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An approach to the Weinstein conjecture via J-holomorphic curves

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Introduction

The aim of this work is to give a concise introduction to the *Weinstein conjecture* and to analyze a proof of the conjecture in a particular case. The Weinstein conjecture is a meeting point between two important fields of mathematics: dynamical systems and contact-symplectic geometry. On a closed compact odd-dimensional manifold Σ, endowed with a contact 1-form α, it is well-defined a nowhere vanishing vector field R_α, called the *Reeb vector field* of α. The Weinstein conjecture claims that R_α has a periodic solution.

As regard the dynamical point of view, when $\Sigma = S^3$ we can interpret this statement as a particular case of the *Seifert conjecture*, which asserts that every nowhere vanishing smooth vector field on S^3 has a periodic orbit. The Seifert conjecture was disproven in 1994 by K. Kuperberg, who showed that nowhere vanishing vector fields without periodic orbits do exist on any compact closed odd-dimensional manifold.

If we restrict the class of vector field a little more, entering in the realm of symplectic geometry, we come to the Hamiltonian vector field. Suppose that Σ can be embedded in a symplectic manifold (M, ω) and that there exists a function $H\colon M \to \mathbb{R}$, such that $\Sigma = H^{-1}(0)$ (i.e. Σ is the 0 energy level) and 0 is a regular value for H. Then the Hamiltonian vector field X_H on M associated to H, resticts to a nowhere vanishing vector field on Σ. The corresponding existence conjecture for vector fields of this kind is called the *Hamiltonian Seifert conjecture* and was disproved in 1999 by Herman when the dimension of M is strictly bigger than 4.

On the other hand, positive results under additional hypotheses were known from the end of the Seventies. In 1978 Alan Weinstein proved that if the energy level Σ is the boundary of a convex domain in $\mathbb{R}^{2n}$, then it carries periodic orbits. In the same year Rabinowitz generalized this theorem proving that it is sufficient to suppose that Σ is the boundary of a star-shaped domain. These achievements deeply impressed the mathematicians who worked on Hamiltonian dynamics. Many thought that these theorems could prelude to further developments. However, as Weinstein himself pointed out, the hypotheses used to prove the existence of periodic orbits were not satisfactory. The dynamics of a Hamiltonian system in a symplectic manifold is invariant under diffeomorphisms which preserve the symplectic structure, hence the notion of having a periodic orbit is invariant under the action of this group. On the contrary both the convexity and the star-like assumptions are not invariant. Weinstein introduced in 1979 a property that on the one hand could generalized the star-shaped hypothesis and on the other hand were well-defined in an abstract symplectic context. This is the notion of *hypersurfaces of contact type*, that allowed Weinstein to state his famous conjecture:

(Original Weinstein Conjecture). *Let (M, ω) be a symplectic manifold and $H : M \to \mathbb{R}$ a smooth function. Suppose that 0 is regular value for H and $\Sigma := H^{-1}(0)$ a hypersurface of contact type, with $H^1(\Sigma, \mathbb{R}) = 0$. Then the Hamiltonian field on Σ carries a periodic orbit.*

Nowadays the homological hypothesis has been abandoned since reputed unnecessary and the problem has been reformulated within a genuine contact geometric framework in the following way:

(Weinstein Conjecture). *Let Σ be a compact closed manifold endowed with a contact form α. manifold. The Reeb vector field of α carries a periodic orbit.*

The conjecture in this generality is still open. In this thesis we are going to prove only a particular case.

Main Theorem. *Every compact hypersurface, which is restricted contact type and displaceable in an exact and convex at infinity symplectic manifold carries a closed Reeb orbit.*

For the convenience of the reader we include here a short summary of the content of each chapter.

In the first chapter we give an introduction to basic notions in contact and symplectic geometry and describe some concrete and important examples, where the conjecture is mainly studied: Stein manifold and the particle in a magnetic field are the two most relevant instances.

In Chapter 2, we give an account of the approaches to the proof, which have been developed so far. In particular we dwell on methods based on a theorem of existence on almost every energy level, due to Hofer and Zehnder. The interest to this technique relies on the fact that the hypotheses at the ground can be compared to those of the Main THeorem introduced above.

The proof of the Main Theorem itself is developed from Chapter 3 to 6.

In chapter 3 we define $\mathbb{A}$, the Hamiltonian action functional on E_0, the space of loops with values in M and with arbitrary period. $\mathbb{A}$ was exploited by Rabinowitz in the proof of his already mentioned theorem. It is interesting for the following reason: its nontrivial critical points are the periodic orbits we seek. In order to study the critical set we consider the space $\mathcal{M}$, composed by paths w from $\mathbb{R}$ to E_0, solving a gradient-like equation

$$\frac{dw}{ds} = -\nabla \mathbb{A}(w), \qquad (*)$$

and satisfying particular boundedness conditions for the derivative and with a prescribed behaviour at infinity. Expliciting $(*)$ we find that it is an order 0 perturbation of the equation of **J-holomorphic curves** from the cylinder $\mathbb{T} \times \mathbb{R}$ in M:

$$\partial_s u + J_u \partial_t u = 0,$$

where J is an almost complex structure on M, compatible with ω. This partial derivative equation has been studied for the first time in 1985 by Gromov and its properties are essential throughout the proof.

In the fourth chapter we endow $\mathcal{M}$ with the C^{∞}_{loc}-topology and show that **the topological space we get is sequentially relatively compact**. The calculations needed to arrive to this result are a generalization of those used by Cieliebak and Frauenfelder in 2009 for the definition of the Rabinowitz Floer Homology of an hypersurface. However, our proof is direct and does not require the construction of such homology, which relies on cumbersome transversality arguments.

In Chapter 5 we investigate the asymptotic propertis of elements in $\mathcal{M}$. **Morse-Bott theory** turns out to be applicable in this case.

Finally in chapter 6 we use **Fredholm Theory** to show that $\mathcal{M}$ is not a C^{∞}_{loc}-closed space. Putting together the results from the preceding chapters, we arrive to the existence of a limit point $\widehat{w}$ not belonging to $\mathcal{M}$. Analyzing the behavior at infinity of the function $\widehat{w}$, we succeed in finding a periodic orbit and thus in proving the Main Theorem.

Chapter 1

Preliminaries

This chapter aims to construct the language and the environment needed to understand the conjecture in its full generality. Therefore we begin with an introduction to the basic definitions and guiding examples from symplectic and contact geometry.

1.1 An introduction to symplectic geometry

The Hamiltonian formulation of the dynamics' problem was a fruitful approach in the study of classic physical systems. It is enough to mention here KAM theory ('50 -'60) which has become the cornerstone of the theory of perturbation. Symplectic geometry was born to give a coordinate-free description of the Hamilton equation when the phase space is an abstract manifold and not only a domain in an Euclidean space.

Within this chapter all the objects belong to the *smooth* category.

Definition 1.1.1. A **symplectic manifold** is a couple (M, ω) where M is a manifold and ω is a closed 2-form on M which is nondegenerate, i.e. the following implication holds $\forall z \in M$:

$$\exists v \in T_zM, \forall u \in T_zM \quad \omega_z(u,v) = 0 \quad \Rightarrow \quad v = 0\,.$$

In the following discussion we use the notations:

- If V is a subbundle of TM then V^ω is the subbundle whose fibers are defined by

$$(V^\omega)_z := \{u \in T_zM |\ \forall v \in V_z,\ \omega_z(u,v) = 0\}\,.$$

- If $v \in TM$ and η is a k-form on M, then

$$\iota_v \eta := \eta(v, \cdot)$$

is the $(k-1)$-form obtained by contraction of η on v.

Then the nondegeneracy condition can be written more concisely as

$$(TM)^\omega = 0 \qquad \text{or} \qquad \iota_v\omega = 0 \;\Rightarrow\; v = 0$$

and it establishes the following linear isomorphism

$$\begin{aligned} TM &\to T^*M \\ v &\mapsto \iota_v\omega. \end{aligned}$$

Remark 1.1.2. We have defined the form ω by two properties.

a) The nondegeneracy is a *punctual* property. It is a condition for ω_z as a bilinear antisymmetric form on T_zM and can be generalised to arbitrary vector bundles.
We call (E, ω) a **symplectic vector bundle** if $E \to M$ is a vector bundle over a manifold and $\omega\colon E \times E \to \mathbb{R}$ is a bilinear nondegenerate antisymmetric form on each fiber. Since ω is nondegenerate the rank of E is even. Indeed, suppose $E \neq 0$ and fix a point $z \in M$. The dimension of E_z can't be one because every antisymmetric form on $\mathbb{R}$ is zero. So we can pick in E_z two linearly independent vectors u_1, v_1 such that $\omega(u_1, v_1) = 1$. Then the nondegeneracy yields

$$E_z = \operatorname{Span}(u_1, v_1) \oplus \operatorname{Span}(u_1, v_1)^\omega$$

and ω restricted to both this subspaces is nondegenerate. Now the conclusion follows from induction. In this way we get as a byproduct a basis for E_z made by vectors $(u_1, v_1, \ldots, u_n, v_n)$ such that, if $(u^1, v^1, \ldots, u^n, v^n)$ is the dual basis, we can write

$$\omega_z = \sum_{k=1}^{n} u^k \wedge v^k.$$

Since we can perform this construction smoothly in a neighbourhood of z we have found *canonical local frames* in which the symplectic vector bundle has a simple model.
From this model we see that a symplectic vector bundle is orientable (and so the same is true for a symplectic manifold).
In fact $\underbrace{\omega \wedge \omega \wedge \ldots \wedge \omega}_{n \text{ times}}$ is a volume form on E. Its expression using coordinates induced from a local frame is

$$n!\left(u^1 \wedge v^1 \wedge \cdots \wedge u^n \wedge v^n\right),$$

which is nowhere vanishing.

b) The closedness of ω is a *local* property. It describes how the forms on each fiber fit together and it is responsible for the existence of *canonical local coordinates*. Namely it is possible to choose the frames described above as coordinate vectors frames. This is the content of Darboux's Theorem.

Theorem 1.1.3 (Darboux). *Let (M, ω) be a symplectic manifold and $z \in M$. Then there exists coordinates $(p^1, q^1, \ldots, p^n, q^n)$ in a neighbourhood U of z such that*

$$\omega_{|U} = \sum_{k=1}^{n} dp^k \wedge dq^k.$$

Darboux's Theorem says that there is a unique local model for symplectic manifold. So now we will take a closer look to this standard structure.

Example 1.1.4. Consider $\mathbb{C}^n$ as a complex vector space. The moltiplication by a scalar is made componentwise. Let us denote by J the moltiplication by the imaginary unit: it is a $\mathbb{C}$-linear automorphism of $\mathbb{C}^n$ such that $J^2 = -1$. Consider now the n standard coordinate vectors ∂_{z^k} and their dual basis dz^k so that a vector can be written as $u = \sum_k dz^k(u)\partial_{z^k}$. Then define

$$dp^k := \Re(dz^k) \quad \text{and} \quad dq^k := \Im(dz^k)$$

and an $\mathbb{R}$-linear isomorphism with $\mathbb{R}^{2n}$ as follows:

$$u \mapsto (dp^1(u), dq^1(u), \cdots, dp^n(u), dq^n(u)).$$

If we set

$$\partial_{p^k} := \partial_{z^k} \quad \text{and} \quad \partial_{q^k} := J\partial_{z^k} \ ,$$

then this isomorphism gives the coordinates of a vector in $\mathbb{C}^n$ with respect to this $\mathbb{R}$-basis. From now on we always consider $\mathbb{C}^n$ as a real vector space equipped with an endomorphism J that acts on it as follows:

$$J\partial_{p^k} = \partial_{q^k} \qquad J\partial_{q^k} = -\partial_{p^k}.$$

Consider the following two additional structure on $\mathbb{C}^n$.

1. *Euclidean*: a real scalar product

$$g(u, v) = \sum_k \Big(dp^k(u)dp^k(v) + dq^k(u)dq^k(v) \Big).$$

2. *Symplectic*: a bilinear antisymmetric form

$$\omega(u, v) = \sum_k dp^k \wedge dq^k(u, v).$$

The complex structure relates these bilinear forms by the formula

$$g(u, v) = \omega(Ju, v).$$

So we only need two among $g, \omega and J$ in order to find the last one.
This construction was made for a finite dimensional vector space but we can

take an open set $V \subset \mathbb{C}^n$, regarded as a real manifold, and use the canonical isomorphism between T_zV and $\mathbb{C}^n$ in order to transfer the above structures on TV (here we mean the real tangent space). If z^k are the complex coordinates and $p^k := \Re(z^k)$, $q^k := \Im(z^k)$ are the real coordinates then the notations used above for vectors and forms fits with the usual meaning those symbols have in differential geometry, for example dp^k indicates the differential of the real function p^k.
The form ω we obtain becomes a symplectic form on TV. Indeed, since w is constant, $d\omega = 0$.

In this case we have found that the symplectic form and the complex structure on V are compatible in some sense. This can be generalized as follows.

Definition 1.1.5. Let (M, ω) be a symplectic manifold and $J : TM \to TM$ an **almost complex structure**, i.e. J is a bundle map such that $J^2 = -\,\mathrm{id}_{TM}$. J is said to be **compatible with** ω if

$$g_z(u, v) := \omega_z(J_z u, v), \quad u, v \in T_zM$$

is a metric on M (in other words (M, g) becomes a Riemannian manifold).

For every fixed symplectic manifold (M, ω) the set

$$\mathcal{J}_\omega := \{J \text{ is compatible with } \omega\}$$

is nonempty and contractible, so TM is a well-defined complex vector bundle (see (34) for further details). Every complex manifold M carries a natural almost complex structure (and if a map J arises in this way is said **integrable**), however if M is also symplectic, this does not imply that the two structures are compatible in the sense given above. If this turns out to be the case M is called a **Kähler manifold**. A distinguished class of Kähler manifolds is described in the next example.

Example 1.1.6 (*Stein manifolds)***.** Let V be a complex open manifold and let J be the associated integrable structure on TV. A function $f : V \to \mathbb{R}$ is *exhausting* if it is proper and bounded from below and is *strictly plurisubharmonic* if the exact 2-form $\omega = d\,(df \circ J)$ is such that

$$\omega_z(J_z v, v) > 0, \quad \forall v \in T_zV, \ v \neq 0.$$

If V admits an exhausting strictly plurisubharmonic function f then it is called a **Stein manifold** and we will write (V, J, f) to denote it.
Observe that the above inequality implies that ω is nondegenerate and, since it is also exact, it is actually a symplectic form and hence V is a symplectic manifold. Since J is integrable ω is of type $(1,1)$ with respect to the splitting of $T_{\mathbb{C}}V$ induced by J. Then J is ω-compatible since

$$\omega(Ju, Jv) = \omega(u, v).$$

In $\mathbb{C}^n$ the function $z \mapsto |z|^2$ is an exhausting plurisubharmonic function. Indeed

$$d\big(d|z|^2 \circ J\big) = d\big((2pdp)\circ J + (2qdq)\circ J\big) = 2d\big(pdq - qdp\big) = 4dp \wedge dq.$$

Therefore up to a constant factor we get the standard symplectic form.

Let us continue now with an example from classical physics.

Example 1.1.7 (*Cotangent bundles*)**.** Let M be a smooth manifold and $\pi\colon T^*M \to M$ the cotangent vector bundle. We define a 1-form λ on $T(T^*M)$ as follows:

$$\forall \eta \in T^*M,\ \forall v \in T_\eta\left(T^*M\right), \quad \lambda_\eta(v) = \eta_{\pi(\eta)}(d_\eta \pi(v)).$$

λ is characterised by the following property:

$$\forall \eta\colon M \to T^*M, \quad \eta^*(\lambda) = \eta.$$

Then $(T^*M, d\lambda)$ is a symplectic manifold. Indeed $d\lambda$ is a closed form and if we choose coordinates (p^k, q^k) on T^*M that are induced from coordinates (q^k) on M then we find that locally $\lambda = \sum_k p^k dq^k$. Its differential is locally $\sum_k dp^k \wedge dq^k$, which we have seen to be nondegenerate.
This class of examples encloses also the case of $\mathbb{C}^n$ because $\mathbb{R}^{2n} \cong T^*\mathbb{R}^n$.

In the previous examples the symplectic form was actually exact. This additional property will be relevant in what follows and so we include it in a definition.

Definition 1.1.8. A symplectic manifold (M, ω) is said to be exact if exists a 1-form λ on TM (called a **Liouville form**), such that $\omega = d\lambda$. Since often the 1-form itself is more important than its symplectic differential we shall denote an exact manifold by (M, λ) rather than $(M, d\lambda)$.

Remark 1.1.9. The exactness of ω implies the exactness of ω^n, which is a volume form on M. This fact implies that an exact manifold can't be closed. For the same reason if we rotate the perspective, a closed manifold with $H^2_{dR} = 0$ cannot carry any symplectic structure.

We define now the diffeomorphisms and the vector fields compatible with the symplectic structure.

Definition 1.1.10. A diffeomorphism $F : M \to M'$ between (M, ω) and (M', ω') is a **symplectomorphism** (or is *symplectic*) if

$$F^*\omega' = \omega.$$

A vector field X on (M, ω) is a **symplectic vector field** if

$$\mathcal{L}_X \omega = 0.$$

Here F^* is the *pullback* by the function F and $\mathcal{L}$ denotes the *Lie derivative.*

Remark 1.1.11. We make the following two observations regarding this definition.

- Darboux's Theorem is equivalent to saying that locally every symplectic manifold is symplectomorphic to an open set in $\mathbb{C}^n$ with the standard symplectic structure. So from a local point of view all symplectic manifolds look the same.

- Vector fields can be seen as the infinitesimal counterpart of diffeomorphism. For every real t we can consider Φ_t the flow at time t associated to X. This is a diffeomorphism between two open sets in M (possibly empty) and is symplectic if and only if X is symplectic too. Indeed, if $t \geq 0$ and z is a point in the domain of Φ_t, then is in the domain of Φ_s for $0 \leq s \leq t$, too. Since $\Phi_0 = \mathrm{Id}$, Φ_0 is obviously symplectic. So,

$$\begin{aligned} \forall t \ \left(\Phi_t^* \omega\right)_z = \omega_z \quad &\Longleftrightarrow \quad \tfrac{d}{dt} \left(\Phi_t^* \omega\right)_z = 0 \\ &\Longleftrightarrow \quad \Phi_t^* \left((\mathcal{L}_X \omega)_{\Phi_t(z)} \right) = 0 \\ &\Longleftrightarrow \quad \mathcal{L}_X \omega = 0. \end{aligned}$$

  Moreover *Cartan's formula* yields:

$$\mathcal{L}_X \omega = \iota_X d\omega + d(\iota_X \omega) = d(\iota_X \omega)\,.$$

  This allows us to rewrite the condition of being symplectic:

$$\mathcal{L}_X \omega = 0 \qquad \Longleftrightarrow \qquad d\left(\iota_X \omega\right) = 0\,.$$

  At the beginning of this section we have pointed out that ω establishes an isomorphism between vector fields and 1-forms. Therefore if we want to construct a symplectic vector field we only need to pick a closed form η and then get X from the equality $\eta = \iota_X \omega$.
  The easiest closed forms are the differentials of functions on M. This will give the vector fields which we are interested in.

Definition 1.1.12. Let (M, ω) be a symplectic manifold and $H \colon M \to \mathbb{R}$ a function on it. We call the vector field X_H defined by

$$\iota_{X_H} \omega = -dH$$

an *Hamiltonian vector field* and H the *Hamiltonian* of the system.

Then the equation

$$\dot{z} = X_H(z). \tag{1.1}$$

represents the Hamiltonian formulation of the problem of dynamics for a classical physical system.

The function H can be viewed as the energy of the system and it is preserved during the motion. Indeed, if z is a trajectory, then

$$\frac{dH}{dt}(z(t)) = dH\left(X_H(z(t))\right) = -\omega\left(X_H(z(t)), X_H(z(t))\right) = 0.$$

In this sense we say that autonomous Hamiltonian systems are *conservative*.

The study of Equation 1.1 can be carried out along different lines depending on what is the goal one person has in mind. For example one may be concerned with quantitative estimates as well as stability issues or topological properties of trajectories. The focus of our enquiry will be on the last class of problems. In particular we shall investigate which general hypotheses can be imposed in order to guarantee

the existence of periodic solutions for the ordinary differential equation (1.1) **associated to an Hamiltonian** H **in a given energy level.**

However before starting with an analysis of the problem from an abstract point of view we will dwell a little more on the connection between symplectic geometry and physics.

1.2 From Newton's law to the Hamilton equations

Consider a particle (or a physical system) that moves in a Riemannian manifold (M, g) under the action of a force f, where $f : TM \to TM$ is an arbitrary function. If we set ∇ for the Levi-Civita connection on $\pi : TM \to M$ induced by g, then an admissible trajectory $\gamma : (a, b) \to M$ satisfies the Newton's law:

$$\nabla_{\dot\gamma}\dot\gamma = f(\dot\gamma), \tag{1.2}$$

where $\nabla_{\dot\gamma}$ is the covariant derivative for vector fields along γ. This is a second order differential equation for curves on M, but we can find an equivalent first order equation for its velocity $\dot\gamma$.

On $T(TM)$ is canonically defined the *vertical subbundle* $\mathcal{V}$ whose fiber at $v \in T_qM$ is the image of the injective linear maps

$$\begin{aligned} I_v : T_qM &\to T_v(TM) \\ u &\mapsto \frac{d}{dt}\Big|_{t=0}(v + tu). \end{aligned}$$

Equivalently $\mathcal{V}$ is the kernel of the bundle map $d\pi : T(TM) \to TM$.

Moreover the connection gives rise to a subbundle $\mathcal{H}$ of $T(TM)$ which is called the *horizontal subbundle* and which is a direct summand of $\mathcal{V}$, i.e. $T(TM) = \mathcal{V} \oplus \mathcal{H}$. It can be defined through the injective maps

$$\begin{aligned} L_v : T_qM &\to T_v(TM) \\ u &\mapsto d\tilde v(u) - I_v(\nabla_u \tilde v), \end{aligned}$$

where $\tilde{v}: M \to TM$ is an arbitrary extension of v to a vector field on M. Then

$$\mathcal{H}_v := L_v(T_q M)$$

and L_v is a right inverse for $d_v\pi$, namely

$$d_v\pi \circ L_v = \mathrm{id}_{T_q M}\,. \tag{1.3}$$

Set now $v := \dot{\gamma}$ and apply $I_{\dot{\gamma}}$ to both sides of Equation (1.2) obtaining the equivalent equation

$$I_v f(v) = I_v(\nabla_v v) = dv(v) - L_v(v), \tag{1.4}$$

where we have substituted for $\nabla_v v$ using the definition of L_v. Set

$$F(v) := I_v f(v)$$

and define the *geodesic vector field* $G: TM \to T(TM)$ as

$$G(v) := L_v(v).$$

Then (1.4) can be rearranged into

$$\dot{v} = G(v) + F(v). \tag{1.5}$$

Remark 1.2.1.

- Observe that, since G is horizontal and F is vertical, the vector field on the right hand side of (1.5) respects the splitting on $T(TM)$ induced by g.
- Furthermore if $F = 0$ the solutions are precisely the geodesics of (M, g), hence the adjective '*geodesic*' for G.

It is interesting to notice that g gives rise to the bundle isomorphisms

$$T^*M \xrightarrow{\sharp} TM, \quad TM \xrightarrow{\flat} T^*M.$$

Then we can

- endow T^*M with the pullback metric $k := \sharp^* g$,
- obtain an equation for $\eta := \flat v$ on T^*M that is equivalent to (1.5),

$$\dot{\eta} = d_{(\sharp\eta)}\flat(G(\sharp\eta) + F(\sharp\eta)) = d_{(\sharp\eta)}\flat(G(\sharp\eta)) + d_{(\sharp\eta)}\flat(F(\sharp\eta)). \tag{1.6}$$

As is clear from (1.6) we can analyse the pushforward of F and G separately. For brevity we set

$$\begin{cases} \hat{G}(\eta) := d_{(\sharp\eta)}\flat(G(\sharp\eta)), \\ \hat{F}(\eta) := d_{(\sharp\eta)}\flat(F(\sharp\eta)). \end{cases}$$

The crucial point is that these vector fields are indeed Hamiltonian with respect to the standard structure we defined in Example 1.1.7.

Proposition 1.2.2. *Let* $(T^*M, d\lambda)$ *be the standard symplectic structure on the cotangent bundle* $\hat{\pi}\colon T^*M \to M$ *of a Riemannian manifold* (M, g).
Using the previous notations we define $K(\eta) := \frac{1}{2}k_{\hat{\pi}(\eta)}(\eta, \eta)$ *and we suppose that* $f = -(\nabla V) \circ \pi$, *where* $V\colon M \to \mathbb{R}$ *is a real function. Then*

$$\hat{G} = X_K, \quad \hat{F} = X_{V\circ\hat{\pi}}.$$

Setting $H := K + V\circ\hat{\pi}$, *we see that* (1.5) *can be written as*

$$\dot{\eta} = X_H(\eta).$$

Remark 1.2.3. K represents the *kinetic energy* of the system. K is convex along the fibers.
$V \circ \hat{\pi}$ represents the *potential energy*. When f admits a potential physicists say that the force is *conservative*.

If we write down this equation using local coordinates (p, q), we recover the Hamilton equation of classical physics. The following identity holds:

$$\iota_{X_H} d\lambda = \iota_{X_H}(dp \wedge dq) = dp(X_H) \cdot dq - dq(X_H) \cdot dp.$$

Furthermore,

$$-dH = -\frac{\partial H}{\partial p} \cdot dp - \frac{\partial H}{\partial q} \cdot dq.$$

From these equations we obtain the components of X_H. Substituting in (1.1) we get the familiar

$$\begin{cases} \dot{p} &= -\dfrac{\partial H}{\partial q} \\ \dot{q} &= \dfrac{\partial H}{\partial p} \end{cases}.$$

During the Eighties the case of a charged particle immersed in a magnetic field became the subject of an intensive research. The Lagrangian (which we will not discuss here) and the Hamiltonian approach were carried out by Novikov and Tamainov, who used a generalization of Morse theory to multivalued functionals (37; 38), and by Arnol'd, who in addition exploited techniques from symplectic geometry (6). Their research was continued further by scholars such as V. Ginzburg (21; 22), G. Contreras (12) and G. P. Paternain (13). Since the Weinstein conjecture has been proven positively for systems belonging to this category, now we shall describe shortly what the problem is about.

Example 1.2.4 (*Particle in a magnetic field*)**.** In the three-dimensional Euclidean space the Maxwell equations for the magnetic field B yield

$$\mathrm{div} B = 0.$$

If we define

$$\sigma := \iota_B(dq^1 \wedge dq^2 \wedge dq^3),$$

then σ is a closed 2-form on $\mathbb{R}^3$.
Moreover a particle which has a unitary charge is subject to the *Lorentz force*

$$f(\dot{q}) = \dot{q} \times B.$$

A simple calculation shows

$$\dot{q} \times B = \sharp\left(\iota_{\dot{q}}\sigma\right).$$

So we can generalize this situation to an arbitrary triple (M, g, σ), where (M, g) is a Riemannian manifold and σ is a closed 2-form on M. With the notation as above the corresponding vector field $\hat{F}$ on T^*M is given by

$$\hat{F} = \hat{I}_\eta(\iota_{\sharp\eta}\sigma),$$

where $\hat{I}$ is the vertical lift from M to the cotangent bundle T^*M. Furthermore we can use σ to define the **twisted** 2-form on T^*M

$$\omega_\sigma := d\lambda - \hat{\pi}^*\sigma,$$

which is easily seen to be symplectic. The following proposition shows the connection between magnetic fields and symplectic geometry.

Proposition 1.2.5. *Let (M, g, σ) be defined as before and consider a charged particle on M subjected to a force of the form*

$$f(v) := -\nabla V(\pi(v)) + \sharp\left(\iota_v\sigma\right). \tag{1.7}$$

Then the corresponding Newton's equation is equivalent to a Hamilton equation with respect to the twisted *symplectic structure (T^*M, ω_σ). The Hamiltonian of the system is $H = K + V \circ \hat{\pi}$.*

Moreover if the magnetic field is exact, i.e. $\sigma = d\alpha$, the following translation map is a symplectomorphism

$$\begin{aligned} \Psi_\alpha \colon (T^*M, d\lambda) &\to (T^*M, \omega_{d\alpha}) \\ \eta &\mapsto \eta + \alpha_{\hat{\pi}(\eta)}. \end{aligned}$$

Therefore we get an equivalent Hamiltonian system on $(T^*M, d\lambda)$ with the Hamiltonian function obtained by substitution

$$H_\alpha(\eta) := K(\eta + \alpha(\hat{\pi}(\eta))) + V(\hat{\pi}(\eta)).$$

Proof. First notice that for every vertical vector field X on T^*M, we have

$$\begin{cases} \iota_X(\hat{\pi}^*\sigma)_\eta &= 0\,, \\ \iota_X(d\lambda)_\eta &= \hat{\pi}^*(\hat{I}_\eta^{-1}(X))\,. \end{cases} \tag{1.8}$$

Furthermore from Equation (1.3) we find

$$\begin{aligned} \left(\iota_{\hat{G}}\hat{\pi}^*\sigma\right)_\eta &= (\hat{\pi}^*\sigma)_\eta\,(\hat{G}(\eta),\cdot) \\ &= \sigma_{\hat{\pi}(\eta)}(d_\eta\hat{\pi}(\hat{G}(\eta)), d_\eta\hat{\pi}(\cdot)) \\ &= \sigma_{\hat{\pi}(\eta)}(d_{(\sharp\eta)}\pi(G(\sharp\eta)), d_\eta\hat{\pi}(\cdot)) \\ &= \sigma_{\hat{\pi}(\eta)}(\sharp\eta, d_\eta\hat{\pi}(\cdot)) \\ &= \hat{\pi}^*\left(\iota_{(\sharp\eta)}\sigma\right)_\eta\,. \end{aligned}$$

We calculate now $\iota_{\hat{G}+\hat{F}}\omega_\sigma$.

$$\begin{aligned} \iota_{\hat{G}+\hat{F}}\omega_\sigma = \iota_{\hat{G}}\omega_\sigma + \iota_{\hat{F}}\omega_\sigma &= \iota_{\hat{G}}d\lambda - \iota_{\hat{G}}\hat{\pi}^*\sigma + \iota_{\hat{F}}d\lambda - \iota_{\hat{F}}\omega_\sigma \\ &= -dK - \iota_{\hat{G}}\hat{\pi}^*\sigma + \iota_{\hat{I}_{(\cdot)}(\iota_{\sharp(\cdot)}\sigma)}d\lambda - d(V\circ\hat{\pi}) \\ &= -dK - \hat{\pi}^*\big(\iota_{(\sharp\cdot)}\sigma\big) + \hat{\pi}^*\left(\hat{I}_{(\cdot)}^{-1}\hat{I}_{(\cdot)}(\iota_{\sharp(\cdot)}\sigma)\right) - d(V\circ\hat{\pi}) \\ &= -dK - d(V\circ\hat{\pi}) \end{aligned}$$

Suppose now that $\sigma = d\alpha$. First we find that

$$\begin{aligned} \left(\Psi_\alpha^*\lambda\right)_\eta(\xi) &= \lambda_{\Psi_\alpha(\eta)}(d_\eta\Psi_\alpha(\xi)) \\ &= \Psi_\alpha(\eta)(d_{\Psi_\alpha(\eta)}\hat{\pi}d_\eta\Psi_\alpha(\xi)) \\ &= \Psi_\alpha(\eta)(d_\eta\hat{\pi}(\xi)) \\ &= \eta(d_\eta\hat{\pi}(\xi)) + \alpha_{\hat{\pi}(\eta)}(d_\eta\hat{\pi}(\xi)) \\ &= (\lambda + \hat{\pi}^*\alpha)_\eta\,(\xi). \end{aligned}$$

Using this identity we get

$$\begin{aligned} \Psi_\alpha^*(\omega_{d\alpha}) &= \Psi_\alpha^*\,(d\lambda) - \Psi_\alpha^*\,(\hat{\pi}^*\sigma) \\ &= d\,(\Psi_\alpha^*\lambda) - (\Psi_\alpha\circ\hat{\pi})^*\sigma \\ &= d(\lambda + \hat{\pi}^*\alpha) - \hat{\pi}^*\sigma \\ &= d\lambda. \end{aligned}$$

□

Remark 1.2.6. The first part of the proposition indicates that the introduction of a magnetic term in the force affects the symplectic geometry of the cotangent bundle while the Hamiltonian function remains unchanged.

As regard the second part we find the following byproduct: if α is closed, i.e. $\sigma = 0$, Ψ_α is a symplectomorphism from the standard symplectic structure $(T^*M, d\lambda)$ to itself.

1.3 The contact hypothesis

As we have said at the end of the previous section we are looking for periodic solutions of Equation (1.1). The first task will be to describe the additional hypotheses Weinstein included in the formulation of his conjecture about the existence of a periodic orbit.
We have observed before that H is a constant of the motion. Thus we can focus our attention on a fixed set $\Sigma_c := \{H = c\}$, because it is invariant under the flow of X_H.
The first hypothesis on Σ_c that seems reasonable to include is its **compactness**. In fact if we consider on $\mathbb{C}^n$ the function

$$H(p, q) := q^1,$$

we get the following vector field, whose orbits are open:

$$X_H = \partial_{p^1}.$$

Furthermore we would like to remain in the smooth category in order to use techniques coming from differential geometry. Therefore we assume that c is a **regular value** for H. Then Σ_c is a smooth submanifold by the implicit function theorem. On the contrary if c would be a critical value on the one hand we would have $z_0 \in \Sigma_c$ such that $d_(z_0)H = 0$. Then $X_H(z_0) = 0$ and we would have the trivial solution $z(t) \equiv z_0$. On the other hand the complement of critical points would be invariant under the flow and noncompact. So, as we have said above, we cannot expect the existence of periodic orbits in general.

The next step is to take a closer look to the relationship between ω and Σ_c. The nondegeneracy of ω implies that

$$\mathcal{R} := (T\Sigma_c)^\omega$$

is a one-dimensional subbundle of TM. Since the dimension of Σ_c is odd, the restriction ω' of ω to $T\Sigma_c$ is degenerate. Thus its kernel must be $\mathcal{R}$ and so $\mathcal{R} \subset T\Sigma_c$. The importance of this bundle relies in the next result.

Proposition 1.3.1. *If c is a regular value of an Hamiltonian function H and Σ_c and $\mathcal{R}$ are defined as above, then*

$$X_H \in \mathcal{R}.$$

Therefore periodic orbits correspond to closed leaves of the distribution $\mathcal{R}$, i.e. embeddings $\gamma : S^1 \to \Sigma_c$ such that $\dot{\gamma} \in \mathcal{R}$.

Proof. Let $v \in T\Sigma_c$. Then $dH(v) = 0$ yields $-\omega(X_H, v) = 0$ and so $X_H \in \mathcal{R}$. Clearly a periodic orbit is a closed leaf: X_H never vanishes on Σ_c and if we have an autointersection point, then the two tangent vectors are equal at the intersection since the system (1.1) is autonomous.

