_i - x_{i-1})) - \Psi_0(e(z_1 - x_{i-1}))\right).$$

Note that $g(x_i) > g(z_1)$ (see Figure 8). Moreover, since Ψ_0 is convex, $\Psi_0(0) = 0$ and $\Psi_0(+\infty) = +\infty$, the function Ψ_0 is increasing. Hence, $\Psi_0(e(x_i - x_{i-1})) \geq \Psi_0(e(z_i - x_{i-1}))$

because $x_i - x_{i-1} \geq z_i - x_{i-1}$. Thus $F_i(x_i) - F_i(z_1) > 0$. However, it is a contradiction to the assumption that x_i is a minimizer for F_i. Thus we must have $x_i \leq z_1$ for all $i = 1, 2, ..., N$.

Step 3. Fix $\delta > 0$. We show that if $e = \varepsilon/\tau$ is large enough, then $|x_i - x_{i-1}| \leq \delta$ for all $i = 1, 2, ..., N-1$. We have

$$F_i(x_i) - F_i(x_{i-1}) = t_i(g(x_i) - g(x_{i-1})) - x_i + |x_i - x_{i-1}| + e^{-1}\Psi_0(e(x_i - x_{i-1})).$$

Since x_{i-1} and x_i belong to $[0, z_1]$, we have

$$|t_i(g(x_i) - g(x_{i-1})) - x_i + |x_i - x_{i-1}|| \leq C := 2\left(\sup_{x\in[0,z_1]} |g(x)| + z_1\right).$$

On the other hand, since $\lim_{|x|\to\infty} \Psi_0(x)/|x| = \infty$, if $|x_i - x_{i-1}| \leq \delta$, by choosing $e > 0$ large enough (dependent only on δ, and independent of x_i and x_{i-1}), we have

$$e^{-1}\Psi_0(e(x_i - x_{i-1})) \geq C + 1.$$

We thus obtain $F_i(x_i) - F_i(x_{i-1}) \geq 1 > 0$, which is a contradiction to the assumption that x_i is a minimizer for F_i.

Thus we must have $|x_i - x_{i-1}| \leq \delta$ for all $i = 1, 2, \ldots, N-1$, provided that $e = \varepsilon/\tau$ is large enough.

Step 4. Now we use the same argument of the proof of BV solutions constructed by epsilon-neighborhood solutions. Let $j_\delta \in \mathbb{N}$ be the largest number satisfying

$$\min\{|z_{2j_\delta-1} - z_{2j_\delta}|, |z_{2j_\delta-1} - z_{2j_\delta-2}|\} \geq \delta.$$

We shall show that $x_i \leq z_{2j_\delta-1}$ for all $i = 1, 2, \ldots, N$.

Assume by contradiction that there exists $i \in \{1, 2, \ldots, N\}$ such that $x_{i-1} \leq z_{2j_\delta-1} < x_i$. Then

$$F_i(x_i) - F_i(z_{2j_\delta-1}) = t_i(g(x_i) - g(z_{2j_\delta-1})) + e^{-1}\left(\Psi_0(e(x_i - x_{i-1})) - \Psi_0(e(z_{2j_\delta-1} - x_{i-1}))\right).$$

Since $|x_i - x_{i-1}| \leq \delta$, we have

$$x_{i-1}, x_i \in [z_{2j_\delta-1} - \delta, z_{2j_\delta-1} + \delta] \subset [z_{2j_\delta}, z_{2j_\delta-2}].$$

Since the function $g(x)$ has the unique minimizer $z_{2j_\delta-1}$ over $x \in [z_{2j_\delta}, z_{2j_\delta-2}]$, we get $g(x_i) > g(z_{2j_\delta-1})$. Moreover, $\Psi_0(e(x_i - x_{i-1})) \geq \Psi_0(e(z_{2j_\delta-1} - x_{i-1}))$ since Ψ_0 is increasing and $x_i - x_{i-1} \geq z_{2j_\delta-1} - x_{i-1}$. Thus $F_i(x_i) > F_i(z_{2j_\delta-1})$. However, it is a contradiction to the assumption that x_i is a minimizer for F_i.

Thus we must have $x_i \leq z_{2j_\delta-1}$ for all $i = 1, 2, \ldots, N$.

Step 5. After passing to the limit as $\tau \to 0$, $\varepsilon \to 0$ and $e = \varepsilon/\tau \to \infty$, we obtain the BV solution $x(\cdot)$ satisfying $0 \leq x(t) \leq z_{2j_\delta-1}$ for all $t \in [0, 1]$. Because this bound holds true for all $\delta > 0$ and $\lim_{\delta\to 0} z_{2j_\delta-1} = 0$, we can conclude that $x(t) = 0$ for all $t \in [0, 1]$. □

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