Conversely assume that γ is a closed leaf and regard γ as a 1-periodic function. $\dot{\gamma}$ and X_H are parallel and we assume that they point in the same direction by changing the orientation of γ if necessary. Then exists a positive 1-periodic function f such that

$$f(t)\dot{\gamma}(t) = X_H(\gamma(t)).$$

Then we consider the real function g defined by the following equations

$$\begin{cases} \dfrac{dg}{ds}(s) &= f(g(s)) \\ g(0) &= 0\,. \end{cases}$$

Since f is positiveand bounded, g is a diffeomorphism defined on all $\mathbb{R}$. Then

$$\frac{d\gamma}{ds}(g(s)) = f(g(s))\dot{\gamma}(g(s)) = X_H(\gamma(g(s))).$$

Therefore $\gamma(g(s))$ is a periodic solution. Its period is the smallest positive value s_0 such that $g(s_0) = 1$. □

The proposition shows that the existence problem can be formulated only in terms of the relative position between ω and Σ_c. However it has been proved that we cannot solve the problem in the affirmative for a generic hypesurface $(\Sigma, \mathcal{R}) \subset (M, \omega)$: see for example (25; 20). Weinstein's point of view is a compromise between the approach based upon Hamiltonian equations and the one which relies exclusively on the distribution $\mathcal{R}$. Its success is rooted in its connection with another important field: contact geometry. Therefore we begin with some introductory definitions from the contact setting.

Definition 1.3.2. A *contact form* α on a manifold Σ is a nowhere vanishing 1-form on $T\Sigma$ such that $d\alpha$ is a symplectic form on the subbundle $\xi := \ker \alpha$.

Remark 1.3.3. The definition immediately implies that Σ is odd-dimensional and, since $d\alpha$ is symplectic on ξ,

$$T\Sigma = (T\Sigma)^{d\alpha} \oplus \xi.$$

We can choose a generator R of $(T\Sigma)^{d\alpha}$ by requiring that $\alpha(R) = 1$.
R is uniquely determined by the conditions

$$\begin{cases} \iota_R d\alpha &= 0\,, \\ \alpha(R) &= 1\,. \end{cases}$$

R is called the **Reeb vector field** of α.

The definitions and propositions below lie the ground for the connection between symplectic and contact geometry.

Definition 1.3.4. Let $\Sigma \subset (M,\omega)$ be a hypersurface in a symplectic manifold. A vector field Y defined in a neighbourhood of Σ and transverse to Σ is a **Liouville vector field for** Σ if

$$\mathcal{L}_Y \omega = \omega.$$

(In what follows we shall abbreviate the transversality condition as $Y \pitchfork \Sigma$.)

Definition 1.3.5. Let (Σ, α) be a contact manifold and (M,ω) a symplectic manifold. We say that (Σ,α) is a *contact submanifold* of (M,ω), and we write $(\Sigma,\alpha) \subset (M,\omega)$, if there exists an embedding $j : \Sigma \to M$ of Σ as a hypersurface in M such that

$$j^*\omega = d\alpha.$$

From this definition is clear that being a contact submanifold is invariant under symplectomorphism.

Proposition 1.3.6. *If $(\Sigma,\alpha) \subset (M,\omega)$, then $R \in \mathcal{R}$.*

Proof. Since $j^*\omega = d\alpha$ the conclusion follows from the very definitions of R and $\mathcal{R}$. □

Proposition 1.3.7. *Let $\Sigma \subset (M,\omega)$ be a compact hypersurface. The following conditions are equivalent:*

i) there exists a contact form α on Σ such that $(\Sigma,\alpha) \subset (M,\omega)$,

ii) Σ has a Liouville vector field Y,

iii) exists a contact form α on Σ, a neighbourhood U of Σ and a diffeomorphism $\Psi\colon \Sigma \times (-\varepsilon,\varepsilon) \to U$ which is the identity on Σ, such that

$$\Psi^*\omega = d(e^t\alpha).$$

Moreover any of them implies that there is a neighbourhood U of Σ and a function $H\colon U \to \mathbb{R}$ such that 0 is a regular value for H and

$$\Sigma = \{H = 0\}, \ X_H|_\Sigma = R\,.$$

Proof.

i) ⇒ ii) First we observe that an application of the generalized Poincaré lemma gives the equivalence between $i)$ and the apparently stronger condition:

i') there is a neighbourhood of Σ and a 1-form λ on it such that $(\Sigma, j^*\lambda)$ is a contact manifold and $\omega = d\lambda$.

So we can define Y from the equation $\iota_Y\omega = \lambda$. Then

$$\mathcal{L}_Y\omega = \iota_Y d\omega + d(\iota_Y\omega) = d\lambda = \omega.$$

The transversality of Y can be seen as follows. Let R be the Reeb field of $j^*\lambda$. Then $1 = \lambda(R) = \omega(Y, R)$. Unless $Y \notin T\Sigma$, this leads to a contradiction because $(T\Sigma)^\omega = R$. As a byproduct we find the symplectic splitting

$$TM|_\Sigma = \mathrm{Span}(Y, R) \oplus \xi,$$

since $\iota_Y\omega|_\xi = \lambda|_\xi = 0$.

ii) $\Rightarrow$ *iii)* We set $\lambda := \iota_Y\omega$. Then $d\lambda = \omega$ and $\mathcal{L}_Y\lambda = \lambda$.
Let Φ_t be the flow of Y. It is well defined for small t in a neighbourhood U of Σ, since Σ is compact. We can construct the diffeomorphism

$$\begin{aligned} \Psi : \Sigma \times (-\varepsilon_0, \varepsilon_0) &\to U \\ (x, t) &\mapsto \Phi_t(x). \end{aligned}$$

Let ρ_t be the flow of the coordinate vector field ∂_t on $\Sigma \times (-\varepsilon_0, \varepsilon_0)$. Then Ψ carries ∂_t upon Y and coniugates their flows.
Let $\pi : \Sigma \times (-\varepsilon_0, \varepsilon_0) \to \Sigma$ be the projection on the first factor and $j_t : \Sigma \to \Sigma \times (-\varepsilon_0, \varepsilon_0)$ the embedding of Σ at height t, then $j_t = \rho_t j_0$.
Define

$$\alpha := \Psi^*\lambda.$$

Then $\alpha(\partial_t) = \lambda(Y) = 0$ and so $\alpha_{(x,t)} = \pi^* j_t^* \alpha_{(x,t)}$.
Now compute

$$\frac{d}{dt}(j_t^*\alpha) = j_0^* \frac{d}{dt}(\rho_t^*\Psi^*\lambda) = j_0^*\Psi^*\mathcal{L}_Y\lambda = j_0^*\Psi^*\lambda = j_0^*\alpha.$$

Therefore $j_t^*\alpha = e^t j_0^*\alpha$. Applying π^* to this equation we find at last

$$\alpha_{x,t} = e^t (j_0\pi)^*\alpha.$$

Taking the differential on both sides yields the conclusion.

iii) $\Rightarrow$ *i)* It is enough to put $\lambda = (\Psi^{-1})^*(e^t\alpha)$. Then

$$(\Sigma \times 0, \alpha) \subset (\Sigma \times (-\varepsilon, \varepsilon), d(e^t\alpha)) \quad \Rightarrow \quad (\Sigma, \lambda) \subset (M, \omega).$$

In order to finish the proof we have to exhibit the function H. Let π' be the projection upon the second factor in $\Sigma \times (-\epsilon, \epsilon)$. The function H such that

$$H \circ \Psi = \pi'$$

has the desired property. Since Ψ is a symplectomorphism it carries $X_{\pi'}$ to X_H and identifies $\Sigma \times \{0\}$ with Σ embedded in M. Therefore it is enough to show that $X_{\pi'} = R$:

$$\iota_R d(e^t\alpha)_{|t=0} = dt(R) - \alpha(R)dt + \iota_R d\alpha = -dt$$

□

Remark 1.3.8. Condition (ii) implies that being a contact submanifold is property which is resistant to C^1 perturbation, as long as Y remains transverse to the hypersurface.
Condition (iii) is also interesting because gives a neighbourhood of Σ which is foliated by contact hypersurfaces diffeomorphic to Σ. Moreover Reeb vector fields over two of such hypersurfaces are conjugated up to a constant factor and so they share the same dynamical properties. For instance if one of them has a closed characteristic so does the Reeb field over any other leaf of the foliation.

We will now exhibit a relevant class of contact manifolds within the setting already described in Section 1.2.

Example 1.3.9 (Cotangent bundles)**.** If the particle moves freely on (M, g), the only term in the Hamiltonian is the kinetic energy K. Then the zero-section is made by stationary point of the system whereas all the hypersurfaces $\{K = c,\ c > 0\}$ are of contact-type. Indeed, the vertical vector field $Y(\eta) := \hat{I}_\eta(\eta)$ is transverse to each nonzero level since

$$d_\eta K(Y(\eta)) = \frac{d}{dt}\Big|_{t=0} K(\eta + t\eta) = k_{\hat{\pi}(\eta)}(\eta, \eta) = 2K(\eta) \tag{1.9}$$

(N.B. this identity can be seen as an application of Euler's theorem for homogeneous function on vector bundles).
Finally $K(\eta) > 0$ provided $\eta \neq 0$. Using local coordinates we get the two equalities

$$\begin{cases} \iota_Y \lambda &= 0\,, \\ \iota_Y d\lambda &= \lambda\,. \end{cases}$$

The latter is equivalent to $\mathcal{L}_Y d\lambda = d\lambda$, remembering Cartan's formula.
Then one of the criteria in Proposition 1.3.7 is satisfied and so every nonzero energy level is a contact submanifold. Furthermore if R_c is the Reeb vector field at energy c and $\eta \in \{K = c\}$, we know that $d\lambda(Y(\eta), R_c) = 1$. In order to find the relation between X_K and R_c is sufficient to compute

$$d\lambda(Y(\eta), X_K(\eta)) = d_\eta K(Y(\eta)) = 2K(\eta) = 2c.$$

Then,

$$X_K|_{\{K=c\}} = 2cR_c.$$

We point out that not only all levels are of contact-type but also that the dynamics upon them is conjugated up to a constant positive factor. If we call Φ_t the flow of Y at time t, then

$$\mathcal{L}_Y d\lambda = d\lambda \quad \Rightarrow \quad \Phi_t^* (d\lambda) = e^t d\lambda.$$

From this and the homogeneity of K we find

$$\begin{aligned} d\lambda(d_\eta\Phi_t(X_K(\eta)), d_\eta\Phi_t(\xi))_{\Phi_t(\eta)} &= \Phi_t^* d\lambda(X_K(\eta), \xi) \\ &= e^t d\lambda(X_K(\eta), \xi) \\ &= -e^t d_\eta K(\xi) \\ &= -e^t d_\eta(K \circ \Phi_t^{-1} \circ \Phi_t)(\xi) \\ &= -e^t d_{\Phi_t(\eta)}(K \circ \Phi_{-t})(d_\eta\Phi_t(\xi)) \\ &= -e^{-t} d_{\Phi_t(\eta)} K(d_\eta\Phi_t(\xi)). \end{aligned}$$

Therefore from the definition of X_K we finally get

$$X_K(\Phi_t(\eta)) = e^t d_\eta\Phi_t(X_K(\eta)).$$

Introduce now a non-zero magnetic field σ and endow T^*M with the symplectic structure ω_σ as in Example 1.2.4. Then the zero-section is still made by stationary point and Y is still transverse to the other energy levels, however Y fails to satisfy the condition about the Lie derivative. In fact since Y is vertical, from (1.8) we have

$$\mathcal{L}_Y \omega_\sigma = d\lambda$$

and so Y is not a Liouville vector field for ω_σ. One attempt could be to find a vertical vector field $Z(\eta) = \hat{I}_\eta(\alpha(\hat{\pi}(\eta)))$, where $\alpha\colon M \to T^*M$ is a 1-form, such that

$$\begin{cases} (Y + Z) \pitchfork \{K = c\} \, , \\ \mathcal{L}_Z d\lambda = \hat{\pi}^* \sigma \, . \end{cases} \tag{1.10}$$

Mimicking the calculations (1.9), the first condition can be rewritten as

$$k_{\hat{\pi}(\eta)} \left(\eta, \eta + \alpha_{\hat{\pi}(\eta)})\right) \neq 0.$$

Moreover the second equation (1.8) yields

$$\mathcal{L}_Z d\lambda = \hat{\pi}^*(d\alpha) \, .$$

Using the injectivity of $\hat{\pi}^*$ the couple of conditions (1.10) rewrites as

$$\begin{cases} k_{\hat{\pi}(\eta)} \left(\eta, \eta + \alpha_{\hat{\pi}(\eta)}\right) \neq 0 \, , \\ d\alpha = \sigma \, . \end{cases} \tag{1.11}$$

So the second condition forces us to reduce to the case of exact magnetic fields whereas the first one tells us that the system is expected to behave differently on different energy levels. In fact consider an energy level $\{K = c\}$ such that exists a primitive α of σ such that

$$\forall \eta \in \{K = c\}, \quad K(\eta) > K(\alpha_{\hat{\pi}(\eta)}).$$

Then this hypothesis and the Cauchy-Schwarz inequality imply

$$\begin{aligned} k_{\hat{\pi}(\eta)}\left(\eta, \eta + \alpha_{\hat{\pi}(\eta)}\right) &= k_{\hat{\pi}(\eta)}\left(\eta, \eta\right) + k_{\hat{\pi}(\eta)}\left(\eta, \alpha_{\hat{\pi}(\eta)}\right) \\ &> 2K(\eta) - 2\sqrt{K(\eta)}\sqrt{K(\alpha_{\hat{\pi}(\eta)})} > 0 \,. \end{aligned}$$

The quantity that has a crucial role here is the *Mañé critical value* c_0:

$$c_0 = c_0(k, \sigma) := \inf_{\alpha \,|\, d\alpha = \sigma} \Big(\sup_{q \in M} K(\alpha_q) \Big). \tag{1.12}$$

The analysis we have made so far for exact magnetic fields yields

$$c > c_0 \;\Rightarrow\; \{K = c\} \textit{ is contact-type} \,.$$

A detailed analysis about how the dynamics changes with the energy level can be found in the recent article by K. Cieliebak, U. Frauenfelder and G.P. Paternain (10).

The opposite situation, namely the case in which σ is symplectic, was studied by V. Ginzburg and E. Kerman (31) as well. They have studied the existence of periodic orbits on low energy levels, trying to generalize the so-called *Weinstein-Moser conjecture* (47; 35) to this class of twisted cotangent bundles.

Chapter 2

The conjecture

Alan Weinstein proposed his famous conjecture for the first time in 1979 (48) inspired by the recent work of P. Rabinowitz (40), who established the existence of periodic orbits when Σ is the boundary of star-shaped domains in $\mathbb{C}^n$. This result deeply impressed matematicians involved in Hamiltonian systems, however Weinstein was not satisfied with the hypothesis of the theorem since it was not invariant under symplectomorphisms. His intuition was to recognize that the radial vector field $r\partial_r$ was one of the main ingredient of the proof and that the properties of $r\partial_r$, which were essential for the proof, were actually symplectic (i.e. preserved by symplectomorphisms). $r\partial_r$ is the prototype of what we have called a Liouville vector field and turns Σ into a contact hypersurface.

2.1 The statement

We are now in position to state precisely the

(Weinstein conjecture). *Let (M, ω) be a symplectic manifold and $\Sigma \subset M$ a compact hypersurface. If Σ is a contact submanifold of M then it carries a closed characteristic.*

Remark 2.1.1.

i) The conjecture is still open today, although it is commonly believed to be true since it was proven in the affirmative in many particular cases.

ii) The original formulation of the Weinstein conjecture included the additional assumption

$$H^1(\Sigma, \mathbb{R}) = 0.$$

However, subsequently the condition on the first cohomology group was dropped since almost all the approaches to the proof tempted so far do not rely on it.

The presence of the hypothesis on the vanishing of $H^1(\Sigma, \mathbb{R})$ in the early statement is due to the fact that it can be used as a substitutive requirement in some instances as we will say later when we discuss Liouville domains.

iii) The conjecture can be stated equivalently without any reference to the symplectic environment:

Any compact contact manifold (Σ, α) *carries a closed characteristic.*

Indeed every contact manifold can be embedded in its *symplectization*:

$$(\Sigma \times \mathbb{R}, d(e^t \alpha)).$$

iv) The conjecture becomes false if the contact hypothesis is removed without replacing it with something else. M.-R. Herman showed in (25) that exists a proper smooth function on $\mathbb{C}^n$ $(n > 2)$, which has an energy level without closed trajectories. Later the counterexample was refined by Ginzburg and Gürel in (20) exhibiting a C^2 function on $\mathbb{C}^2$ with the same properties.

The conjecture with this degree of generality is still open. However, it was proven to be true for several classes of contact submanifolds. In the next section we shall give a brief account of some of the techniques used through the years.

2.2 Approaches to the proof

One of the main guideline has been to regard the conjecture exclusively as a problem in contact geometry. However since the problem is too general the starting point has been to fix a class $\mathcal{C}$ of manifolds characterized by some properties (of topological nature, for instance) and accordingly a class of contact forms Λ on the elements of $\mathcal{C}$. This method works well with three-dimensional manifolds where contact forms were intensively studied and classified (see Giroux (23) and Eliashberg (17)) and culminated in the full answer given by Taubes in 2007.
Beginning from the early Nineties the Weinstein conjecture has been proven in the affermative for the following cases:

Case 1. Hofer (27):

$$\mathcal{C}_{H^0} = \{\dim \Sigma = 3\}, \quad \Lambda_{H^0} = \{\lambda \mid \ker \lambda \text{ is overtwisted}\}.$$

Case 2. Hofer (27):

$$\mathcal{C}_{H^1} = \{\dim \Sigma = 3, \ \pi_2(\Sigma) \neq 0\}, \quad \Lambda_{H^1} = \{\lambda \mid \ker \lambda \text{ is tight}\}.$$

Case 3. Abbas, Cieliebak and Hofer (1):

$$\mathcal{C}_{ACH} = \{\dim \Sigma = 3\}, \quad \Lambda_{ACH} = \left\{ \lambda \;\middle|\; \begin{array}{c} \ker \lambda \text{ is supported} \\ \text{by a planar open book} \end{array} \right\}.$$

Case 4. Taubes (45):

$$\mathcal{C}_{T} = \{\dim \Sigma = 3\}, \quad \Lambda_{T} = \{\lambda \text{ is an arbitrary contact form}\}.$$

However recently improvements in higher dimensions were made too. The following two results generalize Case 1 and Case 2 respectively.

Case 5. Albers and Hofer (5):

$$\mathcal{C}_{AH} = \{\dim \Sigma = 2n+1\},$$
$$\Lambda_{AH} = \{\lambda | \ker \lambda \text{ is } \textit{Plastikstufe}\text{-overtwisted}\}.$$

Case 6. Niederkrüger and Rechtman (36):

$$\mathcal{C}_{NR} = \{\dim \Sigma = 2n+1\},$$
$$\Lambda_{NR} = \left\{ \lambda \;\middle|\; \begin{array}{c} \exists\, N \hookrightarrow \Sigma \mid 0 \neq [N] \in H_{n+1}(\Sigma, \mathbb{F}_2), \\ N \text{ carries a Legendrian open book} \end{array} \right\}.$$

The following scheme summarizes the implications which hold between the results listed above.

$$\begin{array}{ccccccc} (H^1) & \Leftarrow & (T) & \Rightarrow & (ACH) & \Rightarrow & (H^0) \\ \Uparrow & & & & & & \Uparrow \\ (NR) & & & & & & (AH) \end{array}$$

For further insights the reader can consult Hofer (26) and Hutchings (30).

The other big guiding principle towards a proof of the conjecture is to investigate the presence of periodic orbits for a given Hamiltonian system as the energy level changes. The typical results that are available with this approach are the existence on $\{H = a\}$ for almost all values a, with respect to the Lebesgue measure in $\mathbb{R}$, or for a belonging to a dense subset of $\mathbb{R}$. Theorems of the first kind are called 'almost existence theorems' whereas the others are called 'nearby existence theorems'. These results rely on the definition of **symplectic capacities**. These are symplectic invariants defined axiomatically for symplectic manifolds in the following way.

Definition 2.2.1. A map c which associates to every symplectic manifold of fixed dimension $2n$ a number in $[0, +\infty]$ is a *symplectic capacity* if satisfies the three properties:

C1. Monotonicity: $c(M,\omega) \leq c(M',\omega')$,
if there is a symplectic embedding $(M,\omega) \hookrightarrow (M',\omega')$.

C2. Conformality: $c(M,s\omega) = |s|c(M,\omega), \quad \forall s \in (0,\infty)$.

C3. Nontriviality: $c(B(1),d\lambda) = \pi = c(Z(1),d\lambda)$, where
- $d\lambda$ is the standard contact structure on $\mathbb{R}^{2n}$,
- $B(1) \subset \mathbb{C}^n$ is the open unit ball,
- $Z(1) \subset \mathbb{C}^n$ is the open cylinder $\{(q^1)^2 + (p^1)^2 = 1\}$.

The notion of capacity was introduced by Ekeland and Hofer in 1990 ((14; 15)). In the same year Hofer and Zehnder constructed an explicit capacity c_{HZ} in (29), whose value depends essentially on the existence of periodic solutions of certain Hamiltonian systems on M. Let (M,ω) be a symplectic manifold and denote by $\mathcal{H}(M,\omega)$ the space of real functions H on M satisfying:

P1. there exist U_H open, K_H compact and a constant $m(H)$ such that

$$U_H \subset K_H \subset (M \setminus \partial M), \quad H(U_H) \equiv 0, \quad H(M \setminus K_H) \equiv m(H),$$

P2. $\forall x \in M, \quad 0 \leq H(x) \leq m(H)$.

Here $m(H)$ can be interpreted as the oscillation of the function. Consider the subset $\mathcal{H}_a(M,\omega) \subset \mathcal{H}(M,\omega)$ whose elements are called *admissible* and characterized by the property that all the periodic solutions for the associated Hamiltonian system (1.1) are constant or have period strictly greater than 1. These Hamiltonians can be seen as the ones having periodic solutions with 'bad' properties. In fact it is interesting to know when there are functions on the complement set $\mathcal{H}(M,\omega) \setminus \mathcal{H}_a(M,\omega)$, namely functions that have a periodic solutions with small non zero period T, $0 < T \leq 1$. This information is provided by the **Hofer-Zehnder capacity** defined by

$$c_{HZ}(M,\omega) := \sup_{\mathcal{H}_a(M,\omega)} m(H).$$

In fact if $C \geq 0$, then

$$c_{HZ}(M,\omega) \leq C \quad \iff \quad \Big(\forall\, H \in \mathcal{H}(M,\omega),\ m(H) > C \Rightarrow H \notin \mathcal{H}_a(M,\omega)\Big).$$

Therefore if c_{HZ} is finite H has a fast periodic solution, provided its oscillation is big enough. The connection with the Weinstein conjecture relies on the following

Theorem 2.2.2 (Nearby existence)**.** *Let $\Sigma \subset (M,\omega)$ be a compact hypersurface and let $\Sigma \times (-\varepsilon_0,\varepsilon_0) \hookrightarrow M$ be an embedding onto an open neighbourhood U of Σ, in other words we are choosing a tubular neighbourhood for Σ. Then*

$$c_{HZ}(U,\omega) < \infty \quad \Rightarrow \quad \text{for a.e. } \varepsilon \in (-\varepsilon_0,\varepsilon_0),\ \Sigma \times \{\varepsilon\} \text{ carries a periodic orbit.}$$

Remark 2.2.3. The preceding theorem tells us that the Weinstein conjecture holds true when we make the further assumption that Σ has an open neighbourhood with finite capacity c_{HZ}. Indeed, Remark 1.3.8 implies that the dynamics of the Reeb field on Σ is conjugated, up to a time reparametrization, to the dynamics on $\Sigma \times \{\varepsilon\}$, for every ε. Then, the existence of a periodic orbit on Σ follows from the fact that, thanks to the theorem, there is a periodic orbits on some $\Sigma \times \{\varepsilon_0\}$. This line of reason leads us to consider the larger class of **stable** hypersurfaces, which contains the contact ones.

Definition 2.2.4. An hypersurface Σ is called **stable** if there exists an embedding $\Sigma \times (-\varepsilon_0, \varepsilon_0) \hookrightarrow M$ such that characteristic bundle $\mathcal{R}_\varepsilon$ on $\Sigma \times \{\varepsilon\}$ is independent of ε.

Cieliebak and Mohnke in (11) show that stability is equivalent to the existence of a *stabilizing* 1-form α on Σ, such that

$$\mathcal{R} \subset \Sigma^{d\alpha}, \quad \alpha_{|\mathcal{R}} \neq 0.$$

The discussion made so far proves that

Corollary 2.2.5. *A compact stable hypersurface Σ with finite capacity c_{HZ} carries a closed characteristic.*

Remark 2.2.6. Properties (C.1) and (C.3) implies that every bounded open set in an Euclidean space has finite capacity and so the conjecture is fully established for hypersurfaces in $\mathbb{C}^n$. This result dates back to Viterbo, who however used variational arguments for the proof (46).

Remark 2.2.7. We have seen how the introduction of a special kind of capacity can be a useful tool for a solution of the conjecture. However the capacity is not unique and many deep results in symplectic geometry are enclosed within the properties (C.1)-(C.3): maybe rigidity phenomena for symplectomorphisms are the most important. They were investigated by Gromov (24) and Eliashberg (16) during the Seventies and the Eighties. Furthermore proving the existence of a capacity is in general a difficult task, which requires hard analitycal and variational tecnhiques. See (28) if you want to know more about this topic.

After this short survey (more on the state of art can be found in (19)), let us start with the proof of the Weinstein conjecture, which we have worked on. The main ingredient is the **free period action functional** which was used by Rabinowitz in his already mentioned proof of the conjecture (40). Recently this functional was rediscovered by Cieliebak and Frauenfelder (8) in order to define a Morse-Bott homology for a class of symplectic manifolds. They called this homological theory **Rabinowitz-Floer Homology** (the shorthand is RFH) and used it to find obstructions to certain kind

of embeddings or to prove the existence of closed characteristics. Moreover very soon it was clear that RFH could be applied equally well to solve several classical problems in symplectic geometry. Albers and Frauenfelder exploited it to solve Moser's problem about leafwise intersections (3). Papers by Cieliebak, Frauenfelder and Oancea (9) and by Abbondandolo and Schwartz (2) developed explicit calculations for cotangent bundles finding rehlations with the well known *symplectic (co-)homology.* Finally Cieliebak, Frauenfelder and Paternain extended these results to more general manifolds (the so-called stable tame case) and combined them with the theory of Mañé critical values on twisteed cotangent bundles (10). For a survey about RFH and its applications the reader can see (4). The scheme of the proof that we are going to describe is inspired by these papers (see in particular (3) and Section 4.3 in (10)) and uses ideas from RFH, although is self-contained and does not require the transversality theory which is essential in the construction of RFH.
The first step will be to state what are the additional assumptions we need. The actual line of reasoning will be developed in the subsequent chapters.

2.3 The additional hypotheses

We have highlighted in Remark 2.1.1.*ii* that every contact manifold (Σ, α) can be embedded as a contact submanifold in its symplectization

$$\big(\Sigma\times\mathbb{R}, d(e^t\alpha)\big)\,.$$

However it would be nice if the ambient symplectic manifold for Σ could be chosen with some compactness property. The following definition goes in this direction and sets up a class of manifolds which are interesting for our purposes.

Definition 2.3.1. A compact exact symplectic manifold with boundary (V, λ) is called a **Liouville domain**, if $(\Sigma := \partial V, \alpha := \lambda_{|\partial V})$ is a contact submanifold.

Every Liouville domain carries a Liouville vector field Y defined by the equation $\iota_Y d\lambda = \lambda$. Then the contact condition implies that Y points outwards through Σ and its flow gives coordinates $(x,t) \in \Sigma\times(-\varepsilon, 0]$ on a collar of Σ. $\mathcal{L}_Y\lambda = \lambda$ implies that $\lambda = e^t\alpha$ in these coordinates.
Hence we can paste along the boundary an exterior piece $V_{\text{ext}} := \Sigma\times[0,+\infty)$, define on it the 1-form $\lambda_{\text{ext}} := e^t\alpha$ and construct the **completion** $\hat{V}$ of V, that is the exact symplectic manifold without boundary

$$(\hat{V}, \hat{\lambda}) := (V \amalg_Y V_{\text{ext}},\ \lambda \amalg_Y \lambda_{\text{ext}}).$$

Every $(\Sigma\times\{t\}, e^t\alpha)$ is contact and thus V is the monotone union of Liouville domains. Furthermore the Liouville field is simply ∂_t on the exterior and so its flow is complete on $\hat{V}$ and without critical points in the exterior. These properties characterizes the manifolds that are completions of Liouville domains, as we see in the next proposition which we state without proof.

Definition 2.3.2. Let (M, λ) be an exact symplectic manifold. Then

- if there exists an exhaustion of Liouville domains $(V_k, \lambda_{|V_k})$, such that

$$V_k \subset V_{k+1}, \quad M = \bigcup_{k \in \mathbb{N}} V_k,$$

  then M is called an **exact convex symplectic manifold**,

- if the flow of its Liouville field Y is complete, then M is said to be **complete**;

- if $Y \neq 0$ outside a compact set, then M has **bounded topology**.

Proposition 2.3.3. *An exact convex symplectic manifold is complete and has bounded topology if and only if it is the completion of some Liouville domain.*

Example 2.3.4 (Stein manifolds)**.** A Stein manifold (V, J, f) is a classical example of an exact convex manifold. We have seen in Example 1.1.6 that is exact with Liouville form $\lambda := -df \circ J$. Suppose that a is a regular value and consider the manifold with boundary

$$V_a := \{f \leq a.\}$$

Then V_a is a Liouville domain. This can be seen as follows. Let g be the compatible Riemann metric defined by

$$g(u, v) = d(\lambda)(Ju, v)$$

and compute the Hamiltonian vector field X_f through its very definition:

$$-df(u) = df \circ J(Ju) = \lambda(Ju) = d\lambda(Y, Ju) = -d\lambda(JY, u).$$

So we get

$$X_f = -JY, \quad \nabla f = Y,$$

where ∇f is the gradient of f with respect to g. Hence we find that Y points outward through ∂V_a as we wanted. Since the set of critical values is negligible we find that V is an exact convex manifold. Furthermore if all the critical points of f are cointaned in a single compact set we get also that V has bounded topology. The completeness can always be achieved after a suitable reparametrization $f \mapsto \beta \circ f$ (see Biran and Cieliebak (7)).

It is convenient to define morphisms between exact convex symplectic manifolds that are not merely symplectomorphisms. In fact we shall require that the 1-forms can change only up to a summand that is the differential of a compactly supported function.

Definition 2.3.5. Let $\psi\colon (M,\lambda) \to (M',\lambda')$ be a map between two exact symplectic manifolds. ψ is called **exact** if there exists a compactly supported function h on M, such that

$$\psi^*\lambda' = \lambda + dh.$$

Remark 2.3.6. Since the support of h is assumed to be compact if an exact manifold M embeds through an exact map into an exact convex manifold than M is convex, too. As a result convexity is a property which is well-defined up to exact diffeomorphisms.

The ideal candidate class for the ambient symplectic manifolds are completions of Liouville manifolds since they are exact and they behave nicely at infinity.
The former feature allows for the definition of the period-free action for loops on M and, during the proof, it will give apriori estimates for the first derivative for functions belonging to a specific moduli space $\mathcal{M}$. The latter feature will be important in finding C^0-bounds on the same set $\mathcal{M}$.

Remark 2.3.7. Every compact hypersurface Σ in an exact convex symplectic manifold M can be embedded in $\hat{V}_{\Sigma,M}$ the completion of a Liouville manifold in such a way that the neighbourhoods of Σ (in M and in $\hat{V}_{\Sigma,M}$) are isomorphic. Indeed, it suffices to choose $V := V_k$ with k sufficiently large. So we can work in the larger class of exact convex symplectic manifold.

Now that we have said what the ambient manifold looks like we have to impose some further condition on Σ. We actually ask for two kinds of properties. The former is needed to develop tools necessary for the proof, such as the *defining Hamiltonian* and the *action-period equality*. The latter is composed by the *displaceability* condition only. It reflects a symplectic geometry relationship between Σ and M and in fact it is related to other symplectic quantities such as c_{HZ}.

Restricted contact type submanifolds

As far as the first kind of properties is concerned, we have found out in Proposition 1.3.7 that if a hypersurface Σ in a symplectic manifold (M,ω) is contact then there exists a neighbourhood U of Σ such that:

- ω is exact on U with a primitive λ which is a contact form on Σ,

- there exists a proper function $H\colon U \to (-\varepsilon_0, \varepsilon_0)$ such that
$$\Sigma = \{H = 0\} \text{ and } R = X_H.$$

The hypersurfaces we are looking for are those for which λ and H are globally defined so that the free period action functional can be calculated for loops with values in the whole M. In other words we can pick $U = M$ above.

Definition 2.3.8. An hypersurface Σ in an exact convex symplectic manifold (M, λ) is called of **restricted contact type** if there exists an exact embedding of a Liouville domain (V, λ') in (M, λ), with $\Sigma = \partial V$.

This is equivalent to saying that

i) Σ is *bounding*, i.e. $M \backslash \Sigma$ is made by two connected componets and one of them has compact closure. We call this one the *interior* of Σ, the other the *exterior*;

ii) there exists a compactly supported function h on M such that
$$\big(\Sigma, (\lambda + dh)_{|\Sigma}\big) \text{ is of contact type.}$$

So if Σ is restricted contact type the first point tells us that the function H provided by Proposition 1.3.7 can be extended from a small neighbourhood of Σ to the whole M in such a way that

- H is proper,
- $H < 0$ on the interior, $H > 0$ on the exterior,
- dH is compactly supported.

One such function is called a **defining Hamiltonian for** Σ. In order to fulfill this requirement take simply $H\colon \Sigma \times (-\varepsilon_0, \varepsilon_0) \to (-\varepsilon_0, \varepsilon_0)$ that is the projection on the second factor. Then extend smoothly on the complement of $\Sigma \times (-\varepsilon_0, \varepsilon_0)$, putting
$$H \equiv -\varepsilon_0 \ \text{ in the interior and } H \equiv \varepsilon_0 \ \text{ in the exterior.}$$

The point *b)* gives a globally defined 1-form $\hat{\lambda} := \lambda + dh$ which is contact on Σ and which still makes M into an exact convex manifold. By the means of $\hat{\lambda}$ we can define the free period action functional $\mathbb{A}$ for a loop $\gamma := \mathbb{R}/T\mathbb{Z} \to M$ of arbitrary period T as follows:
$$\gamma \mapsto \int_{\mathbb{R}/T\mathbb{Z}} \gamma^* \hat{\lambda} - \int_{\mathbb{R}/T\mathbb{Z}} H \circ \gamma \, dt.$$

Then if, γ is a curve on Σ which satisfies $\dot{\gamma} = X_H(\gamma)$,
$$\mathbb{A}(\gamma) = \int_{\mathbb{R}/T\mathbb{Z}} \hat{\lambda}_{\gamma(t)}\big(\dot{\gamma}(t)\big) \, dt - \int_{\mathbb{R}/T\mathbb{Z}} 0 \, dt$$

$$= \int_{\mathbb{R}/T\mathbb{Z}} \hat{\lambda}_{\gamma(t)}\big(X_H(\dot{\gamma}(t))\big)\, dt$$
$$= \int_{\mathbb{R}/T\mathbb{Z}} \hat{\lambda}_{\gamma(t)}\big(R(\dot{\gamma}(t))\big)\, dt$$
$$= \int_{\mathbb{R}/T\mathbb{Z}} 1\, dt = T.$$

Hence we have got the *action-period equality* for closed orbits:

$$\mathbb{A}(\gamma) = T. \tag{2.1}$$

Remark 2.3.9. If $(\Sigma, \alpha) \subset (M, \lambda)$ is a contact submanifold then the following couple of homological conditions is sufficient in order to guarantee that Σ is of restricted contact type.

- $0 = [\Sigma] \in H_{2n-1}(M, \mathbb{R})$: this implies that Σ is bounding. In codimension 1 singular homology is the same as the cobordism category. So there exists a smooth compact $2n$ manifold N which realizes the homology of Σ to 0: in other words $\Sigma = \partial N$. The other component is simply $M \setminus N$, which is unbounded.

- $H^1_{dR}(\Sigma, \mathbb{R}) = 0$ (this is the condition Weinstein included in the original statement of the conjecture). Condition *i')* in Proposition 1.3.7 yields a 1-form λ' on a neighbourhood U of Σ such that

  $$d\lambda' = \omega = d\lambda \tag{2.2}$$

  and λ' is contact on Σ. Then Equation (2.2) implies that

  $$d(\lambda' - \lambda) = d\lambda' - d\lambda = \omega - \omega = 0.$$

  The vanishing of the *first de Rham cohomology group* therefore yields a function h such that $\lambda' = \lambda + dh$. Multiplying h by a function χ that is equal to 1 near Σ and compactly supported in U gives the function $\hat{h} := \chi h$ which is defined on the whole M and compactly supported. Finally $\hat{\lambda} := \lambda + d\hat{h}$ is the required 1-form.

Displaceability

An important subset of symplectomorphisms are those which can be written as time 1-maps of Hamiltonian flows. We are interested in having a large set available and so we allow for non-autonomous Hamiltonian functions, even though with a periodic dependance on the parameter.

Definition 2.3.10. Let $\Phi : (M,\omega) \to (M,\omega)$ be a symplectic diffeomorphism. Φ is called **Hamiltonian** if there exists a function $H\colon [0,1]\times M \to \mathbb{R}$ such that:

a) there exists a compact set K of M, such that, for every t, H_t has support in K;

b) if Φ_H is the flow at time 1 of X_H, then $\Phi = \Phi_H$.

In the following discussion we will assume that H can be extended to a function $H : {}^{\mathbb{R}}\!/\!_{\mathbb{Z}} \times M \to \mathbb{R}$, since every Hamiltonian diffeomorphism arises from a periodic Hamiltonian. If $H\colon [0,1]\times M \to \mathbb{R}$ is a generic function such that $\Phi = \Phi_H$, then we can define $\widehat{H}(t,z) := h(t)H(t,z)$, where $h\colon [0,1] \to \mathbb{R}$ is a non-negative function, with support in $(0,1)$ and $\int_0^1 h dt = 1$. This last condition implies $\Phi_{\widehat{H}} = \Phi_H = \Phi$. The condition on the support tells us that $\widehat{H}$ has a periodic extension.

We will denote by $\mathcal{H}_c(M)$ the set of functions that satisfy *a)* and by $\mathrm{Ham}(M,\omega)$ the set of Hamiltonian diffeomorphisms. Then *b)* gives a surjective map

$$\begin{array}{rcl} \pi\colon \mathcal{H}_c(M) & \to & \mathrm{Ham}(M,\omega) \\ H & \mapsto & \Phi_H. \end{array}$$

The fiber upon a diffeomorphism represents the possible ways to realize it as a periodic mechanical movement. The energy of such a movement can be defined using the associated Hamiltonian.

Definition 2.3.11. Let $H \in \mathcal{H}_c(M)$ and define the function $\mathrm{osc}(H)$ as follows.

$$\begin{array}{rcl} \mathrm{osc}(H)\colon \ {}^{\mathbb{R}}\!/\!_{\mathbb{Z}} & \to & [0,+\infty) \\ t & \mapsto & \max\limits_{z\in M} H_t(z) - \min\limits_{z\in M} H_t(z). \end{array}$$

Then define,

$$\|H\| := \int_{{}^{\mathbb{R}}\!/\!_{\mathbb{Z}}} \mathrm{osc}(H)\, dt. \tag{2.3}$$

$\|\cdot\|$ induces a corresponding function on $\mathrm{Ham}(M,\omega)$ through the map π:

$$\|\Phi\| := \inf_{H\in\mathcal{H}_c(M)} \left\{ \|H\| \,\middle|\, H \in \pi^{-1}(\Phi) \right\}. \tag{2.4}$$

So $\|\Phi\|$ expresses the 'minimum' amount of energy which makes the mechanical movement Ψ possible. We call this new function the **Hofer's norm**. We stress the fact that this is *not* a norm (since $\mathrm{Ham}(M,\omega)$ is not a vector space). However $\|\Phi\|$ represents the distance between the identity map and Φ when we endow $\mathrm{Ham}(M,\omega)$ with a suitable distance, called the *Hofer's metric*. An account of the properties of this metric can be found in (28) as well as in the monograph by L. Polterovich (39).

We are now ready to give the definition of displaceability.

Definition 2.3.12. Let A be a subset of (M, ω). The **displacement energy** of A is given by

$$e_\omega(A) := \inf_{\Phi \in \mathrm{Ham}(M,\omega)} \{ \|\Phi\| | \ \Phi(A) \cap A = \emptyset \}. \tag{2.5}$$

A is called **displaceable** if $e_\omega(A) < +\infty$, namely there exists Φ such that $\Phi(A) \cap A = \emptyset$.

Remark 2.3.13. Here are some observations about the displacement energy.

- Since the Hamiltonian functions considered are compactly supported, a displaceable set is *bounded*, i.e. contained in a compact subset.
- The displacement energy decreases under the action of symplectic embeddings. Suppose $\Psi : (M, \omega) \hookrightarrow (M', \omega')$ is one such embedding and Φ is in $\mathrm{Ham}(M, \omega)$. Then $\Psi \circ \Phi \circ \Psi^{-1}$ defined on the image of Ψ can be extended to an element Φ' of $\mathrm{Ham}(M', \omega')$ simply imposing

  $$\Phi'(z) = z, \ z \notin \Psi(M).$$

  This new element satisfies $\|\Phi'\| \leq \|\Phi\|$ because we have also an extension map $\mathcal{H}_c(M) \to \mathcal{H}_c(M')$ which maps H to an H' defined in the obvious way. Then $\|H\| = \|H'\|$ and the commutativity relation

  $$\pi'(H') = (\pi(H))'$$

  yields $\|\Phi'\| \leq \|\Phi\|$. Furthermore if Φ displaces A, then Φ' displaces $\Psi(A)$ and so

  $$e_\omega(A) \geq e_{\omega'}(\Psi(A)).$$
- In a fixed symplectic manifold (M, ω) the displacement energy is monotone:

  $$A \subset B \quad \Rightarrow \quad e_\omega(A) \leq e_\omega(B).$$
- The Hofer's norm and, hence, the displacement energy are positively homogeneous with respect to the symplectic form:

  $$\forall a > 0, \ e_{a\omega} = |a| e_\omega.$$
- The displacement energy is outer regular. Namely if $e_\omega(A) < +\infty$ and $\varepsilon > 0$ is fixed, then there exists a neighbourhood U_ε of A such that

  $$e_\omega(U_\varepsilon) < e_\omega(A) + \varepsilon.$$

- As we have mentioned few pages ago the displacement energy is tied to another important geometric quantity, namely the Hofer-Zehnder capacity. This is done via the *energy-capacity inequality.* Several results of this kind are obtained under distinct assumptions. F. Schlenk studied this problem in (43). One of the corollaries he gets is the following one.

  Theorem 2.3.14. *Let (M, ω) be a symplectic manifold geometrically bounded (M the completion of a Liouville domain is sufficient). If A is a subset of M, then*

$$c_{HZ}(A) \leq 4e_\omega(A).$$

Example 2.3.15 (Bounded sets in linear spaces)**.** Every bounded set B in $\mathbb{C}^n$ is easily seen to be displaceable. Any translation by a vector v where v is of the form $v = \sum_k v^k \partial_{q^k}$ is in $\mathrm{Ham}(\mathbb{C}^n, d\lambda)$. It is enough to take

$$H(p,q) = \sum_k v^k q^k.$$

Call Φ_t the flow of X_H. In order to find a *compactly supported* function, whose flow at time 1 displaces B, simply multiply H by a cut-off function which is constantly equal to 1 in a neighbourhood of the bounded set

$$\bigcup_{t\in[0,1]} \Phi_t(B).$$

2.4 The main theorem

We are now ready to state the theorem we are going to prove in the subsequent chapters.

Theorem 2.4.1. *Let (M, λ) be an exact convex symplectic manifold and let Σ be a compact hypersurface contained in M. If Σ is* ***restricted contact type*** *and* ***displaceable*** *then it carries a contractible closed characteristic whose period is smaller than $e_{d\lambda}(\Sigma)$.*

The manifolds which best suit the hypotheses of the theorem are **subcritical Stein manifolds**. For a generic Stein manifold (V, J, f) it is possible to choose f as a Morse function whose critical points have index less or equal to half the dimension of V. If the inequality is strict, then V is called *subcritical.* These manifolds has been studied by Biran and Cieliebak (7), who discovered that every compact subset is displaceable.

Remark 2.4.2. In Remark 2.3.13 we have mentioned the *energy-capacity inequality.* This inequality allows for a comparison between the theorem presented here and the theorem of *nearby existence* developed by Hofer and Zehnder, which is easily seen to be stronger. Indeed,

- $e_\omega(\Sigma) < +\infty \quad \Rightarrow \quad c_{HZ}(\Sigma) < +\infty$,
- Σ restricted contact type $\quad \Rightarrow \quad \Sigma$ stable submanifold.

Therefore the hypotheses of Theorem 2.4.1 implies those of Corollary 2.2.5, which was a consequence of the Nearby Existence Theorem 2.2.2. On the other hand, recently Cieliebak, Frauenfelder and Paternain have succeeded in extending the definition of RFH to the larger class of stable tame manifolds. As a byproduct they improved Theorem 2.4.1 substituting the restricted contact type hypothesis with the slightly relaxed stable tame hypothesis. However the gap between the energy-capacity inequality methods and those based on the free period action functional is still wide and it is likely to remain so. We have decided to not present the theorem in this strong and up-to-date version because new ideas come into play in its proof that are not merely a generalization of the simple case.

Chapter 3

The free period action functional

We have described in the first chapter how Newton's physics can be encoded in the language of Hamiltonian systems. The latter formulation presents some advantages respect to an approach merely based on the Second Law of Dynamics: there is a group of transformation which preserves the dynamics (symplectic diffeomorphism) and many stability results are known. But perhaps the most appealing feature is the possibility to get Hamilton equations via a variational argument. The 'admissable' or physical motions are characterized by the fact that they are critical points of a suitable functional defined on a space of smooth paths in the configuration space. However, since the domain of the functional is infinite-dimensional, establishing the existence of critical points is quite a difficult task. Several properties were singled out which are sufficient for a functional in order to have critical points (the most important are probably the *direct method* and the *minimax method*), but unfortunately these do not apply directly to the action functional of classical mechanics on the space of loops. The major difficulty is that the critical points of the action do not have finite Morse index. Rabinowitz was the first in 1978 (40) to circumvent the problem and to exploit variational properties of the action. However it was only with the work of A. Floer that a general theory has been available. Floer in (18) constructed an homology theory, whose complex is generated by critical points. Therefore if we can compute the homology, we will gain information also about the critical points. Although we will not construct an homology theory for the action *à la Floer*, the proof will share some basic lemmas with Floer's theory. In this first chapter we will define a family of free period action functionals, see that they admit a gradient-like system and establish some properties of the solutions with finite energy.

3.1 The space of loops

Suppose that the hypotheses of Theorem 2.4.1 are fulfilled. From the observations in Remark 2.3.7 and Remark 2.3.13 follows that we can work with the completion of a Liouville manifold as ambient space. The following notations are fixed till the end of this exposition. Let (V, λ) be a Liouville domain, $M := \hat{V}$ its completion, Y the Liouville vector field and the function ρ defined on its exterior

$$\begin{array}{rccc} \rho\colon & V_{\text{ext}} & \to & \mathbb{R} \\ & (x,t) & \mapsto & t. \end{array}$$

It is convenient to give a name also to the exhaustion of Liouville domains whose union is M:

$$V_a := V \cup \{\rho \leq a\}, \quad a \geq 0. \tag{3.1}$$

On M we can construct an almost complex structure J compatible with $d\lambda$ and with the further property that

$$d\rho \circ J = \lambda, \quad \text{on } V_{\text{ext}}. \tag{3.2}$$

To this aim is sufficient to choose J as the direct sum $J_1 \oplus J_2$ with respect to the splitting

$$T_{(x,t)}M = \xi \oplus \text{Span}(X_\rho, Y), \quad \xi := \ker \lambda_{\big| T\{\rho = t\}}.$$

J_1 is an almost complex structure compatible with $d\lambda_\xi$ and J_2 acts in the following way:

$$J_2 X_\rho = Y, \quad J_2 Y = -X_\rho.$$

Then Equation (3.2) is easily seen to be true separatedly for ξ, Y and X_ρ.
Let $g(\cdot,\cdot) = d\lambda(J\cdot,\cdot)$ the Riemannian metric associated with $d\lambda$ and J and remember that g has an extension to the whole tensor algebra of T_zM. Moreover let Σ be a hypersurface of restricted contact type in M so that λ is a contact form when restricted to Σ and let $H : M \to \mathbb{R}$ be a defining Hamiltonian for Σ chosen as in Section 2.3. From that discussion is clear that the support of dH can be made arbitrarily close to Σ. This is important because we can suppose that any displacing Hamiltonian F for Σ displaces the support of dH as well (see Section 4.6.3).
At the end of the previous chapter we have defined the free period action functional for a loop $\gamma\colon {}^{\mathbb{R}}\!/_{T\mathbb{Z}} \to M$ in the following way

$$\mathbb{A}(\gamma) := \int_{{}^{\mathbb{R}}\!/_{T\mathbb{Z}}} \gamma^*\lambda - \int_{{}^{\mathbb{R}}\!/_{T\mathbb{Z}}} H \circ \gamma \, dt.$$

However we would like to have a functional defined on loops with fixed period. To this purpose consider the standard one-dimensional torus $\mathbb{T} := {}^{\mathbb{R}}\!/_{\mathbb{Z}}$ and the diffeomorphism

$$\begin{array}{rccc} \phi_T\colon & \mathbb{T} & \to & {}^{\mathbb{R}}\!/_{T\mathbb{Z}} \\ & y & \mapsto & Ty. \end{array}$$

Setting $u\colon \mathbb{T} \to M$ as $u := \gamma \circ \phi_T$ we get

$$\begin{aligned}
\mathbb{A}(\gamma) &= \int_{\mathbb{R}/T\mathbb{Z}} \gamma^*\lambda - \int_{\mathbb{R}/T\mathbb{Z}} H \circ \gamma \, dt \\
&= \int_{\mathbb{R}/T\mathbb{Z}} (\phi_T^{-1})^* \, (u^*\lambda) - \int_{\mathbb{R}/T\mathbb{Z}} H \circ u \circ \phi_T^{-1} \, dt \\
&= \int_{\mathbb{T}} u^*\lambda - T \int_{\mathbb{R}/T\mathbb{Z}} (\phi_T^{-1})^* \, (H \circ u \, dt) \\
&= \int_{\mathbb{T}} u^*\lambda - T \int_{\mathbb{T}} H \circ u \, dt.
\end{aligned}$$

Then we can define $\mathbb{A}$ on $E_0 := \Lambda_0 \times \mathbb{R}$, where $\Lambda_0 \subset \Lambda := \mathbb{C}^\infty(\mathbb{T}, M)$ is the space of contractible loops:

$$\begin{array}{rccl}
\mathbb{A}\colon & E_0 & \to & \mathbb{R} \\
& (u, T) & \mapsto & \displaystyle\int_{\mathbb{T}} u^*\lambda - T \int_{\mathbb{T}} H \circ u \, dt.
\end{array}$$

Take now a closer look to the loop space. On Λ we put the C^∞-topology. A prebase is made by the sets $\mathcal{U}(u, \psi, \psi', K, \varepsilon, m)$, where $u \in \Lambda$, (V, ψ) and (V', ψ') are coordinate charts in $\mathbb{T}$ and M respectively, $K \subset V$ is a compact set such that $u(K) \subset V'$, ε is a positive real number and m is a natural number. Then

$$\mathcal{U}(u, \psi, \psi', K, \varepsilon, m) := \left\{ v \in \Lambda \,\middle|\, \begin{array}{l} v(K) \subset V', \ \forall k \leq m, \\ \left\| \dfrac{d^k}{dt^k}(\psi' \circ v \circ \psi^{-1}) - \dfrac{d^k}{dt^k}(\psi' \circ u \circ \psi^{-1}) \right\| < \varepsilon \end{array} \right\}.$$

Alternatively we can embed M in $\mathbb{R}^N$, thanks to the *Whitney embedding theorem*, and regard Λ as a closed subset of $\mathbb{C}^\infty(\mathbb{T}, \mathbb{R}^N)$, which is a Fréchet space. In any case Λ_0 is easily seen to be a connected component of Λ.

E_0 is equipped with the product topology, but it has some kind of weak differentiable structure. This structure is specified by assigning to each element of E_0 a set of admissible variations.

Definition 3.1.1. Let $w = (u, T) \in E_0$. An **admissible variation** for w is a couple of smooth functions

$$\hat{w} := \Big(\hat{u}\colon \mathbb{T} \times (-\varepsilon, \varepsilon) \to M, \quad \hat{T}\colon (-\varepsilon, \varepsilon) \to \mathbb{R} \Big)$$

such that

$$\hat{u}(t, 0) = u(t), \quad \hat{T}(0) = T.$$

The variation gives also a path $(-\varepsilon, \varepsilon) \to E_0$, that we still call $\hat{w}$ with a little abuse of notation:

$$\hat{w}(s) = (\hat{u}(\cdot, s), \hat{T}(s)).$$

We can associate to $\hat{w}$ the element

$$\frac{d\hat{w}}{ds}(0) := \Big(\frac{d}{ds}\Big|_{s=0} \hat{u}(t,s), \frac{d}{ds}\Big|_{s=0} \hat{T}(s) \Big).$$

This is an element of $\Gamma(u^*TM)\times\mathbb{R}$, where the first factor is the space of smooth sections of the pull-back bundle u^*TM.
We call $T_wE_0 := \Gamma(u^*TM)\times\mathbb{R}$ **the tangent space** at w.

Remark 3.1.2. Here are some observations about the notions just introduced.

- Every element (X,η) in T_wE_0 comes from an admissible variation. It is enough to consider the maps (well-defined for small s)

  $$\hat{u}\colon (t,s) \mapsto \exp_{u(t)}(sX), \qquad \hat{T}\colon s \mapsto T + s\eta.$$

  The claim follows from the fact that $d_O \exp_z = id_{T_zM}$.

- We can endow T_wE_0 with an L^2-scalar product using the metric g.

  $$\langle (X,\eta), (X',\eta')\rangle_w := \int_{\mathbb{T}} g(X,X')\,dt + \eta\cdot\eta' \tag{3.3}$$

  and we denote by $\|\cdot\|_w$ the induced norm. Then $\langle\cdot,\cdot\rangle$ induces an injective map:

  $$\begin{aligned} \flat\colon\ T_wE_0 &\to \operatorname{Hom}_{\mathbb{R}}(T_wE_0,\mathbb{R}) \\ (X,\eta) &\mapsto \langle (X,\eta),\cdot\rangle_w. \end{aligned}$$

3.2 Closed characteristics as critical points

Now we can test the differentiability of functionals on E_0 using admissible variations. A functional f is *Gateaux differentiable* at a point w, if there exists a linear map $d_wf : T_w\Lambda_0 \to \mathbb{R}$ such that, for every variation $\hat{w}$, the function $s \mapsto f(\hat{w}(s))$ defined in an open neighbourhood of $0\in\mathbb{R}$ is differentiable at 0 and the following relation holds:

$$\frac{d}{ds}\Big|_{s=0} f\big(\hat{w}(s)\big) = d_wf\left(\frac{d\hat{w}}{ds}(0)\right).$$

A point w such that $d_wf = 0$ is a **critical point** for f.

Proposition 3.2.1. *$\mathbb{A}$ is Gateaux differentiable at every point of E_0 and*

$$d_w\mathbb{A}(X',\eta') = \langle \Big(J_u\big(\dot{u} - TX_H(u)\big), \int_{\mathbb{T}} H(u(t))\,dt \Big), \Big(X',\eta'\Big)\rangle_w.$$

So d_wf is in the image of $\flat$ and we set

$$\nabla\mathbb{A}(w) := \flat^{-1}(d_w\mathbb{A}) = \left(J_u(\dot{u} - TX_H(u)), -\int_{\mathbb{T}} H(u(t))\,dt \right).$$

*We call $\nabla\mathbb{A}\colon E_0 \to E_0$ the **gradient** of $\mathbb{A}$.*

Proof. Since $\mathbb{A}$ is the sum of two pieces, we make two separate estimates. Let $\hat{w}$ be an admissible variation and compute

$$\int_{\mathbb{T}} (\hat{u}(s))^* \lambda. \tag{3.4}$$

On $\mathbb{T}$ there is a global form dt, therefore the 1-form in (3.4) is equal to

$$(\hat{u}(s))^* \lambda(\partial_t)\, dt. \tag{3.5}$$

The real number $(\hat{u}(s))^* \lambda(\partial_t)$ is a function of the two variables (t, s) and a moment's thought shows that it is equal to

$$\hat{u}^* \lambda(\partial_t), \tag{3.6}$$

where $\hat{u}$ and ∂_t are defined on a open neighbourhood of $\mathbb{T}\times\{0\}\subset\mathbb{T}\times\mathbb{R}$. Let Φ_s be the flow of the vector field ∂_s, defined on a smaller neighbourhood of $\mathbb{T}\times\{0\}$, then $(t, s) = \Phi_s(t, 0)$. Differentiating (3.6) with respect to s yields

$$\begin{aligned}
\frac{d}{ds}\Big|_{s=0} (\hat{u}^*\lambda)_{(t,s)} \left(\partial_t|_{(t,s)}\right) &= \frac{d}{ds}\Big|_{s=0} (\hat{u}^*\lambda)_{\Phi_s(t,0)} \left(d_{(t,0)}\Phi_s \partial_t|_{(t,0)}\right) \\
&= \mathcal{L}_{\partial_s} \left(\hat{u}^*\lambda\right)_{(t,0)} (\partial_t) \\
&= \hat{u}^* \left(d\lambda\right) (\partial_s, \partial_t) + d\left(\hat{u}^*\lambda\right)_{(t,0)} (\partial_t) \\
&= d\lambda\left(\frac{\partial \hat{u}}{\partial s}(0), \dot{u}\right) + d\left(u^*\lambda\right)(\partial_t)
\end{aligned}$$

Using the fact that the derivative commutes with the integral sign we find that the function (3.4) is differentiable for $s = 0$ and its derivative is equal to

$$\int_{\mathbb{T}} d\lambda\left(\frac{\partial \hat{u}}{\partial s}(0), \dot{u}\right) dt, \tag{3.7}$$

since $d\left(u^*\lambda\right)(\partial_t)dt$ is exact on $\mathbb{T}$. The computation of the second summand is easier

$$\begin{aligned}
\frac{d}{ds}\Big|_{s=0} -\hat{T}(s)\int_{\mathbb{T}} H(\hat{u}(t,s))dt &= -\frac{d\hat{T}}{ds}(0)\int_{\mathbb{T}} H\circ u\, dt - T\int_{\mathbb{T}} dH(\frac{\partial \hat{u}}{\partial s}(0))dt \\
&= -\frac{d\hat{T}}{ds}(0)\int_{\mathbb{T}} H\circ u\, dt + T\int_{\mathbb{T}} d\lambda\left(X_H(u), \frac{\partial \hat{u}}{\partial s}(0)\right) dt.
\end{aligned}$$

Putting all together we find

$$\int_{\mathbb{T}} d\lambda\left(\frac{\partial \hat{u}}{\partial s}(0), \dot{u} - TX_H(u)\right) dt - \frac{d\hat{T}}{ds}(0)\int_{\mathbb{T}} H\circ u\, dt. \tag{3.8}$$

Alternatively using the scalar product g,

$$\int_{\mathbb{T}} g\left(\frac{\partial \hat{u}}{\partial s}(0), J(\dot{u} - TX_H(u))\right) dt - \frac{d\hat{T}}{ds}(0)\int_{\mathbb{T}} H\circ u\, dt. \tag{3.9}$$

Recalling the definition of $\langle\cdot,\cdot\rangle$, the proposition is thus proved. □

The previous proposition allows for a simple calculation of the critical points of $\mathbb{A}$.

Corollary 3.2.2. *The critical points of* $\mathbb{A}$ *are of two kinds:*

a) $(z, 0)$, *with* z *a constant path on* Σ *and* $\mathbb{A}((z, 0)) = 0$,

b) (u, T), *with* $T \neq 0$ *and* $u \circ \phi_{1/T}$ *is a periodic orbit of* X_H *contained in* Σ *and* $\mathbb{A}((u, T)) = T$.

Proof. Proposition 3.2.1 shows that

$$d_w f\left(\frac{\partial \hat{u}}{\partial s}(0), \frac{d\hat{T}}{ds}(0)\right) = \langle \nabla\mathbb{A}(w), \left(\frac{\partial \hat{u}}{\partial s}(0), \frac{d\hat{T}}{ds}(0)\right)\rangle.$$

Moreover the first observation in Remark 3.1.2 says that $\left(\frac{\partial \hat{u}}{\partial s}(0), \frac{d\hat{T}}{ds}(0)\right)$ in (3.9) can be any element in $T_w E_0$. So the fact that $\langle\cdot,\cdot\rangle$ is nondegenerate implies that

$$d_w\mathbb{A} = 0 \quad \Longleftrightarrow \quad \nabla\mathbb{A}(w) = 0.$$

$\mathbb{A}(w) = 0$ is in turn equivalent to the couple of equations

$$\begin{cases} \dot{u} &= TX_H(u), \\ 0 &= \int_{\mathbb{T}} H(u(t))\, dt. \end{cases}$$

Let's consider separatedly the cases $T = 0$ and $T \neq 0$.

- $T = 0$. The first equation becomes $\dot{u} = 0$ and hence $u \equiv z \in M$ is constant. Hence the second equation is simply $H(z) = 0$, which implies $z \in \Sigma$.

- $T \neq 0$. The first equation implies that $u\left(\frac{t}{T}\right)$ is a closed orbit of period T. The energy conservation then yields $H(u\left(\frac{t}{T}\right)) \equiv h \in \mathbb{R}$, i.e. $H(u)$ is constant. Then the second equation implies that $h = 0$ and, therefore, u is a loop on Σ.

□

The Corollary 3.2.2 shows that inside the critical set, the points of type *a)* form a copy of the hypersurface Σ. On the other hand we are interested in the existence of points of type *b)*. When a functional f is defined on a finite dimensional manifold N, one of the standard techniques in order to find critical points is to consider the **gradient flow** of f with respect to some metric μ. The gradient vector field is defined as before using the map $\flat$:

$$\nabla f = \flat^{-1}(df).$$

The gradient flow is generated by the ordinary equation

$$\dot{z} = -\nabla f(z). \tag{3.10}$$

Then one sees how the topology of energy levels $\{f = a\}$ depends on a. In particular if one knows that two levels are not homeomorphic, this forces the existence of a critical point. Refined arguments are provided by Morse theory, which guarantees that under an apriori non-degeneracy assumption for the critical points, the cardinality of the critical set is bounded from below by the sum of the Betti numbers.

However for $\mathbb{A}$ things are quite different. Its domain is neither finite dimensional nor at least is a manifold modeled on some Banach space, where an ODE theory is still available. On the contrary if we consider in this case the equation

$$\frac{dw}{ds} = -\nabla \mathbb{A}(w), \tag{3.11}$$

we saw that the right hand side can be defined, however the only way to define the left hand side we have found so far is by the means of admissible variations. We defined a variation as a couple of functions and one of them depends on two variables: therefore we must shift from an ODE-based theory to a PDE-based theory. For this reason we shall say that a couple of smooth functions

$$w = \big(u\colon \mathbb{T}\times (a,b) \to M,\ T\colon (a,b) \to \mathbb{R}\big)$$

is a solution of (3.11) if and only if

$$\left\{\begin{array}{rcl} \dfrac{\partial u}{\partial s} + J_u\left(\dfrac{\partial u}{\partial t} - TX_H(u)\right) & = & 0, \\ \dfrac{dT}{ds} + \displaystyle\int_{\mathbb{T}} H(u)\, dt & = & 0. \end{array}\right. \tag{3.12}$$

Remark 3.2.3.

- Obviously such a couple gives rise in a natural way also to a continuous curve $w\colon (a,b) \to E_0$ (the naturality justifying the little abuse of notation).
- The first equation in (3.12) is a perturbation of order 0 of the **J-holomorphic curves equation**

  $$\frac{\partial u}{\partial s} + J_u \frac{\partial u}{\partial t} = 0. \tag{3.13}$$

  The solutions of this equation are a generalization of holomorphic curves to the case of a non-integrable J since the operator

  $$\overline{\partial}_J := \frac{\partial}{\partial s} + J\frac{\partial}{\partial t}.$$

is the analogous of the classic $\overline{\partial}$ operator for maps between complex manifolds and shares with it some important regularity properties, which will be crucial in the proof.

The analogy with the finite-dimensional case pushes us to focus the attention on Equation (3.12), but now we have to understand how solutions of this equation can reveal something about the structure of the critical set. Continuing further the analogy we observe that when the underlying manifold N is compact and the function f is Morse, all the solutions of (3.10) are defined for all $s \in \mathbb{R}$ and they tend to a pair of critical points z_+ and z_- as s goes to $-\infty$ and $+\infty$ respectively. Therefore it is convenient to group the solutions using couples of critical points and define the sets

$$\mathcal{M}(f, g, z_+, z_-) := \{z \in C^\infty(\mathbb{R}, N) \mid \dot{z} = -\nabla f(z),\ z(\pm\infty) = z_\pm\}.$$

For a generic metric g these sets (also called **moduli spaces** turn out to be smooth finite-dimensional manifolds and, what is extremely important, they interact together by the means of a phoenomenon called the **breaking of gradient flow lines**, which reflects the fact that Equation (3.10) is preserved under C^∞_{loc}-limits and time shifts (if z satisfies (3.10), then so does $z(\cdot + \sigma)$ while the boundary conditions are not. In fact when a moduli space is not compact a sequence of points (z_k) happens to exist in $\mathcal{M}(f, g, z_+, z_-)$ that tends in the C^∞_{loc}-topology to a solution z which belongs to another moduli space. If this is the case, then there is a positive natural number m such that:

a) there exist m couples of critical points $(z^1_-, z^1_+), \ldots, (z^m_-, z^m_+)$ with

$$z^1_- = z_-,\ z^1_+ = z^2_-, \quad \ldots \quad z^h_+ = z^{h+1}_-, \quad \ldots \quad z^1_+ = z_+.$$

b) there exist m sequences of time shifts

$$\left(\sigma^h_k\right)^{1 \le h \le m}_{k \in \mathbb{N}} \quad \text{and } \sigma^{h_0}_k \equiv 0 \ \text{ for some } h_0,\ 1 \le h_0 \le m.$$

Furthermore these sequences have a growth that increases as h ranges from 1 to m:

$$\lim_{k \to +\infty} (\sigma^{h+1}_k - \sigma^h_k) = +\infty,$$

c) the sequences

$$z^h_k := z_k(\cdot + \sigma^h_k)$$

tend in the C^∞_{loc}-topology to trajectories

$$z^h := \lim_{k \to +\infty} z^h_k,$$

which belong to $\mathcal{M}(f, g, z^h_-, z^h_+)$.

One can visualize this behaviour thinking that the sequence of whole lines z_k comes nearer and nearer to the set of critical point described in *a)* breaking eventually in a chain of several lines which connect the couple of original critical points (z_-, z_+). This phenomenon hidden in the loss of compactness is revealed by suitable shifts in time.

The important principle to retain from the preceding reasoning is that the lack of compactness for a moduli space implies the existence of other critical points. In the case of the free period functional we know that there is a trivial critical subset isomorphic to Σ, therefore our aim is to use it in order to build a noncompact moduli space and hope that this will give rise to a break of the flow lines just as in the finite-dimensional case. However in our case there are some additional difficulties to overcome.

First of all even if we are interested in a noncompact moduli space we want that its C^∞_{loc}-closure is compact in order to find a candidate sequence z_k that breaks. In order to achieve this compactness we need different ingredients such as the exactness of the symplectic form, the contact hypothesis and the structure at infinity of M.

Secondly once a suitable sequence is available, additional hypotheses must be fulfilled in order to have the breaking. In the finite-dimensional theory the common assumption that one makes is that the functional is Morse. This implies, for instance, that the critical set is discrete. However the free period functional does not meet this requirement. On the one hand we have noticed that the trivial critical points form a copy of Σ on the other hand, since the system is autonomous, the nontrivial critical points are divided into subsets and each of them is isomorphic to S^1 (every such subset is simply made by the time shifts of a fixed closed characteristics). Therefore the components of $\operatorname{Crit}\mathbb{A}$ are manifolds of positive dimension and so $\mathbb{A}$ is necessarily not Morse. However it still satisfies a weaker condition, which is enough to break the flow lines. In fact we will show in a subsequent chapter that the trivial critical points of $\mathbb{A}$ form a **Morse-Bott component**: in short this means that the flow lines come from and go to the trivial critical set fast and transversally.

3.3 The moduli space

Now that we have established the guiding principles to follow, it is time to construct explicitly the moduli space. The first thing to do is to use the displaceability condition in order to define an homotopy of functionals $\mathbb{A}_\beta$ such that $\operatorname{Crit}\mathbb{A}_0 \simeq \operatorname{Crit}\mathbb{A}$ and $\mathbb{A}_1$ is a functional without critical points.
By assumption Σ is displaced by $F \in \operatorname{Ham}(M, d\lambda)$. Furthermore we claim that we can pick F such that $F(\cdot, t) = 0$ for all t whose fractional part is in $[0, \frac{1}{2}]$ without changing the integral which defines the Hofer norm (2.3).

Consider a function $\chi\colon \mathbb{R} \to \mathbb{R}$ such that

- $\chi(t) = 0, \quad \text{when } t \in [0, \frac{1}{2}]\,,$ • $\frac{d\chi}{dt} \geq 0\,,$ • $\chi(t+1) = \chi(t) + 1\,.$

χ passes to a map from $\mathbb{T}$ to itself, which we shall denote with the letter χ, too. Then define

$$F^{\chi}(t,z) := \frac{d\chi}{dt}(t) F(\chi(t), z).$$

A simple calculation shows that

- $\Phi_{F^{\chi}} = \Phi_F\,,$ • $\|F^{\chi}\| = \|F\|$

and the claim is thus proved. In what follows we will indicate the displacing function with this additional property simply by F. In a similar fashion we can find a function $\tilde{\chi}$ such that

- $\tilde{\chi}(t) = 1, \quad \text{when } t \in [\frac{1}{2}, 1]\,,$ • $\frac{d\tilde{\chi}}{dt} \geq 0\,,$ • $\tilde{\chi}(t+1) = \tilde{\chi}(t) + 1\,.$

Set

$$\tilde{H}(t,z) := \frac{d\tilde{\chi}}{dt}(t) H(z)$$

and for every β in $[0,1]$ define the functional

$$\mathbb{A}_{\beta}(w) := \int_{\mathbb{T}} u^*\lambda - T\int_{\mathbb{T}} \tilde{H}(t,u)\,dt - \beta \int_{\mathbb{T}} F(t,u)\,dt. \tag{3.14}$$

Proposition 3.3.1. *Each $\mathbb{A}_{\beta}$ is differentiable and furthermore admits the gradient*

$$\nabla \mathbb{A}_{\beta}(w) := \left(J_u\big(\dot{u} - TX_{\tilde{H}}(t,u) - \beta X_F(t,u)\big), - \int_{\mathbb{T}} \tilde{H}(t, u(t))\,dt \right) \tag{3.15}$$

Furthermore:

- *there is a one-to-one correspondence between* $\operatorname{Crit} \mathbb{A}$ *and* $\operatorname{Crit} \mathbb{A}_0$
$$(u(t), T) \to (u(\tilde{\chi}(t)), T),$$
- $\operatorname{Crit} \mathbb{A}_1$ *is empty.*

Proof. The first part of the proposition can be proven in the same way as Proposition 3.2.1. Therefore the critical points of $\mathbb{A}_0$ can be described equally well by Corollary 3.2.2, if we substitute H with $\tilde{H}$. Then we observe that if $u\colon \mathbb{R} \to M$ is an integral curve for X_H, then $u \circ \tilde{\chi}$ is an integral curve for $X_{\tilde{H}}$ and

$$u(k) = u(\tilde{\chi}(k)), \quad \forall k \in \mathbb{Z} \tag{3.16}$$

and all the integral curves are of this kind. In fact suppose that $\tilde{u}$ satisfies

$$\dot{\tilde{u}} = X_{\tilde{H}}(t, \tilde{u}),$$

with $\tilde{u}(0) = u_0$. If u is an integral curve for X_H, such that $u(0) = u_0$, then

$$\tilde{u} = u \circ \tilde{\chi}.$$

Once this correspondence has been established we notice that solutions with integer period are carried to solutions with the same integer period bijectively because of Equation 3.16. It remains to show that $\mathbb{A}_1$ does not have critical points. The equations in this case read

$$\begin{cases} \dot{u} &= TX_{\tilde{H}}(t,u) + X_F(t,u), \\ 0 &= -\int_{\mathbb{T}} \dot{\tilde{\chi}}(t) H(u(t))\, dt \end{cases}$$

Now we use the fact that for $t \in [0, \frac{1}{2}] + \mathbb{Z}$, $F(t, \cdot) = 0$ and for $t \in [\frac{1}{2}, 1] + \mathbb{Z}$, $\tilde{H}(t, \cdot) = 0$ and consider the equations separatedly on this two intervals of times. What we obtain are the following equations for a couple of functions

$$\left(u_1 \colon [0, \frac{1}{2}] \to M, \quad u_2 \colon [\frac{1}{2}, 1] \to M \right),$$

$$\begin{cases} \dot{u}_1(t) &= TX_{\tilde{H}}(t, u_1(t)), \\ 0 &= -\int_0^{\frac{1}{2}} \dot{\tilde{\chi}}(t) H(u_1(t))\, dt; \end{cases} \qquad \begin{cases} \dot{u}_2(t) &= X_F(t, u_2(t)), \\ 0 &= 0; \end{cases}$$

with the boundary conditions

$$u_1(0) = u_2(1), \qquad u_1(\frac{1}{2}) = u_2(\frac{1}{2}).$$

The first set implies that u_1 lies entirely on Σ, whereas the second set yields $u_2(1) = \Phi_F\left(u_2(\frac{1}{2})\right)$. This means that

$$u_1(0) = \Phi_F\left(u_1(\frac{1}{2})\right) \in \Sigma \cap \Phi_F(\Sigma).$$

Since Φ_F displaces Σ the critical set of $\mathbb{A}_1$ is empty and thus the proposition is proved. □

Now that we have the homotopy of functionals we must use it to construct a correspondent homotopy of gradient-like equations. We refer to this as a **stretching-the-neck** homotopy, since not only the functional changes with the parameter, but also the times during which a functional of the family operates dynamically through its gradient. For this purpose we need a function

$$\begin{array}{rcl} \beta \colon [0, +\infty) \times \mathbb{R} & \to & [0,1] \\ (r\,,\, s) & \mapsto & \beta_r(s) \end{array}$$

with the following properties

1. $\forall s \geq 0, \quad \dfrac{d\beta_r}{ds} \leq 0;$
2. $\forall s \leq 0, \quad \dfrac{d\beta_r}{ds} \geq 0;$
3. $\forall r \geq 1,$
   - if $|s| \leq r-1, \quad \beta_r(s) = 1,$
   - if $|s| \geq r, \quad \beta_r(s) = 0;$
4. $\forall r \leq 1,$
   - if $|s| \geq 1, \quad \beta_r(s) = 0,$
   - $\beta_r(s) \leq r.$

The existence of such a function β (or of a smooth family of functions β_r) is easy and can be achieved for example taking dilations and scalar multiple of a fixed bump function.

Now we are in position to define the gradient-like equation that characterizes the moduli space.

Definition 3.3.2. Let $r \in [0, +\infty)$ be a real number and we consider the set F_0 made by couples of smooth functions

$$w = \big(u\colon \mathbb{T}\times \mathbb{R} \to M,\ T\colon \mathbb{R} \to \mathbb{R}\big).$$

We endow both $C^\infty(\mathbb{T}\times \mathbb{R}, M)$ and $C^\infty(\mathbb{R}, \mathbb{R})$ with the topology of the uniform convergence of all derivatives on every compact subset. This is the so-called C^∞_{loc}*-topology*. Then F_0 is given the product topology.
$w \in F_0$ is said **to satisfy the r-Equation** or to be a **r-Solution** if and only if

$$\frac{dw}{ds}(s) = -\nabla \mathbb{A}_{\beta(r,s)}\,(w(s)) \tag{3.17}$$

holds. The r-Equation can be expanded into the couple

$$\frac{\partial u}{\partial s}(t,s) + J_{u(t,s)}\bigg(\frac{\partial u}{\partial t}(t,s) - T(s) X_{\tilde{H}}(t, u(t,s)) - \beta(r,s) X_F(t, u(t,s))\bigg) = 0,$$
$$\frac{dT}{ds}(s) - \int_{\mathbb{T}} \tilde{H}(t, u(t,s))\, dt = 0. \tag{3.18}$$

Remark 3.3.3. The 0-Equation reduces to the Equation(3.11) for $\mathbb{A}_0$. As r increases the interval of times during which the Equation 3.11 is perturbed widens and its width is roughly proportional to r. However for every r the solutions of the r-Equation satisfy the gradient equation for the functional $\mathbb{A}_0$ as s approaches infinity.

From now on let z_0 be a distinguished point on Σ. Then the moduli space $\mathcal{M}$ we are interested in is so defined:

$$\mathcal{M} := \left\{ (r,w) \in [0,+\infty)\times F_0 \;\middle|\; \begin{array}{l} w \textit{ satisfies the } r\textit{-Equation}, \\ w(-\infty) := \lim_{s\to-\infty} w(s) = (z_0, 0), \\ w(+\infty) := \lim_{s\to+\infty} w(s) \in \Sigma\times 0. \end{array} \right\} \tag{3.19}$$

As we notice in Remark 3.2.3 every element of F_0 gives rise to a path in E_0 and thus **the limits in** (3.19) **are intended in the topology of** E_0. Furthermore $\mathcal{M}$ inherits from $[0,+\infty)\times F_0$ the product topology, the topology of F_0 being the C^∞_{loc}-topology described above.
Obviously the subset of r-Solutions is closed in $[0,+\infty)\times F_0$, namely if $(r_k,w_k)\to(r,w)$ then

$$w_k \text{ satisfies the } r_k\text{-Equation} \quad \Rightarrow \quad w \text{ satisfies the } r\text{-Equation.}$$

However $\mathcal{M}$ is not a closed subspace since the boundary conditions are not always preserved. Nevertheless the asymptotic behaviour can still be controlled although in a weaker sense. This is achieved by introducing an important quantity, called *the energy*. On the one hand the limit of a C^∞_{loc}-convergent sequence of maps with bounded energy has finite energy and on the other hand we will see that in some cases a map with finite energy admits asymptots, which are critical points. This last phaenomenon will be investigated in Chapter 5.

3.4 Energy

Definition 3.4.1. Let w be a map in F_0. Its energy is defined by the formula

$$E(w) := \int_{\mathbb{R}} \left\| \frac{dw}{ds}(s) \right\|^2_{w(s)} ds \ \in [0,+\infty], \tag{3.20}$$

or after expanding the norm in the integral

$$E(w) = \int_{\mathbb{R}} \left(\int_{\mathbb{T}} \left| \frac{\partial u}{\partial s}(t,s) \right|^2_{u(t,s)} dt \right) ds + \int_{\mathbb{R}} \left| \frac{dT}{ds}(s) \right|^2 ds. \tag{3.21}$$

The next proposition establishes some inequalities for the energy of r-Solutions, which demonstrate as the action and the energy are linked together.

Proposition 3.4.2. *Let w be an r-Solution. Then,*

$$\mathbb{A}_0 \circ w\big|_{\{s\le -r\}},\ \mathbb{A}_0 \circ w\big|_{\{s\ge r\}} \qquad \textit{are non-increasing functions.}$$

Moreover if we define

$$\mathbb{A}_0(w_\pm) := \lim_{s\to\pm\infty} \mathbb{A}_0(w(s)),$$

the following inequalities hold:

$$E(w) \le \mathbb{A}_0(w_-) - \mathbb{A}_0(w_+) + \|F\|, \tag{3.22}$$

$$\forall s\in\mathbb{R}, \quad |\mathbb{A}_{\beta(r,s)}(w(s))| \le \max\{\mathbb{A}_0(w_-), -\mathbb{A}_0(w_+)\} + \|F\|. \tag{3.23}$$

Proof. Let $s_0 \leq s_1$ and set

$$\Delta(s_0, s_1) := \mathbb{A}_{\beta(r,s_0)}(w(s_0)) - \mathbb{A}_{\beta(r,s_1)}(w(s_1)).$$

Then compute

$$\begin{aligned}\Delta(s_0, s_1) &= -\int_{s_0}^{s_1} \frac{d\mathbb{A}_{\beta(r,s)}(w(s))}{ds}(s)\, ds \\ &= -\int_{s_0}^{s_1} \left(\frac{\partial \mathbb{A}_{\beta(r,s)}}{\partial s}(s)\right)(w(s))\, ds - \int_{s_0}^{s_1} d_{w(s)}\mathbb{A}_{\beta(r,s)}\left(\frac{dw}{ds}(s)\right) ds.\end{aligned}$$

Writing explicitly $\dfrac{\partial \mathbb{A}_{\beta(r,s)}}{\partial s}$ and using the fact that w is an r-Solution we find

$$\Delta(s_0, s_1) = \int_{s_0}^{s_1} \frac{d\beta_r}{ds}(s)\left(\int_{\mathbb{T}} F(t, u(t,s))dt\right) ds + \int_{s_0}^{s_1} \left\|\frac{dw}{ds}(s)\right\|_{w(s)}^2 ds. \quad (3.24)$$

Set

$$\theta(s_0, s_1) := \int_{s_0}^{s_1} \frac{d\beta_r}{ds}(s)\left(\int_{\mathbb{T}} F(t, u(t,s))dt\right) ds$$

and first show that

$$-\theta(s_0, s_1) \leq \|F\|. \quad (3.25)$$

We consider separatedly the cases $s_0 \leq s_1 \leq 0$ and $0 \leq s_0 \leq s_1$ since $\frac{d\beta_r}{ds}$ has constant sign on the positive ray and on the negative ray. For the first case we find

$$\begin{aligned}-\theta(s_0, s_1) &= \int_{s_0}^{s_1} -\frac{d\beta_r}{ds}(s)\left(\int_{\mathbb{T}} F(t, u(t,s))dt\right) ds \\ &\leq \int_{s_0}^{s_1} -\frac{d\beta_r}{ds}(s)\left(\int_{\mathbb{T}} \min_{z\in M} F(t,z)dt\right) ds \\ &= (\beta_r(s_1) - \beta_r(s_0)) \int_{\mathbb{T}} -\min_{z\in M} F(t,z)dt\end{aligned}$$

The second possibility yields

$$-\theta(s_0, s_1) \leq (\beta_r(s_0) - \beta_r(s_1)) \int_{\mathbb{T}} \max_{z\in M} F(t,z)dt.$$

Recall now the definition of the Hofer's norm and keep in mind that in any case

$$|\beta_r(s_0) - \beta_r(s_1)| \leq 1.$$

Then for the two cases considered above (3.25) follows immediately. If s_0 and s_1 have different signs then (3.25) follows again by the splitting $\theta(s_0, s_1) = \theta(s_0, 0) + \theta(0, s_1)$. This concludes the proof of the inequality 3.25.
For the proof of the first inequality we let $s_0 \to -\infty$ and $s_1 \to +\infty$ and get

$$\mathbb{A}_0(w_-) - \mathbb{A}_0(w_+) = \theta(-\infty, +\infty) + E(w) \geq E(w) - \|F\|.$$

For the second inequality we get from (3.24)

$$\theta(s_0, s_1) \leq \Delta(s_0, s_1), \tag{3.26}$$

Now pick $s' \in \mathbb{R}$ and make the two different substitutions in (3.26):

$$\big(s_0 \to -\infty,\ s_1 = s'\big), \qquad \big(s_0 = s',\ s_1 \to +\infty\big).$$

As a result we obtain the couple of inequalities

$$\begin{cases} \theta(-\infty, s') \leq \mathbb{A}_0(w_-) - \mathbb{A}_{\beta(r,s')}(s') \leq \max\{\mathbb{A}_0(w_-), -\mathbb{A}_0(w_+)\} - \mathbb{A}_{\beta(r,s')}(s'), \\ \theta(s', +\infty) \leq \mathbb{A}_{\beta(r,s')}(s') - \mathbb{A}_0(w_+) \leq \max\{\mathbb{A}_0(w_-), -\mathbb{A}_0(w_+)\} + \mathbb{A}_{\beta(r,s')}(s'). \end{cases}$$

That can be rearranged into

$$\begin{cases} \mathbb{A}_{\beta(r,s')}(s') \leq \max\{\mathbb{A}_0(w_-), -\mathbb{A}_0(w_+)\} - \theta(-\infty, s'), \\ -\mathbb{A}_{\beta(r,s')}(s') \leq \max\{\mathbb{A}_0(w_-), -\mathbb{A}_0(w_+)\} - \theta(s', +\infty)\}. \end{cases}$$

Using again (3.25) we get the desired inequality. □

We know from the very definition of $\mathcal{M}$ that w has limits for s that tends to infinity. So the continuity of $\mathbb{A}_0$ on E_0 implies that, if $(r, w) \in \mathcal{M}$, then

$$\mathbb{A}_0(w_\pm) = \mathbb{A}_0(w(\pm\infty)) = 0.$$

Therefore we have the uniform estimates on $\mathcal{M}$

$$E(w) \leq \|F\|, \tag{3.27}$$

$$\forall s \in \mathbb{R}, \quad |\mathbb{A}_{\beta(r,s)}(w(s))| \leq \|F\|. \tag{3.28}$$

We have said that the energy has a better behaviour under C^∞_{loc}-limits than the asymptotic conditions. This is the content of the next simple proposition that closes this chapter. In the fourth chapter we will focus on the compactness property of $\mathcal{M}$ and we will prove that the moduli space is relatively compact, despite not being closed.

Proposition 3.4.3. *Suppose $w_k \to w$ in the C^∞_{loc}-topology. Then*

$$E(w) \leq \liminf_{k\to+\infty} E(w_k) \tag{3.29}$$

Proof. The proposition follows from the calculation:

$$\begin{aligned} E(w) &= \lim_{a\to+\infty} \int_{-a}^{a} \left\| \frac{dw}{ds}(s) \right\|^2_{w(s)} ds \\ &= \lim_{a\to+\infty} \lim_{k\to+\infty} \int_{-a}^{a} \left\| \frac{dw_k}{ds}(s) \right\|^2_{w_k(s)} ds \end{aligned}$$

$$
\begin{aligned}
&= \lim_{a\to+\infty} \liminf_{k\to+\infty} \int_{-a}^{a} \left\| \frac{dw_k}{ds}(s) \right\|_{w_k(s)}^2 ds \\
&\leq \lim_{a\to+\infty} \liminf_{k\to+\infty} \int_{\mathbb{R}} \left\| \frac{dw_k}{ds}(s) \right\|_{w_k(s)}^2 ds = \liminf_{k\to+\infty} E(w_k).
\end{aligned}
$$

$\square$

Chapter 4

C^∞_{loc}-compactness

In the preceding chapter we have defined the family of r-Equations 3.17

$$\frac{dw}{ds}(s) = -\nabla \mathbb{A}_{\beta(r,s)}(w(s))$$

and starting from them a moduli space $\mathcal{M}$ of solutions:

$$\mathcal{M} := \left\{ (r,w) \in [0,+\infty) \times F_0 \;\middle|\; \begin{array}{l} w \textit{ satisfies the } r\textit{-Equation,} \\ w(-\infty) := \lim_{s\to-\infty} w(s) = (z_0, 0), \\ w(+\infty) := \lim_{s\to+\infty} w(s) \in \Sigma \times 0. \end{array} \right\}$$

An element $(r,w) \in \mathcal{M}$ is composed by a positive real number r and a couple of smooth functions $w = \big(u\colon \mathbb{T}\times\mathbb{R} \to M,\ T\colon \mathbb{R} \to \mathbb{R}\big)$. We have called the space of these couples F_0 and we have endowed it with the C^∞_{loc}-topology.

The purpose of the present chapter is to prove that $\mathcal{M}$ **is relatively compact with respect to the product topology of** $[0,+\infty) \times F_0$. This result will follow from a more general compactness theorem, whose proof is the main content of this chapter.

4.1 Bounded solutions

We need a refinement of the concept of r-Solution, introduced in the preceding chapter.

Definition 4.1.1. Let $r \in [0,+\infty)$. We call w a **bounded r-Solution** if

- w is an r-Solution;
- there exists a compact set K_w, such that the image of u is contained in K_w for large s;
- the asymptotic values of the action $\mathbb{A}_0$ are finite, i. e.

$$\max\{|\mathbb{A}_0(w_-)|, |\mathbb{A}_0(w_+)|\} < +\infty.$$

In the following discussion suppose we are given a generic set $\mathcal{N}$, whose elements are of the form (r, w), where w is a bounded r-Solution for some $r \in [0, \infty)$ (we point out that r is not fixed on this set so that two elements of $\mathcal{N}$ can have different values of the parameter) with *uniform bounds on the asymptotic values of the action.* This means that there exists $A \geq 0$ such that, for every $(r, w) \in \mathcal{N}$, we have

$$\max\{|\mathbb{A}_0(w_-)|, |\mathbb{A}_0(w_+)|\} \geq A.$$

Then Proposition 3.4.2 implies that

$$\bullet\ E(w) \leq 2A + \|F\|, \qquad \bullet\ |\mathbb{A}_{\beta_r}(w)| \leq A + \|F\|.$$

Clearly $\mathcal{M}$ belongs to the class of sets just defined, hence a compactness theorem for a generic $\mathcal{N}$ will apply also $\mathcal{M}$.

The main theorem relies on the elliptic estimates for the Cauchy-Riemann operator and the joint work of *Sobolev embeddings* and the *Arzelà-Ascoli* theorem. However in order to make the mechanism start working we need as an input *a priori* estimates for low derivatives. In our case we have to prove three kinds of uniform estimates for an element $(r, (u, T))$ of $\mathcal{N}$:

1. C^0-bound for u,
2. C^0-bound for the period T,
3. C^1-bound for u.

Obviously it is understood that the bounds does not depend on the particular w and the constant are universal in $\mathcal{N}$. Once this estimates are proven we will need a little additional argument in order to control also the parameter r: this will be the content of the fifth section of this chapter. First we make use of the elliptic regularity of the classic $\overline{\partial}$ operator, in order to prove a corresponding regularity theorem for u.

4.2 Sobolev estimates

We have to recall the three cornerstones on which we are going to build this section: Arzelà-Ascoli Theorem, a version of the Sobolev Embedding Theorem and the elliptic estimates for the operator $\overline{\partial}$.

Theorem 4.2.1. *Let $u_\nu : (X_0, d_0) \to (X_1, d_1)$ be a sequence of continuous functions between two metric spaces, such that X_0 is compact and X_1 is complete. Assume that*

- *there exists a compact set $K \subset X_1$ such that*

$$u_\nu(X_0) \subset K;$$

- *the sequence is uniformly equicontinuous. In other words, $\forall \varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that*

$$d_0(x, x') < \delta_\varepsilon \quad \Longrightarrow \quad d_1(u_\nu(x), u_\nu(x')) < \varepsilon.$$

Then there exists a subsequence converging uniformly on X_0.

Let p be a real number such that $1 \leq p < +\infty$ and let $U \subset \mathbb{C}$ be a bounded open set with smooth boundary. Consider the Sobolev spaces $W^{k,p}(U, \mathbb{R}^{2n})$ for every $k \in \mathbb{N}$. Each of these spaces is the completion of $C^\infty(\overline{U}, \mathbb{R}^{2n})$ with respect to the norm

$$\|u\|^p_{W^{k,p}(U)} := \int_U \left(\sum_{|\alpha| \leq k} |D^\alpha u|^p \right) dsdt,$$

where α is a multiindex. Moreover denote by $\|\cdot\|_{C^k(\overline{U})}$ the norm defined for functions in $C^k(\overline{U})$ by the formula

$$\|u\|_{C^k(\overline{U})} := \sum_{|\alpha| \leq k} \sup_{x \in U} |D^\alpha u(x)|.$$

Often we will use the shorthand $\|\cdot\|_{k,p}$ and $\|\cdot\|_{C^k}$ when the domain is clear from the context and we will indicate simply by $\|\cdot\|_\infty$ the norm in C^0.

Theorem 4.2.2. *If $p > 2$ and U is a bounded open subset of $\mathbb{C}$ with smooth boundary. Then there exist constants $B_{k,p,U}$ such that*

$$\|u\|_{C^{k-1}(U)} \leq B_{k,p,U} \|u\|_{W^{k,p}(U)}.$$

Furthermore the inclusion $W^{k,p}(U) \hookrightarrow C^{k-1}(\overline{U})$ is compact.

The elliptic regularity for the Cauchy-Riemann operator in the integrable case reads in the following way.

Theorem 4.2.3. *Let J_0 be any constant complex structure on $\mathbb{R}^{2n}$ and let us denote by $\overline{\partial}_{J_0}$ the usual Cauchy-Riemann operator associated with J_0*

$$\overline{\partial}_{J_0} = \frac{\partial}{\partial s} + J_0 \frac{\partial}{\partial t}.$$

Then, for every $k \in \mathbb{N}$ and $p \in \mathbb{R}$ such that $1 < p < +\infty$, there exists a constant $A_{k,p}$ such that, for every smooth function with compact support $u \colon \mathbb{C} \to \mathbb{R}^{2n}$, we have

$$\|u\|_{W^{k+1,p}(\mathbb{C})} \leq A_{k,p} \|\overline{\partial}_{J_0} u\|_{W^{k,p}(\mathbb{C})}. \tag{4.1}$$

The proof of these results can be found in many textbook and will not included here: for example the reader can consult Appendix B in (33) for the last two results.
With this theorem at our disposal we wish to prove a regularity theorem for J-holomorphic curves, when J is not constant. The next lemma helps us in finding a useful inequality that goes in this direction. In the following discussion we suppose $p > 2$.

Lemma 4.2.4. *Let Q be a bounded open set in $\mathbb{C}$ and W an open set of $\mathbb{C}^n$, endowed with an almost complex structure J. Fix $\alpha\colon Q \to [0,1]$ a smooth function with compact support. Let (k,p) be a couple defined as before and let J_0 be a constant almost complex structure on W. Then, there exists a constant $C_{k,p,Q,\alpha}$, such that, for every smooth function $u\colon Q \to W$, we have*

$$\begin{aligned}(1-A_{k,p}\|J_0-J_u\|_\infty)\|\alpha u\|_{k+1,p} \leq C_{k,p,Q,\alpha}(1+\|J_u\|_{k,p})\|u\|_{C^1}+\\ + C_{k,p,Q,\alpha}(1+\|J_u\|_\infty)\|u\|_{k,p}+A_{k,p}\|\overline{\partial}_J(u)\|_{k,p}.\end{aligned}$$

Proof. In what follows the first inequality is given by Theorem 4.2.3 and the same symbol C is used to indicate a generic constant which can depend on Q, α, k and p.

$$\begin{aligned}\|\alpha u\|_{k+1,p} &\leq A_{k,p}\|\overline{\partial}_{J_0}(\alpha u)\|_{k,p}\\ &\leq C\|u\|_{k,p} + A_{k,p}\|\alpha\overline{\partial}_{J_0}u\|_{k,p}\\ &\leq C\|u\|_{k,p} + A_{k,p}\|\alpha\overline{\partial}_J(u)\|_{k,p} + A_{k,p}\|\alpha(J_0-J_u)\partial_t u\|_{k,p}\\ &= C\|u\|_{k,p} + C\|\overline{\partial}_J(u)\|_{k,p}+\\ &\quad + A_{k,p}\|(J_0-J_u)\partial_t(\alpha u) - \partial_t\alpha(J_0-J_u)u\|_{k,p}\\ &\leq C\|u\|_{k,p} + C\|\overline{\partial}_J(u)\|_{k,p}+\\ &\quad + A_{k,p}\|(J_0-J_u)\partial_t(\alpha u)\|_{k,p} + A_{k,p}\|\partial_t\alpha(J_0-J_u)u\|_{k,p}.\end{aligned}$$

We make two separate calculations for the last terms

$$\theta_1 := \|(J_0-J_u)\partial_t(\alpha u)\|_{k,p}, \qquad \theta_2 := \|\partial_t\alpha(J_0-J_u)u\|_{k,p}.$$

We use the following inequality for a product of two functions

$$\|\phi\psi\|_{k,p} \leq \|\phi\|_\infty\|\psi\|_{k,p} + \|\psi\|_\infty\|\phi\|_{k,p}.$$

$$\begin{aligned}\theta_1 &\leq \|J_0-J_u\|_\infty\|\partial_t(\alpha u)\|_{k,p} + \|J_0-J_u\|_{k,p}\|\partial_t(\alpha u)\|_\infty\\ &\leq \|J_0-J_u\|_\infty\|\alpha u\|_{k+1,p} + C(1+\|J_u\|_{k,p})\|u\|_{C^1}.\end{aligned}$$

Whereas for the second term we have

$$\theta_2 \leq C\|(J_0-J_u)u\|_{k,p}$$

$$
\begin{aligned}
&\le C\|u\|_\infty \|J_0 - J_u\|_{k,p} + C\|u\|_{k,p}\|J_0 - J_u\|_\infty \\
&\le C\|u\|_\infty (\|J_u\|_{k,p} + 1) + C(1 + \|J_u\|_\infty)\|u\|_{k,p} \\
&\le C\left(\|u\|_{C^1}(\|J_u\|_{k,p} + 1) + (1 + \|J_u\|_\infty)\|u\|_{k,p}\right).
\end{aligned}
$$

Putting these two inequalities in the preceding calculation we get the desired inequality. □

Now we are ready to state the regularity theorem. Since its natural formulation is for curves whose domain is contained in an arbitrary Riemann surface, first we need to generalize the notion of Cauchy-Riemann operator to this case. Indeed, we point out that the expression

$$\overline{\partial}_J := \frac{\partial}{\partial s} + J\frac{\partial}{\partial t}$$

is meaningful only in a coordinate chart. The corresponding global object is described by the next definition.

Definition 4.2.5. Let G be a Riemann surface endowed with complex structure $\jmath$ and M a manifold endowed with an almost complex structure J. For each $u \in C^\infty(G, M)$, is defined $\partial_J(u)$, an antilinear form on G, with values in the bundle u^*TM:

$$\overline{\partial}_J u := du + J \circ du \circ \jmath \;\in \Omega^{0,1}(G, u^*TM). \tag{4.2}$$

Remark 4.2.6. If (t, s) are holomorphic coordinates, then (4.2) becomes

$$\overline{\partial}_J u = \Big(\partial_s u + J_u \partial_t u\Big) ds + \Big(\partial_t u - J_u \partial_s u\Big) dt$$

and by antilinearity

$$\overline{\partial}_J u = 0 \quad \iff \quad \partial_s u + J_u \partial_t u = 0.$$

Theorem 4.2.7. *Let G be a Riemannian surface without boundary and let U_ν be an increasing sequence of open sets whose union is G.*
Let $\ell \in \mathbb{N}_{\ge 1} \cup \{+\infty\}$ and $u_\nu : U_\nu \to M$ be a sequence of C^ℓ-functions with values in a manifold M and let J^ν be a sequence of almost complex structure on M of class C^ℓ. Suppose that there exists an almost complex structure J of class C^0 such that on every compact set

$$J^\nu \xrightarrow{C^0} J.$$

Furthermore the following assumptions hold

1. *there exists a compact set $K \subset M$ such that $u_\nu(U_\nu) \subset K$;*
2. *there is $b > 0$ such that $\|du_\nu\|_\infty \le b$, for all ν;*

3. u_ν *is a* J^ν*-holomorphic curve*

$$\overline{\partial}_{J^\nu} u_\nu = 0. \tag{4.3}$$

4. every z *in* M *has a local coordinate chart* W^z *such that if* $w\colon U \to W^z$ *is a* C^ℓ*-function from some open subset of* $\mathbb{C}$*, we have*

$$\|J^\nu_w\|_{W^{k,p}(U)} \le c_{k,p,x}(1 + \|w\|_{W^{k,p}(U)}). \tag{4.4}$$

Then for every point $x \in G$ *there exists a neighbourhood* Q^x *of* x *and a subsequence* $u^x_{\nu_\mu}$ *such that*

$$\|u^x_{\nu_\mu}\|_{W^{\ell+1,p}(Q_x)} \le C_{\ell,p,x}. \tag{4.5}$$

As a consequence of this, there exists a subsequence u_{ν_μ} *converging to a* J*-holomorphic curve* $u \in C^\ell(G, M)$ *in the* C^ℓ_{loc}*-topology.*

Proof. Let's fix $x \in G$, then the first two assumptions allows for an application of the *Arzelá-Ascoli theorem*. We get a subsequence, which we will still denote by u_ν, that is uniformly convergent on some compact neighbourhood of Q^x_0 to a continuous function u. Letting $W := W^{u(x)}$ the coordinate chart given by Assumption *4*, we can shrink Q^x_0 and suppose that $u_\nu(Q^x) \subset W$, for every ν. Then we can find a sequence of compact neighbourhoods $Q^x_k \subset G,\ 0 \le k \le \ell$, such that

- $Q^x_{k+1} \subset \mathring{Q}^x_k$ and there exists Q^x a compact neighbourhood of x such that $Q^x \subset \mathring{Q}^x_k$ for every $k \le \ell$;
- if $A_{k,p}$ is the constant contained in Theorem 4.2.3 then

$$\frac{1}{2A_{k,p}} \ge \|J_{u(x)} - J^\nu_{u_\nu}\|_{C^0(Q^x_0)};$$

The last point stems out from the fact that $J_{\nu_{u_\nu}}$ converges uniformly to J_u, and J_u is uniformly continuous. Furthermore by Assumption *4* there is a constant $c_{k,p,x}$ such that

$$\|J^\nu_{u_\nu}\|_{W^{k,p}(Q^x_k)} \le a_{k,p,x}(\|u\|_{W^{k,p}(Q^x_k)} + 1).$$

Now we use Assumption *3* and apply Lemma 4.2.4 with $Q = Q^x_k$ and $\alpha = \alpha_k$ having the additional property: $\alpha_k \equiv 1$ on Q^x_{k+1}. We get

$$\begin{aligned}\|u_\nu\|_{W^{k+1,p}(Q^x_{k+1})} \le 2C_{k,p,x}\|u_\nu\|_{C^1}(1 + \|u_\nu\|_{W^{k,p}(Q^x_k)}) +\\ +2C_{k,p,x}(1 + \|J_\nu\|_\infty)\|u_\nu\|_{W^{k,p}(Q^x_k)}.\end{aligned} \tag{4.6}$$

Since the terms $\|u_\nu\|_{C^1}$ and $\|J_\nu\|_\infty$ are uniformly bounded too and $\|u_\nu\|_{W^{1,p}(Q^x)}$ is bounded by Assumption *1*, a repeated use of (4.6) yields

$$\|u_\nu\|_{W^{k+1,p}(Q^x)} \leq C'_{k,p,x}, \quad 1 \leq k \leq \ell. \tag{4.7}$$

This establishes the first part of the theorem. As regard the second statement observe that, thanks to (4.7), an application of Theorem 4.2.2 yields a convergent subsequence on Q^x. This is sufficient to finish the proof. Indeed, as a second step we can choose an exhaustion of G by compact sets K_j. The compactness of each K_j guarantees that the theorem holds for K_j. Finally the theorem is proved by extracting a diagonal subsequence from the subsequences we have found for each K_j.

□

Now we will see how a clever trick allows for an an application of the preceding theorem to the case of a sequence of perturbed J-holomorphic equation. The hypotheses are the same except those regarding the sequence of almost complex structures J^ν.

Corollary 4.2.8. *Let G be a Riemannian surface without boundary and let U_ν be an increasing sequence of open sets whose union is G.*
Let $\ell \in \mathbb{N}_{\geq 1} \cup \{+\infty\}$ and $u_\nu : U_\nu \to M$ be a sequence of C^ℓ-functions with values in a manifold M and let J be an almost complex structure on M of class C^ℓ. Let $\Lambda^\nu : U_\nu \times M \in TM$ be a sequence of C^ℓ-maps such that for every $x \in \mathbb{C}$, $\Lambda_\nu(x, \cdot)$ is a section of TM. Suppose that there exists a continuous map $\Lambda \colon G \times M \in TM$ such that on every compact set

$$\Lambda^\nu \xrightarrow{C^0} \Lambda.$$

Furthermore the following assumptions hold

1. *there exists a compact set $K \subset M$ such that $u_\nu(U_\nu) \subset K$;*
2. *there is $b > 0$ such that $\|du_\nu\|_\infty \leq b$, for all ν;*
3. *u_ν satisfies the perturbed J-holomorphic equation*

$$\overline{\partial}_J u_\nu + \Lambda^\nu(\cdot, u_\nu) = 0. \tag{4.8}$$

4. *for every (x, z) in M there exist local coordinate charts U^x, W^z around these points such that if $w \colon U' \to W^z$ is a C^ℓ-function from some open set $U' \subset U^x$, we have*

$$\|\Lambda^\nu(\cdot, w)\|_{W^{k,p}(U')} \leq c_{k,p,x}(1 + \|w\|_{W^{k,p}(U')}). \tag{4.9}$$

Then for every point $x \in G$ there exists a neighbourhood Q^x of x and a subsequence $u^x_{\nu_\mu}$ such that

$$\|u^x_{\nu_\mu}\|_{W^{\ell+1,p}(Q_x)} \leq C_{\ell,p,x}. \tag{4.10}$$

As a consequence of this, there exists a subsequence u_{ν_μ} converging to $u \in C^\ell(G, M)$ in the C^ℓ_{loc}-topology. u satisfies the equation

$$\overline{\partial}_J u + \Lambda(\cdot, u) = 0. \tag{4.11}$$

Proof. We want to apply Theorem 4.2.7 and so we need a little trick in order to transform perturbed J-holomorphic equations into genuine J^ν-equations. We consider the manifold $\mathbb{C}\times M$ endowed with the almost complex structures

$$J^\nu_{(x,z)}(h_t, h_s, v) := \Big(-h_s, h_t, J_z v + h_t\Lambda^\nu(x,z) + h_s J_z\Lambda_\nu(x,z)\Big).$$

Setting $w_\nu(t,s) = (t, s, u_\nu(t,s))$, a simple calculation shows

$$\overline{\partial}_{J^\nu} w_\nu = \Big(0, 0, \overline{\partial}_J u_\nu + \Lambda^\nu(\cdot, u_\nu)\Big). \tag{4.12}$$

Then

$$J^\nu \xrightarrow{C^0} \widehat{J}$$

where $\widehat{J}_{(x,z)}(h_t, h_s, v) = \Big(-h_s, h_t, J_z(v) + h_t\Lambda(x,z) + h_s J_z\Lambda(x,z)\Big)$.
The Assumption *4* for Λ^ν implies that J^ν satisfies Assumption *4* in 4.2.7. Finally assumption *3* of the preceding theorem is fulfilled since (4.12) implies that w_ν is a J_ν-holomorphic curve. Hence Theorem 4.2.7 gives a subsequence w_{ν_μ} that satisfies

$$\|w_{\nu_\mu}\|_{W^{\ell+1,p}(Q_x)} \leq C_{\ell,p,x}.$$

This implies a similar estimate for u_{ν_μ} and therefore first statement of the theorem is proved. Then the reasoning for the second assertion goes like before. □

Remark 4.2.9. We can substitute the fourth assumptions in the preceding theorems respectively with the stronger hypotheses

$$J^\nu \xrightarrow{C^\ell} J \qquad\qquad \Lambda^\nu \xrightarrow{C^\ell} \Lambda.$$

Now we can start with the first estimate concerning low derivatives. It relies on a maximum principle for subharmonic functions.

4.3 Uniform estimates for u

Since $X_{\tilde{H}}$ and X_F are compactly supported in M, uniformly in t, there exists $b > 0$ such that on the complement of V_b

$$\bullet\ X_{\tilde{H}} \equiv 0, \qquad \bullet\ X_F \equiv 0$$

and b is the smallest real number with this property. Let w be an r-Solution. In particular w satisfies the first equation in (3.18). Thus on the open set

$$U_b := u^{-1}(M \setminus V_b) \subset \mathbb{T} \times \mathbb{R},$$

u satisfies the Cauchy-Riemann equation

$$\frac{\partial u}{\partial s}(t,s) + J_{u(t,s)} \frac{\partial u}{\partial t}(t,s) = 0.$$

Consider the real function $u_\rho := \rho \circ u$ defined on U_b. Then u_ρ is subharmonic.

Lemma 4.3.1. *The function $u_\rho = \rho \circ u$ satisfies*

$$\Delta u_\rho \geq 0, \quad \text{on } U_b.$$

Proof. For the inequalities we need two ingredients. First recall that the complex structure on M is of a very special kind. J satisfies the Equation 3.2

$$d\rho \circ J = \lambda, \quad \text{on } V_{\text{ext}}.$$

Secondly we have the following identity for a 1-form α and vectors (v_1, v_2)

$$d\alpha(v_1, v_2) = v_1\big(\alpha(v_2)\big) - v_2\big(\alpha(v_1)\big) - \alpha\left([v_1, v_2]\right).$$

Now we can begin

$$\begin{aligned}
\Delta u_\rho &= \frac{\partial^2 u_\rho}{\partial t^2} + \frac{\partial^2 u_\rho}{\partial s^2} \\
&= \frac{\partial}{\partial t}\left(\frac{\partial u_\rho}{\partial t}\right) + \frac{\partial}{\partial s}\left(\frac{\partial u_\rho}{\partial s}\right) \\
&= \frac{\partial}{\partial t}\left(d_u\rho\left(\frac{\partial u}{\partial t}\right)\right) + \frac{\partial}{\partial s}\left(d_u\rho\left(\frac{\partial u}{\partial s}\right)\right) \\
&= \frac{\partial}{\partial t}\left(\lambda_u\left(-J_u\frac{\partial u}{\partial t}\right)\right) + \frac{\partial}{\partial s}\left(\lambda_u\left(-J_u\frac{\partial u}{\partial s}\right)\right) \\
&= \frac{\partial}{\partial t}\left(\lambda_u\left(\frac{\partial u}{\partial s}\right)\right) - \frac{\partial}{\partial s}\left(\lambda_u\left(\frac{\partial u}{\partial t}\right)\right) \\
&= \frac{\partial}{\partial t}\left(u^*\lambda\left(\frac{\partial}{\partial s}\right)\right) - \frac{\partial}{\partial s}\left(u^*\lambda\left(\frac{\partial}{\partial t}\right)\right) \\
&= d\left(u^*\lambda\right)\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right)
\end{aligned}$$

$$
\begin{aligned}
&= u^* d\lambda \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial s} \right) \\
&= d\lambda_u \left(\frac{\partial u}{\partial t}, \frac{\partial u}{\partial s} \right) \\
&= d\lambda_u \left(J_u \frac{\partial u}{\partial s}, \frac{\partial u}{\partial s} \right) \\
&= g_u \left(\frac{\partial u}{\partial s}, \frac{\partial u}{\partial s} \right) \geq 0.
\end{aligned}
$$

□

With this lemma the uniform bound follows.

Proposition 4.3.2. *There exists a positive number b, such that for any $(r, w) \in \mathcal{N}$ we have*

$$u(\mathbb{T} \times \mathbb{R}) \subset V_b.$$

Proof. Let $w = (u, T)$ be an r-Solution. As we have done before we can associate with u the open set U_b. If on the one hand this set is empty then $u(\mathbb{T} \times \mathbb{R}) \cap (M \setminus V_b) = \emptyset$ and therefore

$$u(\mathbb{T} \times \mathbb{R}) \subset V_b.$$

On the other hand suppose that U_b is not empty. We can apply the previous lemma and find that u_ρ is subharmonic on U_b. Moreover we know that U_b is bounded by the assumption we made on $\mathcal{N}$. Then we can apply the *maximum principle* to u_ρ and find that it attains its maximum on the boundary of U_b. This means that

$$\rho \circ u \equiv b, \quad \text{on } U_b,$$

hence the thesis. □

In other words Proposition 4.3.2 tells us that there exists a fixed compact set V_b, which contains every cylinder u.

We can deal now with the second estimate: it is a variant of results that was established for the first time in (8). The contact hypothesis will make the argument work.

4.4 Uniform estimates for the period

We observed immediately before Remark 2.3.9 that for critical points we have the *action-period equality* (2.1). Since the critical points are characterized by the vanishing of the gradient, we hope that when the gradient is small we can still control the size of the period with the action. This in turn was proved to be uniformly bounded in Proposition 3.4.2.

On the other hand when the gradient is large the period is bounded since the amount of time in which we have $\|\nabla \mathbb{A}_{\beta(r,s)}(w(s))\| \geq \varepsilon$ is controlled by the energy via a *Markov inequality* and the magnitude of period derivative is controlled by the second equation in (3.18).

Small gradient: period-action inequality

The main result of this paragraph is the following one.

Proposition 4.4.1. *There exist $\varepsilon > 0$ and a positive constant C_a depending on $a \geq 0$ such that if β is a positive real number and $w = (u, T) \in E_0$, with $u(\mathbb{T}) \subset V_a$, the following implication holds*

$$\|\nabla \mathbb{A}_\beta(w)\| \leq \varepsilon \quad \Longrightarrow \quad |T| \leq 2(|\mathbb{A}_\beta(w)| + \|F\| + C_a).$$

For the proof we need two lemmata. In the following discussion we use the notation $U_\delta := \{H \in (-\delta, \delta)\}$.

Lemma 4.4.2. *For every $\delta > 0$ there exists an $\varepsilon > 0$ such that for every $w = (u, T) \in E_0$ we have*

$$\|\nabla \mathbb{A}_\beta(w)\| \leq \varepsilon \quad \Longrightarrow \quad u(t) \in U_\delta, \ \forall t \in [0, \frac{1}{2}] + \mathbb{Z}.$$

Lemma 4.4.3. *There exists $\delta > 0$ such that if $w = (u, T) \in E_0$ with*

$$\bullet\ u(\mathbb{T}) \subset V_a, \qquad \bullet\ u(t) \in U_\delta \quad \forall t \in [0, \frac{1}{2}] + \mathbb{Z},$$

then $\forall \beta > 0$

$$|T| \leq 2|\mathbb{A}_\beta(w)| + 2\|\lambda\|_{L^\infty(V_a)}\|\nabla \mathbb{A}_\beta(w)\| + 2\|F\| + 2\|\lambda\|_{L^\infty(V_a)}\|X_F\|_{L^\infty(\mathbb{T}\times V_a)}.$$

Proof of Lemma 4.4.2. We prove the equivalent implication:

$$\exists \bar{t} \in [0, \frac{1}{2}] + \mathbb{Z},\ |H(u(\bar{t})| > \delta \quad \Rightarrow \quad \|\nabla \mathbb{A}_\beta(w)\| > \varepsilon.$$

Suppose $H(u(\bar{t})) > \delta$, the other case is completely analoguous. There are two possibilities:

1. $\forall t \in [0, \frac{1}{2}],\ H(u(t)) > \frac{\delta}{2}$.

On the other hand if there exists $\tilde{t} \in [0, \frac{1}{2}]$, such that $H(u(\tilde{t})) \leq \frac{1}{2}$, then the connected component of the set $\{t \in [0, \frac{1}{2}] \mid H(t) > \frac{\delta}{2}\}$, which passes through $\bar{t}$ is an interval I' and one of its extreme points t' is not 0 nor $\frac{1}{2}$. Then $H(t') = \frac{\delta}{2}$. Hence if we consider the interval I with extreme points $\bar{t}, t'$ we see that, if the posiibility 1. does not hold, then

2. there exists an interval $I = [t_0, t_1] \subset [0, \frac{1}{2}]$ such that

$$\bullet\ \forall t \in I,\ H(u(t)) \geq \frac{\delta}{2}, \qquad \bullet\ \big|H(u(t_1)) - H(u(t_0))\big| \geq \frac{\delta}{2}.$$

In the first case we use the second summand in the gradient

$$\|\nabla \mathbb{A}_\beta(w)\| \geq \left| \int_{\mathbb{T}} \dot{\chi}(t) H(u(t)) dt \right| > \frac{\delta}{2}.$$

While in the second case we use the first summand

$$\begin{aligned}
\|\nabla \mathbb{A}_\beta(w)\| &\geq \int_{t_0}^{t_1} \Big|\dot{u}(t) - \dot{\chi}(t) X_H(u(t))\Big| dt \\
&\geq \frac{1}{\|X_H\|_\infty} \int_{t_0}^{t_1} \Big|\dot{u}(t) - \dot{\chi}(t) X_H(u(t))\Big| \cdot \Big|J_{u(t)} X_H(u(t))\Big| dt \\
&\geq \frac{1}{\|X_H\|_\infty} \left| \int_{t_0}^{t_1} g_{u(t)}\big(\dot{u}(t) - \dot{\chi}(t) X_H(u(t))), \nabla H(u(t))\big) dt \right| \\
&= \frac{1}{\|X_H\|_\infty} \left| \int_{t_0}^{t_1} g_{u(t)}(\dot{u}(t), \nabla H(u(t))) dt \right| \\
&= \frac{1}{\|X_H\|_\infty} \left| \int_{t_0}^{t_1} d_{u(t)} H(\dot{u}(t)) dt \right| \\
&= \frac{1}{\|X_H\|_\infty} \big|H(u(t_1)) - H(u(t_0))\big| \\
&\geq \frac{1}{\|X_H\|_\infty} \cdot \frac{\delta}{2},
\end{aligned}$$

where $|X_H|_\infty := \|X_H\|_{L^\infty(M)}$. To sum up the lemma holds with

$$\varepsilon := \frac{\delta}{2} \cdot \min \left\{ 1, \frac{1}{\|X_H\|_\infty} \right\}.$$

□

Proof of Lemma 4.4.3.

$$\begin{aligned}
|\mathbb{A}_\beta(w)| &= \left| \int_{\mathbb{T}} u^*\lambda - T \int_{\mathbb{T}} \tilde{H}(t,u)\, dt - \beta \int_{\mathbb{T}} F(t,u)\, dt \right| \\
&\geq \left| \int_{\mathbb{T}} \lambda_u(\dot{u}) dt \right| - |T| \int_{\mathbb{T}} \dot{\chi}(t) |H(u)| dt - \|F\| \\
&\geq \left| \int_{\mathbb{T}} \lambda_u(\dot{u}) dt \right| - \delta |T| - \|F\| \\
&= \left| \int_{\mathbb{T}} \lambda_u(\dot{u} - T X_{\tilde{H}}(t,u) - \beta X_F(t,u)) + \lambda_u(T X_{\tilde{H}}(t,u) + \beta X_F(t,u)) dt \right| + \\
&\quad - \delta |T| - \|F\|
\end{aligned}$$

$$
\begin{aligned}
&\geq |T|\left(\left|\int_{\mathbb{T}} \dot{\tilde{\chi}}(t)\lambda_u(X_H(u))dt\right| - \delta\right) - \|F\| - \|\lambda\|_{L^\infty(V_a)}\|X_F\|_{L^\infty(\mathbb{T}\times V_a)} + \\
&\quad - \left|\int_{\mathbb{T}} \lambda_u(\dot{u} - TX_{\tilde{H}}(t,u) - \beta X_F(t,u))dt\right| \\
&\geq |T|\left(\left|\int_{\mathbb{T}} \dot{\tilde{\chi}}(t)\lambda_u(X_H(u))dt\right| - \delta\right) + \\
&\quad - \|\lambda\|_{L^\infty(V_a)}\|\nabla\mathbb{A}_\beta(w)\| - \|F\| - \|X_F\|_{L^\infty(\mathbb{T}\times V_a)}.
\end{aligned}
$$

Now the contact hypothesis implies that on Σ, $\lambda(X_H) = 1$ and so if δ is sufficiently small the following inequalities hold

$$
\bullet\ \delta < \frac{1}{4}, \qquad \bullet\ \lambda_u(X_H(u)) > \frac{3}{4}, \text{ on } U_\delta.
$$

They give

$$
\left|\int_{\mathbb{T}} \dot{\tilde{\chi}}(t)\lambda_u(X_H(u))dt\right| - \delta \geq \left|\int_{\mathbb{T}} \frac{3}{4}\,\dot{\tilde{\chi}}(t)dt\right| - \frac{1}{4} = \frac{1}{2}.
$$

Substituting in the preceding chain of inequalities and rearranging the terms the first lemma is proved. □

Remark 4.4.4. Before proving the proposition we need to highlight a byproduct of Lemma 4.4.2 which we will use later on in order to establish the bound for the parameter r.
If t_0 and t_1 are numbers in $[0, \frac{1}{2}]$, then the last chain of inequalities implies

$$
\big|H(u(t_1)) - H(u(t_0))\big| \leq \|X_H\|_\infty \left(\int_{\mathbb{T}} \left|\dot{u} - X_{\tilde{H}}(t,u) - X_F(t,u)\right|^2 dt\right)^{\frac{1}{2}}. \tag{4.13}
$$

Proof of Proposition 4.4.1. Choose δ as in Lemma 4.4.3 and use Lemma 4.4.2 to find a corresponding ε. Then if $w \in E_0$ is such that $u(\mathbb{T}) \subset V_a$ and $\|\nabla\mathbb{A}_\beta(w)\| \leq \varepsilon$, Lemma 4.4.2 applies to w and thus it satisfies the hypotheses of Lemma 4.4.3. This gives the inequality

$$
\begin{aligned}
|T| &\leq 2|\mathbb{A}_\beta(w)| + 2\|F\| + 2\|\lambda\|_{L^\infty(V_a)}\|\nabla\mathbb{A}_\beta(w)\| + 2\|\lambda\|_{L^\infty(V_a)}\|X_F\|_{L^\infty(\mathbb{T}\times V_a)} \\
&\leq 2|\mathbb{A}_\beta(w)| + 2\|F\| + 2\|\lambda\|_{L^\infty(V_a)} \cdot \varepsilon + \|\lambda\|_{L^\infty(V_a)}\|X_F\|_{L^\infty(\mathbb{T}\times V_a)} \\
&\leq 2(|\mathbb{A}_\beta(w)| + \|F\| + C_a),
\end{aligned}
$$

where we have set

$$
C_a := \|\lambda\|_{L^\infty(V_a)}(\varepsilon + \|X_F\|_{L^\infty(\mathbb{T}\times V_a)}).
$$

□

Large gradient: Markov inequality

Lemma 4.4.5. *Let w be an r-Solution with finite energy and let $\varepsilon > 0$ be an arbitrary real number. Then we have the Markov inequality*

$$\text{measure}\left\{s \in \mathbb{R} \mid \|\nabla \mathbb{A}_\beta(w(s))\| > \varepsilon\right\} \leq \frac{E(w)}{\varepsilon^2}.$$

Proof. Integrating the following pointwise inequality between functions

$$\varepsilon^2 \cdot \mathbf{1}_{\left\{s\in\mathbb{R} \mid \|\nabla \mathbb{A}_\beta(w(s))\|(s)>\varepsilon\right\}}(s) \leq \left\|\frac{dw}{ds}(s)\right\|^2,$$

(with $\mathbf{1}_B$ we denote the characteristic function of the set B) we get the estimate

$$\varepsilon^2 \cdot \text{measure}\left\{s \in \mathbb{R} \mid \|\nabla \mathbb{A}_\beta(w(s))\| > \varepsilon\right\} \leq \int_{\mathbb{R}} \left\|\frac{dw}{ds}(s)\right\|^2 ds = E(w).$$

□

Now we can put together the results of the preceding paragraph and come up with the bound for the period.

Proposition 4.4.6. *There exists a constant C, such that for any $(r, w) \in \mathcal{N}$ we have*

$$\|T\|_\infty \leq C.$$

Proof. Proposition 4.4.1 give an $\varepsilon > 0$ such that, $\|\nabla \mathbb{A}_\beta(w(s))\| \leq \varepsilon$ implies

$$|T(s)| \leq 2(|\mathbb{A}_\beta(w(s))| + \|F\| + C_b) \leq 2A + 4\|F\| + 2C_b,$$

where b is given by Proposition 4.3.2 and the second inequality is given by the discussion immediately after the definition of $\mathcal{N}$. On the other hand if s' is such that $\|\nabla \mathbb{A}_\beta(w(s'))\| > \varepsilon$, then there exists an interval I such that

$$\text{measure}(I) \leq \frac{E(w)}{2\varepsilon^2}$$

and one extreme is s' and the other is a point s'' such that $\|\nabla \mathbb{A}_\beta(w(s''))\| \leq \varepsilon$. This is a consequence of Lemma 4.4.5. Then the second equation in (3.18) yields

$$\left|\frac{dT}{ds}(s)\right| = \left|\int_{\mathbb{T}} \dot{\chi}(t) H(u(t)) dt\right| \leq \|H\|_\infty.$$

Thus we get

$$|T(s'') - T(s')| = \left|\int_I \frac{dT}{ds}(s) ds\right| \leq \text{measure}(I) \cdot \|\frac{dT}{ds}\|_\infty \leq \|H\|_\infty \cdot \frac{2A + \|F\|}{2\varepsilon^2},$$

where in the last inequality we have used the bound for $E(w)$ in $\mathcal{N}$. Then $\forall s \in \mathbb{R}$ we have

$$|T(s)| \leq 2A + 4\|F\| + 2C_b + \|H\|_\infty \cdot \frac{2A + \|F\|}{2\varepsilon^2}. \tag{4.14}$$

□

This proposition concludes the period estimates: now it is the turn of the first derivatives of u.

4.5 Uniform bounds for ∇u

In this section the exactness of the symplectic form plays a crucial role. Actually the asphericity of ω would have been enough in order to carry on the argument. ω is said to be **aspherical** if for every smooth map $u\colon S^2 \to M$ we have

$$\int_{S^2} u^*\omega = 0. \tag{4.15}$$

The relevance of this hypothesis becomos clear in the light of the next result.

Proposition 4.5.1. *Let $u\colon N \to (M, \omega)$ be a J-holomorphic curve from a Riemannian surface to a symplectic manifold M endowed with a compatible almost complex structure J. Let g be the associated metric on M. Then*

$$(u^*w)_{(t,s)} = \left|\frac{\partial u}{\partial t}(t,s)\right|^2 dt \wedge ds = \left|\frac{\partial u}{\partial s}(t,s)\right|^2 dt \wedge ds. \tag{4.16}$$

If furthermore ω is aspherical and $N = S^2$, then u is a constant map.

Proof. The second equality in (4.16) stems out from the fact that J is an orthogonal map with respect to g. The first follows simply from the definition of J-holomorphic curves

$$u^*\omega\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right) = \omega\left(\frac{\partial u}{\partial t}, \frac{\partial u}{\partial s}\right) = \omega\left(J\frac{\partial u}{\partial s}, \frac{\partial u}{\partial s}\right) = g\left(\frac{\partial u}{\partial s}, \frac{\partial u}{\partial s}\right).$$

Use (4.15) to conclude the proof:

$$0 = \int_{S^2} u^*\omega = \int_{S^2} \left|\frac{\partial u}{\partial t}\right|^2 dt \wedge ds = \int_{S^2} \left|\frac{\partial u}{\partial s}\right|^2 dt \wedge ds.$$

This implies $du \equiv 0$ and finishes the proof. □

In our case we will use a slight modification of this argument due to the fact that ω is exact.

Proposition 4.5.2. *Let $u\colon \mathbb{C} \to (M,\omega)$ be a J-holomorphic curve, where J is an almost complex structure on M. Suppose that the image of u is contained in some compact set, that ω is exact and that the energy of u is finite, i.e.*

$$\int_{\mathbb{C}} u^*\omega = \int_{\mathbb{C}} \left|\frac{\partial u}{\partial s}\right|^2 dt \wedge ds < +\infty.$$

Then u is a constant map.

Proof. Using polar coordinates we can rewrite the energy as

$$\int_0^{+\infty} 2\pi m \left(\int_{\mathbb{T}} \left|\frac{\partial u}{\partial s}(\gamma_m(\theta))\right|^2 d\theta \right) dm,$$

where γ_m is the curve defined by $\gamma_m(\theta) = m\cos(2\pi\theta) + m\sin(2\pi\theta)$. The finiteness of the energy then implies that the function $f\colon [0,+\infty) \to [0,+\infty)$ defined by

$$f(m) := 2\pi m \int_{\mathbb{T}} \left|\frac{\partial u}{\partial s}(\gamma_m(\theta))\right|^2 d\theta$$

is integrable. Hence $\forall \varepsilon > 0$ the following inequality holds for large m:

$$f(m) \leq \frac{\varepsilon}{m}$$

(because $\frac{\varepsilon}{m}$ is not integrable). Therefore there is a sequence (m_j) such that

$$m_j \nearrow +\infty, \quad (2\pi m_j)^2 \int_{\mathbb{T}} \left|\frac{\partial u}{\partial s}(\gamma_m(\theta))\right|^2 d\theta \to 0.$$

Using Jensen inequality this implies

$$0 \leq 2\pi m_j \int_{\mathbb{T}} \left|\frac{\partial u}{\partial s}(\gamma_{m_j}(\theta))\right| d\theta \leq \left((2\pi m_j)^2 \int_{\mathbb{T}} \left|\frac{\partial u}{\partial s}(\gamma_{m_j}(\theta))\right|^2 d\theta \right)^{\frac{1}{2}} \longrightarrow 0.$$

Let D_j be the closed ball in $\mathbb{C}$ of radius m_j centered in 0. Then γ_{m_j} is a curve which parametrizes ∂D_j and a simple calculation yields

$$\left|\frac{d(u \circ \gamma_{m_j})}{d\theta}\right| = 2\pi m_j \left|\frac{\partial u}{\partial s}\right| \circ \gamma_{m_j}. \tag{4.17}$$

Stoke's Theorem finishes the work:

$$\begin{aligned} \left|\int_{D_j} u^*(d\lambda)\right| = \left|\int_{\partial D_j} u^*\lambda\right| &= \left|\int_{\mathbb{T}} \lambda \left(\frac{d(u\circ\gamma_{m_j})}{d\theta}\right) d\theta\right| \\ &\leq \|\lambda\|_{L^\infty(u(\partial D_j))} 2\pi m_j \int_{\mathbb{T}} \left|\frac{\partial u}{\partial s}(\gamma_{m_j})\right|. \end{aligned}$$

Letting j goes to infinity we get that the energy is zero. □

In order to prove a C^1 bound for u in $\mathcal{M}$ we argue by contradiction. Assuming that there exists a sequence of functions u_j whose first derivative norm goes to infinity as j goes to infinity. Then a clever use of the Compactness Theorem 4.2.7 will give a nonconstant limit function that satisfies the hypotheses of Proposition 4.5.2. This contradiction will finish the proof.

Proposition 4.5.3. *There exists a positive constant b such that for any $\big(r,(u,T)\big) \in \mathcal{N}$ we have*

$$\|du\|_\infty \leq b \tag{4.18}$$

Proof. Within this proof we will consider functions on the cylinder $\mathbb{T} \times \mathbb{R}$ as function in $\mathbb{R}^2$ that are 1-periodic in the t variable. As we have outlined before we assume by contradiction that there exist a sequence $\big(r_\nu,(u_\nu,T_\nu)\big)$ and a corresponding sequence of points (t_ν, s_ν) such that

$$|du_\nu(t_\nu, s_\nu)| \longrightarrow +\infty. \tag{4.19}$$

For each ν we make a translation of the domain so that the the point in which the derivate blows up remains fixed. Define

$$\tilde{u}_\nu(t,s) := u_\nu(t+t_\nu, s+s_\nu).$$

Then

$$a_\nu := |d\widetilde{u}_\nu(0,0)| \longrightarrow +\infty.$$

Since u_ν satisfy a perturbed J-holomorphic equation, the same is true of $\widetilde{u}_\nu$. If we define

$$\Lambda_\nu\big((t,s),z\big) := -J_z\left(T_\nu(s)X_{\tilde{H}}(t,z) + \beta(r_\nu,s)X_F(t,z)\right),$$

and the translated operators

$$\widetilde{\Lambda}_\nu\big((t,s),z\big) := \Lambda_\nu\big((t+t_\nu, s+s_\nu),z\big)$$

then we have

$$\overline{\partial}_J u_\nu + \Lambda_\nu\big((t,s),u_\nu\big) = 0, \tag{4.20}$$

$$\overline{\partial}_J \widetilde{u}_\nu + \widetilde{\Lambda}_\nu\big((t,s),u_\nu\big) = 0. \tag{4.21}$$

Since the first derivative diverges the $\overline{\partial}_J u_\nu$ term in (4.21) dominates the term of order 0, (remember that we have already bounded the periods in $\mathcal{M}$). This suggests to perform a rescaling of the functions $\tilde{u}_\nu$ in order to find a further sequence of functions

$$\widehat{u}_\nu(t,s) = \widetilde{u}\left(\frac{1}{a_\nu}(t,s)\right). \tag{4.22}$$

Then $\widehat{u}_\nu$ satisfies the equation

$$\overline{\partial}_J \widehat{u}_\nu + \widehat{\Lambda}_\nu\big((t,s), u_\nu\big) = 0, \tag{4.23}$$

with $\widehat{\Lambda}_\nu\big((t,s),z\big) := \frac{1}{a_\nu}\widetilde{\Lambda}_\nu\big(\frac{1}{a_\nu}(t,s),z\big)$. We want to apply the compactness corollary 4.2.8 with $\ell = 1$ and so we need to fulfill the assumptions contained therein. We claim that the maps

$$\widehat{\Lambda}_\nu = -\frac{1}{a_\nu} J_z\Big(T_\nu(\frac{s}{a_\nu}+s_\nu)X_{\tilde{H}}(\frac{t}{a_\nu}+t_\nu,u_\nu) + \beta(r_\nu,\frac{s}{a_\nu}+s_\nu)X_F(\frac{s}{t_\nu}+t_\nu,u_\nu)\Big)$$

converge in the C^1-topology to 0. Indeed, since a_ν diverges then

$$\widehat{\Lambda}_\nu \xrightarrow{C^0} 0.$$

The only thing to check for the C^1 estimate is that

$$\frac{1}{a_\nu^2}\frac{dT_\nu}{ds} \xrightarrow{C^0} 0.$$

However this is true since $\dfrac{dT_\nu}{ds}$ is uniformly bounded by the equation

$$\frac{dT_\nu}{ds}(s) = -\int_{\mathbb{T}} \tilde{H}(t,u_\nu(t,s))\,dt.$$

The only thing that remains to establish is the uniform boundedness of $d\widehat{u}$. Since $|d\tilde{u}_\nu(0,0)| = a_\nu$, we have a bound $|du_\nu(t,s)| \leq 2a_\nu$, when $|(t,s)| \leq \varepsilon_\nu$. We want ε_ν to satisfy the crucial property $\varepsilon_\nu a_\nu \to +\infty$, so that $\widehat{u}_\nu$ will satisfy $|d\widehat{u}_\nu| \leq 2$ on an exhausting sequence of open balls whose union is the whole plane (with the notation of Theorem 4.2.7 we have $U_\nu := B_{\varepsilon_\nu a_\nu}(0,0)$ and we need $G := \mathbb{C}$ in order to apply Proposition 4.5.2). To achieve this we need a lemma which yields a sequence ε_ν with the desired property, although it might change the blow-up points (t_ν, s_ν). The proof is contained in the sixth chapter of (28).

Lemma 4.5.4. *Let (X,d) be a complete metric space and $g\colon X \to [0,+\infty)$ a continuous map. Assume $x_0 \in X$ and $\varepsilon_0 > 0$ are given. Then there exists $x \in X$ and $\varepsilon > 0$ such that*

- $0 < \varepsilon \leq \varepsilon_0$*;*
- $g(x)\varepsilon \geq g(x_0)\varepsilon_0$*;*
- $d(x,x_0) \leq 2g(x)$ *for all y satysfying $d(y,x) \leq \varepsilon$.*

Let's apply Lemma 4.5.4 with $g = |d\widetilde{u_\nu}|$ and $\varepsilon_0 = 1$. Then we replace the old blow-up points with the new ones but we keep the notation and symbols used so far as if these new points were chosen from the beginning of our discussion. Then $\widehat{u}_\nu$ and $\widehat{\Lambda}_\nu$ satisfy the hypotheses of Corollary 4.2.8. Thus a subsequence of $\widehat{u}_\nu$ (which we will still denote by $\widehat{u}_\nu$) converges to a J-holomorphic plane $\widehat{u}$ in the C^1_{loc}-topology. We claim that the energy of this plane is finite. This is due to the fact that the energy behaves well with respect to translations and rescaling in the domain. Let K be an arbitrary compact subset of $\mathbb{C}$. Then

$$\begin{aligned}\int_K \left|\frac{\partial \widehat{u}}{\partial s}\right|^2 dsdt &= \lim_{\nu\to+\infty} \int_K \left|\frac{\partial \widehat{u}_\nu}{\partial s}\right|^2 dsdt \\ &= \lim_{\nu\to+\infty} \int_{\frac{K}{a_\nu}} \left|\frac{\partial \widetilde{u}_\nu}{\partial s}\right|^2 dtds \\ &= \lim_{\nu\to+\infty} \int_{\frac{K}{a_\nu}+(t_\nu,s_\nu)} \left|\frac{\partial u_\nu}{\partial s}\right|^2 dtds \\ &\le \limsup_{\nu\to+\infty} E(u_\nu) \\ &\le 2A + \|F\|.\end{aligned}$$

The hipotheses of Proposition 4.5.2 are satisfied and therefore $\widehat{u}$ is constant. On the other hand

$$|d\widehat{u}(0,0)| = \lim_{\nu\to+\infty} |d\widehat{u}_\nu(0,0)| = 1$$

gives a contradiction. The proposition is thus proved. □

4.6 An upper bound for the parameter r

The main tool is the following proposition. It strengthens the fact that $\operatorname{Crit}\mathbb{A}_1 = \emptyset$. This in turn was proved making use of displaceability.

Proposition 4.6.1. *There exists a positive constant μ, such that for any $w \in E_0$*

$$\|\nabla \mathbb{A}_1(w)\| \ge \mu.$$

We begin with a lemma. Let

$$\mathfrak{S} := \operatorname{supp}(X_H),$$

namely the closure of the points $z \in M$, such that $X_H(z) \neq 0$. By hypothesis this is a compact set, futhermore its complement $M \setminus \mathfrak{S}$ is disjoint from Σ, since Σ is a regular hypersurface. Therefore

$$\delta_H := \inf_{z\in M\setminus\mathfrak{S}} |H(z)| > 0$$

and if we construct the defining Hamiltonian H as we did in the discussion following Definition 2.3.8 this is nothing but the supremum of H:

$$\delta_H = \|H\|_\infty.$$

Finally we can assume

$$\Phi_F(\mathfrak{S}) \cap \mathfrak{S} = \emptyset \tag{4.24}$$

as we have noticed in Section 3.1.

Lemma 4.6.2. *There exists $\varepsilon_0 > 0$ such that if $(u, T) \in E_0$ satisfies*

$$\Big(u(\frac{1}{2}), u(1)\Big) \in \mathfrak{S} \times \mathfrak{S},$$

then

$$\big\|\dot{u} - TX_{\tilde{H}}(t, u) - X_F(t, u)\big\| \geq \varepsilon_0.$$

Proof. We use F to define a new metric on M. If v belongs to T_zM, then

$$|v|_z^F := \min_{t\in[0,1]} |d_z\Phi_F^t v|_{\Phi_F^t(z)}. \tag{4.25}$$

Remember that $|v|_z = \sqrt{g_z(v, v)}$ and Φ_F^t is the flow of X_F starting at time 0 and ending at time t.
This new metric induces a distance on M in the usual way

$$d^F(z_0, z_1) = \inf_{\gamma\in\Gamma_{z_0}^{z_1}} \int_I |\dot{\gamma}|_{\gamma(t)}^F dt, \tag{4.26}$$

where $\Gamma_{z_0}^{z_1}$ is the space of smooth path from some interval I in M, which connects the points z_0 and z_1. Since F is 1-periodic, (4.24) is equivalent to

$$\mathfrak{S} \cap \Phi_F^{-1}(\mathfrak{S}) = \emptyset. \tag{4.27}$$

Since these two sets are compact, (4.27) implies that their distance is a positive number ε_0. In other words

$$(z_0, z_1) \in \mathfrak{S} \times \Phi_F^{-1}(\mathfrak{S}) \quad\Longrightarrow\quad d^F(z_0, z_1) \geq \varepsilon_0, \tag{4.28}$$

and ε_0 is the largest number with this property. Use $u|_{[\frac{1}{2},1]}$ to construct

$$\begin{array}{rcl} \tilde{u}\colon [\frac{1}{2}, 1] & \to & M \\ t & \mapsto & \big(\Phi_F^t\big)^{-1}(u(t)). \end{array}$$

This is a path that connects the points

- $\tilde{u}(\frac{1}{2}) = \big(\Phi_F^{\frac{1}{2}}\big)^{-1}(u(\frac{1}{2})) = u(\frac{1}{2})$ (recall that $F \equiv 0$ on $[0, \frac{1}{2}] + \mathbb{Z}$) and
- $\tilde{u}(1) = \Phi_F^{-1}(u(1))$ (recall that the periodicity implies $\big(\Phi_F^1\big)^{-1} = \Phi_F^{-1}\big)$.

The hypothesis of the lemma gives

$$\Big(\tilde{u}(\frac{1}{2}), \tilde{u}(1)\Big) \in \mathfrak{S} \times \Phi_F^{-1}(\mathfrak{S}).$$

Therefore (4.28) implies that

$$d^F(\tilde{u}(\frac{1}{2}), \tilde{u}(1)) \geq \varepsilon_0.$$

Now using the definition of $\tilde{u}$ and the formula

$$\frac{d}{dt}\big(\Phi_F^t\big)^{-1}(z)\Big|_{t=t_0} = -d_z\big(\Phi_F^{t_0}\big)^{-1} X_F(t_0, z),$$

We differentiate and get

$$\frac{d\tilde{u}}{dt}(t) = d_{u(t)}\big(\Phi_F^t\big)^{-1}\Big(\dot{u}(t) - X_F(t, u(t))\Big).$$

Then

$$\left|\frac{d\tilde{u}}{dt}\right|^F = \min_{t' \in [0,1]} \left| d\Phi_F^{t'}\big(d\Phi_F^t\big)^{-1}\Big(\dot{u}(t) - X_F(t, u(t))\Big)\right| \leq \big|\dot{u}(t) - X_F(t, u(t))\big|, \tag{4.29}$$

having chosen $t' = t$. The definition of the distance d^F then gives

$$\begin{aligned} \varepsilon_0 \leq d^F\big(\tilde{u}(\frac{1}{2}), \tilde{u}(1)\big) \leq & \int_{\frac{1}{2}}^1 \left|\frac{d\tilde{u}}{dt}\right|^F dt \leq \int_{\frac{1}{2}}^1 |\dot{u}(t) - X_F(t, u(t))| dt \\ & \leq \int_0^1 |\dot{u}(t) - TX_{\tilde{H}}(t, u(t)) - X_F(t, u(t))| dt \\ & \leq \big\|\dot{u}(t) - TX_{\tilde{H}}(t, u(t)) - X_F(t, u(t))\big\| \end{aligned}$$

□

Proof of Proposition 4.6.1. Take ε_0 from the previous lemma and suppose

$$\big\|\dot{u} - TX_{\tilde{H}}(t, u) - X_F(t, u)\big\| \leq \varepsilon' := \min\left\{\varepsilon_0, \frac{\delta_H}{2\|X_H\|_\infty}\right\}. \tag{4.30}$$

Then Remark 4.4.4 tells us that, for t_0, t_1 in $[0, \frac{1}{2}]$,

$$\big|H(u(t_1)) - H(u(t_0))\big| \leq \frac{\delta_H}{2},$$

and Lemma 4.6.2 yields

$$\max\left\{|H(u(0))|, |H(u(\frac{1}{2}))|\right\} \geq \delta_H.$$

Combining these formulae we get for t_0, t_1 in $[0, \frac{1}{2}]$

$$|H(u(t))| \geq \frac{\delta_H}{2}.$$

This in turn implies that the second part of the gradient satisfies

$$\left| \int_{\mathbb{T}} \tilde{H}(t, u) dt \right| \geq \frac{\delta_H}{2}.$$

Therefore the proposition holds if we set

$$\mu := \min \left\{ \varepsilon', \frac{\delta_H}{2} \right\}.$$

Indeed $\left\| \dot{u} - T X_{\tilde{H}}(t, u) - X_F(t, u) \right\| \geq \mu$ easily implies $\|\nabla \mathbb{A}_1(w)\| \geq \mu$. Whereas $\left\| \dot{u} - T X_{\tilde{H}}(t, u) - X_F(t, u) \right\| < \mu \leq \varepsilon'$ yields

$$\left| \int_{\mathbb{T}} \tilde{H}(t, u) dt \right| \geq \frac{\delta_H}{2} \geq \mu \quad \Rightarrow \quad \|\nabla \mathbb{A}_1(w)\| \geq \mu.$$

□

As a corollary we get the bound on the parameter.

Proposition 4.6.3. *Let (r, w) be an element of $\mathcal{N}$. Then*

$$r \leq \frac{2A + \|F\|}{2\mu^2} + 1 \tag{4.31}$$

Proof. For r greater than one we have

$$2A + \|F\| \geq E(w) \geq \int_{-(r-1)}^{r-1} \|\nabla \mathbb{A}_1(w(s))\|^2_{w(s)} ds \geq 2\mu^2(r-1).$$

Rearranging the terms we get what we need. □

4.7 The relative compactness of $\mathcal{N}$

In this final section we will prove the compactness theorem for the abstract space $\mathcal{N}$ and discuss some consequences descending from it. In order to simplify the notation in the proofs, every time we pass to a subsequence and discard the whole sequence in the subsequent discussion we will not change the indexing and no additional subscript will be added.

Theorem 4.7.1. *Let $(r_\nu, w_\nu))$ be a sequence in $\mathcal{N}$ and $s_\nu \to \overline{s} \in [-\infty, +\infty]$ and $t_\nu \to \overline{t}$, two sequences of real numbers. Define the translated sequence*

$$\widehat{w}_\nu(t, s) := (\widehat{u}_\nu(t, s), \widehat{T}_\nu(s)) := (u_\nu(t + t_\nu, s + s_\nu), T_\nu(s + s_\nu))$$

Then there exists a subsequence $(r_{\nu_\mu}, \widehat{w}_{\nu_\mu})$ *such that*

$$(r_{\nu_\mu}, \widehat{w}_{\nu_\mu}) \xrightarrow{C^\infty_{\text{loc}}} (r, \widehat{w}).$$

Moreover if $\bar{t} = 0$ *and* $\bar{s} = 0$*, then* $\widehat{w}$ *is an* r*-Solution, whereas if* $\bar{t} = 0$ *and* $\bar{s} \in \{-\infty, +\infty\}$*,* $\widehat{w}$ *is a* 0*-Solution. In any case*

$$E(\widehat{w}) \leq 2A + \|F\|.$$

Proof. We will prove only the case $t_\nu \equiv 0$, $s_\nu \equiv 0$. The general case is conceptually identical since the only additional feature is that the Hamiltonian terms depend on ν in the following way: $H^\nu(t,z) := H(t+t_\nu, z)$, $\beta_\nu(s)F^\nu(t,z) := \beta(r_\nu, s+s_\nu)F(t+t_\nu, z)$. However since H and βF have uniform bounds this is not an obstacle to get the estimates we need. Now we can start the argument.

Proposition 4.6.3 implies that r_ν is bounded and therefore we can assume $r_\nu \to r$. Proposition 4.4.6 and the second equation in (3.18) imply that T_ν is uniformly bounded with its first derivatives and therefore using Arzelà-Ascoli theorem, we can assume that

$$T_\nu \xrightarrow{C^0} T.$$

Define the maps

- $\Lambda^\nu\big((t,s), z\big) := -J_z\Big(T_\nu(s)X_{\tilde{H}}(t,z) + \beta_{r_\nu}(s)X_F(t,z)\Big),$ (4.32)
- $\Lambda\big((t,s), z\big) := -J_z\Big(T(s)X_{\tilde{H}}(t,z) + \beta_r(s)X_F(t,z)\Big).$ (4.33)

Then u_ν and Λ^ν satisfy

$$\overline{\partial}_J u_\nu + \Lambda^\nu(\cdot, u_\nu) = 0, \qquad \Lambda^\nu \xrightarrow{C^0} \Lambda.$$

Furthermore another application of the Arzelá-Ascoli theorem yields a subsequence u_ν converging C^0_{loc} to a continuous function u.

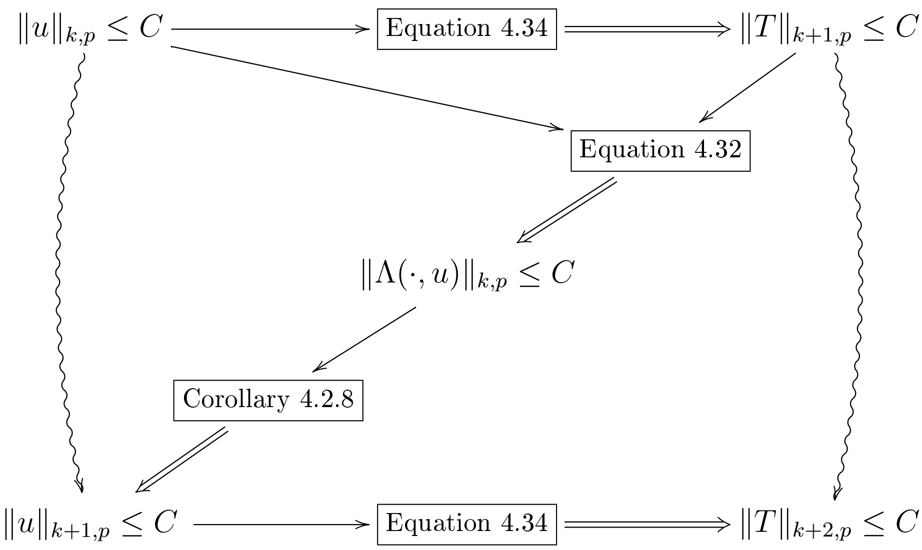
In order to prove the theorem we wish to have the following estimates for each compact subset $K \in \mathbb{C}$ and every natural number k

$$\|u_\nu\|_{W^{k,p}(K)} \leq C_{k,p,K}.$$

We aim to use Corollary 4.2.8. However we have to be careful since the regularity of Λ^ν depends essentially on the regularity of T_ν and this in turn relies on the regularity of u_ν via the equation

$$\frac{dT_\nu}{ds}(s) = -\int_{\mathbb{T}} \tilde{H}(t, u_\nu(t,s))\, dt. \tag{4.34}$$

Therefore we have to work following the inductive scheme represented below.



We cannot use immediately the second part of Corollary 4.2.8, but before we use the estimate (4.10) into the above scheme and only at the end, when the regularity is higher enough, we can apply the part of the corollary concerning the uniform convergence on compact sets.
Thus we want to find $W^{k,p}$-boundsnear a point $(\bar{t}, \bar{s})$. Let $I_{\overline{s}} \subset$ be a compact neighbourhood for $\overline{s}$. The first difficulty we encounter is that (4.34) shows that the estimate for T_ν on $I_{\overline{s}}$ depends on the value of u on the set $\mathbb{T} \times I_{\overline{s}}$, which contains points that are far from our fixed point $(\bar{t}, \bar{s})$. This fact is unpleasant since there might not exist a single local chart in M, containing all the images of $u_\nu(\mathbb{T} \times I_{\overline{s}})$. However since $\mathbb{T} \times I_{\overline{s}}$ is compact we can cover it with a finite number of open sets of the form $U_j \times I_{\overline{s}}$, where j is an index ranging within a finite set. Then we can suppose that for each j, $u_\nu(U_j \times I_{\overline{s}})$ is contained in some chart W_j and we can try to estimate the Sobolev norms of u_ν on all these sets simultaneously. This will be possible since a bound for T_ν on $I_{\overline{s}}$ gives bounds for Λ^ν over each U_j.
By assumption we have the initial estimates

$$\sum_j \|u_\nu\|_{W^{1,p}(U_j \times I_{\overline{s}})} \leq C, \quad \|T_\nu\|_{W^{1,p}(I_{\overline{s}})} \leq C.$$

Then let's work along the lines of the schemes represented above. In the following discussion C denotes a generic positive constant.
Start with $\sum_j \|u_\nu\|_{W^{k,p}(U_j \times I_{\overline{s}})} \leq C$. Then we find

- $\|T_\nu\|_{W^{k+1,p}(I_{\overline{s}})} \leq C$.

Set

$$f^{\nu}_{k+1} := \left(\int_{I_{\overline{s}}} \left| \frac{d^{k+1} T_\nu}{ds^{k+1}} \right|^p \right)^{\frac{1}{p}} \qquad g^{\nu}_k(t,s) = \frac{d^k}{ds^k} \Big(X_{\tilde{H}}(t, u_\nu(t,s)) \Big).$$

Thus we have

$$\begin{aligned} f^{\nu}_{k+1} &= \left(\int_{I_{\overline{s}}} \left| \int_{\mathbb{T}} g^{\nu}_k(t,s) dt \right|^p ds \right)^{\frac{1}{p}} \\ &\leq \sum_j \left(\int_{I_{\overline{s}}} \left| \int_{U_j} g^{\nu}_k(t,s) dt \right|^p ds \right)^{\frac{1}{p}} \\ &\leq C \sum_j \left(\int_{U_j \times I_{\overline{s}}} |g^{\nu}_k(t,s)|^p \, dt ds \right)^{\frac{1}{p}} \\ &\leq C \sum_j \|g^{\nu}_k\|_{L^p(U_j \times I_{\overline{s}})}. \end{aligned}$$

Then a simple inspection shows

$$\|g^{\nu}_k\|_{L^p(U_j \times I_{\overline{s}})} \leq C \|u_\nu\|_{W^{k,p}(U_j \times I_{\overline{s}})}.$$

- $\|\Lambda^\nu(t, u_\nu)\|_{W^{k+1,p}(U_j \times I_{\overline{s}})} \leq C.$

  The only quantity that need a careful estimate is

  $$h^{\nu}_k := \|T_\nu J_{u_\nu} X_{\tilde{H}}(t, u_\nu)\|_{W^{k,p}(U_j \times I_{\overline{s}})}.$$

  $$\begin{aligned} h^{\nu}_k &\leq C \|T_\nu\|_{C^k(I_{\overline{s}})} \|J_{u_\nu} X_{\tilde{H}}(t, u_\nu)\|_{W^{k,p}(U_j \times I_{\overline{s}})} \\ &\leq C \|T_\nu\|_{C^k(I_{\overline{s}})} \big(\sup_{t \in U_j} \|J_z X_{\tilde{H}}(t,z)\|_{C^k(W_j)} \big) (1 + \|u_\nu\|_{W^{k,p}(U_j \times I_{\overline{s}})}). \end{aligned}$$

  The uniform bound now follows, since the preceding point and the Sobolev inequality give a bound for $\|T_\nu\|_{C^k(I_{\overline{s}})}$.

- $\|u_\nu\|_{W^{k+1,p}(U_j \times I_{\overline{s}})} \leq C.$

  We are in position to apply Corollary 4.2.8, with $\ell := k$. Here there is another subtlety, since the corollary gives the estimate for the $W^{k+1,p}$-norm of u_ν on a smaller neighbourhood U'_j, but we want that the new sets still cover $\mathbb{T}$. However, if we look at the proof of Theorem 4.2.7, the shrinking of the neighbourhood is needed for the construction of the chain Q^x_k and the difference between U'_j and U_j can be made arbitrarily small, so that U'_j still cover $\mathbb{T}$.

Then we have completed the inductive step of the scheme and we have uniform bounds for the derivative of w_ν in each order. An application of the Sobolev Embedding Theorem 4.2.2 then gives a convergent subsequence. Passing to the limit in

$$\overline{\partial}_J u_\nu + \Lambda^\nu(\cdot, u_\nu) = 0$$

and in Equation 4.34 we find that the limit function is an r-Solution. □

Corollary 4.7.2. *For each couple of natural numbers (h, k) and each natural number m, there exist positive constants $C_{h,k}$ and C_m such that, for every $(r, (u, T)) \in \mathcal{N}$,*

$$|\partial_s^h \partial_t^k u(t, s)| \le C_{h,k}, \qquad \left| \frac{d^m T}{ds^m} \right| \le C_m. \tag{4.35}$$

Proof. Arguing by contradiction, there exist a multiindex $(\overline{h}, \overline{k})$, a sequence of functions (u_ν, T_ν) in $\mathcal{N}$ and a sequence of points (t_ν, s_ν) such that

$$\lim_{\nu \to +\infty} |\partial_s^{\overline{h}} \partial_t^{\overline{k}} u_\nu(t_\nu, s_\nu)| = +\infty.$$

By compactness of $\mathbb{T} \times [-\infty, +\infty]$ we can suppose that $(t_\nu, s_\nu) \to (\overline{t}, \overline{s})$. Hence applying the preceding theorem we find that the translated sequence has a subsequence $(\widehat{u}_\nu, \widehat{T}_\nu)$ converging on compact sets. However this is a contradiction since

$$|\partial_s^{\overline{h}} \partial_t^{\overline{k}} \widehat{u}_\nu(0)| = |\partial_s^{\overline{h}} \partial_t^{\overline{k}} u_\nu(t_\nu, s_\nu)| \to +\infty.$$

□

Corollary 4.7.3. *Let $(r_\nu, w_\nu))$ be a sequence in $\mathcal{M}$ and $s_\nu \to \overline{s} \in [-\infty, +\infty]$ and $t_\nu \to \overline{t}$, two sequences of real numbers. Define the translated sequence*

$$\widehat{w}_\nu(t, s) := (\widehat{u}_\nu(t, s), \widehat{T}_\nu(s)) := (u_\nu(t + t_\nu, s + s_\nu), T_\nu(s + s_\nu))$$

Then there exists a subsequence $(r_{\nu_\mu}, \widehat{w}_{\nu_\mu})$ such that

$$(r_{\nu_\mu}, \widehat{w}_{\nu_\mu}) \xrightarrow{C^\infty_{\text{loc}}} (r, \widehat{w}).$$

Moreover if $\overline{t} = 0$ and $\overline{s} = 0$, then $\widehat{w}$ is an r-Solution, whereas if $\overline{t} = 0$ and $\overline{s} \in \{-\infty, +\infty\}$, then $\widehat{w}$ is a 0-Solution. In any case

$$E(\widehat{w}) \le \|F\|.$$

Corollary 4.7.4. *For each couple of natural numbers (h, k) and each natural number m, there exist positive constants $C_{h,k}$ and C_m such that, for every $(r, (u, T)) \in \mathcal{M}$,*

$$|\partial_s^h \partial_t^k u(t, s)| \le C_{h,k}, \qquad \left| \frac{d^m T}{ds^m} \right| \le C_m. \tag{4.36}$$

Proof of Corollaries 4.7.3 and 4.7.4. As we have pointed out before we can take $\mathcal{N} := \mathcal{M}$. □

Let us tell something about the asymptotic behavior of elements in $\mathcal{N}$. In particular the next proposition shows that every bounded r-Solution yields a number of critical points for $\mathbb{A}_0$. We hope to find Reeb orbits among them.

Proposition 4.7.5. *Let $w = (u, T)$ be a bounded r-Solution. Moreover let $s_\nu \to +\infty$ (the case $s_\nu \to -\infty$ is identical) and let $t_\nu \to \bar{t}$. Then there exists a subsequence s_{ν_μ} and (u, T) a constant path in E_0 (i.e. $\big(u(t, s), T(s)\big) \equiv \big(u(t), T\big)$), such that*

$$\Big(u(\cdot + t_{\nu_\mu}, \cdot + s_{\nu_\mu}), T(\cdot + s_{\nu_\mu})\Big) \xrightarrow{C^\infty_{\mathrm{loc}}} (u, T).$$

This implies in particular that

$$\Big(u(\cdot + t_{\nu_\mu}, s_{\nu_\mu}), T(s_{\nu_\mu})\Big) \xrightarrow{E_0} (u, T).$$

If furthermore $\bar{t} = 0$, $(u, T) \in \operatorname{Crit} \mathbb{A}_0$ and the following equality holds

$$\mathbb{A}_0(u, T) = \mathbb{A}_0(w_+).$$

Proof. Set $w_\nu(t, s) := w(t + t_\nu, s + s_\nu)$. Then we can apply the point 2 of Theorem 4.7.1 to $\mathcal{N} := \{w\}$ and find $w_{\nu_\mu} \xrightarrow{C^\infty_{\mathrm{loc}}} \widehat{w}$. We claim that $\widehat{w}$ is constant in the variable s. If K is a compact subset of $\mathbb{C}$, then, since the energy is finite:

$$\begin{aligned}\int_K |\partial_s \widehat{w}|^2 dtds &= \lim_{\mu \to +\infty} \int_K |\partial_s w_{\nu_\mu}|^2 dtds \\ &= \lim_{\mu \to +\infty} \int_{K + (t_{\nu_\mu}, s_{\nu_\mu}} |\partial_s w|^2 dtds = 0.\end{aligned}$$

So

$$\widehat{w}(s) \equiv \widehat{w}_0 = (\widehat{u}_0, \widehat{T}_0).$$

The uniform convergence of w_{ν_μ} on the compact set $\mathbb{T} \times 0$ yields the desired conclusion on the convergence in E_0.

The statement regarding the case $\bar{t} = 0$ is obvious. □

Corollary 4.7.6. *Let $w = (u, T)$ be a bounded r-Solution and set*

$$a_- := \mathbb{A}(w_-), \qquad a_+ := \mathbb{A}(w_+).$$

Let $\overline{s} \geq \frac{\max\{|a_-|, |a_+|\} + \|F\|}{2\mu^2} + 1$ be a real number, then

1. *for any* $\varepsilon > 0$ *and any couple of integers* $h \geq 1$, $k \geq 0$, *there exists* $\delta_{h,k} > 0$ *depending only on* a_-, a_+, h, k, *such that if*

$$\mathbb{A}_0(-\overline{s}) \geq a_- - \delta_{h,k}, \qquad \mathbb{A}_0(\overline{s}) \leq a_+ + \delta_{h,k},$$

hold true, then

$$\sup_{t \in \mathbb{T} \times \{|s| \geq \overline{s}\}} |\partial_s^h \partial_t^k u(t,s)| \leq \varepsilon; \tag{4.37}$$

2. *for any* $\varepsilon > 0$ *and any integer* $h \geq 1$, *there exists* $\delta_h > 0$ *depending only on* a_-, a_+, h, *such that if*

$$\mathbb{A}_0(-\overline{s}) \geq a_- - \delta_h, \qquad \mathbb{A}_0(\overline{s}) \leq a_+ + \delta_h,$$

hold true, then

$$\sup_{s \geq \overline{s}} \left| \frac{d^h T}{ds^h}(s) \right| \leq \varepsilon; \tag{4.38}$$

3. *for any* U_{a_-}, U_{a_+} *couple of neighborhoods of* $\operatorname{Crit} \mathbb{A}_0 \cap \mathbb{A}_0^{-1}(a_-)$ *and* $\operatorname{Crit} \mathbb{A}_0 \cap \mathbb{A}_0^{-1}(a_+)$ *respectively, there exists* $\delta_{U_{a_-}, U_{a_+}} > 0$ *depending only on* U_{a_-}, U_{a_+}, *such that if*

$$\mathbb{A}_0(-\overline{s}) \geq a_- - \delta_{U_{a_-}, U_{a_+}}, \qquad \mathbb{A}_0(\overline{s}) \leq a_+ + \delta_{U_{a_-}, U_{a_+}},$$

hold true, then

$$w(s) \in U_{a_-}, \text{ for } s \leq -\overline{s}, \qquad w(s) \in U_{a_+}, \text{ for } s \geq \overline{s}. \tag{4.39}$$

Proof. We consider only the case of positive values of s. We argue by contradiction and suppose that for some couple (h,k), there exist ε_0 and sequences r_ν, w_ν and $s_\nu \geq \overline{s}$ such that

- $r_\nu \leq \max\{|a_-|, |a_+|\} + \|F\| \leq \overline{s}$,
- w_ν is a r_ν-Solution,
- $\mathbb{A}_0(w_{\nu-}) = a_-$, $\mathbb{A}_0(w_{\nu+}) = a_+$,
- $\lim_{\nu \to +\infty} \mathbb{A}_0(w(s_\nu)) = a_+$,
- $s_\nu \to \overline{s} \in [s_0, +\infty]$,
- $|\partial_s^h \partial_t^k u_\nu(t,s)| \geq \varepsilon_0$.

By Corollary 4.7.3 the translated sequence $w_\nu(\cdot + s_\nu)$ admit a convergent subsequence $w_{\nu_\mu} \to w$. Since w is a C^∞_{loc}-limit we have

$$|\partial_s^h \partial_t^k u(t,s)| \geq \varepsilon_0, \quad \mathbb{A}_0(w(0)) = a_+$$

and since $s_\nu \geq \bar{s}$, w is a 0-solution for positive values of s. Furthermore if $s \geq 0$, then

$$\mathbb{A}_0(w(s)) = \lim_{\mu \to +\infty} \mathbb{A}_0(w_{\nu_\mu}(s + s_{\nu_\mu})) \geq a_+.$$

Then $\mathbb{A}_0(w(s)) \equiv a_+$, for $s \geq 0$. This implies

$$0 = \frac{d}{ds}\mathbb{A}_0(w(s)) = -\|\partial_s w\|^2.$$

Finally $\partial_s w \equiv 0$ implies the contradiction

$$0 = |\partial_s^h \partial_t^k u_\nu(t,s)| \geq \varepsilon_0.$$

The estimates for the derivatives of T can be found following the same recipe.

As regard the last point of the corollary we observe that arguing by contradiction one more time we find a function w such that for $s \geq 0$,

- w is a 0-Solution, • $w \equiv w(0) \notin U_{a_+}$, • $\mathbb{A}_0(w) \equiv a_+$.

The first and third point imply that $w(0) \in \operatorname{Crit} \mathbb{A}_0 \cap \mathbb{A}_0^{-1}(a_+) \subset U_{a_+}$, which contadicts the second point. □

Remark 4.7.7. It is not true without further assumptions that there exists $(u_\pm, T_\pm) \in \operatorname{Crit} \mathbb{A}_0$ such that

$$\big(u(\cdot + t_{\nu_\mu}, s), T(s)\big) \xrightarrow{E_0} (u_\pm, T_\pm), \text{ as } s \to \pm\infty.$$

The problem is that even if w gets closer and closer to the critical subsets $\operatorname{Crit} \mathbb{A}_0 \cap \mathbb{A}_0^{-1}(a)$ and $\operatorname{Crit} \mathbb{A}_0 \cap \mathbb{A}_0^{-1}(b)$ it may winds tangentially around them without converging to a specific critical point. We will see in the next chapter that this problem can be fixed by assuming that $\operatorname{Crit} \mathbb{A}_0 \cap \mathbb{A}_0^{-1}(a)$ and $\operatorname{Crit} \mathbb{A}_0 \cap \mathbb{A}_0^{-1}(b)$ are Morse-Bott component for the functional $\mathbb{A}_0$.

Chapter 5

Morse-Bott theory

Using the results from the preceding chapter we wish to study the asymptotic behavior of the class $\mathcal{N}_0$ of smooth functions $w = (u\colon \mathbb{T} \times \mathbb{R} \to M, T\colon \mathbb{R} \to \mathbb{R})$, having the properties:

1. w is a bounded r-Solution,
2. $\mathbb{A}(w_+) = \mathbb{A}(w_-) = 0$.

Obviously we have $\mathcal{M} \subset \mathcal{N}_0$, hence all the statements we are going to prove for elements in $\mathcal{N}_0$, are true also for elements in $\mathcal{M}$.

The results of Section 4.7 apply to $\mathcal{N}_0$ and they will be important in several points of the discussion. However the convergence results we are going to find rely on an additional crucial property, namely the fact that 0 is a **Morse-Bott critical value** for $\mathbb{A}_0$. This is a generalization of the notion of *Morse critical value*.

5.1 Generalities

We say that b is a Morse critical value for a functional ϕ if, at the critical subset $\operatorname{Crit}\phi \cap \phi^{-1}(b)$, the Hessian of ϕ is nondegenerate. This can be seen as a particular case of the following notion.

Definition 5.1.1. Let $\phi\colon N \to \mathbb{R}$ a functional of class C^2 on some Banach manifold. A real number $b \in \mathbb{R}$ is called **a Morse-Bott critical value**, if the set $N_b := \operatorname{Crit}\phi \cap \phi^{-1}(b)$ is a Banach submanifold of N and for every $q \in N_b$

$$\ker \mathcal{H}_\phi(q) = T_q N_b,$$

where $\mathcal{H}_\phi$ is the Hessian of ϕ. In this case N_b is called **a Morse-Bott component for** ϕ.

The fact that 0 is a Morse-Bott critical value for $\mathbb{A}_0$ corresponding to the component $\Sigma \times 0$ is the essential ingredient to prove the main theorem of this chapter.

Theorem 5.1.2. *For each $w \in \mathcal{N}_0$ there are two points z_- and z_+ in Σ, such that*

$$w \overset{E_0}{\longrightarrow} (z_\pm, 0), \qquad as\ s \to \pm\infty.$$

Moreover there exist three positive constants $\overline{\delta}, C, a$ not depending on w and $U_- \subset M$, $U_+ \subset M$ two coordinate neighborhoods of z_- and z_+ respectively, such that, if for some $\overline{s} \geq r$ the conditions

$$\mathbb{A}_0(w(-\overline{s})) \geq -\overline{\delta}, \qquad \mathbb{A}_0(w(\overline{s})) \leq \overline{\delta},$$

hold true, then

$$u\big|_{\mathbb{T}\times\{\pm s \geq \overline{s}\}} \subset U_\pm.$$

and we have the exponential decay

$$\max\left\{|u - z_\pm|, |T|, |\partial_s u|, |\partial_t u|, \left|\frac{dT}{ds}\right|\right\} \leq Ce^{\frac{a}{2}(\overline{s}-|s|)}, \quad for\ \pm s \geq \overline{s}. \tag{5.1}$$

We will carry out the discussion for the positive asymptot only. The other case can be treated in a similar fashion.

As a first step observe that for some $\overline{y} > 0$, Σ has an open neighborhood in M of the form $\{H \in (-\overline{y}, \overline{y})\}$ and such that

$$\{H \in (-\overline{y}, \overline{y})\} = \bigcup_{z\in\Sigma} U_z.$$

For each $z \in \Sigma$, $U_z \subset M$ is a coordinate neighborhood of z, diffeomorphic to $\tilde{U}_z \times (-\overline{y}, \overline{y}) \subset \mathbb{R}^{2n-1} \times \mathbb{R}$ and such that the coordinate map extends to a neighborhood of the closure of U_z. Furthermore if x is the coordinate on the $\mathbb{R}^{2n-1}$-factor and y the coordinate on the $\mathbb{R}$-factor, then the following three conditions hold

$$\bullet\ U \cap \Sigma = \{y = 0\}, \qquad \bullet\ H(x, y) = y, \qquad \bullet\ J_z = J_0,$$

where J_0 is the standard complex structure in $\mathbb{R}^{2n}$.

Remark 5.1.3. Observe that the y-coordinate of a point in U_z does not depend on z.

Suppose we are given an element $w = (u, T)$ in $\mathcal{N}_0$ and an interval $I = [s_0, s_1]$, such that $u(\mathbb{T} \times I) \subset U_z$. Since by Proposition 4.6.3

$$r \leq \frac{\|F\|}{2\mu^2} + 1,$$

we assume from now on that $s_0 \geq \frac{\|F\|}{2\mu^2} + 1$, so that w is a 0-solution on I. Then using the coordinates on U_z, we split u in its components (u_x, u_y) and write the 0-Equation in these coordinates:

$$\left(\partial_s u + J_u \partial_t u - T f \partial_y, \frac{dT}{ds} - \int_{\mathbb{T}} u_y f\, dt\right) = (0, 0) \tag{5.2}$$

(where $f := \dot{\chi}$ and ∂_y is the y-coordinate vector).

In this equation we isolate the terms involving s-derivates from the others. Then if we consider w as a path $w : I \to C^\infty(\mathbb{T}, U_z) \times \mathbb{R}$ we see that the latter terms operate on each $w(s) \in C^\infty(\mathbb{T}, \mathbb{R}^{2n}) \times \mathbb{R}$ separatedly. Thus we are led to consider the path of linear maps, defined for $s \in I$,

$$\begin{aligned} A(s) \colon C^\infty(\mathbb{T}, \mathbb{R}^{2n}) \times \mathbb{R} &\to C^\infty(\mathbb{T}, \mathbb{R}^{2n}) \times \mathbb{R} \\ (v, S) &\mapsto \left(J_u \dot{v} - S f \partial_y, -\int_{\mathbb{T}} v_y f dt \right). \end{aligned} \tag{5.3}$$

We notice that the dependance on s is due to the fact that the matrix J_u is dependent on u. Since u takes value in U_z (in other words near z), we hope that investigating the properties of the single operator

$$\begin{aligned} A_0 \colon C^\infty(\mathbb{T}, \mathbb{R}^{2n}) \times \mathbb{R} &\to C^\infty(\mathbb{T}, \mathbb{R}^{2n}) \times \mathbb{R} \\ (v, S) &\mapsto \left(J_0 \dot{v} - S f \partial_y, -\int_{\mathbb{T}} v_y f dt \right) \end{aligned} \tag{5.4}$$

will give enough information on this path of operators.

5.2 The Hessian operator A_0

The first thing to do is to extend $A(s)$ to a continuous linear map between two suitable Hilbert space completions of $C^\infty(\mathbb{T}, \mathbb{R}^{2n}) \times \mathbb{R}$. The norms that we will use to define the completions have to

- take into account also the derivatives of w (in view of (5.1)),
- be induced by a scalar product (in order to write explicitly their derivatives).

Let $k \in \mathbb{N}$ and endow $C^\infty(\mathbb{T}, \mathbb{R}^{2n}) \times \mathbb{R}$ with the $W^{k,2}$-scalar product:

$$\langle (u_1, T_1), (u_1, T_2) \rangle := \int_{\mathbb{T}} \sum_{j \le k} g_0\left(\frac{d^j u_1}{dt^j}, \frac{d^j u_2}{dt^j}\right) dt + T_1 T_2,$$

where $g_0(\cdot, \cdot)$ is a scalar product compatible with J_0 (the standard scalar product on $\mathbb{R}^{2n}$ will do):

$$g_0(J_0 u_1, J_0 u_2) = g_0(u_1, u_2).$$

Then, we choose $W^{k+1,2}(\mathbb{T}, \mathbb{R}^{2n}) \times \mathbb{R}$ as the domain and $W^{k,2}(\mathbb{T}, \mathbb{R}^{2n}) \times \mathbb{R}$ as codomain, for some $k \in \mathbb{N}$ (we will see that $k = 2$ will suffice). We denote still by $A(s)$ and A_0 the extended operators and notice that we can regard them both as continuous linear map between these two spaces, and as unbounded operators in $W^{k,2}(\mathbb{T}, \mathbb{R}^{2n}) \times \mathbb{R}$ defined on the dense domain $W^{k+1,2}(\mathbb{T}, \mathbb{R}^{2n}) \times \mathbb{R}$. A_0 and $A(s)$ belong to an important class of linear operators, they are *Fredholm operators*.

Definition 5.2.1. Let $L\colon E_1 \to E_2$ be a continuous linear operator between two Banach spaces. L is said to be a **Fredholm operator** if the following three conditions hold:

- $\dim\ker L < \infty$,
- $\operatorname{im} L$ is closed,
- $\dim\operatorname{coker} L < \infty$.

We can associate to each Fredholm operator an integer number $\imath(L)$ called the **Fredholm index** of L:

$$\imath(L) := \dim\ker L - \dim\operatorname{coker} L.$$

In the next proposition we collect all the facts we need about these operators. For a proof of these statements the reader can consult (32).

Proposition 5.2.2. *Let E_1, E_2 be two Banach spaces and denote by $\mathcal{F}(E_1, E_2)$ the set of Fredholm operators from E_1 to E_2. Then*

- *$\mathcal{F}(E_1, E_2)$ is an open subset of all the linear operators from E_1 to E_2 with respect to the topology of uniform convergence,*
- *the index function is continuous with respect to the uniform topology, hence constant on the connected components of $\mathcal{F}(E_1, E_2)$,*
- *if K is a compact operator and $L \in \mathcal{F}(E_1, E_2)$, then*

$$F + K \in \mathcal{F}(E_1, E_2), \qquad \imath(F + K) = \imath(F),$$

- *$F \in \mathcal{F}(E_1, E_2)$ if and only if there exist L_1 and L_2, bounded operators from E_2 to E_1, and two compact operatorsi $K_1\colon E_1 \to E_1$, $K_2\colon E_2 \to E_2$, such that*

$$L_1 F = \mathrm{id}_{E_1} + K_1 \qquad F L_2 = \mathrm{id}_{E_2} + K_2.$$

We are now ready to prove the following statement about A_0.

Lemma 5.2.3. *Let A_0 be the operator defined above. Then*

1. *A_0 is symmetric with respect to the $W^{k,2}$-scalar product,*
2. $\ker A_0 = \left\{(v, 0) \mid v \equiv (x_0, 0) \in \mathbb{R}^{2n-1} \times 0\right\}$,

$$\operatorname{im} A_0 = \ker A_0^{\perp} = \left\{\int_{\mathbb{T}} v_x dt = 0\right\},$$

where $\ker A_0^{\perp}$ is the orthogonal in $W^{k,2}(\mathbb{T}, \mathbb{R}^{2n}) \times \mathbb{R}$,

3. *A_0 is an invertible operator between the Banach spaces $\ker A_0^{\perp} \cap W^{k+1,2}$ and $\operatorname{im} A_0$.*

Proof. Let $(v_1, S_1),\ (v_2, S_2) \in W^{k+1,2}(\mathbb{T}, \mathbb{R}^{2n}) \times \mathbb{R}$, then

$$
\begin{aligned}
\langle A_0(v_1, S_1),(v_2, S_2)\rangle &= \sum_{j\le k}\Big(\int_{\mathbb{T}} g_0\big(\frac{d^j}{dt^j}(J_0\dot v_1 - S_1 f\partial_y), \frac{d^j}{dt^j}v_2\big)dt\Big) - S_2\int_{\mathbb{T}}(v_1)_y f\,dt \\
&= \sum_{j\le k}\Big(\int_{\mathbb{T}} g_0\big(J_0\frac{d^{j+1}}{dt^{j+1}}v_1, \frac{d^j}{dt^j}v_2\big)dt\Big) + \\
&\quad - \int_{\mathbb{T}} g_0\big(S_1 f\partial_y, v_2\big)dt - S_2\int_{\mathbb{T}}(v_1)_y f\,dt \\
&= \sum_{j\le k}\Big(\int_{\mathbb{T}} g_0\big(J_0\frac{d^{j+1}}{dt^{j+1}}v_1, \frac{d^j}{dt^j}v_2\big)dt\Big) + \\
&\quad - S_1\int_{\mathbb{T}}(v_2)_y f\,dt - S_2\int_{\mathbb{T}}(v_1)_y f\,dt.
\end{aligned}
$$

The symmetry in the second and third term is clear. The symmetry in the first summatory is a consequence of the compatibility between g_0 and J_0:

$$
\begin{aligned}
\int_{\mathbb{T}} g_0\big(J_0\frac{d^{j+1}}{dt^{j+1}}v_1, \frac{d^j}{dt^j}v_2\big)dt &= \int_{\mathbb{T}}\frac{d}{dt}\Big(g_0\big(J_0\frac{d^j}{dt^j}v_1, \frac{d^j}{dt^j}v_2\big)\Big)dt + \\
&\quad - \int_{\mathbb{T}} g_0\big(J_0\frac{d^j}{dt^j}v_1, \frac{d^{j+1}}{dt^{j+1}}v_2\big)dt \\
&= 0 + \int_{\mathbb{T}} g_0\big(\frac{d^j}{dt^j}v_1, J_0\frac{d^{j+1}}{dt^{j+1}}v_2\big)dt.
\end{aligned}
$$

Now calculate $\ker A_0$. $(v, S) \in \ker A_0$ if and only if

$$
\begin{cases}
0 = \dot v - S f\partial_y \\
0 = -\displaystyle\int_{\mathbb{T}} v_y f\,dt.
\end{cases}
\tag{5.5}
$$

Integrating the first equation in (5.5) we find

$$
v(t) = v(0) + S\Big(\int_0^t f(t')dt'\Big)\partial_y.
$$

Bearing in mind that $\int_{\mathbb{T}} f = 1$ and $v(0) = v(1)$, this implies $S = 0$ and hence $v(t) \equiv v_0 = (x_0, y_0)$. Then the second equation in (5.5) becomes

$$
0 = y_0\int_{\mathbb{T}} f = y_0.
$$

Thus we arrive to the conclusion

$$
\ker A_0 = \big\{(v, 0) \mid v \equiv (x_0, 0) \in \mathbb{R}^{2n-1} \times 0\big\}.
$$

We claim that the symmetry implies the inclusion $\operatorname{im} A_0 \subset \ker A_0^{\perp}$. Indeed, if $(v,S) \in W^{k+1,2}(\mathbb{T},\mathbb{R}^{2n}) \times \mathbb{R}$ and $(\tilde{v},\tilde{S}) \in \ker A_0 \cap W^{k+1,2}(\mathbb{T},\mathbb{R}^{2n}) \times \mathbb{R}$, then

$$\langle A_0(v,S),(\tilde{v},\tilde{S})\rangle = \langle (v,S), A_0(\tilde{v},\tilde{S})\rangle = 0.$$

Since $\ker A_0 \cap W^{k+1,2}(\mathbb{T},\mathbb{R}^{2n}) \times \mathbb{R} = \ker A_0$, the claim is proven.
We wish to show that the injective operator

$$A_0\big|_{\ker A_0^{\perp}} : \ker A_0^{\perp} \cap W^{k+1,2}(\mathbb{T},\mathbb{R}^{2n}) \times \mathbb{R} \to \ker A_0^{\perp}$$

is invertible. To this purpose is enough to prove that A_0 is Fredholm with index zero. Using Proposition 5.2.2, the following couple of facts is sufficient:

- $(v,S) \mapsto (J_0\dot{v},0)$ is Fredholm with index 0,
- $(v,S) \mapsto (-Sf\partial_y, -\int_{\mathbb{T}} v_y f dt)$ is compact.

The former point is proven by an explicit calculation and the latter by the Arzelà-Ascoli and Sobolev Embedding Theorem.
Finally let us characterize the elements (v,S) in $\ker A_0^{\perp}$:

$$\begin{aligned} 0 = \langle (v,S),((x_0,0),0)\rangle &= \int_{\mathbb{T}} g_0(v,(x_0,0))dt \\ &= g_0\left(\int_{\mathbb{T}} vdt,(x_0,0)\right), \quad \forall x_0 \in \mathbb{R}^{2n-1}. \end{aligned}$$

This chain of equalities implies

$$\ker A_0^{\perp} = \left\{\int_{\mathbb{T}} v_x dt = 0\right\}. \tag{5.6}$$

□

Remark 5.2.4. It is easy to show that A_0 is the Hessian operator of $\mathbb{A}_0$, as soon as we express the elements of E_0 near the constant loop $(z,0)$ using the coordinate chart U_z. Then, the previous proposition tells us that $\ker A_0$ is exactly the tangent space of the trivial critical set of $\mathbb{A}_0$. Hence 0 is a Morse-Bott critical value for $\mathbb{A}_0$.

Denote by P_0 the orthogonal projection on $\ker A_0$ and by $Q_0 := 1 - P_0$ the projection on $\ker A_0^{\perp}$. Then we have

$$P_0(v,S) = \left(\int_{\mathbb{T}} v_x dt, 0\right)$$

and there exists $a > 0$ such that

$$\|A_0Q_0(v,S)\|_k \geq a\|Q_0(v,S)\|_{k+1}, \tag{5.7}$$

where $\|\cdot\|_k$ is the norm in the space $W^{k,2}(\mathbb{T},\mathbb{R}^{2n})\times\mathbb{R}$. Furthermore for every $s\in I$ we have $\ker A_0\subset\ker A(s)$. This can be stated using one of the equivalent equations

$$A(s)P_0=0,\qquad A(s)=A(s)Q_0.$$

5.3 An application of the Maximum Principle

Let us resume the notation of the first section. We have a path $w: I\to C^\infty(\mathbb{T},U_z)\times\mathbb{R}$, which solves the 0-Equation. Composing with the inclusion $C^\infty(\mathbb{T},\mathbb{R}^{2n})\times\mathbb{R}\hookrightarrow W^{k+1,2}(\mathbb{T},\mathbb{R}^{2n})\times\mathbb{R}$ we get a differentiable path $w\colon\mathbb{R}\to W^{k+1,2}(\mathbb{T},\mathbb{R}^{2n})\times\mathbb{R}$. The 0-Equation for w rewrites as

$$\frac{dw}{ds}+A(s)w=0. \tag{5.8}$$

Define the following function

$$\begin{aligned}\varphi\colon I &\to [0,+\infty)\\ s &\mapsto \frac{1}{2}\|Q_0w\|_k^2,\end{aligned} \tag{5.9}$$

We aim to find a differential inequality for φ. In order to do this we will see that we must have a control on the following quantity:

$$\begin{aligned}\Theta_w(s):=\ &\|\partial_sA(s)\|_{k+1,k}+\|A(s)\|_{k+1,k}\|(A(s)-A_0)^*\|_{k+1,k}\\ &+\|A_0\|_{k+1,k}\|A(s)-A_0\|_{k+1,k}\end{aligned}$$

(where $(A(s)-A_0)^*$ is the adjoint with respect to the $W^{k,2}$-scalar product and $\|\cdot\|_{k+1,k}$ is the uniform norm for operators from $W^{k+1,2}(\mathbb{T},\mathbb{R}^{2n})\times\mathbb{R}$ to $W^{k,2}(\mathbb{T},\mathbb{R}^{2n})\times\mathbb{R}$). Let us analyze separatedly the different terms in Θ_w.

1. $\|A_0\|_{k+1,k}$ is a constant, which does not depend on w or s.
2. $A(s)$ depends on s in the term $J_u\partial_t$ only. The norm of this piece involves the t-derivatives of u up to order k. By Corollary 4.7.2 these are uniformly bounded and therefore are independent of s.
3. The norm of $\partial_sA(s)$ is bounded by a sum of terms of the form

   $$C\|\partial_s\partial_t^hu\|_\infty\|\partial_t^{h'}u\|_\infty,\qquad 0\le h\le k,\ 1\le h'\le k.$$

   Combining Corollaries 4.7.2 and the first point in 4.7.6 we get that this quantity is small if $\mathbb{A}_0(s_0)$ is sufficiently near to 0.
4. $(A(s)-A_0)(v,S)=((J_u-J_0)\dot v,0)$. The first factor is the composition of two bounded operators:

- $v_1 \mapsto \dot{v}_1$, from $W^{k+1,2}(\mathbb{T}, \mathbb{R}^{2n})$ to $W^{k,2}(\mathbb{T}, \mathbb{R}^{2n})$ and is independent of s;
- $v_2 \mapsto (J_u - J_0)v_2$, from $W^{k,2}(\mathbb{T}, \mathbb{R}^{2n})$ into itself. Its norm is bounded by the sum

$$\|J_u - J_0\|_\infty + \sum_{1 \le h \le k} C \|\partial_t^h u\|_\infty.$$

  We have $\|J_u - J_0\|_\infty \le C|u - z|_\infty$. Thus, this number is small provided the diameter of U_z is sufficiently small. Moreover by the third point of Corollary 4.7.6 we know that $\|\partial_t^h u\|_\infty$ is small if $\mathbb{A}_0(s_0)$ is sufficiently small, since $\{\|\partial_t^h u\|_\infty < \varepsilon\}$ is an open neighborhood of $\Sigma \times 0$ in E_0.

5. Finally we deal with $(A(s) - A_0)^*$. We have to study the adjoint of

$$v \mapsto B\dot{v}, \qquad B := J_u - J_0$$

with respect to the $W^{k,2}(\mathbb{T}, \mathbb{R}^{2n})$-scalar product. As we have said before, this operator is a composition. Hence,

$$\begin{aligned} \langle v_1, B\frac{d}{dt}v_2\rangle &= \langle B^* v_1, \frac{d}{dt}v_2\rangle \\ &= \langle -\frac{d}{dt}\left(B^* v_1\right), v_2\rangle \\ &= \langle -\partial_t\left(B^*\right) v_1 - B^* \frac{d}{dt}v_1, v_2\rangle. \end{aligned}$$

We claim that $\partial_t\left(B^*\right) = \left(\partial_t B\right)^*$. Indeed,

$$\begin{aligned} \langle v_1, \partial_t\left(B^*\right) v_2\rangle &= -\langle v_1, B^* \dot{v}_2\rangle - \langle \dot{v}_1, B^* v_2\rangle \\ &= -\langle B v_1, \dot{v}_2\rangle - \langle B\dot{v}_1, v_2\rangle \\ &= \langle \frac{d}{dt}\left(B v_1\right), v_2\rangle - \langle B\dot{v}_1, v_2\rangle \\ &= \langle \left(\partial_t B\right) v_1, v_2\rangle + \langle B\dot{v}_1, v_2\rangle - \langle B\dot{v}_1, v_2\rangle \\ &= \langle \left(\partial_t B\right) v_1, v_2\rangle. \end{aligned}$$

Then we have to bound the norm of $-\left(\partial_t B\right)^* - B^* \frac{d}{dt}$ from $W^{k+1,2}(\mathbb{T}, \mathbb{R}^{2n})$ to $W^{k,2}(\mathbb{T}, \mathbb{R}^{2n})$.

$$\begin{aligned} \left\| -\left(\partial_t B\right)^* - B^* \frac{d}{dt} \right\|_{k+1,k} &\le \|\left(\partial_t B\right)^*\|_{k+1,k} + \left\| B^* \frac{d}{dt} \right\|_{k+1,k} \\ &\le \|\left(\partial_t B\right)^*\|_{k,k} + \|B^*\|_{k,k} \left\| \frac{d}{dt} \right\|_{k+1,k} \\ &= \|\partial_t B\|_{k,k} + \|B\|_{k,k}. \end{aligned}$$

The latter term has been studied in the preceding point and the former can be treated in a similar way. To sum up also in this case $\|A(s)^* - A_0\|_{k+1,k}$ is small provided the diameter of U_z and $\mathbb{A}_0(s_0)$ are sufficiently small.

Thus we get the following lemma.

Lemma 5.3.1. *Let a be the positive constant introduced in* (5.7). *There exist a positive constant δ_a, which is universal in $\mathcal{N}_0$, such that if the diameter of U_z is sufficiently small (and hence $\overline{y}$) and $w = (u, T) \in \mathcal{N}_0$ is such that $u(\mathbb{T} \times [s_0, s_1]) \subset U_z$, then*

$$\mathbb{A}_0(w(s_0)) \leq \delta_a \implies \Theta_w(s) \leq \frac{a^2}{2}. \tag{5.10}$$

In the subsequent discussion we will always assume that U_z is small enough, so that we can apply the preceding lemma. This allows to prove a crucial estimate for φ.

Proposition 5.3.2. *Let $I = [s_0, s_1]$ be an interval, $z \in \Sigma$. Suppose that $w \in \mathcal{N}_0$ is such that $w_{|I}$ is a 0-Solution and $\mathbb{A}_0(w(s_0)) \leq \delta_a$. Then we have*

$$\varphi'' \geq a^2 \varphi,$$

where φ is the function defined in (5.9).
This implies that

$$\varphi(s) \leq \max\{\varphi(s_0), \varphi(s_1)\} \frac{\cosh(a(s - \frac{s_0+s_1}{2}))}{\cosh(a\frac{s_1-s_0}{2})}. \tag{5.11}$$

Proof. A derivation under the integral sign yields

$$\varphi'' = \|Q_0 w'\|^2_{k+1} + \langle Q_0 w, Q_0 w'' \rangle_{k+1} \geq \langle Q_0 w, Q_0 w'' \rangle_{k+1}.$$

Then using the following three facts (see point 1 in Proposition 5.2.3, the discussion following Remark 5.2.4 and Equation 5.8):

- A_0 is symmetric,
- $A(s) = A(s)Q_0$ and $\partial_s A(s) = \partial_s A(s) Q_0$,
- $w' = -A(s)w$ and differentiating,

$$w'' = -A(s)w' - (\partial_s A(s))\, w$$

and the inequalities (see (5.7) and the preceding lemma)

$$\bullet\ a^2 \|Q_0 w\|^2_{k+1} \leq \|A_0 Q_0 w\|^2_k, \qquad \bullet\ \Theta_w \leq \frac{a^2}{2},$$

we find

$$
\begin{aligned}
\langle Q_0 w, Q_0 w'' \rangle &= \langle Q_0 w, (-A(s)Q_0 w' - (\partial_s A(s))Q_0 w) \rangle \\
&= - \langle Q_0 w, (A(s) - A_0)Q_0 w' \rangle - \langle Q_0 w, A_0 Q_0 w' \rangle + \\
&\quad - \langle Q_0 w, (\partial_s A(s))Q_0 w \rangle \\
&= - \langle (A(s) - A_0)^* Q_0 w, Q_0 w' \rangle + \langle A_0 Q_0 w, A(s) Q_0 w \rangle + \\
&\quad - \langle Q_0 w, (\partial_s A(s))Q_0 w \rangle \\
&= - \langle (A(s) - A_0)^* Q_0 w, A(s) Q_0 w \rangle + \langle A_0 Q_0 w, A_0 Q_0 w \rangle + \\
&\quad + \langle A_0 Q_0 w, (A(s) - A_0) Q_0 w \rangle - \langle Q_0 w, (\partial_s A(s)) Q_0 w \rangle \\
&\geq - \|(A(s) - A_0)^*\|_{k+1,k} \|A(s)\|_{k+1,k} \|Q_0 w\|_{k+1}^2 + \\
&\quad + \|A_0 Q_0 w\|_k^2 - \|A_0\|_{k+1,k} \|A(s) - A_0\|_{k+1,k} \|Q_0 w\|_{k+1}^2 + \\
&\quad - \|\partial_s A(s)\|_{k+1,k} \|Q_0 w\|_{k+1}^2 \\
&= \|A_0 Q_0 w\|_k^2 - \Theta_w \|Q_0 w\|_{k+1}^2 \\
&\geq (a^2 - \Theta_w) \|Q_0 w\|_{k+1}^2 \\
&\geq \frac{a^2}{2} \|Q_0 w\|_{k+1}^2 \\
&\geq \frac{a^2}{2} \|Q_0 w\|_k^2 \\
&= a^2 \varphi.
\end{aligned}
$$

Set

$$
\psi(s) := \max\{\phi(s_0), \phi(s_1)\} \frac{\cosh(a(s - \frac{s_0+s_1}{2}))}{\cosh(a\frac{s_1-s_0}{2})}.
$$

Then $\psi'' = a^2 \psi$ and, since cosh is an even function,

$$
\psi(a) = \psi(b) = \max\{\phi(s_0), \phi(s_1)\}.
$$

Thus the function $\widehat{\varphi} = \varphi - \psi$ still satisfies $\widehat{\varphi}'' \geq a^2 \widehat{\varphi}$ and furthermore is not positive on the boundary of I. Then the maximum of $\widehat{\varphi}$ cannot be positive. Arguing by contradiction, if the point of maximum $\tilde{s}$ were in the interior of I and $\widehat{\varphi}(\tilde{s}) > 0$, we would have the impossible inequality

$$
0 \geq \widehat{\varphi}''(\tilde{s}) \geq \widehat{\varphi}(\tilde{s}) > 0.
$$

Therefore we get the desired inequality

$$
0 \geq \widehat{\varphi} = \varphi - \psi, \qquad \text{on } I.
$$

□

The previous proposition allows to give a bound on the space u travels during the interval I.

Proposition 5.3.3. *With the notation as above we have*

$$|u(t,s) - u(t,s_0)| \leq \frac{C}{a} \max \left\{ \|Qw(s_0)\|_k, \|Qw(s_1)\|_k \right\}, \tag{5.12}$$

for some constant $C > 0$.

Proof. Remember that $Q_0 w = (u - \int_{\mathbb{T}} u_x dt, T)$, then

$$|T| \leq \|Q_0 w\|_k$$

and, when $k \geq 2$, the Sobolev Embedding Theorem yields also a constant C such that

$$|\partial_t u| = |\partial_t Q_0 w| \leq C \|Q_0 w\|_k.$$

Now get $\partial_s u$ from Equation 5.2 and use the two estimates just found in order to obtain

$$|u(s) - u(s_0)| \leq \int_{s_0}^{s} |\partial_s u(s')| \mathrm{d}s' \leq C' \int_{s_0}^{s} \|Q_0 w(s')\|_k \mathrm{d}s'. \tag{5.13}$$

Using (5.11) we know that

$$\|Q_0 w(s)\|_k \leq \max\{\|Q_0 w(s_0)\|_k, \|Q_0 w(s_1)\|_k\} \sqrt{\frac{\cosh(a(s - \frac{s_0+s_1}{2}))}{\cosh(a\frac{s_1-s_0}{2})}}.$$

The subadditive inequality

$$\sqrt{b_1 + b_2} \leq \sqrt{b_1} + \sqrt{b_2}, \qquad b_1, b_2 \geq 0$$

yields

$$\bullet \ \sqrt{\cosh s} \leq \sqrt{2} \cosh \frac{s}{2}, \qquad \bullet \ \sinh s \leq \frac{1}{\sqrt{2}} \sqrt{\cosh(2s)}.$$

Then we get the bound

$$\begin{aligned} \int_{s_0}^{s} \sqrt{\cosh\left(a\left(s' - \frac{s_0+s_1}{2}\right)\right)} \mathrm{d}s' &\leq \sqrt{2} \int_{s_0}^{s} \cosh\left(\frac{a}{2}\left(s' - \frac{s_0+s_1}{2}\right)\right) \mathrm{d}s \\ &= \frac{4\sqrt{2}}{a} \sinh\left(\frac{a}{2}\frac{s_1 - s_0}{2}\right) \\ &\leq \frac{4}{a} \sqrt{\cosh\left(a\frac{s_1 - s_0}{2}\right)}. \end{aligned}$$

Continuing the chain of inequality in (5.13), we get the thesis

$$\begin{aligned} |u(s) - u(s_0)| &\leq \max\{\|Q_0 w(s_0)\|_k, \|Q_0 w(s_1)\|_k\} \frac{4C'}{a} \frac{\sqrt{\cosh(a\frac{s_1-s_0}{2})}}{\sqrt{\cosh(a\frac{s_1-s_0}{2})}} \\ &= \frac{4C'}{a} \max\{\|Q_0 w(s_0)\|_k, \|Q_0 w(s_1)\|_k\}. \end{aligned}$$

□

5.4 Exponential decay

This last section will be entirely devoted to the proof of Theorem 5.1.2.

Lemma 5.4.1. *For every real number $\widehat{y}$ with $0 \le \widehat{y} < \overline{y}$ there exists a real number $c > 0$ such that if we have a point $z_0 \in M$ with $|H(z_0)| \le \widehat{y}$, then there exist $z \in \Sigma$ such that*

$$\bullet \; z_0 \in U_z, \qquad \bullet \; \inf_{z' \in \partial U_z} |z_0 - z'| \ge c,$$

where $|\cdot|$ is the standard Euclidean metric in the coordinate induced by U_z.

Proof. The set $\{H \in [-\widehat{y}, \widehat{y}]\}$ admits the open cover

$$\Big\{U_z \cap \{H \in [-\widehat{y}, \widehat{y}]\}\Big\}_{z \in \Sigma}.$$

Since $\{H \in [-\widehat{y}, \widehat{y}]\}$ is compact, the open cover admits a positive *Lebesgue number*. This fact and $\overline{y} - \widehat{y} > 0$ together imply that there exists $c > 0$ such that the ball centered in z_0 with radius c is compactly contained in some U_z. Since the metric of M restricted to U_z and the standard Euclidean metric, which the coordinates (x, y) bring on U_z, differ by a constant factor independent of z, the lemma follows. □

Proof of Theorem 5.1.2. First we apply Lemma 5.4.1 with $\widehat{y} := \frac{\overline{y}}{2}$ and get a positive constant c. Then we observe that, for every $\varepsilon > 0$, the set

$$\{w \in E_0 \mid u \subset U_z \text{ for some } z \in \Sigma, \; \|Q_0 w\|_k \le \varepsilon\}$$

is a neighborhood of $\Sigma \times 0$ in E_0. By Corollary 4.7.6, there exists $\delta_\varepsilon > 0$ such that if $\mathbb{A}_0(w(s_0)) \le \delta_\varepsilon$, for some $s_0 \ge r$, then, for every $s \ge s_0$, there exist U_z (that may depend on s), such that $u(s)$ lies in U_z and $\|Q_0 w(s)\|_k \le \varepsilon$. Since we have the bound

$$\max\{|\dot{u}|, |u_y|, |T|\} \le C\|Q_0 w\|_k, \tag{5.14}$$

then, for ε sufficiently small,

$$|u_y(t, s)| \le \frac{\overline{y}}{2}, \quad \text{for } s \ge s_0.$$

This means that there exists $\varepsilon_0 > 0$ sufficiently small and a corresponding δ_{ε_0}, such that if $\mathbb{A}_0(w(s_0)) \le \delta_{\varepsilon_0}$ holds, then

1. there exists $z \in \Sigma$ such that $u(s_0) \subset U_z$ and

$$\inf_{z' \in \partial U_z} |u(t, s_0) - z'| \ge c;$$

2. $\varepsilon_0 \leq \dfrac{ac}{2C}$, with C the constant contained in Proposition 5.3.3;

3. $\|Q_0 w\|_k \leq \varepsilon_0$.

Suppose now that, for every $w \in \mathcal{N}_0$, an s_0 is chosen in such a way that $\mathbb{A}_0(w(s_0)) \leq \delta_{\varepsilon_0}$ (observe that s_0 may depend on w). We claim that for every $s \geq s_0$, $u(s) \subset U_z$, z being given from the first point of the preceding list. U_z will be the neighborhood U_+ mentioned in the statement of the theorem. Assume by contradiction that $u(s)$ exits U_z, for some $s \geq s_0$. Then there exists a couple $(\tilde{t}, \tilde{s})$, with $\tilde{s} \geq s_0$ such that

- $u(t,s) \in U_z$, for $t \in \mathbb{T}$, $s_0 \leq s < \tilde{s}$,
- $u(\tilde{t}, \tilde{s}) \in \partial U_z$.

Then we can use Proposition 5.3.3 with $I = [s_0, s]$, $s < \tilde{s}$, finding

$$|u(\tilde{t}, s) - u(\tilde{t}, s_0)| \leq \frac{C}{a} \max\left\{\|Qw(s_0)\|_k, \|Qw(s)\|_k\right\} \leq \frac{C}{a}\varepsilon_0 \leq \frac{c}{2}.$$

Taking the limit $s \to \tilde{s}$ we get

$$|u(\tilde{t}, \tilde{s}) - u(\tilde{t}, s_0)| \leq \frac{c}{2}.$$

This is a contradiction because $u(\tilde{t}, \tilde{s}) \in \partial U_z$. Now that we have proven that $u(\cdot, s) \subset U_z$ for every $s \geq s_0$, the function $\|Q_0 w(s)\|_k$ is well defined for $s \geq s_0$. By Corollary 4.7.6 we know that $\|Q_0 w(s)\|_k$ tends to zero as s goes to $+\infty$. If s_1, $s_2 \geq s_0$, we can apply once more Proposition 5.3.3, with $I = [s_1, s_2]$, and get

$$|u(t, s_1) - u(t, s_2)| \leq \frac{C}{a} \max\left\{\|Q_0 w(s_1)\|_k, \|Q_0 w(s_2)\|_k\right\}.$$

Since $\|Q_0 w(s)\|_k$ tends to zero, we have

$$u(t,s) \to \widehat{u}(t) \text{ and } T(s) \to 0, \qquad \text{as } s \text{ goes to } +\infty.$$

By Proposition 4.7.5 we have that $\widehat{u}(t) \equiv z' \in \Sigma \cap U_z$ and

$$w(s) \xrightarrow{E_0} (z', 0).$$

Let us study now the asymptotic behaviour of $\|Q_0 w(s)\|_k$. The hypotheses of Proposition 5.3.2 are satisfied for every $I = [s_0, s_1]$, with $s_1 \geq s_0$. Thus we get

$$\|Q_0 w(s)\|_k \leq \max\{\|Q_0 w(s_0)\|_k, \|Q_0 w(s_1)\|_k\} \sqrt{\frac{\cosh(a(s - \frac{s_0+s_1}{2}))}{\cosh(a(\frac{s_1-s_0}{2}))}}.$$

Letting s_1 go to $+\infty$ we have

- $\max\{\|Q_0 w(s_0)\|_k, \|Q_0 w(s_1)\|_k\} \longrightarrow \|Q_0 w(s_0)\|_k$

- $$\frac{\cosh(a(s-\frac{s_0+s_1}{2}))}{\cosh(a\frac{s_1-s_0}{2})} = e^{a(\frac{s_1+s_0}{2}-s)}e^{-a\frac{s_1-s_0}{2}} + \rho(s_1) = e^{a(s_0-s)} + \rho(s_1)$$

and $\rho(s_1) \to 0$, as $s_1 \to +\infty$. Hence we get the exponential decay:

$$\|Q_0 w(s)\|_k \leq \|Q_0 w(s_0)\|_k e^{\frac{a}{2}(s_0-s)}. \tag{5.15}$$

The exponential decay of $\dot{u}$, u_y and T follows from the inequality 5.14. The exponential decay of these three quantities imply that of $\partial_s u$ and $\frac{dT}{ds}$ via the 0-Equation 5.2. At last, the exponential decay of $u - z'$ follows integrating the inequality

$$|\partial_s u(t,s)| \leq Ce^{\frac{a}{2}(s_0-s)}$$

we have just found. □

Chapter 6

A noncompactness theorem: the conclusion of the argument

In this final chapter we shall prove the Main Theorem 2.4.1, stated at the end of the second chapter. The next step to reach our goal is to give $\mathcal{M}$ an alternative topology τ_p that turns it into a noncompact set. In the end this result will be combined with the C^∞_{loc}-relative compactness and Morse-Bott theory in order to conclude the argument. For the sake of simplicity, in this chapter we will carry out the details when M is $\mathbb{R}^{2n}$.

6.1 The Sobolev setting

In general, $\mathcal{M}$ can be seen as the zero set of a map $F\colon [0,+\infty)\times \widehat{X} \to E$, where $E \to \widehat{X}$ is a Banach vector bundle and each slice

$$F_r := F_{\big|\{r\}\times \widehat{X}}$$

is a Fredholm section of this bundle. Then τ_p is simply the topology that $[0,+\infty)\times \widehat{X}$ induces on its subset $\mathcal{M}$. The topological features of $(\mathcal{M},\tau_p)$ can be investigated making use of some important properties of Fredholm maps. When M is $\mathbb{R}^{2n}$, the space $\widehat{X}$ is simply chosen as

$$\widehat{X} := \Sigma \times X,$$

where X is some Banach space, the bundle E is trivial and its fibers are isomorphic to a Banach space Y. Hence F can be seen as a Fredholm map $F\colon [0,+\infty)\times \widehat{X} \to Y$ and $\mathcal{M}$ becomes the counterimage of the value 0. Let us now describe in a precise manner this analytical setting.

Let a be the constant introduced in (5.7) of the preceding chapter and, for each $p>2$, set

$$b_p := \frac{ap}{4}.$$

Then define

$$X := W^{1,p}(\mathbb{T}\times\mathbb{R},\mathbb{R}^{2n}; e^{b_p|s|}dtds)\times W^{1,p}(\mathbb{R},\mathbb{R}; e^{b_p|s|}ds),$$

(the need for an exponential weight will be clear in the proof of Lemma 6.3.2). Moreover let

$$Y := L^p(\mathbb{T}\times\mathbb{R},\mathbb{R}^{2n}; e^{b_p|s|}dtds)\times L^p(\mathbb{R},\mathbb{R}; e^{b_p|s|}ds).$$

We can endow these Sobolev spaces with the obvious product norms:

- $\|(u,T)\|_X^p := \|u\|_{1,p}^p + \|T\|_{1,p}^p$, where

$$\|u\|_{1,p}^p := \int_{\mathbb{T}\times\mathbb{R}}\Big(|u|^p + |\partial_t u|^p + |\partial_s u|^p\Big)e^{b_p|s|}dtds,$$
$$\|T\|_{1,p}^p := \int_{\mathbb{R}}\Big(|T|^p + \left|\frac{dT}{ds}\right|^p\Big)e^{b_p|s|}ds;$$

- $\|(u,T)\|_Y^p := \|u\|_p^p + \|T\|_p^p$, where

$$\|u\|_p^p := \int_{\mathbb{T}\times\mathbb{R}}|u|^p e^{b_p|s|}dtds, \qquad \|T\|_p^p := \int_{\mathbb{R}}|T|^p e^{b_p|s|}ds.$$

We can suppose after a suitable translation that the distinguished point used in the definition of $\mathcal{M}$ is $z_0 = 0$. Furthermore, let us consider a smooth step function $\sigma\colon\mathbb{R}\to[0,1]$ such that $\sigma(s)=0$, for $s\le 0$ and $\sigma(s)=1$, for $s\ge 1$. Then every element (z,u,T) of the set

$$\widehat{X} := \Sigma\times X$$

gives rise to a couple of continuous maps (this is a consequence of $p>2$ and the Sobolev Embedding Theorem 4.2.2)

$$(z,u,T)\mapsto\Big(u_z,T\Big) := \Big(\sigma(s)z + u(t,s), T(s)\Big).$$

The function $\sigma(s)z+u(t,s)$ is a cylinder in $\mathbb{R}^{2n}$, whose uniform limit at $-\infty$ is the constant path $z_0=0$ and whose uniform limit at $+\infty$ is the constant path z. T belongs to the class of real continuous functions, that go to 0 at infinity. We notice that also the elements of $\mathcal{M}$ are continuous and satisfy the same asymptotic conditions. They can be written as $(r,\sigma z+u,T)$, for some $z\in\Sigma$. By Theorem 5.1.2 we know that there exists $\overline{s}>0$, (possibly depending on the element of $\mathcal{M}$ we are considering) such that

$$\max\left\{|u|,|\partial_s u|,|\partial_t u|,|T|.\left|\frac{dT}{ds}\right|\right\}\le Ce^{\frac{a}{2}(\overline{s}-|s|)},\qquad |s|\ge\overline{s}. \tag{6.1}$$

This imply $\|(u,T)\|_X < +\infty$. Then, we can identify $\mathcal{M}$ with a subset of $[0,+\infty)\times\widehat{X}$.

Remark 6.1.1. We point out that in the case $M = \mathbb{R}^{2n}$, is not necessary to pick $p > 2$. The argument works in the simpler Hilbert case $p = 2$, as well. However in general we assume higher integrability because, in order to construct an atlas for the manifold $\widehat{X}$, we need to deal with continuous functions. See (42) for an overview of the general construction.

In the realm of smooth functions, $\mathcal{M}$ is characterized as being the set of solutions of the r-Equations

$$\frac{dw}{ds}(s) + \nabla \mathbb{A}_{\beta(r,s)}(w(s)) = 0,$$

with certain asymptotic properties. We can regard the last equation as the defining equation for the zero set of a map G_r, indexed by a parameter r

$$G_r(u,T) := \Big(\partial_s u + J_u\big(\partial_t u - T X_{\tilde{H}}(t,u) - \beta_r X_F(t,u)\big), T' - \int_{\mathbb{T}} \tilde{H}(t,u)dt\Big) \quad (6.2)$$

(here T' indicates the derivative of T). We see that G_r can be defined using the same formula (substituting u with u_z) as a function between the spaces $\widehat{X}$ and Y. This family of maps can be gathered in a single one:

$$\begin{array}{rcl} G\colon [0,+\infty) \times \widehat{X} & \to & Y \\ (r,z,u,T) & \mapsto & G_r(u_z,T). \end{array} \quad (6.3)$$

Define the zero set of G:

$$\mathcal{M}_p := G^{-1}(0) = \bigcup_{r \in [0,+\infty)} \{r\} \times G_r^{-1}(0) \quad (6.4)$$

We know that $\mathcal{M} \subset \mathcal{M}_p$ and we may wonder if the inclusion is strict or not. We claim that

$$\mathcal{M} = \mathcal{M}_p.$$

In order to prove this, we must show that if $w \in \widehat{X}$ solves $G_r(w) = 0$, then w is indeed smooth. This follows from a regularity theorem similar to 4.2.7. We will not prove this result and invite the interested reader to read the Appendix B in (33). The precise statement is the following.

Theorem 6.1.2. *Let $w = (z,u,T) \in \widehat{X}$ a solution of the equation*

$$G_r(w) = 0.$$

Then u_z and T are smooth functions such that

$$\big(u_z(\cdot,s), T(s)\big) \xrightarrow{E_0} (0,0), \quad \text{as } s \to -\infty,$$

$$\big(u_z(\cdot,s), T(s)\big) \xrightarrow{E_0} (z,0), \quad \text{as } s \to +\infty$$

and therefore $(r,u_z,T) \in \mathcal{M}$.

To sum up we have found that $\mathcal{M}$ is also the zero set $\mathcal{M}_p$ of the function G. Thus, we can endow it with the topology τ_p induced by $[0,+\infty)\times\widehat{X}$. When this is the case, we will use the notation $\mathcal{M}_p$ instead of $\mathcal{M}$. We wish to investigate the topological properties of $\mathcal{M}_p$. We already know that is *closed* (since it is the zero set of a continuous function) and we aim to show that is *noncompact*. This is the content of the next two sections.

6.2 The Implicit Function Theorem and the Sard-Smale Theorem

First we notice that a simple calculation shows that G_r is of class C^1. Its differential at a point $w=(z,(u,T))$ acts in the following way on a vector $\big(v,(\xi,\eta)\big)\in T_z\Sigma\times X$:

$$d_wG_r\big(v,\xi,\eta\big)=\Big(d_w^1G_r(v,\xi,\eta),d_w^2G_r(v,\xi,\eta)\Big),$$

We find that

$$\begin{aligned}d_w^1G_r\big(v,\xi,\eta\big)=\sigma'v+\partial_s\xi+J_{u_z}\Big(\partial_t\xi-\eta X_{\tilde H}(t,u_z)-T(d_{(t,u_z)}X_{\tilde H})(\xi+\sigma v)\Big)+\\ -J_{u_z}\Big(\beta_r(d_{(t,u_z)}X_F)(\xi+\sigma v)\Big)+\\ +(d_{u_z}J)(\xi+\sigma v)\Big(\partial_t u-TX_{\tilde H}(t,u_z)-\beta_rX_F(t,u_z)\Big).\end{aligned}$$

Rewrite the equation as

$$\begin{aligned}d_w^1G_r\big(\rho,v,(\xi,\eta)\big)=\partial_s\xi+J_{u_z}\Big(\partial_t\xi-\eta X_{\tilde H}(t,u_z)-T(d_{(t,u_z)}X_{\tilde H})(\xi+\sigma v)\Big)+\\ +(d_{u_z}J)(\xi+\sigma v)\Big(\partial_t u-TX_{\tilde H}(t,u_z)\Big)+S^r_{(t,s,u_z,T)}(v,\xi),\end{aligned}$$

where

$$\begin{aligned}S^r_{(t,s,u_z,T)}(v,\xi)\colon=\sigma'v-J_{u_z}\Big(\beta_r(d_{(t,u_z)}X_F)(\xi+\sigma v)\Big)+\\ -(d_{u_z}J)(\xi+\sigma v)\Big(\beta_rX_F(t,u_z)\Big)\end{aligned}$$

is a term that vanish identically for s large.
For the second factor we have simply

$$d_w^2G_r(v,\xi,\eta)=\eta'-\int_{\mathbb{T}}(d_{(t,u_z)}\tilde H)(\xi+\sigma v)dt.$$

Since the dependance on the parameter r is smooth, the fact that G_r is of class C^1 implies that G has the same regularity.

From the finite dimensional analysis we know that a way to investigate the properties of a zero set of a *continuously differentiable map* is to study its differential at the points of the zero set. If the differential is surjective then by the implicit function theorem we can deduce that this set is actually a smooth manifold. Therefore our first task will be to study dG. In short we will see that it turns out to be a Fredholm operator.

Thus is convenient to give the nonlinear counterpart of Definition 5.2.1.

Definition 6.2.1. A map $\Lambda : N_1 \to N_2$ of class C^1 between two Banach manifolds modeled on the Banach spaces E_1, E_2 is said to be a **Fredholm map** if at every point $q \in N_1$ its differential

$$d_q\Lambda \colon T_qN_1 \to T_{\Lambda(q)}N_2$$

is a Fredholm operator.

From Proposition 5.2.2 descends that the index of $d_q\Lambda$ is locally constant in q, hence on every connected component $N_1' \subset N_1$ the index of the map Λ is well defined. This index $\imath(\Lambda)$ is simply defined as $\imath(d_q\Lambda)$, where q is an arbitrary point in N_1'. For Fredholm maps an infinite dimensional analogue of the implicit function theorem is available. Before we need a definition.

Definition 6.2.2. Let $\Lambda \colon N_1 \to N_2$ a map of class C^1 between two Banach manifolds. A point $q \in N_1$ is called **a regular point** if $d_q\Lambda$ has a right inverse. A point $p \in N_2$ is called **a regular value** if every $q \in \Lambda^{-1}(p)$ is a regular point.

A proof of the next theorem can be found in the Appendix A of (33).

Theorem 6.2.3 (Implicit Function Theorem)**.** *Let $\Lambda \colon N_1 \to N_2$ be a Fredholm map of class C^1, let N_1 be connected (so that the index $\imath(\Lambda)$ is well defined) and let $p \in N_2$ be a regular value. Then $\Lambda^{-1}(p)$ is a smooth submanifold of N_1. Its dimension is $\imath(\Lambda)$.*

At this point it might seem strange that we have defined the larger class of Fredholm map, when the implicit function theorem we need holds only if the differential has a right inverse. However we will see that the theorems at our disposal yield only the Fredholm property of dG and not the existence of a right inverse. Luckily this gap is bridged by an analogue of Sard's Theorem for Fredholm maps, proved by Smale in (44). Before, we need to recall the *Baire's Category Theorem.*

Definition 6.2.4. Let N be a topological space. A set $N' \subset N$ is called **a residual set in** N if it contains a countable intersections of open dense subsets of N.

Theorem 6.2.5 (Baire's Category Theorem)**.** *Every residual set in a complete metric space is dense.*

Theorem 6.2.6 (Sard-Smale Theorem). *Let $\Lambda\colon N_1 \to N_2$ a Fredholm map of class C^k between two separable Banach manifolds. Let N_1 be connected. If $k \geq \max\{1, \imath(\Lambda)+1\}$, then the set of regular values of Λ is residual in N_2.*

Remark 6.2.7. Since every Banach manifold is locally homeomorphic to a complete metric space, using Baire's Category Theorem, we get that *the set of regular values of* Λ is actually *dense* in N_2.

In order to apply Theorem 6.2.6 and hence Theorem 6.2.3, we have to prove that dG is Fredholm. This will be the content of the next section

6.3 The Fredholm property

In this section we prove the following statement.

Proposition 6.3.1. *Let $G\colon [0,+\infty) \times \widehat{X} \to Y$ defined by formula 6.3. Then*

1. $d_{(0,(0,0))}G_0\colon T_{(0,0)}\widehat{X} \to T_{(0,0)}Y$ *is bijective,*
2. *G is a Fredholm map,*
3. $\imath(G) = 1$.

We begin with a preliminary result.

Lemma 6.3.2. *dG_r is a Fredholm operator and $d_{(0,(0,0))}G_0$ is bijective.*

Proof. Fix some $r \in [0,+\infty)$ and consider the operator $d_{(z,u,T)}G_r$ restricted to $0 \times X \subset T_z\Sigma \times X$. It has the form

$$D_r(\xi,\eta) := (\partial_s\xi, \eta') + A_{t,s,r}(\xi,\eta).$$

Since the remaining factor $T_z\Sigma \times 0$ is finite dimensional, $d_{(z,u,T)}G_r$ is Fredholm if and only if D_r is Fredholm.
The operator $A_{t,s,r}$ tends to A_0 when $s \to \infty$. We know that A_0 is not invertible, however the introduction of the exponential weight in the definition of the Sobolev spaces allows to construct an isomorphism with the standard spaces

$$W^{1,p}(\mathbb{T}\times\mathbb{R}, \mathbb{R}^{2n}; dtds)\times W^{1,p}(\mathbb{R},\mathbb{R};ds), \quad L^p(\mathbb{T}\times\mathbb{R}, \mathbb{R}^{2n}; dtds)\times L^p(\mathbb{R},\mathbb{R};ds).$$

The isomorphism is obtained simply by mapping (u,T) in $(e^{\phi(s)}u, e^{\phi(s)}T)$, where ϕ is a smooth function coinciding with $\frac{a}{4}|s|$ for $|s|$ large. We obtain an operator, conjugated with D_r, between these new spaces. It is of the form

$$\widehat{D}_r(\xi,\eta) := (\partial_s\xi, \eta') + A_{t,s,r}(\xi,\eta,v) + \phi'\cdot(\xi,\eta). \tag{6.5}$$

The new limit operators are

$$A_0 - \frac{a}{4}, \qquad A_z + \frac{a}{4}$$

and are invertible from $W^{1,2}(\mathbb{T},\mathbb{R}^{2n})\times\mathbb{R}$ to $L^2(\mathbb{T},\mathbb{R}^{2n})\times\mathbb{R}$. Indeed, $A_0-\frac{b_p}{p}$ preserves the splitting $\operatorname{im} Q_0 \oplus \operatorname{im} P_0$. The inequality 5.7 yields that

$$A_0 Q_0 - \frac{a}{4} Q_0 \quad \text{is invertible on the image of } Q_0.$$

On the second factor the map is simply the scalar multiplication by $-\frac{a}{4}$, which is invertible, since $a \neq 0$.

Linear mappings of the kind 6.5 were intensively studied, for instance in the second lecture of D. Salamon in (42). The discussion contained therein implies that **an operator of this type is Fredholm and hence so is** dG_r, because the Fredholm property is preserved by conjugacy.

Prove now the second part of the lemma and start analyzing $d_{(0,(0,0))}G_0$. The operator $\widehat{D}_0$ in this case is simply

$$\widehat{D}_0(\xi,\eta) = (\partial_s \xi, \eta') + A_0(\xi,\eta) + \phi' \cdot (\xi,\eta).$$

We aim to find its index. Still referring to Salamon's lectures, we know that

$\imath(\widehat{D}_0)$ is minus **the spectral flow** of the path of operators $A_0+\phi'$, considered as self-adjoint operator on $L^2(\mathbb{T},\mathbb{R}^{2n})\times\mathbb{R}$.

Intuitively, the spectral flow counts with multeplicity the number of times the following situation occurs:

an eigenvalue $\gamma(s)$ for $A_0+\phi'$ is negative in $s\in(s_0-\varepsilon,s_0)$ and positive in $(s_0,s_0+\varepsilon)$, for some real numbers $\varepsilon>0, s_0$.

Many things should be checked in order to prove that this definition makes sense. The most evident is that one is able to select all the eigenvalues of $(A_0+\phi')(s)$ in a smooth way with respect to the s-variable, in such a way that functions $s\mapsto\gamma(s)$ are defined and they describe all the eigenvalues of $(A_0+\phi')(s)$ (this is essentially the content of Kato Selection Theorem). For a rigourous treatment of the spectral flow and its application to the setting we are dealing with, we suggest to take a look at (41).

Here we need only some basic properties that can be inferred from the following discussion and that we give for granted. Thus, let us begin the line of reasoning.

Since each element of the path preserves the splitting $\operatorname{im} Q_0 \oplus \operatorname{im} P_0$ the spectral flow we need is the sum of the spectral flows on the two separated factors. We need to calculate the spectral flow of $A_0+\phi'$ and of ϕ' on $\operatorname{im} Q_0$ and $\operatorname{im} P_0$ respectively.

In order to compute the former we use the fact that the spectral flow is preserved under homotopies, which leave the endpoints in the space of self adjoint invertible operators. Then we can define a homotopy of 6.5 depending on a parameter δ:

$$A_0 + \delta Q_0 \phi'.$$

As before we can see that as δ goes to 0, the limit operators remain invertible. Since the new path we obtain is constant, the spectral flow is zero.

As regard the path ϕ' on $\operatorname{im} P_0$, we notice that the limit operators are simply $-\frac{a}{4}$ and $\frac{a}{4}$, so that all the $2n$ eigenvalues pass from being negative to being positive. Thus the spectral flow is $2n$. Putting all together:

$$\imath(D_0) = \imath(\widehat{D}_0) = -2n. \tag{6.6}$$

Furthermore we claim that D_0 is injective. Indeed, if $D_0(\xi, \eta) = 0$, then

$$(\partial_s \xi, \eta') = -A_0(\xi, \eta) \tag{6.7}$$

implies that (ξ, η) is smooth. We thus can apply Proposition 5.3.2, get the inequality 5.11 for an arbitrary $I = [s_0, s_1]$ and then I go to the whole $\mathbb{R}$. As a result we find that $Q_0(\xi, \eta) = 0$. Moreover taking the projection P_0 in (6.7), the function $\psi := P_0(\xi, \eta)$ satisfies the equation

$$\psi' = 0.$$

Since ψ tends to zero at infinity, we get $\psi \equiv 0$ and therefore

$$(\xi, \eta) = Q_0(\xi, \eta) + P_0(\xi, \eta) = 0 + 0 = 0.$$

Observe now that:

$$d_{(0,(0,0))} G_0(v, \xi, \eta) = D_0(\xi, \eta) + \sigma' v.$$

The map $\jmath \colon v \mapsto \sigma' v$ is clearly injective. This implies that its range is $2n$-dimensional. We claim that $\operatorname{im} \jmath \cap \operatorname{im} D_0 = 0$. This yields that $d_{(0,(0,0))} G_0$ is bijective. In order to prove the claim consider the linear continuous map from Y to $\mathbb{R}^{2n}$

$$(\xi, \eta) \mapsto \int_{\mathbb{R}} P_0 \xi ds.$$

Notice that it is well defined because P_0 makes sense also when $p \neq 2$ and $P_0 \xi$ is in $L^1(\mathbb{R}, \mathbb{R}^{2n})$ since it belongs to $L^p(\mathbb{R}, \mathbb{R}^{2n}; e^{\frac{bp}{\top}|s|} ds)$. We have

- $$\int_{\mathbb{R}} P_0 \left((\partial_s \xi, \eta') + A_0(\xi, \eta) \right) ds = \int_{\mathbb{R}} (P_0 \xi)' ds = 0,$$
- $$\int_{\mathbb{R}} P \jmath v ds = \int_{\mathbb{R}} \sigma' v ds = v.$$

This equalities yields the claim and prove the lemma. □

Proof of Proposition 6.3.1. The first point was proven in the Lemma 6.3.2.

The second point stems out from the fact that dG_r is the restriction of dG to a closed subspace of finite codimension.

Finally, the index of G can be computed on an arbitrary point. We choose $(0,(0,(0,0)))$. Then

$$\operatorname{coker} d_{(0,(0,0))}G_0 = 0, \quad \Longrightarrow \quad \operatorname{coker} d_{(0,(0,(0,0)))}G = 0.$$

If $w \in \widehat{X}$ and $c \in \mathbb{R}$ we have

$$\begin{aligned} d_{(0,(0,(0,0)))}G(c\partial_r, w) \quad &\Longleftrightarrow 0 \quad = c\partial_r G + d_{(0,(0,0))}G_0 w \\ &\Longleftrightarrow w \quad = -c\left(d_{(0,(0,0))}G_0\right)^{-1}\partial_r G. \end{aligned}$$

Thus $\ker d_{(0,(0,(0,0)))}G = \operatorname{Span}\left(\left(d_{(0,(0,0))}G_0\right)^{-1}\partial_r G\right)$ has dimension one. □

6.4 A topological obstruction

In this section we use the previous results in order to prove that $\mathcal{M}_p$ is not a compact set. First we need to investigate a bit further Fredholm maps.

Definition 6.4.1. Let $\Lambda : N_1 \to N_2$ be a continuous map between two topological spaces. Λ is said **locally proper** if for each point $q \in N_2$ there exists U_q a neighbourhood of q such that $\Lambda_{|U_q}$ is a proper map (i.e. if $K \subset N_2$ is compact, then $\Lambda^{-1}(K)$ is compact as well).

Remark 6.4.2. If Λ is locally proper then each compact set $K' \subset N_1$ has a neighbourhood U_K such that $\Lambda_{|U_K}$ is proper.

Lemma 6.4.3. *Fredholm maps are locally proper.*

Proof. Let $\Lambda \colon E_1 \to E_2$ a Fredholm map. Fix a point $q_0 \in E_1$. Without loss of generality we can assume $q_0 = 0$, $\Lambda(q_0) = 0$. Since $d_0\Lambda$ is Fredholm we know by Proposition 5.2.2 that there exist a bounded operator $L \colon E_2 \to E_1$ and a compact operator $K \colon E_1 \to E_1$ such that

$$L d_0 \Lambda = \mathrm{id}_{E_1} + K. \tag{6.8}$$

Define the map

$$\begin{aligned} \Gamma \colon E_1 \quad &\to \quad E_1 \\ q \quad &\mapsto \quad L\Lambda(q) - Kq. \end{aligned} \tag{6.9}$$

Γ is of class C^1 and by (6.8) we have

$$d_0\Gamma = \mathrm{id}_{E_1}.$$

The inverse function theorem yields a neighbourhood U of $0 \in E_1$ and a neighbourhood V of $0 \in E_1$ such that Γ is a homeomorphism between U and V. Furthermore we can choose both neighbourhoods to be bounded and closed.

The lemma follows once we show that $\widehat{\Lambda} := \Lambda \circ \Gamma^{-1}$ is proper. Substituting $q = \Gamma^{-1}(p)$ in the definition of Γ (6.9), we find the equation

$$p = L\widehat{\Lambda}(p) - \widehat{K}(p), \qquad \widehat{K} := K \circ \Gamma^{-1}. \tag{6.10}$$

Thus

$$L\widehat{\Lambda} = \mathrm{id}_{E_1} + \widehat{K}.$$

Since V and U are bounded, the map $\widehat{K}$ is compact, i.e. the image of every subset is relatively compact in E_1 and hence in V (since V is closed). Consider now $C \subset E_2$ a compact set. We aim to show that $\widehat{\Lambda}^{-1}(C)$ is compact. As a preliminary observation we find that for every set $A \subset E_1$

$$(\mathrm{id} + \widehat{K})^{-1}(A) \subset A - \widehat{K}\left((\mathrm{id} + \widehat{K})^{-1}(A)\right).$$

Indeed, let $x \in V$ be such that

$$p + K(p) = p' \in A.$$

Then $p = p' - K(p) \in A - K\left((\mathrm{id} + \widehat{K})^{-1}(A)\right)$. Now compute

$$\begin{aligned} \widehat{\Lambda}^{-1}(C) \subset \widehat{\Lambda}^{-1}(L^{-1}LC) &= (L\widehat{\Lambda})^{-1}(LC) \\ &= (\mathrm{id} + \widehat{K})^{-1}(LC) \\ &\subset LC - \widehat{K}\left((\mathrm{id} + \widehat{K})^{-1}(LC)\right) \\ &\subset LC - \overline{\widehat{K}\left((\mathrm{id} + \widehat{K})^{-1}(LC)\right)}. \end{aligned}$$

Both LC and $\overline{\widehat{K}\left((\mathrm{id} + \widehat{K})^{-1}(LC)\right)}$ are compact and therefore also their difference is compact. Thus we have found that $\widehat{\Lambda}^{-1}(C)$ is closed and contained in a compact set, hence it is compact. □

With this lemma at our disposal we can prove the desired proposition.

Proposition 6.4.4. *$\mathcal{M}_p$ is not compact.*

Proof. Let $\mathcal{M}_p'$ the connected component of $\mathcal{M}_p$ passing through the point $(0,(0,(0,0)))$. Assume that $\mathcal{M}_p'$ is compact: we will see that this leads to a contradiction. By Lemma 6.4.3 and Remark 6.4.2 there exists a closed neighbourhood U of $\mathcal{M}_p'$ such that $G_{|U}$ is proper. Since $d_{(0,(0,0))}G_0$ is invertible we can shrink U if necessary and suppose that G_0 is bijective on $U \cap 0 \times \widehat{X}$. Observe that

$$\mathcal{M}_p' \cap \left(\overline{\partial U \setminus 0 \times \widehat{X}}\right) = \emptyset. \tag{6.11}$$

By Sard-Smale Theorem 6.2.6 and the remark following it, we can find a sequence $c_\nu \subset Y$, such that

- $c_\nu \to 0$,
- c_ν is a regular value for G,
- there exists a unique $q_\nu \in U \cap 0 \times \widehat{X}$, such that $G(q_\nu) = c_\nu$.

Consider the following sequence of decreasing compact subsets of Y:

$$C_\nu := \{0\} \cup \bigcup_{\mu \geq \nu} \{c_\mu\}.$$

Then $C'_\nu := \left(\overline{\partial U \setminus 0 \times \widehat{X}} \right) \cap G^{-1}_{|U}(C_\nu)$ is a decreasing sequence of compact subsets in $[0, +\infty) \times \widehat{X}$. They are such that

$$\bigcap_\nu C'_\nu = \left(\overline{\partial U \setminus 0 \times \widehat{X}} \right) \cap \bigcap_\nu C_\nu = \left(\overline{\partial U \setminus 0 \times \widehat{X}} \right) \cap G^{-1}(0) = \emptyset.$$

Therefore there exists $\overline{\nu}$ such that $C'_{\overline{\nu}} = \emptyset$. This implies that

$$\begin{aligned} G^{-1}_{|U}(c_{\overline{\nu}}) \cap \partial U &= F^{-1}_{|U}(c_{\overline{\nu}}) \cap \left(\overline{\partial U \setminus 0 \times \widehat{X}} \right) \ \cup \ G^{-1}_{|U}(c_{\overline{\nu}}) \cap 0 \times \widehat{X} \\ &= \emptyset \cup \{q_\nu\} \\ &= \{q_\nu\}. \end{aligned}$$

By the Implicit Function Theorem 6.2.3 we find that $G^{-1}_{|U}(c_{\overline{\nu}})$ is a compact manifold of dimension 1, whose boundary has a single element $q_{\overline{\nu}}$. However every compact manifold of dimension 1 is homeomorphic to a disjoint union of closed segments and circles, hence the cardinality of its boundary must be even. This contradiction proves the theorem. □

6.5 Conclusion

We begin with the fundamental proposition.

Proposition 6.5.1. *Let (r_ν, w_ν) a sequence of elements in $\mathcal{M}$ such that*

$$(r_\nu, w_\nu) \xrightarrow{C^\infty_{\text{loc}}} (r, w).$$

The following alternative holds

1. *there exists δ_0, a subsequence $(r_{\nu'}, w_{\nu'})$ and a sequence of time $s_{\nu'}$ positively or negatively diverging such that*

$$|\mathbb{A}_0(w_{\nu'}(s_{\nu'}))| \geq \delta_0.$$

In this case w is a bounded r-Solution, with

$$\delta_0 \leq \max\{|\mathbb{A}_0(w_-)|, |\mathbb{A}_0(w_+)|\} \leq \|F\|. \tag{6.12}$$

2. for every $\delta > 0$, there exists $\overline{s}_\delta \geq \frac{\|F\|}{2\mu^2} + 1$, such that

$$\max\{\mathbb{A}_0(w_\nu(-\overline{s}_\delta)), \mathbb{A}_0(w_\nu(\overline{s}_\delta))\} \leq \delta, \qquad \forall \nu \in \mathbb{N}. \tag{6.13}$$

In this case

$$\bullet\ (r,w) \in \mathcal{M}, \quad \bullet \lim_{\nu \to +\infty} w_\nu(+\infty) = w(+\infty), \quad \bullet\ (r_\nu, w_\nu) \xrightarrow{\tau_p} (r,w)$$

(where the last convergence is in the space $[0,+\infty) \times \widehat{X}$).

Proof. Let us deal with the first case. By the C^∞_{loc}-convergence we know that w is a bounded r-Solution. Fix $s \in \mathbb{R}$, with $|s| \geq \frac{\|F\|}{2\mu^2} + 1$. Then Proposition 3.4.2 implies that

$$|\mathbb{A}_0(w_\nu(s))| \leq \|F\|.$$

Passing to the limit in ν we get

$$|\mathbb{A}_0(w(s))| \leq \|F\|.$$

Letting $|s|$ go to infinity we have the right inequality in 6.12.

Suppose now without loss of generality that $s_{\nu'} \to +\infty$. If $s \geq \frac{\|F\|}{2\mu^2} + 1$, we have $s \leq s_{\nu'}$ for ν' bigger than some natural number $\nu'(s)$. Then, since $\mathbb{A}_0(w)$ is decreasing on the ray $[s, +\infty)$, we get

$$\mathbb{A}_0(w(s)) = \lim_{\nu' \to +\infty} \mathbb{A}_0(w_{\nu'}(s)) \geq \limsup_{\nu' \to +\infty} \mathbb{A}_0(w_{\nu'}(s_{\nu'})) \geq \delta_0.$$

Letting s go to $+\infty$, we get the left inequality.

Examine now the second case. Set $w = (u, T)$. Remembering the discussion in the first section of this chapter, we know that the elements of $\mathcal{M}$ can be written as quadruple $(r_\nu, w_\nu) = (r_\nu, u_\nu(+\infty), u_\nu, T_\nu)$. Theorem 5.1.2 and the hypothesis of this case yield a $\delta_1 > 0$ and an $s_{\delta_1} \geq \frac{\|F\|}{2\mu^2} + 1$ that does not depend on ν, such that we have

$$\max\{-\mathbb{A}_0(w_\nu(-s_{\delta_1})), \mathbb{A}_0(w_\nu(s_{\delta_1})\} \leq \delta,$$

and

$$\max\left\{|u_\nu|, |T_\nu|, |\partial_s u_\nu|, |\partial_t u_\nu|, \left|\frac{dT_\nu}{ds}\right|\right\} \leq C e^{\frac{a}{2}(s_{\delta_1} - |s|)}, \quad \text{for } \pm s \geq s_{\delta_1}.$$

We point out that in the terms involving u_ν, in the preceding inequality we can substitute the norm induced by the local charts $U_\pm$ (as it was required by Theorem 5.1.2) with the Euclidean norm on $M = \mathbb{R}^{2n}$ and on the fiber of $TM \equiv \mathbb{R}^{2n} \times \mathbb{R}^{2n}$. The price we pay is a constant factor that can be absorbed in C.

The second and third point of the statement we aim to prove are equivalent to showing that any subsequence of (r_ν, w_ν) has a further subsequence such that the two limits hold. We suppose from now on to have fixed a subsequence of (r_ν, w_ν) and we shall denote this new sequence by the same subscripts ν. Then, since Σ is compact, after passing to a subsequence ν' we can suppose that $u_{\nu'}(+\infty) \to z$. This means that $(u_{\nu'}, T_{\nu'})$ converges to $(u-\sigma z, T)$ with respect to the C^∞_{loc}-topology Indeed, we know by assumption that $u_{\nu'} + \sigma u_{\nu'}(+\infty)$ is convergent in the C^∞_{loc}-topology to u.

On the other hand we claim that $(u_{\nu'}(+\infty), u_{\nu'}, T_{\nu'})$ is convergent to $(z, u-\sigma z, T)$ also in the topology of $\widehat{X}$ and thus $(z, u-\sigma z, T) \in \widehat{X}$. Indeed, we know already that $u_{\nu'}(+\infty) \to z$. Then, $(u_{\nu'}, T_{\nu'})$ is a Cauchy sequence in X. In order to show this we have to study the behavior of the norms

$$\|(u_{\nu'} - u_{\nu''}, T_{\nu'} - T_{\nu''})\|_X, \qquad \nu', \nu'' \text{ sufficiently large.}$$

Compute for example:

$$\|u_{\nu'} - u_{\nu''}\|^p_{1,p} = \int_{\mathbb{T}\times\mathbb{R}} \Big(|u_{\nu'} - u_{\nu''}|^p + |\partial_t (u_{\nu'} - u_{\nu''})|^p + |\partial_s (u_{\nu'} - u_{\nu''})|^p \Big) e^{b_p |s|} dtds.$$

Let us deal with the first summand only:

$$\Delta_0(\nu', \nu'') := \int_{\mathbb{T}\times\mathbb{R}} |u_{\nu'} - u_{\nu''}|^p e^{b_p|s|} dtds.$$

If $\widehat{s} \geq s_{\delta_1}$, then

$$\begin{aligned}
\Delta_0(\nu', \nu'') &= \int_{\mathbb{T}\times\mathbb{R}} |u_{\nu'} - u_{\nu''}|^p e^{b_p|s|} dtds \\
&= \int_{\mathbb{T}\times\{|s|\leq \widehat{s}\}} |u_{\nu'} - u_{\nu''}|^p e^{b_p|s|} dtds \;+\; \int_{\mathbb{T}\times\{|s|\geq \widehat{s}\}} |u_{\nu'} - u_{\nu''}|^p e^{b_p|s|} dtds \\
&\leq 2\widehat{s} e^{b_p \widehat{s}} \sup_{\mathbb{T}\times\{|s|\leq\widehat{s}\}} |u_{\nu'} - u_{\nu''}|^p \;+\; 2^p \int_{\mathbb{T}\times\{|s|\geq\widehat{s}\}} \Big(|u_{\nu'}|^p + |u_{\nu''}|^p \Big) e^{b_p|s|} dtds \\
&\leq 2\widehat{s} e^{b_p \widehat{s}} \sup_{\mathbb{T}\times\{|s|\leq\widehat{s}\}} |u_{\nu'} - u_{\nu''}|^p \;+\; 2^{p+1} C^p e^{\frac{ap}{2} s_{\delta_1}} \int_{\mathbb{T}\times\{|s|\geq\widehat{s}\}} e^{(b_p - \frac{ap}{2})|s|} dtds \\
&\leq 2\widehat{s} e^{b_p \widehat{s}} \sup_{\mathbb{T}\times\{|s|\leq\widehat{s}\}} |u_{\nu'} - u_{\nu''}|^p \;+\; \frac{2^{p+4}}{ap} C^p e^{\frac{ap}{2} s_{\delta_1}} e^{-\frac{ap}{4}\widehat{s}}.
\end{aligned}$$

For every $\varepsilon > 0$ we can choose $\widehat{s}$ sufficiently large in order to make the latter summand smaller than $\frac{\varepsilon}{2}$. Then we exploit the C^∞-convergence on the compact set $\mathbb{T} \times \{|s| \leq \widehat{s}\}$ and find a ν_ε such that if $\nu', \nu'' \geq \nu_\varepsilon$ then also the former summand is smaller than $\frac{\varepsilon}{2}$. Arguing in a similar manner for all the other terms we find that $(u_{\nu'}, T_{\nu'})$ is a Cauchy sequence in X. Since

$$(u_{\nu'}, T_{\nu'}) \xrightarrow{C^\infty_{\text{loc}}} (u - \sigma z, T)$$

we deduce

$$\bullet\ (u - \sigma z, T) \in X, \qquad \bullet\ (u_{\nu'}, T_{\nu'}) \xrightarrow{\ X\ } (u - \sigma z, T).$$

From this we find that

- $w = (z, u - \sigma z, T) \in \widehat{X}$,
- $w(+\infty) = (z, 0) = \lim\limits_{\nu' \to +\infty} (u_{\nu'}(+\infty), T_{\nu'}(+\infty))$,
- $(r_{\nu'}, w_{\nu'}) \xrightarrow{\tau_p} (r, w)$.

The proposition is thus proven. □

Before proving the Main Theorem we need a last result, which holds under general assumptions.

Proposition 6.5.2. *Let Σ a compact closed manifold and $W \in \Gamma(T\Sigma)$, a vector field without zeros. Then there exists $\eta_0 \in (0, +\infty)$ such that every periodic orbit has period bigger than η_0. As a consequence the set*

$$\mathcal{P}_W := \{\eta > 0 \mid \textit{there exists a closed orbit of period } \eta\}$$

is closed.

Proof. Consider a sequence of orbits $\gamma_\nu : \mathbb{R}/\eta_\nu\mathbb{Z} \to \Sigma$ with period η_ν and $\eta_\nu \to \eta$. Definining the reparametrized curves $\gamma_\nu^*(t) := \gamma\left(\frac{t}{\eta_\nu}\right)$ we get a sequence of 1-periodic functions $\gamma_\nu^* : \mathbb{T} \to \Sigma$ from the standard torus in Σ. They satisfy

$$\dot{\gamma}_\nu^* = \eta_\nu W(\gamma_\nu^*). \tag{6.14}$$

Since Σ is compact and the sequence γ_ν^* is equicontinuous, by the *Arzelà-Ascoli Theorem*, after extracting a subsequence we can suppose $\gamma_\nu^* \to \gamma^*$. Using (6.14), we find that also $\dot{\gamma}_\nu^*$ converges uniformly. Then, γ^* is differentiable and $\dot{\gamma}_\nu^* \to \dot{\gamma}^*$. Passing to the limit in Equation 6.14, we get

$$\dot{\gamma}^* = \eta W(\gamma^*). \tag{6.15}$$

Thus $\eta \in \mathcal{P}_W$, provided $\eta \neq 0$. In order to show this last statement argue by contradiction and assume $\eta = 0$. This implies that $\gamma^* \equiv z$, for some $z \in \Sigma$ and hence $\dot{\gamma}^* \equiv 0$. By hypothesis $W(z) \neq 0$ and therefore there exists a coordinate neighborhood of z, U_z, such that W is a coordinate vector field in U_z. This implies that all the flow lines of W, and hence of $\eta_\nu W$, are open on U_z. This is a contradiction since, when ν is large enough, γ_ν^* is contained in U_z and it is a closed trajectory of the vector field $\eta_\nu W$. □

Now we can easily get the proof of the main theorem.

Proof of Theorem 2.4.1. We know by Proposition 6.4.4, that $\mathcal{M}$ with the τ_p-topology is non-compact. Hence there exists a sequence $(r_\nu, w_\nu) \in \mathcal{M}$ without τ_p-convergent subsequences. Combining this fact with Corollary 4.7.3 we see that there exist a subsequence $(r_{\nu'}, w_{\nu'})$, an $r \in \mathbb{R}$ and a bounded r-Solution w, such that

$$(r_{\nu'}, w_{\nu'}) \xrightarrow{C^\infty_{\text{loc}}} (r, w)$$

and $(r_{\nu'}, w_{\nu'})$ is not convergent in the τ_p-topology. Then we see that for $(r_{\nu'}, w_{\nu'})$ the second alternative in Proposition 6.5.1 cannot occur. As a result the first alternative tells us that one among $|\mathbb{A}_0(w_-)|$ and $|\mathbb{A}_0(w_+)|$ is different from zero and smaller than $\|F\|$. Proposition 4.7.5 yields $(u, T) \in \operatorname{Crit} \mathbb{A}_0$, whose action is non-zero and with modulus smaller than $\|F\|$. The Action-Period Equality in 3.2.2 implies that (u, T) is a Reeb orbit with period smaller than $\|F\|$.

Consider the set

$$\mathcal{P} := \{T > 0 \mid \text{there exists a Reeb orbit on } \Sigma \text{ with period } T\}.$$

By Proposition 6.5.2, the very definition of displacement energy (2.5) and the discussion just made, we know that

$$\min \mathcal{P} = \inf \mathcal{P} \leq \inf\{\|F\| \mid \Phi_F \text{ displaces } \Sigma\ \} = e_{d\lambda}(\Sigma). \tag{6.16}$$

The Reeb orbit corresponding to the minimum of $\mathcal{P}$, satisfies the requirements of the Main Theorem. This concludes the proof and hence our exposition □

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